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HIGHER-ORDER COMPACT EMBEDDINGS OF FUNCTION SPACES ON CARNOT–CARATHÉODORY SPACES

MARTIN FRANCŮ

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Abstract. A sufficient condition for higher-order compact embeddings on bounded domains in Carnot–Carathéodory spaces is established for the class of rearrangement-invariant function spaces. The condition is expressed in terms of compactness of a suitable 1-dimensional integral operator depending on the isoperimetric function relative to the Carnot–Carathéodory structure of the relevant sets. The general result is then applied to particular Sobolev spaces built upon Lebesgue and Lorentz spaces.

1. Introduction

One of the most important characteristics of Sobolev spaces is how they relate to other spaces. This sort of information is usually expressed in terms of (continuous) embeddings, and compact embeddings. Compact embeddings are of particular interest from the point of view of applications of Sobolev spaces in mathematical physics, calculus of variations, economical sciences, and probability theory. A compact embedding can be used to point toward a solution to a given partial differential equation or to show the discreteness of the spectra of linear elliptic partial differential operators defined over bounded domains.

One of the first classical compactness results originated in a lemma by Rellich [33] and was later proved specifically for Sobolev spaces by Kondrachov [24]. These results of course found their way to classical textbooks, such as the one by Adams [1]. Ever since then, compact embeddings of Sobolev spaces have been

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the subject of extensive research as a very important topic in functional analysis. Obtained results extend far beyond the original context of underlining measurable space $\mathbb{R}^n$ to various classes or domains in different measurable spaces.

It has been understood that the quality of an embedding of a Sobolev space into another appropriate space is closely connected to the isoperimetric profile of the underlying domain, and even that Sobolev embeddings can be derived from isoperimetric inequalities. The problem of Sobolev inequalities and function spaces embeddings can thus be approached through isoperimetric inequality. This deep connection was observed in the early 1960s by Maz'ya [26] and [27] and also by Federer and Fleming [13].

The approach to Sobolev spaces via isoperimetric inequalities allows one to consider Sobolev embeddings from a much wider perspective than that of the classical Euclidean setting. Examples of important non-Euclidean embeddings include, for instance, the Gaussian–Sobolev embeddings studied in the connection with the so-called *logarithmic Sobolev inequalities* (see e.g. [16], [2]), a central subject in the investigation of hypercontractive semigroups. On the other hand, investigation of Sobolev embeddings has been carried out on Carnot–Carathéodory spaces, where Sobolev spaces are built with respect to a different differential operator, and whose pivotal example is the Heisenberg chain. While the Sobolev embeddings on Carnot–Carathéodory spaces have been studied to some extent (see, e.g., [6], [8]–[12], [15], [17], [18], [20], [28], [25], [31]), very little is known about *compactness* of such embeddings. In this paper we concentrate on this problem.

The isoperimetric approach was successfully applied in the context of Carnot–Carathéodory spaces to the problem of establishing higher-order Sobolev-type embeddings in [14]. Our main aim here is to determine when a Sobolev embedding on a Carnot–Carathéodory space is compact. We intend to work under the quite general setting of rearrangement-invariant spaces.

The Carnot–Carathéodory spaces (also known as *sub-Riemannian spaces*) possess an exciting range of applications ranging from quantum mechanics (where we can also find the origin of the most famous example, the Heisenberg group), through the control theory to exotic applications such as automatic animation of physically plausible trajectories via computer graphics for passenger vehicles [23]. The present article continues in this trend by applying the readily prepared tools in [7], [14], and [15] to adapting state-of-art proofs from [35] to Carnot–Carathéodory spaces settings. One of the main advantages of the isoperimetric approach to embeddings of Sobolev spaces is the possibility of extending the embeddings to the classes of rearrangement-invariant function spaces and higher-order embeddings. Moreover, it allows us to reduce sufficient condition on embeddings over Carnot–Carathéodory spaces to condition on certain 1-dimensional operators over $\mathbb{R}$.

This article is structured as follows. In Section 2 we collect necessary background material. In particular, we fix all the indispensable basic notions concerning Carnot–Carathéodory spaces, the rearrangement spaces, and the isoperimetric inequalities (as our approach is built on the combination of these three topics).

In Section 3 we state the main theorems. In Section 4, for the readers' convenience, the author collects known results that will be used in the proof. In the final section we present the proofs of the main results.

2. Settings

Let X be a system of vector fields $X_1, \dots, X_m$ such that

$$X_i = \sum_{j=1}^{n} b_{i,j}(x) \frac{\partial}{\partial x_j},$$

where $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and $b_{i,j} \colon \mathbb{R}^n \to \mathbb{R}$, $b_i j \in \mathcal{C}^\infty(\mathbb{R}^n)$ (with respect to the classical Euclidean topology). The simplest choice of $\{X_j = \frac{\partial}{\partial x_j}, j = 1, \dots, n\}$ would yield the classical Euclidean case. A piecewise $\mathcal{C}^1$-curve $\gamma \colon [0,T] \to \mathbb{R}^n$, $T > 0$, is called *horizontal* if whenever $\gamma'(t)$ exists, then

$$\gamma'(t) = \sum_{j=1}^{m} c_j(t) X_j\bigl(\gamma(t)\bigr),$$

where $c_j \colon (0,T) \to \mathbb{R}$ are measurable and satisfying $\sum_{j=1}^m c_j^2(t) \le 1$ for $0 \le t \le T$. The horizontal length of γ is defined by $l_h(\gamma) = T$. Let us denote by $\mathcal{H}$ the family of all horizontal curves. The distance function corresponding to X is defined by

$$d(x,y) = \inf\bigl\{l_h(\gamma) \colon \gamma \in \mathcal{H}, \gamma(0) = x, \gamma\bigl(l_h(\gamma)\bigr) = y\bigr\}, \quad x, y \in \mathbb{R}^n.$$

If $d(x,y)$ is a metric, then $\mathbb{R}^n$ equipped with $d(x,y)$ as metric is considered the Carnot–Carathéodory space, generated by the system X.

Throughout this paper we assume that the distance function is a metric, especially that $d(x,y) < \infty$ for all $x, y \in \mathbb{R}^n$, and that the topology generated by it is the same as the classical Euclidean topology. It is known that this is ensured if the system X enjoys the so-called *Hörmander finite-rank condition.* We also assume throughout this article that $\Omega \subset \mathbb{R}^n$ is open with $|\Omega| < \infty$, where $|\cdot|$ denotes the n-dimensional Lebesgue measure.

For a function $f \in L^1_{\mathrm{loc}}(\Omega)$, its distributional derivative along the vector field X_j, $X_j f$, is defined by the identity

$$\langle X_j f, \phi \rangle = \int_\Omega f X_j^* \phi \, dx \quad \text{for every } \phi \in \mathcal{C}_0^\infty(\Omega),$$

where $X_j^*(\cdot) = -\sum_{k=1}^n \frac{\partial}{\partial x_k}(b_{j,k} \, \cdot)$ denotes the formal adjoint of X_j. Throughout this article, if f is a nonsmooth function, $X_j f$ will be meant in the distributional sense. If derivatives $X_1 f, X_2 f, \dots, X_m f$ exist, then the vector of X-*gradient* of a function f is defined by

$$X\nabla f = (X_1 f, X_2 f, \dots, X_m f).$$

Moreover, let us introduce the *higher-order derivatives* as

$$XD_\alpha(\cdot) = X_{\alpha_1}\bigl(X_{\alpha_2}\bigl(\dots X_{\alpha_k}(\cdot)\dots\bigr)\bigr),$$

where $\alpha = (\alpha_1, \ldots, \alpha_k) \in \{1, \ldots, m\}^k$. Provided that $XD_\alpha f$ exists for all $\alpha \in \{1, \ldots, m\}^k$, the X*-gradient of order* k is defined as a vector of length m^k of the following form:

$$X\nabla^k f = \big(XD_\alpha(f) \colon \alpha \in \{1, \ldots, m\}^k\big).$$

Naturally, the norm of the X-gradient of order k reads as

$$|X\nabla^k f|^2 = \sum_{\alpha \in m^k} \big(XD_\alpha(f)\big)^2.$$

The X-variation and the X-perimeter can be defined as follows: if we denote

$$\mathcal{F}_\Omega = \Big\{\phi = \{\phi_1, \phi_2, \ldots, \phi_m\} \in \mathcal{C}_0^1(\Omega \to \mathbb{R}^m) \colon \sup_{x \in \Omega}\Big(\sum_{j=1}^m \big|\phi_j(x)\big|^2\Big)^{\frac{1}{2}} \le 1\Big\},$$

then, for a given $u \in L^1_{\mathrm{loc}}(\Omega)$, the X*-variation* of u with respect to Ω is defined as

$$\mathrm{Var}_X(u, \Omega) = \sup_{\phi \in \mathcal{F}_\Omega} \int_\Omega u(x) \sum_{j=1}^m X_j^* \phi_j(x)\, dx.$$

The set of functions with bounded X-variation is denoted as $BV_X(\Omega)$ and forms a Banach space with respect to the norm

$$\|\cdot\|_{BV_X} = \|\cdot\|_{L_1(\Omega)} + \mathrm{Var}_X(\cdot, \Omega).$$

If $X\nabla f \in L^1(\Omega)$, then

$$\mathrm{Var}_X(f, \Omega) \le \hat{C}\|X\nabla f\|_{L^1}, \tag{2.1}$$

where $\hat{C} > 0$ depends only on m.

If $E \subset \mathbb{R}^n$ is measurable, then the X*-perimeter* of E relative to Ω is defined by

$$P_X(E, \Omega) = \mathrm{Var}_X(\chi_E, \Omega),$$

where χ_E denotes the characteristic function of E. The X*-isoperimetric function* of Ω is given by the formula

$$I_{X,\Omega}(s) = \inf\Big\{P_X(E, \Omega) \colon E \subset \Omega, s \le |E| \le \frac{1}{2}\Big\} \quad \text{for } s \in \Big[0, \frac{1}{2}\Big],$$

and $I_{X,\Omega}(s) = I_{X,\Omega}(1-s)$ if $s \in (\frac{1}{2}, 1]$.

Throughout this paper we will assume a certain regularity of $I_{X,\Omega}(s)$, namely: suppose that there is some nondecreasing function $I \colon [0, 1] \to \mathbb{R}$ satisfying

$$I_{X,\Omega}(s) \ge cI(cs) \quad \text{for } s \in \Big[0, \frac{1}{2}\Big] \tag{2.2}$$

with some constant $c > 0$, and

$$\inf_{t \in (0,1)} \frac{I(t)}{t} > 0. \tag{2.3}$$

In this generality, the isoperimetric function is usually unknown. However, in [15] it was shown that the isoperimetric function can be evaluated if some additional conditions hold.

The first such condition is the following version of the doubling condition: for any set $U \subset \mathbb{R}^n$ with $\operatorname{diam}(U) < \infty$, there exist constants $C_1 > 0$ and $R_0 < \infty$ such that, for $x_0 \in U$ and $0 < R < R_0$, one has

$$\big|B(x_0, 2R)\big| \geq C_1\big|B(x_0, R)\big|. \tag{2.4}$$

It was shown in [30] that the finite-rank Hörmander condition implies the doubling condition.

The second restriction is the following version of the Poincaré inequality: for any set $U \subset \mathbb{R}^n$ with $\operatorname{diam}(U) < \infty$, there exist constants $C_2 > 0$, $R_0 < \infty$, and $\alpha \geq 1$ such that, for $x_0 \in U$, $0 < R < R_0$, and every Lipschitz function u in $\alpha B = B(x_0, \alpha R)$, we have, for any $\lambda > 0$,

$$\left|\left\{x \in B \colon \left|u(x) - \int_B u(x)\,dx\right| > \lambda\right\}\right| \leq \frac{C_2}{\lambda}\int_{\alpha B}\big|X\nabla u(y)\big|\,dy. \tag{2.5}$$

The third restriction is that $(\mathbb{R}^n, d)$ is complete and it is a length-space; that is

$$d(x, y) = \inf l(\gamma_{xy}), \tag{2.6}$$

where γ is a continuous curve joining x to y and where $l(\gamma_{xy})$ denotes its metric length.

Let $U \subset \mathbb{R}^n$, and denote by C the smallest constant in (2.4). Then the homogeneous dimension relative to U (and X) is defined by

$$Q = \log_2(C).$$

Let us recall definitions of John domains and X-PS domains in the following two paragraphs.

An open set $\Omega \subset \mathbb{R}^n$ is called an *X-John domain* if there exist a constant $c \in (0, 1)$ and a point $x_0 \in \Omega$ such that for every $x \in \Omega$ there exists a rectifiable curve $\omega \colon [0, l] \to \Omega$, parameterized by arc-length, such that $\omega(0) = x$, $\omega(l) = x_0$, and

$$d\big(\omega(r), \partial\Omega\big) \geq cr \quad \text{for } r \in [0, l],$$

where $\partial\Omega$ denotes boundary of Ω.

An open set $\Omega \subset \mathbb{R}^n$ is called *X-PS domain* if there exist a covering $\{B\}_{B\in\mathcal{F}}$ of Ω by metric balls and numbers $N > 0$, $\alpha \geq 1$, and $\nu \geq 1$ such that the following hold.

(1) $\sum_{B\in\mathcal{F}} \chi_{(\alpha+1)B}(x) \leq N\chi_\Omega(x)$ for every $x \in \Omega$.
(2) There exists a (central) ball $B_0 \in \mathcal{F}$ such that, for any $B \in \mathcal{F}$, one can find a chain $B_0, B_1, \ldots, B_{s(B)} = B$, with $B_i \cap B_{i+1} \neq \emptyset$ and $|B_i \cap B_{i+1}| \geq \frac{1}{N}\max(|B_i|, |B_{i+1}|)$.
(3) For any $i = 0, \ldots, s(B)$, one has $B \subset \nu B_i$.

Though the class of John domains is better known in the context of Sobolev-type embeddings, we will state our results by means of the notion of X-PS domains. The class of X-PS domains contains that of X-John domains if (2.4) holds, which is always assumed in the present article. On the other hand, if a certain geodesic segment property is satisfied, then the class of X-John domains contains the class of X-PS domains. Both inclusions are shown in [15, Theorem

1.30]. Consequently, both classes coincide if both (2.4) and the geodesic segment property are satisfied.

If in addition the metric space $(\mathbb{R}^n, d)$ is complete and a length-space, then it is shown in [15] that metric balls with small diameter are X-PS domains. The class of X-PS domains is larger than the class of nontangentially accessible domains (introduced in [19]) and the class of extension domains (introduced in [22]). In [29] it is shown that if X is generated by structure of a step two homogeneous group, then any $\mathcal{C}^{1,1}$ domain is an X-NTA domain and consequently an X-PS domain. (More examples of X-NTA and therefore X-PS domains can be found in [4].) However, the task of finding X-PS domains in a general setting is rather nontrivial.

In [15, Theorem 1.18] it was shown that if $\Omega \subset \mathbb{R}^n$ is an X-PS domain (and conditions (2.4) and (2.5) hold), then

$$I_{X,\Omega}(s) \leq \begin{cases} \frac{C}{\operatorname{diam}(\Omega)|\Omega|^{-\frac{1}{Q}}} s^{\frac{Q-1}{Q}}, & \text{for } s \in [0, \frac{1}{2}], \\ \frac{C}{\operatorname{diam}(\Omega)|\Omega|^{-\frac{1}{Q}}} (1-s)^{\frac{Q-1}{Q}}, & \text{for } s \in]\frac{1}{2}, 1], \end{cases} \tag{2.7}$$

where Q is the homogeneous dimension relative to Ω.

Now we turn our attention to the rearrangement-invariant function spaces (the basic references and more recent ones for readers interested in more details are [3], [5], [21], [32], [36]). We first recall the nonincreasing rearrangement and distribution function. Let $u \in \mathfrak{M}(\Omega)$; then

$$\mu_u(t) = \left|\left\{x \in \Omega \colon \left|u(x)\right| > t\right\}\right|, \quad t \in [0, \infty),$$

is the *distribution function* of u. Let (R, λ) and (S, μ) be two measurable spaces. Functions $u \in \mathfrak{M}(R, \lambda)$ and $v \in \mathfrak{M}(S, \mu)$ are called *equimeasurable* if $\mu_u = \mu_v$ (on $\mathbb{R}^+$). In that case, we write $u \sim v$.

The *nonincreasing rearrangement* of function $u \in \mathfrak{M}(R, \lambda)$ is then defined as

$$u^*(t) = \inf\left\{s \geq 0 \colon \mu_u(s) \leq t\right\}, \quad t \in [0, \infty).$$

A mapping $\varrho \colon \mathfrak{M}_+(R, \lambda) \to [0, \infty]$ is called a *Banach function norm* if, for all f, g, and $\{f_n\}_{n\in\mathbb{N}}$ in $\mathfrak{M}_+(R, \lambda)$, every $a \geq 0$, and for all Lebesgue measurable $E \subset \Omega$, the following properties hold.

(1) $\varrho(f) = 0$ if and only if $f = 0$ almost everywhere. Moreover, $\varrho(af) = a\varrho(f)$ and $\varrho(f + g) \leq \varrho(f) + \varrho(g)$.
(2) If $0 \leq g \leq f$ almost everywhere, then $\varrho(g) \leq \varrho(f)$.
(3) If $0 \leq f_n \uparrow f$ almost everywhere, then $\varrho(f_n) \uparrow \varrho(f)$.
(4) If $|E| < \infty$, then $\varrho(\chi_E) < \infty$.
(5) If $|E| < \infty$, then $\int_E f \, d\lambda \leq C_E \varrho(f)$, for some constant C_E, $0 < C_E < \infty$, depending on E and ϱ but independent of f.

If, in addition, ϱ satisfies $\varrho(f) = \varrho(g)$ for every pair of equimeasurable functions f and g in $\mathfrak{M}_+(R, \lambda)$, then ϱ is called a *rearrangement-invariant Banach function norm*.

The collection $\mathbf{X}(R, \mu) = \mathbf{X}_\varrho(R, \mu)$ of all functions $f \in \mathfrak{M}(R, \lambda)$ for which $\varrho(|f|) < \infty$ is called a *rearrangement-invariant Banach function space* (*r.i. space*).

For each $f \in \mathbf{X}(R,\mu)$, define
$$\|f\|_{\mathbf{X}}(R,\mu) = \varrho\big(|f|\big).$$
Let us recall that there is functional ϱ' defined on $\mathfrak{M}_+(R,\lambda)$ by
$$\varrho'(g) = \sup\Big\{\int_R fg\colon f \in \mathfrak{M}_+(R,\lambda), \varrho(f) \le 1\Big\}, \quad g \in \mathfrak{M}_+(R,\lambda),$$
associated with r.i. Banach function norm ϱ. It turns out that ϱ' is an r.i. Banach function norm, called the *associate function norm* of ϱ. We note that if $|\Omega| < \infty$, then for any r.i. space $\mathbf{X}(\Omega)$ there exists a representation rearrangement-invariant Banach function norm
$$\varrho_{\mathbf{X}(\Omega)}\colon \mathfrak{M}_+(0,1) \to [0,\infty]$$
such that
$$\|f\|_{\mathbf{X}(\Omega)} = \varrho_{\mathbf{X}}\big(f^*\big(|\Omega|\cdot\big)\big), \quad f \in \mathbf{X}(\Omega).$$
This allows us to work sometimes with function spaces over simple measurable space $(0,1)$, instead of with function spaces over Ω.

Let us now give some examples of r.i. norms. A basic example are the Lebesgue norms $L^p(0,1)$, $p \in [1,\infty]$, defined for all $f \in \mathfrak{M}(0,1)$ by
$$\|f\|_{L^p(0,1)} = \begin{cases} (\int_0^1 |f(x)|^p\,dx)^{\frac{1}{p}}, & p < \infty, \\ \operatorname{esssup}_{x\in(0,1)} |f(x)|, & p = \infty. \end{cases}$$
The corresponding r.i. spaces $l^p(R,\mu)$ are then called *Lebesgue spaces.*

One can consider also more general functionals $\|\cdot\|_{L^{p,q}(0,1)}$ and $\|\cdot\|_{L^{p,q;\alpha}(0,1)}$. They are given for any $f \in \mathfrak{M}(0,1)$ by
$$\|f\|_{L^{p,q}(0,1)} = \Big\|f^*(s)s^{\frac{1}{p}-\frac{1}{q}}\Big\|_{L^q(0,1)}$$
and
$$\|f\|_{L^{p,q,\alpha}(0,1)} = \bigg\|f^*(s)s^{\frac{1}{p}-\frac{1}{q}}\Big(\log\frac{2}{s}\Big)^{\alpha}\bigg\|_{L^q(0,1)},$$
respectively. Here, we assume that $p \in [1,\infty]$, $\alpha \in \mathbb{R}$, and we use the convention that $\frac{1}{\infty} = 0$. Note that $\|\cdot\|_{L^p(0,1)} = \|\cdot\|_{L^{p,p}(0,1)}$ and $\|\cdot\|_{L^{p,q}(0,1)} = \|\cdot\|_{L^{p,q;0}(0,1)}$ for every such p and q. However, it turns out that under these assumptions on p, q, and α, $\|\cdot\|_{L^{p,q}(0,1)}$ and $\|\cdot\|_{L^{p,q;\alpha}(0,1)}$ do not have to be r.i. norms. To ensure that $\|\cdot\|_{L^{p,q;\alpha}(0,1)}$ is equivalent to an r.i. norm, we need to assume that one of the following conditions is satisfied:
$$p = q = 1, \qquad \alpha \ge 0, \tag{2.8}$$
$$1 < p < \infty, \tag{2.9}$$
$$p = \infty, \qquad q < \infty, \qquad \alpha + \frac{1}{q} < 0, \tag{2.10}$$
$$p = q = \infty, \qquad \alpha \le 0. \tag{2.11}$$
In this case, $\|\cdot\|_{L^{p,q}(0,1)}$ is called a *Lorentz norm*, $\|\cdot\|_{L^{p,q;\alpha}(0,1)}$ is called a *Lorentz–Zygmund norm*, and the corresponding r.i. spaces $L^{p,q}(0,1)$ and $L^{p,q;\alpha}(0,1)$ are called *Lorentz spaces* and *Lorentz-Zygmund spaces*, respectively.

Let $m \in \mathbb{N}$, and let $\mathbf{X}(\Omega)$ be an r.i. space. We define the mth order Sobolev space $V_X^m\mathbf{X}(\Omega)$ as the set of all functions $f \in \mathfrak{M}(\Omega)$ such that $X\nabla^m f$ exists in a distributional sense, and it is represented by locally integrable functions such that $|X\nabla^m f| \in \mathbf{X}(\Omega)$.

In [14] it is proved that if

$$I_{\Omega,X}(s) \geq \bar{C}s \quad \text{for } s \in \left[0, \frac{1}{2}\right] \tag{2.12}$$

with some constant $\bar{C} > 0$, then

$$V_X^m\mathbf{X}(\Omega) \subset V_X^k L^1(\Omega) \quad \text{for } k < m. \tag{2.13}$$

Provided that (2.12) holds, the $V_X^m\mathbf{X}(\Omega)$ forms a normed linear space with respect to the norm

$$\|u\|_{V_X^m\mathbf{X}(\Omega)} = \sum_{k=0}^{m-1} \|X\nabla^k u\|_{L^1(\Omega)} + \|X\nabla^m u\|_{\mathbf{X}(\Omega)}.$$

Moreover, by $W_X^m\mathbf{X}(\Omega)$ we denote the set of all functions $f \in (\Omega)$ such that, for all $k = 0, 1, \ldots, m$, $X\nabla^k f$ exists in a distributional sense and it is represented by locally integrable functions such that $|X\nabla^k f| \in \mathbf{X}(\Omega)$. $W_X^m(\Omega)$ forms a normed linear space with respect to norm

$$\|u\|_{W_X^m\mathbf{X}(\Omega)} = \sum_{k=0}^{m} \|X\nabla^k u\|_{\mathbf{X}(\Omega)}.$$

Let (R, μ) be a measurable space. Given two function spaces $\mathbf{X}(R, \mu)$ and $\mathbf{Y}(R, \mu)$ (not necessarily rearrangement-invariant), the notation

$$\mathbf{X}(R, \mu) \to \mathbf{Y}(R, \mu)$$

represents the fact that there exists a constant C independent of $f \in \mathbf{X}(R, \mu)$ such that

$$\|f\|_{\mathbf{Y}(R,\mu)} \leq C\|f\|_{\mathbf{X}(R,\mu)}.$$

In such a case we say that $\mathbf{X}(R, \mu)$ is *embedded* into $\mathbf{Y}(R, \mu)$. By saying that $\mathbf{Y}(R, \mu)$ is the *optimal target* in $\mathbf{X}(R, \mu) \to \mathbf{Y}(R, \mu)$, we mean that, for any function space $\mathbf{Z}(R, \mu)$ satisfying $\mathbf{X}(R, \mu) \to \mathbf{Z}(R, \mu)$, one necessarily has $\mathbf{Y}(R, \mu) \to \mathbf{Z}(R, \mu)$.

If $\mathbf{X}(R, \mu) \to \mathbf{Y}(R, \mu)$, then the identity operator Id is continuous from $\mathbf{X}(R, \mu)$ to $\mathbf{Y}(R, \mu)$. If it is also compact, then we write

$$\mathbf{X}(R, \mu) \hookrightarrow\hookrightarrow \mathbf{Y}(R, \mu).$$

In such a case we say that function space $\mathbf{X}(R, \mu)$ is *compactly embedded* into $\mathbf{Y}(R, \mu)$. The fact that operator T is compact from function space $\mathbf{X}(R, \mu)$ into $\mathbf{Y}(R, \mu)$ is denoted as

$$T\colon \mathbf{X}(R, \mu) \to\to \mathbf{Y}(R, \mu).$$

Suppose that $\|\cdot\|_{\mathbf{X}(0,1)}$ and $\|\cdot\|_{\mathbf{Y}(0,1)}$ are rearrangement-invariant norms. We say that $\mathbf{X}(R, \mu)$ is *almost-compactly embedded* into $\mathbf{Y}(R, \mu)$ and we write

$$\mathbf{X}(R, \mu) \overset{*}{\hookrightarrow} \mathbf{Y}(R, \mu)$$

if

$$\lim_{k\to\infty} \sup_{\|f\|_{\mathbf{X}(R,\mu)}\le 1} \|\chi_{E_k} f\|_{\mathbf{Y}(R,\mu)} = 0$$

is satisfied for every sequence $(E_k)_{k=1}^{\infty}$ of μ-measurable subsets of R fulfilling $\chi_{E_k} \to 0$ μ-almost everywhere.

3. The main theorems

The connection between Sobolev embeddings and certain Hardy-type operators in the setting of Carnot–Carathéodory spaces is established in [14]. Here, we are going to extend this connection to compactness of Sobolev embeddings. Let $J\colon (0,1] \to (0,\infty)$ be a measurable function satisfying (2.3); we will consider the operator $H_J\colon \mathfrak{M}(0,1) \to \mathfrak{M}(0,1)$ defined by

$$H_J f(t) = \int_t^1 \frac{|f(s)|}{J(s)}\, ds, \quad f \in \mathfrak{M}(0,1) \text{ and } t \in (0,1). \tag{3.1}$$

Furthermore, given $j \in \mathbb{N}$, we define the operator H_J^j by

$$\underbrace{H_J \circ H_J \circ \cdots \circ H_J}_{j\text{-times}}(f). \tag{3.2}$$

Theorem 3.1. *Assume that (2.4), (2.5), and (2.6) are fulfilled. Let $\Omega \subset \mathbb{R}^n$ be open, and let $m \in \mathbb{N}$. Suppose that there is some nondecreasing function $I\colon [0,1] \to \mathbb{R}$ satisfying (2.2) and (2.3), and let $\|\cdot\|_{\mathbf{X}(0,1)}$ and $\|\cdot\|_{\mathbf{Y}(0,1)}$ be rearrangement-invariant norms. Then*

$$H_I^m\colon \mathbf{X}(0,1) \to\to \mathbf{Y}(0,1) \tag{3.3}$$

implies

$$V_X^m \mathbf{X}(\Omega) \hookrightarrow\hookrightarrow \mathbf{Y}(\Omega). \tag{3.4}$$

Remark 3.2. According to [35] the following two conditions are equivalent under the assumptions of Theorem 3.1:

- $H_I^m\colon \mathbf{X}(0,1) \to\to \mathbf{Y}(0,1)$,
- $\lim\limits_{a\to 0+} \sup_{\|f\|_{\mathbf{X}(0,1)}\le 1} \|H_I^m(\chi_{(0,a)} f)\|_{\mathbf{Y}(0,1)} = 0$.

Adopting some additional conditions allows us to reformulate the condition from Theorem 3.1 in terms of a simpler operator which we will denote K_I^m.

Suppose that $I\colon (0,1] \to (0,\infty)$ is a nondecreasing function satisfying (2.3) and let $m \in \mathbb{N}$. Set

$$J(t) = \frac{(I(t))^m}{t^{m-1}}, \quad t \in (0,1]. \tag{3.5}$$

Let us observe that J is measurable on $(0,1]$ and fulfills (2.3). Consider the operator K_I^m defined by

$$K_I^m f(t) = \int_t^1 |f(s)| \frac{s^{m-1}}{(I(s))^m}\, ds, \quad f \in \mathfrak{M}(0,1), t \in (0,1).$$

If, up to multiplicative constants depending on I,

$$\int_0^s \frac{dr}{I(r)} \approx \frac{s}{I(s)}, \quad s \in (0,1), \tag{3.6}$$

then the sufficient condition for (3.4) can be reformulated with operator K_I^m.

Theorem 3.3. *Assume that (2.4), (2.5), and (2.6) are fulfilled. Suppose that there is some nondecreasing function $I\colon [0,1] \to \mathbb{R}$ satisfying (2.2) and (3.6). Let $m \in \mathbb{N}$ and let $\|\cdot\|_{\mathbf{X}(0,1)}$ and $\|\cdot\|_{\mathbf{Y}(0,1)}$ be rearrangement-invariant norms.*

(1) *Suppose that*

$$\lim_{t\to 0_+} \frac{t^{m-1}}{(I(t))^m} \neq 0. \tag{3.7}$$

Then

$$K_I^m\colon \mathbf{X}(0,1) \to\to \mathbf{Y}(0,1) \tag{3.8}$$

implies that

$$V_X^m \mathbf{X}(\Omega) \hookrightarrow\hookrightarrow \mathbf{Y}(\Omega). \tag{3.9}$$

(2) *Suppose that*

$$\lim_{t\to 0_+} \frac{t^{m-1}}{(I(t))^m} = 0. \tag{3.10}$$

Then (3.9) is satisfied for all pairs of rearrangement-invariant norms $\|\cdot\|_{\mathbf{X}(0,1)}$ and $\|\cdot\|_{\mathbf{Y}(0,1)}$.

If we restrict our consideration to X-PS domains, where the isoperimetric function is known, we can use a yet simpler operator. Given $Q > 0$ and $m \in \mathbb{N}$, we define

$$\mathcal{Q}_Q^m f(t) = \int_t^1 \big|f(s)\big| s^{\frac{m}{Q}-1}\, ds, \quad f \in \mathfrak{M}(0,1) \text{ and } t \in (0,1).$$

Theorem 3.4 (Reduction principle for X-PS domains)**.** *Assume that (2.4), (2.5), and (2.6) are fulfilled. Let $m \in \mathbb{N}$ and Ω be a X-PS domain with homogeneous dimension Q. Suppose that $\|\cdot\|_{\mathbf{X}(0,1)}$ and $\|\cdot\|_{\mathbf{Y}(0,1)}$ are r.i. spaces. If $m \le Q$ and*

$$\mathcal{Q}_Q^m\colon \mathbf{X}(0,1) \to\to \mathbf{Y}(0,1), \tag{3.11}$$

then

$$V_X^m \mathbf{X}(\Omega) \hookrightarrow\hookrightarrow \mathbf{Y}(\Omega). \tag{3.12}$$

In particular, the assumption that $Q = m$ implies that (3.12) is satisfied for all $\|\cdot\|_{\mathbf{Y}(0,1)}$ if $\mathbf{X}(0,1) \neq L^1(0,1)$. Furthermore, if $m > Q$ then (3.12) is fulfilled for all choices of $\mathbf{X}(0,1)$ and $\mathbf{Y}(0,1)$.

The principle introduced in the Theorem 3.1 and further developed in Theorems 3.3 and Theorem 3.4 will be applied to the class of Lorentz spaces.

Theorem 3.5. *Assume that (2.4), (2.5), and (2.6) are fulfilled. Let $m \in \mathbb{N}$, and let $\Omega \subset \mathbb{R}^n$ be an X-PS domain. Let $p_1, p_2, q_1, q_2 \in [1,\infty]$ be such that the triples $(p_1, q_1, 0)$ and $(p_2, q_2, 0)$ satisfy one of the conditions (2.8)–(2.11). Let Q denote the homogeneous dimension of Ω, and assume that $Q > 2$ and $m < Q$. Then each of the following conditions*

(1) $p_1 < \frac{Q}{m}$ *and* $p_2 < \frac{p_1}{1-\frac{mp_1}{Q}}$,
(2) $p_1 = \frac{Q}{m}$ *and* $p_2 < \infty$,
(3) $p_1 > \frac{Q}{m}$

ensures that

$$V_X L^{p_1,q_1}(\Omega) \hookrightarrow\hookrightarrow L^{p_2,q_2}(\Omega). \tag{3.13}$$

Let us note that the cases when $m = Q$ and $m \geq Q$ are missing from Theorem 3.5 are already covered in Theorem 3.4. Also, let us explicitly state that the conditions on compact embeddings of Sobolev spaces build upon Lebesgue spaces which are implied by Theorem 3.5.

Corollary 3.6. *Assume that (2.4), (2.5), and (2.6) are fulfilled. Let $m \in \mathbb{N}$, and let $\Omega \subset \mathbb{R}^n$ be an X-PS domain. Let $p_1, p_2 \in [1,\infty]$. Let Q denote the homogeneous dimension of Ω, and assume that $Q > 2$ and $m < Q$. Then each of the following conditions*

(1) $p_1 < \frac{Q}{m}$ *and* $p_2 < \frac{p_1}{1-\frac{mp_1}{Q}}$,
(2) $p_1 = \frac{Q}{m}$ *and* $p_2 < \infty$,
(3) $p_1 > \frac{Q}{m}$,

ensures that

$$V_X L^{p_1}(\Omega) \hookrightarrow\hookrightarrow L^{p_2}(\Omega). \tag{3.14}$$

4. Support theorems

In this section we collect known theorems which will be used in proofs of theorems from Section 3. First, we recall some facts about the operator H_J^j and its connection to compact and almost-compact embeddings of r.i. spaces. Then we shift our attention to compact and almost-compact embeddings of Lebesgue spaces on Carnot–Carathéodory spaces, adapting known results to this settings if necessary. We conclude this section by brief summary of some properties of functions from Sobolev-like spaces over Carnot–Carathéodory spaces.

Assume that $J\colon (0,1] \to (0,\infty)$ is a measurable function satisfying

$$\inf \frac{J(t)}{t} > 0. \tag{4.1}$$

It is proved in [7, Remark 8.2] that

$$H_J^j f(t) = \frac{1}{(j-1)!} \int_t^1 \frac{|f|}{J(s)} \Big(\int_t^s \frac{dr}{J(r)} \Big)^{j-1} ds,$$

for $f \in \mathfrak{M}(0,1), t \in (0,1)$. Let $j \in \mathbb{N}$, and let $\|\cdot\|_{\mathbf{X}(0,1)}$ be an r.i. norm. For every $f \in \mathfrak{M}(0,1)$, we define the functional $\|\cdot\|_{(X^r_{j,J})'(0,1)}$ by

$$\|f\|_{(X^r_{j,J})'(0,1)} = \frac{1}{(j-1)!} \Big\| \frac{1}{J(s)} \int_0^s \Big(\int_t^s \frac{dr}{J(r)} \Big)^{j-1} f^*(t)\, dt \Big\|_{\mathbf{X}'(0,1)}.$$

It is shown in [7, Proposition 8.3] that $\|\cdot\|_{(\mathbf{X}^r_{j,J})'(0,1)}$ is an r.i. norm and its associate norm $\|\cdot\|_{(\mathbf{X}^r_{j,J})(0,1)}$ fulfills

$$H^j_J\colon \mathbf{X}(0,1) \to \mathbf{X}^r_{j,J}(0,1). \tag{4.2}$$

Lemma 4.1. *Let $J\colon (0,1] \to (0,\infty)$ be a measurable function. Then H^1_J is not compact from $L^1(0,1)$ into $L^\infty(0,1)$.*

Proof. We will follow the argument from the end of the proof of Lemma 4.1 in [35]. Since $\frac{1}{J(t)} > 0$, $t \in (0,1)$, there exists $\varepsilon > 0$ and set $M \subset (0,1)$ with $|M| = \frac{1}{2}$ such that $\frac{1}{J(s)} \geq \varepsilon$ for $s \in M$. Let $(x_n)_{n=1}^\infty$ be a sequence of points in $[0,1)$ such that $|(x_n,1) \cap M| = \frac{1}{2^n}$, $n \in \mathbb{N}$, and set

$$f_n(t) = 2^n \chi_{(x_n,x_{n+1})\cap M}(t), \quad \text{for } t \in (0,1), n \in \mathbb{N}.$$

We have

$$\|f\|_{L^1(0,1)} = 2^n \big|(x_n,x_{n+1}) \cap M\big| = 2^n \Big(\big|(x_n,1)\cap M\big| - \big|(x_{n+1},1)\cap M\big|\Big) = \frac{1}{2},$$

and hence $(f_n)_{n=1}^\infty$ is bounded in $L^1(0,1)$.

Now fix $m,n \in \mathbb{N}$, $m < n$. We have

$$\begin{aligned}
&\big\|H^1_J(f_n) - H^1_J(f_m)\big\|_{L^\infty(0,1)}\\
&\quad\geq \big|H_J f_n(x_n) - H_J f_m(x_n)\big|\\
&\quad= \Big|\int_{x_n}^1 \frac{2^n \chi_{(x_n,x_{n+1})\cap M}(s)}{J(s)}\,ds - \underbrace{\int_{x_n}^1 \frac{2^m \chi_{(x_m,x_{m+1})\cap M}(s)}{J(s)}}_{=0 \text{ since } m<n}\,ds\Big|\\
&\quad= 2^n \int_{x_n}^{x_{n+1}} \frac{\chi_M(s)}{J(s)}\,ds \geq 2^n \varepsilon \frac{1}{2^{n+1}} = \frac{\varepsilon}{2}.
\end{aligned}$$

Consequently, the sequence $(H^1_J(f_n))_{n=1}^\infty$ is not a Cauchy sequence in $L^\infty(0,1)$ and H^1_J is not compact from $L^1(0,1)$ into $L^\infty(0,1)$. $\square$

Let us restate the following two characterizations of compactness of operator H^j_J from [35].

Theorem 4.2 ([35, Theorem 4.1 and Theorem 4.2])**.** *Let $J\colon (0,1] \to (0,\infty)$ be a measurable function satisfying (2.3), and let $j \in \mathbb{N}$. Suppose that $\|\cdot\|_{\mathbf{X}(0,1)}$ and $\|\cdot\|_{\mathbf{Y}(0,1)}$ are rearrangement-invariant norms. Consider the following two conditions:*

(a) $H^j_J\colon \mathbf{X}(0,1) \to\to \mathbf{Y}(0,1)$,
(b) $\lim_{a\to 0+} \sup_{\|f\|_{\mathbf{X}(0,1)}\leq 1} \|H^j_J(\chi_{(0,a)}f)\|_{\mathbf{Y}(0,1)} = 0$.

If $\mathbf{X}(0,1) = L^1(0,1)$, $\mathbf{Y}(0,1) = L^\infty(0,1)$, $j = 1$, and

$$\lim_{a\to 0+} \operatorname*{ess\,sup}_{t\in(0,a)} \frac{1}{J(t)} = 0,$$

then (b) is satisfied but (a) is not. In all other cases, (a) holds if and only if (b) holds.

Theorem 4.3. *Let $J\colon (0,1] \to (0,\infty)$ be a measurable function satisfying (2.3), and let $j \in \mathbb{N}$. Suppose that $\|\cdot\|_{\mathbf{X}(0,1)}$ and $\|\cdot\|_{\mathbf{Y}(0,1)}$ are rearrangement-invariant norms. If*

$$\mathbf{Y}(0,1) \neq L^\infty(0,1) \qquad \text{or} \qquad \int_0^1 \frac{dr}{J(r)} = \infty,$$

then the following conditions are equivalent:

(1) $H_J^j\colon \mathbf{X}(0,1) \to\to \mathbf{Y}(0,1)$,

(2) $\mathbf{X}_{j,J}^r(0,1) \overset{*}{\hookrightarrow} \mathbf{Y}(0,1)$.

Assume that $\mathbf{X}(0,1)$ and J are such that, in addition to (4.1), it holds that

$$\Big\| \Big(\int_t^1 \frac{dr}{J(r)} \Big)^j \Big\|_{\mathbf{X}(0,1)} < \infty. \tag{4.3}$$

For every $f \in \mathfrak{M}(0,1)$ define the function $\|\cdot\|_{\mathbf{Y}_{j,J}^d(0,1)}$ by

$$\|f\|_{\mathbf{Y}_{j,J}^d(0,1)} = \sup_{h \sim f} \|H_J^j h\|_{\mathbf{Y}(0,1)} + \|f\|_{L^1(0,1)}.$$

Important properties of $\mathbf{Y}_{j,J}^d(0,1)$ are summarized in the following proposition and theorem from [35].

Proposition 4.4 ([35, Proposition 4.5])**.** *Let $J\colon (0,1] \to (0,\infty)$ be a measurable function satisfying (2.3), and let $j \in \mathbb{N}$. Suppose that $\|\cdot\|_{\mathbf{Y}(0,1)}$ is a rearrangement-invariant norm fulfilling (4.3). Then $\|\cdot\|_{(\mathbf{Y})_{j,J}^d(0,1)}$ is a rearrangement-invariant norm and*

$$H_J^j\colon \mathbf{Y}_{j,J}^d(0,1) \to \mathbf{Y}(0,1).$$

Theorem 4.5 ([35, Theorem 4.6])**.** *Let $J\colon (0,1] \to (0,\infty)$ be a measurable function satisfying (2.3) and let $j \in \mathbb{N}$. Suppose that $\mathbf{X}(0,1)$ is a r.i. function space such that $\mathbf{X}(0,1) \neq L^1(0,1)$. Assume that $\mathbf{Y}(0,1)$ is an r.i. space fulfilling (4.3). Then the following conditions are equivalent:*

(1) $H_J^j\colon \mathbf{X}(0,1) \to\to \mathbf{Y}(0,1)$,

(2) $\mathbf{X}(0,1) \overset{*}{\hookrightarrow} \mathbf{Y}_{j,J}^d(0,1)$.

When certain conditions are satisfied, the norm $\|\cdot\|_{(L^\infty)_{1,I}^d(0,1)}$ can be approximated by a simpler one as is shown in the next lemma from [35].

Lemma 4.6 ([35, Lemma 5.6])**.** *Let $I\colon (0,1] \to (0,\infty)$ be a nondecreasing function satisfying (2.3) and*

$$\int_0^1 \frac{ds}{I(s)} < \infty.$$

Then

$$\|f\|_{(L^\infty)_{1,I}^d(0,1)} \approx \int_0^1 \frac{f^*(s)}{I(s)}\, ds, \quad f \in \mathfrak{M}(0,1)$$

up to multiplicative constants depending on I.

Useful conditions on compactness of operator H_J^j formulated through its boundedness stated in Lemma 4.7 are proved in [35, Remark 4.8].

Lemma 4.7. *Let* $\mathbf{X}(0,1) \neq L^1(0,1)$ *and* $\mathbf{Y}(0,1)$ *be rearrangement-invariant function spaces. Suppose that* $j \in \mathbb{N}$ *and that* $J\colon (0,1] \to (0,\infty)$ *is a measurable function satisfying*

$$\inf_{t\in(0,1]} \frac{J(t)}{t} > 0.$$

Then

$$H_J^j\colon L^1(0,1) \to \mathbf{Y}(0,1)$$

implies that

$$H_J^j\colon \mathbf{X}(0,1) \hookrightarrow\hookrightarrow \mathbf{Y}(0,1).$$

We continue by stating the characterization of almost-compact embeddings of r.i. spaces from [34].

Theorem 4.8 ([34, Theorem 3.1])**.** *Let* $\mathbf{X}(R,\mu)$ *and* $\mathbf{Y}(R,\mu)$ *be Banach function spaces over a totally* σ*-finite measure space* (R,μ)*. Then* $\mathbf{X}(R,\mu) \overset{*}{\hookrightarrow} \mathbf{Y}(R,\mu)$ *if and only if, for every sequence* $(f_n)_{n=1}^\infty$ *of* μ*-measurable functions on* R *satisfying* $\|f_n\|_{\mathbf{X}(R,\mu)} \leq 1$ *and* $f_n \to 0$ μ*-almost everywhere, one has* $\|f_n\|_{\mathbf{Y}(R,\mu)} \to 0$.

The following lemma is an adaptation from [35].

Lemma 4.9 ([35, Lemma 5.5])**.** *Assume that (2.4), (2.5), and (2.6) are fulfilled, assume that* Ω *is an open domain, and assume that* $m \in \mathbb{N}$*. Let* $\|\cdot\|_{\mathbf{X}}$ *be an rearrangement-invariant function space. Then every sequence* $(u_k)_{k=1}^\infty$ *bounded in* $V_X^m\mathbf{X}(\Omega)$ *contains a subsequence* $(u_{k_l})_{l=1}^\infty$ *converging almost everywhere in* Ω.

To prove Lemma 4.9, we need compact embedding $V_X^1 L^1(B_{x_l}) \hookrightarrow\hookrightarrow L^1(B_{x_l})$, which was conveniently already proved in [15].

Theorem 4.10 ([15, Theorem 1.28])**.** *Assume that (2.4), (2.5), and (2.6) are fulfilled, and let* $\Omega \subset \mathbb{R}^n$ *be an* X*-PS domain with* $\operatorname{diam}(\Omega) < \frac{R_0}{2}$*, where* R_0 *is the constant from (2.4). Then, one has the following.*

(1) *The embedding* $BV_X(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ *holds for any* $1 \leq q < \frac{Q}{Q-1}$.
(2) *For any* $1 \leq p < Q$*, the embedding* $V_X^1 L^p(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ *holds provided that* $1 \leq q < \frac{Qp}{Q-p}$.
(3) *For any* $Q \leq p < \infty$ *and any* $1 \leq q < \infty$*, the embedding* $V_X^1 L^p(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ *holds.*

Proof of Lemma 4.9. Since Ω is open, for all $x \in \Omega$ there exists a ball (with respect to metric d) B_x such that $x \in B_x$ and $B_x \subset \Omega$. There is a sequence $(x_l)_l^\infty$ of points in Ω such that $\{B_{x_l}, x_l = 1, 2, \dots\}$ is a covering of Ω because the topology generated by the metric d is equivalent to the Euclidean topology.

Let $(u_k)_{k=1}^\infty$ be bounded in $V_X^m\mathbf{X}(\Omega)$. Balls with respect to metric d are an X-PS domain; such X-PS domains fulfill (2.12) with a specific constant (consequence of Theorem 1.18 in [15]). Proposition 11 in [14] yields that $V_X^1\mathbf{X}(B_{x_l}) \to L^1(Bx_l)$ for all all $l = 1, 2, \dots, \infty$. Moreover, the proof of Proposition 11 yields that the embedding constant is dependent only on the constant from (2.12); therefore there exists a constant which holds for all embeddings $V_X^1\mathbf{X}(B_{x_l}) \to L^1(B_{x_l})$, $l \in \mathbb{N}$. Consequently, there exists $C > 0$ such that

$$\|u_k\|_{L^1(B_{x_l})} \le C\|u_k\|_{V_X^m\mathbf{X}(B_{x_l})} \le C\|u_k\|_{V_X^m\mathbf{X}(\Omega)}$$

for all $l \in \mathbb{N}$. Sequence $(u_k)_{k=1}^\infty$ is therefore bounded in $V_X^1 L^1(B_{x_l})$ with the same bounding constant for all $l \in \mathbb{N}$. Now, for every $l \in \mathbb{N}$, Theorem 4.10 yields that

$$V_X^1 L^1(B_{x_l}) \hookrightarrow\hookrightarrow L^1(B_{x_l}).$$

Consequently, a bounded sequence $(v_k)_{k=1}^\infty$ in $V_X^1 L^1(B_{x_l})$ contains a subsequence $(v_{k_r})_{r=1}^\infty$ such that v_{k_r} converges in $L^1(B_{x_l})$.

We start with selecting a subsequence of $(u_k)_{k=1}^\infty$—let us name it $(u_{k_{1,r}})_{r=1}^\infty$—such that it converges in $L^1(B_{x_1})$. Then there exists a subsequence of $(u_{k_{1,r}})_{r=1}^\infty$, $(u_{k_{2,r}})_{r=1}^\infty$, such that it converges in $L^1(B_{x_2})$ and $L^1(B_{x_1})$. By repeating this step, we get a sequence $(u_{k_{l,r}})_{r=1}^\infty$ for each $l \in \mathbb{N}$ which converges (with respect to r) in $L^1(B_{x_s})$ for all $s \in \mathbb{N}$, $s \le l$. The diagonal sequence, $(u_{k_{l,l}})_{l=1}^\infty$ is then the desired subsequence of $(u_k)_{k=1}^\infty$ which converges almost everywhere on Ω. □

Theorem 4.11 (Reduction theorem for noncompact embeddings)**.** *Assume that $\Omega \subset \mathbb{R}^n$ is open. Suppose that there is some nondecreasing function $I\colon [0,1] \to \mathbb{R}$ satisfying (2.2) and (2.3). Let $m \in \mathbb{N}$, and let $\|\cdot\|_{\mathbf{X}(0,1)}$ and $\|\cdot\|_{\mathbf{Y}(0,1)}$ be rearrangement-invariant function norms. If there exists a constant $C > 0$ such that*

$$\Big\|\int_t^1 \frac{f(s)}{I(s)}\Big(\int_t^s \frac{dr}{I(r)}\Big)^{m-1} ds\Big\|_{\mathbf{Y}(0,1)} \le C\|f\|_{\mathbf{X}(0,1)} \tag{4.4}$$

for every nonnegative $f \in \mathfrak{M}_+(0,1)$, then

$$V_X^m\mathbf{X}(\Omega) \to \mathbf{Y}(\Omega). \tag{4.5}$$

Remark 4.12. The condition (4.4) can be restated as $H_I^m\colon \mathbf{X}(0,1) \to \mathbf{Y}(0,1)$.

Lemma 4.13 ($W_X^1\mathbf{X}(\Omega) = V_X^1\mathbf{X}$)**.** *Assume that $\Omega \subset \mathbb{R}^n$ is open. Suppose that there is some nondecreasing function $I\colon [0,1] \to \mathbb{R}$ satisfying (2.2) and*

$$\int_0^s \frac{dt}{I(t)} < \infty \quad \text{for some } s > 0. \tag{4.6}$$

Let $\Omega \subset \mathbb{R}^n$ is open. Then there exists a constant $C > 0$ such that

$$\|f\|_{\mathbf{X}(\Omega)} \le C\|X\nabla f\|_{\mathbf{X}(\Omega)}, \quad f \in \mathfrak{M}(0,1).$$

Consequently, up to a multiplicative constants, we get

$$\|f\|_{V_X^1\mathbf{X}(\Omega)} \approx \|f\|_{\mathbf{X}(\Omega)} + \|X\nabla f\|_{\mathbf{X}(\Omega)}, \quad f \in V_X^1\mathbf{X}(\Omega).$$

Proof. We will follow the approach of the Proposition 4.5 in [7]. We define

$$J(t) = \begin{cases} I(s), & s \in [0, \frac{1}{3}], \\ I(\frac{1}{3}), & s \in]\frac{1}{3}, 1]. \end{cases}$$

Then $J(t) > cs$ for some constant $c > 0$, thanks to (4.6), and J fulfills

$$I_{\Omega,X}(s) \ge c' J(c's)$$

for some $c' > 0$ and s near zero. A simple computation shows that $H_J^1\colon L^1(0,1) \to L^1(0,1)$ and $H_J^1 L^\infty(0,1) \to L^\infty(0,1)$. The interpolation theorem of Calderón [3, Chapter 3, Theorem 2.12] then yields that $H_J^1\colon \mathbf{X}(0,1) \to \mathbf{X}(0,1)$. Application of Theorem 4.11 then implies the desired embedding

$$V_X^1\mathbf{X}(\Omega) \to \mathbf{X}(\Omega). \qquad \square$$

Let us restate here Lemma 3.5 from [15], which allows us to use the well-known Maz'ya truncation technique.

Lemma 4.14. *Let $1 \le p < \infty$, and let Ω be a bounded open set in $\mathbb{R}^n$. If $u \in W_X^1 L^p(\Omega)$ and $F \in \mathcal{C}^1(\mathbb{R})$, $F' \in L^\infty(\mathbb{R})$, then we have the following.*

(1) *$F \circ u \in W_X^1 L^p(\Omega)$ and*

$$X_j(F \circ u) = (F' \circ u) X_j u \quad \text{in } \mathcal{D}'(\Omega) \text{ for } 1 \le j \le m.$$

(2) *Also, one has $u^+, u^-, |u| \in V_X^1 L^1(\Omega)$ and*

$$X\nabla u^+ = \begin{cases} X\nabla u & \text{almost everywhere on } \{x \in \Omega\colon u(x) \ge 0\}, \\ 0 & \text{otherwise,} \end{cases}$$

$$X\nabla u^- = \begin{cases} -X\nabla u & \text{almost everywhere on } \{x \in \Omega\colon u(x) < 0\}, \\ 0 & \text{otherwise,} \end{cases}$$

$$X\nabla |u| = \begin{cases} X\nabla u & \text{almost everywhere on } \{x \in \Omega\colon u(x)0\}, \\ 0 & \text{almost everywhere on } \{x \in \Omega\colon u(x) = 0\}, \\ -X\nabla u & \text{almost everywhere on } \{x \in \Omega\colon u(x) < 0\}. \end{cases}$$

5. Proof of the main theorem

Proof of Theorem 3.1. First, assume that $\mathbf{Y}(\Omega) \ne L^\infty(\Omega)$ or $\int_0^1 \frac{ds}{I(s)} = \infty$. In this case, Theorem 4.3 yields $\mathbf{X}^r_{m,I}(0,1) \overset{*}{\hookrightarrow} \mathbf{Y}(0,1)$, and consequently $\mathbf{X}^r_{m,I}(\Omega) \overset{*}{\hookrightarrow} \mathbf{Y}(\Omega)$.

Second, assume that $(u_k)_{k=1}^\infty$ is a sequence bounded in $V_X^m\mathbf{X}(\Omega)$. Lemma 4.9 ensures that there is subsequence $(u_{k_l})_{l=1}^\infty$ which converges to some function u almost everywhere on Ω. The embedding (4.2) implies that $H_I^m\mathbf{X}(0,1) \to \mathbf{X}^r_{m,I}(0,1)$. Theorem 4.11 then yields $V_X^m\mathbf{X}(\Omega) \to \mathbf{X}^r_{m,I}(\Omega)$. Hence, $(u_{k_l})_{l=1}^\infty$ is bounded in $\mathbf{X}^r_{m,I}(\Omega)$. Therefore, the almost-compact embedding $\mathbf{X}^r_{m,I}(\Omega) \overset{*}{\hookrightarrow} \mathbf{Y}(\Omega)$ and Theorem 4.8 yield that $u_{k_l} \to u \in \mathbf{Y}(\Omega)$. Thus, $V_X^m\mathbf{X}(\Omega) \hookrightarrow\hookrightarrow \mathbf{Y}(\Omega)$.

In the following, we will focus on the remaining case $\mathbf{Y}(\Omega) = L^\infty(\Omega)$ and $\int_0^1 \frac{ds}{I(s)} < \infty$. Assume first that $m = 1$. Lemma 4.1 ensures that $\mathbf{X}(0,1) \ne L^1(0,1)$, since we are assuming that

$$H_J^1\colon \mathbf{X}(0,1) \to\to L^\infty(0,1).$$

This, together with the observation that

$$\Big\| \int_t^1 \frac{dr}{I(r)} \Big\|_{L^\infty(0,1)} = \int_0^1 \frac{dr}{I(r)} < \infty \tag{5.1}$$

and Theorem 4.5, yields that

$$\mathbf{X}(0,1) \overset{*}{\hookrightarrow} (L^\infty)^d_{1,I}(0,1). \tag{5.2}$$

Moreover, the inequality (5.1) and Lemma 4.4 yields that assumptions of Theorem 4.11 are met with $\mathbf{X}(0,1) = (L^\infty)^d_{1,I}(0,1)$ and $\mathbf{Y}(0,1) = L^\infty(0,1)$. Consequently, we get

$$V^1_X (L^\infty)^d_{1,I}(\Omega) \to L^\infty(\Omega). \tag{5.3}$$

Assume now that $(u_k)_{k=1}^\infty$ in $V^1_X\mathbf{X}(\Omega)$ is a bounded sequence. Since $\int_0^1 \frac{ds}{I(s)} < \infty$, the Lemma 4.13 ensures that $(u_k)_{k=1}^\infty$ is bounded in $W^1_X\mathbf{X}(\Omega)$ as well. Without loss of generality we may assume that

$$\|u_k\|_{W^1_X\mathbf{X}(\Omega)} \le 1, \quad k \in \mathbb{N}. \tag{5.4}$$

Lemma 4.9 then assures that there is a subsequence $(v_k)_{k=1}^\infty$ which converges in measure to some function v. Indeed, our goal is to show that $(v_k)_{k=1}^\infty$ is a Cauchy sequence in $L^\infty(\Omega)$ and that thus it converges to v in $L^\infty(\Omega)$, which will imply that $V^1_X\mathbf{X}(\Omega)$ is compactly embedded into $L^\infty(\Omega)$.

Fix $\varepsilon > 0$ and $k, l \in \mathbb{N}$. Let us introduce the following notation:

$$d(x) = \big|v_k(x) - v_l(x)\big| = \min\Big\{d(x), \frac{\varepsilon}{2}\Big\} + \max\Big\{d(x) - \frac{\varepsilon}{2}, 0\Big\}$$

for $x \in \Omega$. Moreover, let us write $e(x) = \max\{d(x) - \frac{\varepsilon}{2}, 0\}$, $x \in \Omega$. Differentiability of v_k and v_l combined with Lemma 4.14 ensures that $d - \frac{\varepsilon}{2}$ and e are both differentiable almost everywhere in Ω. Because e is being derived from v_k and v_l by subtraction, absolute value operator, and truncation by constant, standard argumentation, accompanied by Lemma 4.14, it holds that

$$\big|X\nabla e(x)\big| = \chi_{\{d \ge \frac{\varepsilon}{2}\}}(x)\big|X\nabla v_k(x) - X\nabla v_l(x)\big|, \tag{5.5}$$

for almost every $x \in \Omega$. Consequently, we have

$$\begin{aligned}
\|d\|_{L^\infty(\Omega)} &\le \Big\|\min\Big\{d(x), \frac{\varepsilon}{2}\Big\}\Big\|_{L^\infty(\Omega)} + \|e\|_{L^\infty(\Omega)} \\
&\le \frac{\varepsilon}{2} + C\|e\|_{W^1_X(L^\infty)^d_{1,I}(\Omega)} \\
&\le \frac{\varepsilon}{2} + C\big\|\chi_{\{d>\frac{\varepsilon}{2}\}}\big|X\nabla(v_k - v_l)\big|\big\|_{(L^\infty)^d_{1,I}(\Omega)} + C\big\|\chi_{\{d>\frac{\varepsilon}{2}\}}|e|\big\|_{(L^\infty)^d_{1,I}(\Omega)},
\end{aligned}$$

where we have used (5.3), Lemma 4.13, and (5.5). Therefore, we get

$$\begin{aligned}
\|d\|_{L^\infty(\Omega)} \le \frac{\varepsilon}{2} &+ C\Big(\big\|\chi_{\{d>\frac{\varepsilon}{2}\}}|X\nabla v_k|\big\|_{(L^\infty)^d_{1,I}(\Omega)} + \big\|\chi_{\{d>\frac{\varepsilon}{2}\}}|X\nabla v_l|\big\|_{(L^\infty)^d_{1,I}(\Omega)} \\
&+ \big\|\chi_{\{d>\frac{\varepsilon}{2}\}}|v_k|\big\|_{(L^\infty)^d_{1,I}(\Omega)} + \big\|\chi_{\{d>\frac{\varepsilon}{2}\}}|v_l|\big\|_{(L^\infty)^d_{1,I}(\Omega)}\Big) \\
= \frac{\varepsilon}{2} &+ C\Big(\big\|\big(\chi_{\{d>\frac{\varepsilon}{2}\}}|X\nabla v_k|\big)^*\big\|_{(L^\infty)^d_{1,I}(0,1)} \\
&+ \big\|\big(\chi_{\{d>\frac{\varepsilon}{2}\}}|X\nabla v_l|\big)^*\big\|_{(L^\infty)^d_{1,I}(0,1)} \\
&\times \|\chi_{\{d>\frac{\varepsilon}{2}\}}v_k^*\|_{(L^\infty)^d_{1,I}(0,1)} + \|\chi_{\{d>\frac{\varepsilon}{2}\}}v_l^*\|_{(L^\infty)^d_{1,I}(0,1)}\Big).
\end{aligned}$$

Since (5.4) holds, we have that

$$\left\|\left(\chi_{\{d>\frac{\varepsilon}{2}\}}|X\nabla v_k|\right)^*\right\|_{(L^\infty)^d_{1,I}(0,1)} \leq \sup_{\|f\|_{\mathbf{X}(0,1)}\leq 1} \|\chi_{(0,|\{d>\frac{\varepsilon}{2}\}|)}f^*\|_{(L^\infty)^d_{1,I}(0,1)}$$

and

$$\|\chi_{\{d>\frac{\varepsilon}{2}\}}v_k^*\|_{(L^\infty)^d_{1,I}(0,1)} \leq \sup_{\|f\|_{\mathbf{X}(0,1)}\leq 1} \|\chi_{(0,|\{d>\frac{\varepsilon}{2}\}|)}f^*\|_{(L^\infty)^d_{1,I}(0,1)},$$

for $k \in \mathbb{N}$. But (5.2) yields that there is $\delta > 0$ such that

$$\sup_{\|f\|_{\mathbf{X}(0,1)}\leq 1} \|\chi_{(0,\delta)}f^*\|_{(L^\infty)^d_{1,I}(0,1)} < \frac{\varepsilon}{8C}.$$

Since $(v_k)_{k=1}^\infty$ converges in measure to v, we can find $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$

$$\left|\left\{x \in \Omega\colon \left|v_k(x) - v(x)\right| > \frac{\varepsilon}{4}\right\}\right| < \frac{\delta}{2}.$$

Moreover, for all $k, l \geq k_0$, it holds that

$$\left\{x \in \Omega\colon |d| \geq \frac{\varepsilon}{2}\right\} \subset \left\{x \in \Omega\colon \left|v_k(x) - v(x)\right| \geq \frac{\varepsilon}{4}\right\} \cup \left\{x \in \Omega\colon \left|v_l(x) - v(x)\right| \geq \frac{\varepsilon}{4}\right\}$$

and that

$$\left|\left\{x \in \Omega\colon |d| \geq \frac{\varepsilon}{2}\right\}\right| \leq \delta.$$

Consequently, for $k, l > n_0$, we have

$$\begin{aligned}\|d\|_{L^\infty(\Omega)} &\leq \frac{\varepsilon}{2} + 4C \sup_{\|f\|_{\mathbf{X}(0,1)}\leq 1} \|\chi_{(0,|\{d>\frac{\varepsilon}{2}\}|)}f^*\|_{(L^\infty)^d_{1,I}(0,1)} \\ &\leq \frac{\varepsilon}{2} + 4C \sup_{\|f\|_{\mathbf{X}(0,1)}\leq 1} \|\chi_{(0,\delta)}f^*\|_{(L^\infty)^d_{1,I}(0,1)} \leq \varepsilon.\end{aligned}$$

Therefore $(v_k)_{k=1}^\infty$ is a Cauchy sequence in $L^\infty(\Omega)$ and $V_X^1\mathbf{X}(\Omega)$ is compactly embedded into $L^\infty(\Omega)$.

Next, we will deal with the case $m > 1$. (We still assume that $\mathbf{Y}(\Omega) = L^\infty(\Omega)$ and $\int_0^1 \frac{ds}{I(s)} < \infty$.) According to Lemma 4.6, for every $f \in \mathfrak{M}(0,1)$, we have

$$\|g\|_{(L^\infty)^d_{1,I}(0,1)} \approx \int_0^1 \frac{g^*(s)}{I(s)}\,ds = \|H_I g^*\|_{L^\infty(0,1)}$$

up to multiplicative constants depending on I. Thus, whenever $f \in \mathcal{M}(0,1)$ and $a \in (0,1)$, we have

$$\begin{aligned}\left\|H_I^m(\chi_{(0,a)}f)\right\|_{L^\infty(0,1)} &= \left\|H_I\big(H_I^{m-1}(\chi_{(0,a)}f)\big)\right\|_{L^\infty(0,1)} \\ &\approx \left\|H_I^{m-1}(\chi_{(0,a)}f)\right\|_{(L^\infty)^d_{1,I}(0,1)},\end{aligned}$$

up to multiplicative constants depending on I. As it is stated in Remark 3.2, the assumption (3.3) is equivalent to

$$\lim_{a\to 0+} \sup_{\|f\|_{\mathbf{X}(0,1)}\leq 1} \left\|H_I^m(\chi_{(0,a)}f)\right\|_{L^\infty(0,1)} = 0,$$

and hence it is also equivalent to

$$\lim_{a\to 0+} \sup_{\|f\|_{\mathbf{X}(0,1)}\le 1} \left\| H_I^{m-1}(\chi_{(0,a)} f) \right\|_{(L^\infty)^d_{1,I}(0,1)} = 0.$$

In order to use the previously proved case of this proof, we will show that $(L^\infty)^d_{1,I}(0,1) \neq L^\infty(0,1)$. Consider functions $\chi_{(0,a)}$ for $a \in (0,1)$. We note that $\|\chi_{(0,a)}\|_{L^\infty(0,1)} = 1$. On the other hand, up to multiplicative constants depending on I, we have

$$\lim_{a\to 0+} \|\chi_{(0,a)}\|_{(L^\infty)^d_{1,I}(0,1)} \approx \lim_{a\to 0+} \int_0^a \frac{ds}{I(s)} = 0.$$

If $(L^\infty)^d_{1,I}(0,1) = L^\infty(0,1)$, then $\|\cdot\|_{(L^\infty)^d_{1,I}(0,1)}$ must be equivalent to $\|\cdot\|_{L^\infty(0,1)}$ up to multiplicative constants. Consequently, $(L^\infty)^d_{1,I}(0,1) \neq L^\infty(0,1)$, because it is impossible to have constant $c > 0$ such that $\|f\|_{L^\infty(0,1)} \le c\|f\|_{(L^\infty)^d_{1,I}(0,1)}$, for all $f \in L^\infty(0,1)$.

Since $(L^\infty)^d_{1,I}(0,1) \neq L^\infty(0,1)$, the previous part of proof implies that

$$V_X^{m-1}\mathbf{X}(\Omega) \hookrightarrow\hookrightarrow (L^\infty)^d_{1,I}(\Omega). \tag{5.6}$$

Let $(u_k)_{k=1}^\infty$ be a bounded sequence in $V_X^m\mathbf{X}(\Omega)$. Then $(u_k)_{k=1}^\infty$ is bounded in $L^1(\Omega)$, so $(\int_\Omega u_k(x)\,dx)_{k=1}^\infty$ is a bounded sequence of real numbers and we can find a subsequence $(u_k^0)_{k=1}^\infty$ of $(u_k)_{k=1}^\infty$ such that the sequence $(\int_\Omega u_k^0(x)\,dx)_{k=1}^\infty$ is convergent.

Consider sequences $(X_i u_k^0)_{k=1}^\infty$, $X_i \in X, i = 1,\dots,m$. Owing to boundedness of $(u_k^0)_{k=1}^\infty$ in $V_X^m\mathbf{X}(\Omega)$, $(X_i u_k^0)_{k=1}^\infty$ is bounded in $V_X^{m-1}\mathbf{X}(\Omega)$. Using compact embedding (5.6), we can inductively find $(u_k^i)_{k=1}^\infty$, subsequence of $(u_k^{i-1})_{k=1}^\infty$, $i = 1,2,\dots,m$, such that $(X_i u_k^i)_{k=1}^\infty$ is convergent in $(L^\infty)^d_{1,I}(\Omega)$. Consequently, $(X_i u_k^m)_{k=1}^\infty$ is a Cauchy sequence in $(L^\infty)^d_{1,I}(\Omega)$ for every $j = 1,2,\dots,m$.

To conclude the proof, we need to show that $(u_k^m)_{k=1}^\infty$ is a Cauchy sequence in $L^\infty(\Omega)$. Let $\varepsilon > 0$. Assumptions (2.2) and (2.3) ensure (2.12). Therefore the inequality (2.13) with $\mathbf{X}(\Omega) = (L^\infty)^d_{1,I}(\Omega)$ yields that there exists a constant $C > 0$ such that

$$\left\| u - \int_\Omega u(x)\,dx \right\|_{L^\infty(\Omega)} \le C\|X\nabla u\|_{(L^\infty)^d_{1,I}(\Omega)} \le C\sum_{j=1}^m \|X_j u\|_{(L^\infty)^d_{1,I}(\Omega)}.$$

Because $(X_j u_k^m)_{k=1}^\infty$, $j = 1,\dots,m$, is a Cauchy sequence in $(L^\infty)^d_{1,I}(\Omega)$, there exists $k_0 \in \mathbb{N}$ such that

$$\|X_j u_l^m - X_j u_l^m\|_{(L^\infty)^d_{1,I}(\Omega)} \le \frac{\varepsilon}{Cm},$$

for all $j = 1,\dots,m$, whenever $k,l > k_0$. Sequence $(u_k^m - \int_\Omega u_k^m(x)\,dx)_{k=1}^\infty$ is Cauchy sequence in $L^\infty(\Omega)$ since

$$\left\| u_l^m - \int_\Omega u_l^m(x)\,dx - u_k^m - \int_\Omega u_k^m(x)\,dx \right\| \le C\sum_{j=1}^m \|X_j u_l^m - X_j u_l^m\|_{(L^\infty)^d_{1,I}(\Omega)} < \varepsilon,$$

for $k,l > k_0$. As $L^\infty(\Omega)$ is complete, therefore $(u_k^m - \int_\Omega u_k^m(x)\,dx)_{k=1}^\infty$ is a convergent sequence. Sequence $(\int_\Omega u_k^m(x)\,dx)_{k=1}^\infty$ is a subsequence of $(\int_\Omega u_k^0(x)\,dx)_{k=1}^\infty$

which is convergent in $L^\infty(\Omega)$. Therefore $(u_k^m)_{k=1}^\infty$ is convergent sequence in $L^\infty(\Omega)$, as well. This concludes the proof. $\square$

Proof of Theorem 3.3. It is proved in [7, Proposition 8.6] that if (3.6) holds, then for any r.i. space $\mathbf{Y}$ and $f \in \mathfrak{M}(0,1)$ we have

$$\|H_I^m f\|_{\mathbf{Y}(0,1)} \approx \|K_I^m f\|_{\mathbf{Y}(0,1)},$$

up to multiplicative constants depending on m and I. Moreover, it is shown in [35, Proof of Theorem 5.3] that, under the same assumptions, it holds that

$$\left\|H_I^m(\chi_{(0,a)} f)\right\|_{\mathbf{Y}(0,1)} \approx \left\|K_I^m(\chi_{(0,a)} f)\right\|_{\mathbf{Y}(0,1)} \tag{5.7}$$

for a given $a \in (0,1)$, up to multiplicative constants depending on m and I. At the same place, it is shown that in this situation

$$\lim_{t\to 0+} \operatorname{ess\,sup}_{s\in(0,t)} \frac{1}{J(s)} = 0$$

holds if and only if

$$\lim_{t\to 0+} \frac{t^{m-1}}{(I(t))^m} = 0.$$

Therefore, if (3.7) holds, then Theorem 4.2 and the fact that $K_I^m = H_J$ yield that

$$\lim_{a\to 0+} \sup_{\|f\|_{\mathbf{X}(0,1)}\le 1} \left\|K_I^m(\chi_{(0,a)} f)\right\|_{\mathbf{Y}(0,1)} = 0$$

is equivalent to

$$K_I^m \colon \mathbf{X}(0,1) \to\to \mathbf{Y}(0,1).$$

Remark 3.2, together with (5.7), yields that (3.8) implies (3.3). Therefore the assumptions of Theorem 3.1 are satisfied and this ensures that (3.9) holds.

Assume now, that (3.10) is in force. In such a case, Theorem 4.2 yields that

$$\lim_{a\to 0+} \sup_{\|f\|_{L^1(0,1)}\le 1} \left\|H_J^j(\chi_{(0,a)} f)\right\|_{L^\infty(0,1)} = 0.$$

Remark 3.2, together with Theorem 3.1 yields that

$$V_X^m L^1(\Omega) \hookrightarrow\hookrightarrow L^\infty(\Omega).$$

Standard embeddings $\mathbf{X}(\Omega) \hookrightarrow L^1(\Omega)$ and $L^\infty(\Omega) \hookrightarrow \mathbf{Y}(\Omega)$ (which are valid for all rearrangement-invariant spaces $\mathbf{X}(\Omega)$ and $\mathbf{Y}(\Omega)$) then conclude the proof. $\square$

Proof of Theorem 3.4. Consider the function $I(t) = t^{1-\frac{1}{Q}}$. It follows from the fact that Ω is an X-PS domain and from (2.7) that (2.2) holds with such $I(t)$. Simple computation yields that (3.6) holds as well.

An application of the Theorem 3.3 will yield the claim. We have

$$\lim_{t\to 0+} \frac{t^{m-1}}{(I(t))^m} = \lim_{t\to 0+} \frac{t^{m-1}}{t^{m-\frac{m}{Q}}} = \lim_{t\to 0+} t^{\frac{m}{Q}-1}. \tag{5.8}$$

Therefore, if $m > Q$, then $\lim_{t\to 0+} \frac{t^{m-1}}{(I(t))^m} = 0$ and Theorem 3.3 yields the claim.

On the other hand, if $m \le Q$ then $\lim_{t\to 0+} \frac{t^{m-1}}{(I(t))^m} \neq 0$. We have

$$\mathcal{Q}_Q^m f(t) = \int_t^1 \bigl|f(x)\bigr| s^{\frac{m}{Q}-1}\,ds = \int_t^1 \bigl|f(s)\bigr| \frac{s^{m-1}}{(s^{1-\frac{1}{Q}})^m}\,ds = K_I^m f(t),$$

for $f \in \mathfrak{M}(0,1)$ and $t \in (0,1)$. Consequently, if (3.11) holds (and $Q \le m$), then so does (3.8), and Theorem 3.3 yields the required embedding.

If $Q = m$, then we have

$$\mathcal{Q}_m^m f(t) = \int_t^1 \bigl|f(s)\bigr|\,ds = H_1^1 f(t), \quad f \in \mathfrak{M}(0,1).$$

Simple computation now yields $\mathcal{Q}_m^m \colon L^1(0,1) \to L^\infty(0,1)$. Consequently, since $L^\infty(0,1) \hookrightarrow \mathbf{Y}(0,1)$, we get $\mathcal{Q}_m^m \colon L^1(0,1) \to \mathbf{Y}(0,1)$. Application of Lemma 4.7 then yields the result. $\square$

The proof of Theorem3.5 rests on the following characterization of almost-compact embeddings between Lorentz–Zygmund spaces from [35, Proposition 7.12].

Theorem 5.1. *Let $p_1, p_2, q_1, q_2 \in [1,\infty]$, $\alpha_1, \alpha_2 \in \mathbb{R}$ be such that both triples $(p,q,\alpha) = (p_1,q_1,\alpha_1)$ and $(p,q,\alpha) = (p_2,q_2,\alpha_2)$ satisfy one of the conditions (2.8)–(2.11). Then*

$$L^{p_1,q_1;\alpha_1}(0,1) \overset{*}{\hookrightarrow} L^{p_2,q_2;\alpha_2}(0,1)$$

holds if and only of $p_1 > p_2$, or $p_1 = p_2$ and the following conditions are satisfied:

$$\text{if } p_1 = p_2 < \infty \text{ and } q_1 \le q_2, \text{ then } \alpha_1 > \alpha_2;$$
$$\text{if } p_1 = p_2 = \infty \text{ or } q_1 > q_2, \text{ then } \alpha_1 + \frac{1}{q_1} > \alpha_2 + \frac{1}{q_2}.$$

In particular, if $p_1, p_2, q_1, q_2 \in [1,\infty]$ are such that both triplets $(p,q,\alpha) = (p_1,q_1,0)$ and $(p,q,\alpha) = (p_2,q_2,0)$ satisfy one of the conditions (2.8)–(2.11), then

$$L^{p_1,q_1}(0,1) \overset{*}{\hookrightarrow} L^{p_2,q_2}(0,1)$$

holds if and only if $p_1 > p_2$.

Proof of Theorem 3.5. Using Theorem 3.4, we can reduce (3.13) to proving that

$$\mathcal{Q}_Q^m \colon L^{p_1,q_1}(0,1) \to\to L^{p_1,q_1}(0,1). \tag{5.9}$$

We are going to derive the embedding (5.9) from assumption of the Theorem 3.5. Assume first that $p_1 > \frac{Q}{m}$, and then Theorem 5.1 yields that

$$L^{p_1,q_1}(0,1) \overset{*}{\hookrightarrow} L^{\frac{Q}{m},1}(0,1).$$

Since $\int_0^1 \frac{1}{s^{1-\frac{m}{Q}}}\,ds < \infty$, application of Lemma 4.6 yields that

$$\|f\|_{L^{\frac{Q}{m},1}(0,1)} = \int_0^1 \frac{f^*(s)}{s^{1-\frac{m}{Q}}}\,ds \approx \|f\|_{(L^\infty)^d_{1,s^{1-\frac{m}{Q}}}(0,1)}. \tag{5.10}$$

Hence we get

$$L^{p_1,q_1}(0,1) \overset{*}{\hookrightarrow} (L^\infty)^d_{1,s^{1-\frac{m}{Q}}}(0,1). \tag{5.11}$$

Now, we want to use Theorem 4.5 to get required compactness of the operator $\mathcal{Q}^m_Q$. To this end, we need to verify assumptions of the corresponding theorems.

Since $p_1 > \frac{Q}{m}$ and $m < Q$, we have $L^{p_1,q_1}(0,1) \neq L^1(0,1)$. The role of function J in the claim of Theorem 4.5 plays function $J = s^{1-\frac{m}{Q}}(0,1)$, which satisfies condition (2.3). Finally,

$$\Big\| \Big(\int_t^1 \frac{ds}{J(s)} \Big)^1 \Big\|_{L^\infty(0,1)} = \int_0^1 \frac{1}{s^{1-\frac{m}{Q}}}\, ds < \infty,$$

which is the condition (4.3) with $J = s^{1-\frac{m}{Q}}(0,1)$, $j = 1$ and $\mathbf{Y}(0,1) = L^\infty(0,1)$. Theorem 4.5, combined with (5.11), now yields

$$\mathcal{Q}^m_Q \colon L^{p_1,q_1}(0,1) \to\to L^\infty(0,1).$$

The previous result, together with the embedding $L^\infty(0,1) \hookrightarrow L^{p_2,q_2}(0,1)$, now yields the claim.

Suppose now that $p_1 \le \frac{Q}{m}$. Assumptions of this theorem excludes cases when $L^{p_2,q_2}(0,1) = L^\infty(0,1)$ from consideration. This allows us to use the Theorem 4.3 to reduce (5.9) to

$$(L^{p_1,q_1})^r_{1,s^{1-\frac{m}{Q}}}(0,1) \overset{*}{\hookrightarrow} L^{p_2,q_2}(0,1). \tag{5.12}$$

It is shown in [7, Proposition 8.3] that $\mathbf{X}^r_{j,J}(0,1)$ is the smallest rearrangement-invariant space such that

$$H^j_J \colon \mathbf{X}(0,1) \to\to \mathbf{X}^r_{j,J}(0,1)$$

holds. In [7, Theorem 6.8.] the following characterization of such rearrangement-invariant space is given.

$$(L^{p_1,q_1})^r_{1,s^{1-\frac{m}{Q}}}(0,1) = \begin{cases} L^{\frac{p_1}{1-\frac{mp_1}{Q}},q_1}(0,1) & \text{if } \frac{m}{Q} < 1 \text{ and } 1 \le p_1 < \frac{Q}{m}, \\ L^{\infty,q_1;-1}(0,1) & \text{if } \frac{m}{Q} < 1 \text{ and } q_1 > 1, \\ L^\infty(0,1) & \text{otherwise.} \end{cases}$$

Consequently, if $p_1 < \frac{Q}{m}$ and

$$p_2 < \frac{p_1}{1 - \frac{mp_1}{Q}},$$

then the Theorem 5.1 ensures that (5.12) holds.

If $p_1 = \frac{Q}{m}$ then, again, Theorem 5.1 yields that (5.12) holds if $p_2 < \infty$. □

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Department of Mathematical Analysis, Charles University, Sokolovská 83, 186 75 Prague 8, Czech Republic.
E-mail address: martinfrancu@gmail.com