Banach J. Math. Anal. 12 (2018), no. 2, 314–330
https://doi.org/10.1215/17358787-2017-0047
ISSN: 1735-8787 (electronic)
http://projecteuclid.org/bjma

# LOWER AND UPPER LOCAL UNIFORM *K*-MONOTONICITY IN SYMMETRIC SPACES

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Communicated by D. E. Alspach

Abstract. Using the local approach to the global structure of a symmetric space $E$, we establish a relationship between strict $K$-monotonicity, lower (resp., upper) local uniform $K$-monotonicity, order continuity, and the Kadec–Klee property for global convergence in measure. We also answer the question: Under which condition does upper local uniform $K$-monotonicity coincide with upper local uniform monotonicity? Finally, we present a correlation between $K$-order continuity and lower local uniform $K$-monotonicity in a symmetric space $E$ under some additional assumptions on $E$.

## 1. Introduction

The first essential result devoted to *upper local uniform $K$-monotonicity* (ULUKM) was published in [5] by Chilin, Dodds, Sedaev, and Sukochev in 1996. The authors presented a complete characterization of ULUKM written in terms of strict $K$-monotonicity and the Kadec–Klee property for global convergence in measure in symmetric spaces, among others. Recently, many interesting results have appeared in [7], [12], and [11] (see also [4], [14]) exploring the global and local $K$-monotonicity structure of Banach spaces.

The crucial inspiration for our discussion can be found in [8], where we studied an application of strict $K$-monotonicity and $K$-order continuity to the best dominated approximation with respect to the Hardy–Littlewood–Pólya relation $\prec$. (It



Received Mar. 14, 2017; Accepted Apr. 12, 2017.

First published online Dec. 19, 2017.

2010 *Mathematics Subject Classification.* Primary 46E30; Secondary 46B20, 46B42.

*Keywords.* symmetric space, Lorentz space, $K$-order continuity, lower (upper) local uniform $K$-monotonicity, Kadec–Klee property for global convergence in measure.

is worth mentioning, in view of that previous result, that our work [9] will investigate, among other things, the full criteria for $K$-order continuity in symmetric spaces.) The main goal of the present article and our investigation is to develop a complete characterization of strict $K$-monotonicity and $K$-order continuity, as well as upper and lower local uniform $K$-monotonicity in symmetric spaces.

This article is organized as follows. Section 2 contains all the necessary definitions and notation. In Section 3, we focus on a characterization of lower and upper local uniform $K$-monotonicity in symmetric space $E$. First, we investigate a relation between a point of lower local uniform $K$-monotonicity and a point of lower local uniform monotonicity. We also characterize a full correlation between a point of lower local uniform $K$-monotonicity and a conjunction of a point of order continuity and a point of lower $K$-monotonicity and also an $H_g$ point in a symmetric space $E$. Next, we show a correspondence between a point of upper local uniform $K$-monotonicity and a point of upper local uniform monotonicity and also an $H_g$ point in $E$ under some additional assumptions. Our investigation is not restricted only to the local approach to $K$-monotonicity structure; we also discuss as a consequence a complete characterization of global $K$-monotonicity properties in a symmetric space $E$. We answer the crucial question: Under which condition does lower local uniform $K$-monotonicity and upper local uniform $K$-monotonicity coincide in symmetric spaces? In the spirit of the previous result, we also describe an essential connection between a point of $K$-order continuity and a point of lower local uniform $K$-monotonicity and also an $H_g$ point in a symmetric space $E$. It is worth noting that several results and examples concerning respective global properties are also presented in this section.

## 2. Preliminaries

Let $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{N}$ be the sets of reals, nonnegative reals, and positive integers, respectively. In a Banach space $(X, \|\cdot\|_X)$, we use the notation $S(X)$ (resp., $B(X)$) for the unit sphere (resp., closed unit ball). A nonnegative mapping $\phi$ given on $\mathbb{R}^+$ is called *quasiconcave* if $\phi(t)$ is increasing and $\phi(t)/t$ is decreasing on $\mathbb{R}^+$ and also $\phi(t) = 0 \Leftrightarrow t = 0$. Denote as usual by $\mu$ the Lebesgue measure on $I = [0, \alpha)$, where $\alpha = 1$ or $\alpha = \infty$, and denote by $L^0$ the set of all (equivalence classes of) extended real-valued Lebesgue measurable functions on $I$. We also use the notation $A^c = I \backslash A$ for any measurable set $A$. Let us recall that a Banach lattice $(E, \|\cdot\|_E)$ is said to be a *Banach function space* (or a *Köthe space*) if it is a sublattice of $L^0$ satisfying the following conditions.

(1) If $x \in L^0$, $y \in E$, and $|x| \leq |y|$ almost everywhere, then $x \in E$ and $\|x\|_E \leq \|y\|_E$.
(2) There exists a strictly positive $x \in E$.

In addition, we employ in our investigation the symbol $E^+ = \{x \in E : x \geq 0\}$.

An element $x \in E$ is said to be a *point of order continuity* if, for any sequence $(x_n) \subset E^+$ with $x_n \leq |x|$ and $x_n \to 0$ almost everywhere, we have $\|x_n\|_E \to 0$. A Banach function space $E$ is called *order continuous* ($E \in$ (OC) for short) if any element $x \in E$ is a point of order continuity (see [18]). It is said that a Banach function space $E$ has the *Fatou property* when for every $(x_n) \subset E^+$,

$\sup_{n\in\mathbb{N}} \|x_n\|_E < \infty$, and $x_n \uparrow x \in L^0$, we have $x \in E$ and $\|x_n\|_E \uparrow \|x\|_E$. In addition, we assume that $E$ has the Fatou property, unless mentioned otherwise.

An element $x \in E^+$ is called a *point of upper local uniform monotonicity* (resp., a *point of lower local uniform monotonicity*) or a *ULUM point* (resp., an *LLUM point*) if for any $(x_n) \subset E$ such that $x \le x_n$ and $\|x_n\|_E \to \|x\|_E$ (resp., $x_n \le x$ and $\|x_n\|_E \to \|x\|_E$), we get $\|x_n - x\|_E \to 0$. Let us recall that if each point of $E^+ \setminus \{0\}$ is a ULUM point (resp., an LLUM point), then we say that $E$ is *upper local uniformly monotone*, or $E \in$ (ULUM) (resp., *lower local uniformly monotone*, or $E \in$ (LLUM)).

An element $x \in E$ is said to be an $H_g$ *point* (resp., an $H_l$ *point*) in $E$ if for any sequence $(x_n) \subset E$ with $x_n \to x$ globally in measure (resp., locally in measure) and $\|x_n\|_E \to \|x\|_E$, then $\|x_n - x\|_E \to 0$. Let us recall that the space $E$ has the *Kadec–Klee property for global convergence in measure* (resp., *Kadec–Klee property for local convergence in measure*) if any element $x \in E$ is an $H_g$ point (resp., an $H_l$ point) in $E$ (see [5], [12]).

For any function $x \in L^0$, we define its *distribution function* by

$$d_x(\lambda) = \mu\{s \in [0,\alpha) : |x(s)| > \lambda\}, \quad \lambda \ge 0.$$

The *decreasing rearrangement* for any element $x \in L^0$ is given by

$$x^*(t) = \inf\{\lambda > 0 : d_x(\lambda) \le t\}, \quad t \ge 0.$$

Throughout the article, we use the notation $x^*(\infty) = \lim_{t\to\infty} x^*(t)$ if $\alpha = \infty$ and $x^*(\infty) = 0$ if $\alpha = 1$. For any function $x \in L^0$, we denote the *maximal function* of $x^*$ by

$$x^{**}(t) = \frac{1}{t}\int_0^t x^*(s)\,ds.$$

We mention that for any function $x \in L^0$, it is well known that $x^* \le x^{**}$, $x^{**}$ is decreasing, continuous, and subadditive. (For more details on $d_x$, $x^*$, and $x^{**}$, see [1], [17].)

We say that two functions $x, y \in L^0$ are *equimeasurable* ($x \sim y$ for short) if $d_x = d_y$. A Banach function space $(E, \|\cdot\|_E)$ is called *symmetric* or *rearrangement invariant* (r.i. for short) if for any $x \in L^0$ and $y \in E$ with $x \sim y$, we have $x \in E$ and $\|x\|_E = \|y\|_E$. In a symmetric space $E$, we denote by $\phi_E$ the *fundamental function* given by $\phi_E(t) = \|\chi_{(0,t)}\|_E$ for any $t \in [0,\alpha)$ (see [1]). For any two functions $x, y \in L^1 + L^\infty$, the *Hardy–Littlewood–Pólya relation* $\prec$ is defined by

$$x \prec y \Leftrightarrow x^{**}(t) \le y^{**}(t) \quad \text{for all } t > 0.$$

A symmetric space $E$ is called $K$*-monotone* ($E \in$ (KM) for short) if for any $x \in L^1 + L^\infty$ and $y \in E$ with $x \prec y$, we have $x \in E$ and $\|x\|_E \le \|y\|_E$. It is well known that a symmetric space is $K$-monotone if and only if $E$ is an exact interpolation space between $L^1$ and $L^\infty$. It is worth noting that a symmetric space $E$ equipped with an order continuous norm or with the Fatou property is $K$-monotone (see [17]).

An element $x \in E$ is said to be a *point of lower $K$-monotonicity* (an *LKM point* of $E$ for short) if for any $y \in E$, $x^* \neq y^*$ and $y \prec x$, we have $\|y\|_E < \|x\|_E$.

We note that a symmetric space $E$ is called *strictly $K$-monotone* ($E \in$ (SKM) for short) if any element of $E$ is an LKM point.

An element $x \in E$ is called a *point of $K$-order continuity* of $E$ if for any sequence $(x_n) \subset E$ with $x_n \prec x$ and $x_n^* \to 0$ almost everywhere, we have $\|x_n\|_E \to 0$. Recall that a symmetric space $E$ is said to be *$K$-order continuous* ($E \in$ (KOC) for short) if every element $x$ of $E$ is a point of $K$-order continuity.

An element $x \in E$ is said to be a *point of upper local uniform $K$-monotonicity* of $E$ (a *ULUKM point* for short) if for any $(x_n) \subset E$ such that $x \prec x_n$ for every $n \in \mathbb{N}$ and $\|x_n\|_E \to \|x\|_E$, we have $\|x^* - x_n^*\|_E \to 0$. An element $x \in E$ is said to be a *point of lower local uniform $K$-monotonicity* of $E$ (an *LLUKM point* for short) if for any $(x_n) \subset E$ with $x_n \prec x$ for all $n \in \mathbb{N}$ and $\|x_n\|_E \to \|x\|_E$, we have $\|x^* - x_n^*\|_E \to 0$. A symmetric space $E$ is said to be *upper local uniformly $K$-monotone* or $E \in$ (ULUKM) (resp., *lower local uniformly $K$-monotone* or ($E \in$ (LLUKM)) if every element of $E$ is a ULUKM point (resp., an LLUKM point). (We refer the reader to [5], [7]–[9], [14] for more details.)

Recall that the Marcinkiewicz function space $M_\phi^{(*)}$ (resp., $M_\phi$), where $\phi$ is a quasiconcave function on $I$, is a subspace of $L^0$ such that for all $x \in M_\phi^{(*)}$ (resp., $x \in M_\phi$),

$$\|x\|_{M_\phi^{(*)}} = \sup_{t>0}\{x^*(t)\phi(t)\} < \infty$$

$$\big(\text{resp., } \|x\|_{M_\phi} = \sup_{t>0}\{x^{**}(t)\phi(t)\} < \infty\big).$$

Obviously, $\|x\|_{M_\phi^{(*)}} \le \|x\|_{M_\phi}$ for all $x \in M_\phi$, that is, the embedding of $M_\phi$ in $M_\phi^{(*)}$ has norm 1 ($M_\phi \hookrightarrow M_\phi^{(*)}$ for short). Moreover, it should be noted that the Marcinkiewicz space $M_\phi^{(*)}$ (resp., $M_\phi$) is an r.i. quasi-Banach function space (resp., r.i. Banach function space) with the fundamental function $\phi$ on $I$. Let us also recall that for any symmetric space $E$ with the fundamental function $\phi$, we have the embedding $E \hookrightarrow M_\phi$ with norm 1 (see [1], [17]).

Given $0 < p < \infty$ and a locally integrable weight function $w \ge 0$, we define the Lorentz space $\Lambda_{p,w}$ as a subspace of $L^0$ such that

$$\|x\|_{\Lambda_{p,w}} = \Big(\int_0^\alpha \big(x^*(t)\big)^p w(t)\,dt\Big)^{1/p} < \infty,$$

where $W(t) = \int_0^t w < \infty$ for any $t \in I$ and $W(\infty) = \infty$ in the case when $\alpha = \infty$. It is worth mentioning that the spaces $\Lambda_{p,w}$ were introduced by Lorentz in [19], and the space $\Lambda_{p,w}$ is a norm space (resp., quasinorm space) if and only if $1 \le p < \infty$ and $w$ is decreasing (see [16]) (resp., $W$ satisfies the condition $\Delta_2$; see [21], [16]). It is also known that for any $0 < p < \infty$, if $W$ satisfies the condition $\Delta_2$ and $W(\infty) = \infty$, then the Lorentz space $\Lambda_{p,w}$ is an order continuous r.i. quasi-Banach function space (see [16]).

For $0 < p < \infty$ and $w \in L^0$ a nonnegative locally integrable weight function, we consider the Lorentz space $\Gamma_{p,w}$, that is, a subspace of $L^0$ such that

$$\|x\|_{\Gamma_{p,w}} = \|x^{**}\|_{\Lambda_{p,w}} = \Big(\int_0^\alpha x^{**p}(t) w(t)\,dt\Big)^{1/p} < \infty.$$

Unless stated otherwise, we suppose that $w$ belongs to the class $D_p$; that is,

$$W(s) := \int_0^s w(t)\,dt < \infty \quad \text{and} \quad W_p(s) := s^p \int_s^\alpha t^{-p} w(t)\,dt < \infty$$

for all $0 < s \le 1$ if $\alpha = 1$ and for all $0 < s < \infty$ otherwise. It is easy to observe that if $w \in D_p$, then the Lorentz space $\Gamma_{p,w}$ is nontrivial. Moreover, it is clear that $\Gamma_{p,w} \subset \Lambda_{p,w}$. On the other hand, the following inclusion $\Lambda_{p,w} \subset \Gamma_{p,w}$ holds if and only if $w \in B_p$ (see [15]). Let us also recall that $(\Gamma_{p,w}, \|\cdot\|_{\Gamma_{p,w}})$, introduced by Calderón in [3], is an r.i. quasi-Banach function space with the Fatou property. It is well known that in the case when $\alpha = \infty$, the Lorentz space $\Gamma_{p,w}$ has order continuous norm if and only if $\int_0^\infty w(t)\,dt = \infty$ (see [15]). It is also well known that by the Lions–Peetre $K$-method (see [2], [17]), the space $\Gamma_{p,w}$ is an interpolation space between $L^1$ and $L^\infty$. (For more details about the properties of the spaces $\Lambda_{p,w}$ and $\Gamma_{p,w}$, we refer the reader to [7], [10], [12], [15], [16].)

## 3. Lower and upper local uniform $K$-monotonicity in symmetric spaces

In this section, we investigate a connection between lower local uniform $K$-monotonicity and lower local uniform monotonicity in symmetric spaces. We also present a complete characterization of an LLUKM point in terms of a point of order continuity and an LKM point.

**Lemma 3.1.** *Let $E$ be a symmetric space. If $x \in E$ is an LLUKM point, then $x^*(\infty) = 0$.*

*Proof.* Suppose on the contrary that $x^*(\infty) > 0$. Define $x_n = x^*\chi_{[0,n]}$ for any $n \in \mathbb{N}$. Then, for any $n \in \mathbb{N}$, we have $0 \le x_n \le x^*$ and also $x_n \prec x$. It is clear that $x_n \uparrow x^*$ almost everywhere and $\sup_{n\in\mathbb{N}} \|x_n\|_E \le \|x\|_E < \infty$. Hence, by the Fatou property, we conclude that $\|x_n\|_E \to \|x\|_E$. Consequently, by the assumption that $x$ is an LLUKM point, it follows that

$$\|x_n^* - x^*\|_E \to 0.$$

Since $x^*(\infty) > 0$, we obtain $\chi_I \in E$, whence for any $n \in \mathbb{N}$,

$$\|x_n^* - x^*\|_E = \|x^*\chi_{(n,\infty)}\|_E \ge \left\|x^*(\infty)\chi_{(n,\infty)}\right\|_E = x^*(\infty)\|\chi_I\|_E > 0.$$

So, we get a contradiction which finishes the proof. □

**Lemma 3.2.** *Let $E$ be a symmetric space, and let $\phi$ be the fundamental function of $E$. If $x \in E$ is an LLUKM point and $x^*(t)\phi(t) \to 0$ as $t \to 0^+$, then $x$ is a point of order continuity.*

*Proof.* Let us assume, on the contrary, that $x$ is not a point of order continuity in $E$. Then, by Lemma 2.6 in [10] and Proposition 3.2 in [1], there exist $(A_n) \subset I$ a decreasing sequence of measurable sets and $\delta > 0$ such that $A_n \to \emptyset$ and

$$\delta \le \|x^*\chi_{A_n}\|_E \tag{1}$$

for all $n \in \mathbb{N}$. Let $\epsilon \in (0,\delta)$. We claim that there exists $K \in \mathbb{N}$ such that for every $k \geq K$,

$$\|x^*\chi_{[k,\infty)}\|_E < \frac{\epsilon}{2}.$$

Indeed, taking $x_n = x^*\chi_{[0,n)}$ for any $n \in \mathbb{N}$, we have $x_n = x_n^* \uparrow x^*$ and also $\sup_{n\in\mathbb{N}} \|x_n^*\|_E \leq \|x^*\|_E < \infty$. Hence, by the Fatou property and by symmetry of $E$, it follows that $\|x_n\|_E \to \|x\|_E$. Consequently, according to the assumption that $x$ is an LLUKM point, in view of $x_n \prec x$ we obtain our claim. Moreover, it is easy to see that $x^*\chi_{A_n\cap[0,k)} \prec x^*\chi_{[0,\min\{\mu(A_n),k\})}$ for any $k,n \in \mathbb{N}$, whence by symmetry and by the triangle inequality of the norm in $E$, we conclude that

$$\begin{aligned}\|x^*\chi_{A_n}\|_E &\leq \|x^*\chi_{A_n\cap[0,k)}\|_E + \|x^*\chi_{A_n\cap[k,\infty)}\|_E\\ &\leq \|x^*\chi_{[0,\min\{\mu(A_n),k\})}\|_E + \|x^*\chi_{A_n\cap[k,\infty)}\|_E\end{aligned}$$

for any $k,n \in \mathbb{N}$. Hence, since $\mu(A_n) < K$ for sufficiently large $n \in \mathbb{N}$, passing to subsequence and relabeling if necessary, by the claim and by condition (1) we get

$$\delta \leq \|x^*\chi_{A_n}\|_E \leq \|x^*\chi_{[0,\mu(A_n))}\|_E + \|x^*\chi_{A_n\cap[K,\infty)}\|_E \leq \|x^*\chi_{[0,\mu(A_n))}\|_E + \frac{\epsilon}{2}$$

for any $n \in \mathbb{N}$. Therefore, for any $n \in \mathbb{N}$ we have

$$\frac{\delta}{2} \leq \|x^*\chi_{[0,\mu(A_n))}\|_E. \tag{2}$$

Define $t_n = \mu(A_n)$ and $z_n = x^*(t_n)\chi_{[0,t_n)} + x^*\chi_{[t_n,\infty)}$ for all $n \in \mathbb{N}$. Clearly, $z_n = z_n^* \leq x^*$ for every $n \in \mathbb{N}$ and $z_n^* \uparrow x^*$ almost everywhere on $I$. As a consequence, since $\sup_{n\in\mathbb{N}} \|z_n^*\|_E \leq \|x^*\|_E$, by the Fatou property and by symmetry of $E$ this yields $\|z_n\|_E \to \|x\|_E$. Hence, since $z_n \prec x$ for any $n \in \mathbb{N}$ and by the assumption that $x$ is an LLUKM point, there exists $N \in \mathbb{N}$ such that for any $n \geq N$,

$$\left\|\left(x^* - x^*(t_n)\right)\chi_{[0,t_n)}\right\|_E < \frac{\epsilon}{4}.$$

So, by condition (2) and by the triangle inequality of the norm in $E$ we obtain

$$\begin{aligned}\frac{\delta}{2} &\leq \|x^*\chi_{[0,t_n)}\|_E \leq \left\|\left(x^* - x^*(t_n)\right)\chi_{[0,t_n)}\right\|_E + \left\|x^*(t_n)\chi_{[0,t_n)}\right\|_E\\ &\leq \frac{\epsilon}{4} + x^*(t_n)\phi(t_n)\end{aligned}$$

for all $n \geq N$. Consequently, for any $n \geq N$ we have

$$x^*(t_n)\phi(t_n) \geq \delta/4, \tag{3}$$

whence by the assumption that $x^*(t)\phi(t) \to 0$ as $t \to 0^+$ we get a contradiction, which ends the proof. ☐

Now, we answer the crucial question about whether the condition $\phi(t)x^*(t) \to 0$ as $t \to 0^+$ in Lemma 3.2 is necessary and whether it can be avoided. Namely, in the following example we provide a function, in the Lorentz space $\Lambda_{1,\psi'} \cap L^\infty$, that is an LLUKM point and not a point of order continuity.

*Example* 3.3. Let $\psi$ be a strictly concave function such that $\psi(0^+) = 0$ and $\psi(\infty) = \infty$. Consider $E = \Lambda_{1,\psi'} \cap L^\infty$ on $I = [0,1]$, equipped with an equivalent norm given by

$$\|x\|_E = \|x\|_{\Lambda_{1,\psi'}} + \|x\|_{L^\infty}$$

for any $x \in E$. Assuming that $\phi$ is the fundamental function of $E$, we easily observe that $\phi(t) = \psi(t) + 1$ for any $t > 0$. Define $x(t) = (1-t)\chi_{[0,1]}(t)$ for any $t \in I$. First, we prove that the function $x$ is not a point of order continuity in $E$. Indeed, taking $x_n = x\chi_{(0,1/n)}$ for any $n \in \mathbb{N}$, it is easy to see that $x_n \to 0$ almost everywhere and that $x_n \leq x$ for any $n \in \mathbb{N}$. Next, since $\lim_{t\to 0^+} \phi(t)x^*(t) = 1$, by Proposition 5.9 in [1], we have

$$\|x_n\|_E \geq \|x_n\|_{M_\phi} \geq \sup_{t\in(0,1/n]} \big\{(1-t)\big(1+\psi(t)\big)\big\} \geq 1$$

for all $n \in \mathbb{N}$. We claim that $x$ is an LLUKM point in $E$. Since $\psi(\infty) = \infty$ and $\psi(0^+) = 0$, by Proposition 1.4 in [15] it follows that the Lorentz space $\Lambda_{1,\psi'}$ is order continuous. Hence, since $\psi$ is strictly concave, by Theorem 2.11 in [5] we obtain that $\Lambda_{1,\psi'}$ is strictly $K$-monotone and also ULUKM; consequently, by Theorem 3.13, we conclude that $\Lambda_{1,\epsilon^*}$ is LLUKM. Hence, the Lorentz space $E$ endowed with the given norm is strictly $K$-monotone, whence $x$ is an LKM point in $E$. Assume that $(y_n) \subset E$, $y_n \prec x$ for any $n \in \mathbb{N}$ and $\|y_n\|_E \to \|x\|_E$. Then, since $x$ is an LKM point and $x^*(\infty) = 0$, by Theorem 1 in [9] it follows that $y_n^* \to x^*$ globally in measure. Therefore, by property 2.11 in [17] we get $y_n^*(t) \to x^*(t)$ for all $t \in [0,1]$. In consequence, by monotonicity of the decreasing rearrangement $y_n^*$ and by continuity of $x^*$ on $I$, in view of Dini's theorem for monotone functions (see [20, p. 81]) it follows that $y_n^*$ converges to $x^*$ uniformly on $I$; that is,

$$\|x^* - y_n^*\|_{L^\infty} \to 0. \tag{4}$$

So, it is clear that

$$\|y_n\|_{L^\infty} = y_n^*(0) \to x^*(0) = \|x\|_{L^\infty}.$$

Furthermore, by the assumption that $\|y_n\|_E \to \|x\|_E$ and by definition of the norm in $E$, we get $\|y_n\|_{\Lambda_{1,\psi'}} \to \|x\|_{\Lambda_{1,\psi'}}$. Thus, since $y_n \prec x$ for all $n \in \mathbb{N}$ and by the fact that $\Lambda_{1,\psi'}$ is LLUKM, we have

$$\|x^* - y_n^*\|_{\Lambda_{1,\psi'}} \to 0,$$

and consequently, in view of condition (4) and by definition of the norm in $E$, we are done.

**Proposition 3.4.** *Let $E$ be a symmetric space. If $E$ is LLUKM, then $E$ is order-continuous.*

*Proof.* Suppose for the contrary that there exists $x \in E$ that is not a point of order continuity. Let $\phi$ be the fundamental function of $E$. By symmetry of $E$ and by Proposition 5.9 in [1], we have, for any $t > 0$ and $z \in E$,

$$z^*(t)\phi(t) \leq \|z\|_{M_\phi^{(*)}} \leq \|z\|_{M_\phi} \leq \|z\|_E. \tag{5}$$

Next, proceeding similarly as in the proof of Lemma 3.2, in view of conditions (3) and (5) it is easy to see that

$$\frac{\delta}{4} \le \|x\|_{L^\infty}\phi(0^+) \le \|x\|_E.$$

Then, since $\phi(0^+) > 0$, by applying condition (5) for any $z \in E$, we observe that

$$\|z\|_{L^\infty}\phi(0^+) \le \|z\|_E. \tag{6}$$

Define $y = \chi_{[0,1)}$ and $y_n = \chi_{[0,1-1/n)}$ for any $n \in \mathbb{N}$. Obviously, by the Fatou property we get $\|y_n\|_E \to \|y\|_E$. Thus, since $y_n \prec y$ for all $n \in \mathbb{N}$, in view of the assumption that $E$ is LLUKM, we get

$$\|\chi_{[0,1/n)}\|_E = \|y^* - y_n^*\|_E \to 0.$$

Hence, by condition (6) we obtain a contradiction and complete the proof. □

**Theorem 3.5.** *Let $E$ be a symmetric space, and let $\phi$ be the fundamental function of $E$. If $x \in E$ is an LLUKM point and $\lim_{t\to 0^+} x^*(t)\phi(t) = 0$, then $|x|$ is an LLUM point.*

*Proof.* Let $(x_n) \subset E^+$ and $0 \le x_n \le |x|$, $\|x_n\|_E \to \|x\|_E$. Then, by property of the maximal function, we obtain $x_n \prec x$. Hence, by the assumption that $x$ is an LLUKM point, we have

$$\|x_n^* - x^*\|_E \to 0. \tag{7}$$

By Lemma 3.1, we get $x^*(\infty) = 0$, whence by Lemma 2.7 in [10] and by the assumption that $0 \le x_n \le |x|$ for all $n \in \mathbb{N}$, it follows that $x_n$ converges to $x$ in measure. Moreover, since $\lim_{t\to 0^+} x^*(t)\phi(t) = 0$, by Lemma 3.2 this yields that $x$ is a point of order continuity. Consequently, by condition (7) and by Proposition 2.4 in [13], we conclude that

$$\big\|x_n - |x|\big\|_E \to 0. \qquad \square$$

**Theorem 3.6.** *Let $E$ be a symmetric space on $I = [0,1)$, with $\phi$ the fundamental function of $E$. A point $x \in E$ is an LLUKM point and $\lim_{t\to 0^+} x^*(t)\phi(t) = 0$ if and only if $x$ is an LKM point and a point of order continuity.*

*Proof.* Our proof will consist of two parts.

*(Necessity)* Immediately, by Remark 3.1 in [7] and by Lemma 3.2, we complete the proof.

*(Sufficiency)* Let $(x_n) \subset E$, let $x_n \prec x$, and let $\|x_n\|_E \to \|x\|_E$. Since $x$ is a point of order continuity, it is easy to see that $\lim_{t\to 0^+} x^*(t)\phi(t) = 0$. Moreover, since $x^*(\infty) = 0$ and $x$ is an LKM point, by Theorem 1 in [9] we obtain that $x_n^*$ converges to $x^*$ in measure. Hence, by property 2.11 in [17], we get

$$(x_n^* - x^*)^+ \to 0 \quad \text{and} \quad (x^* - x_n^*)^+ \to 0 \tag{8}$$

almost everywhere and in measure on $I$. Note that, for any $n \in \mathbb{N}$, we have

$$(x_n^* - x^*)^+ \le x_n^* \quad \text{and} \quad (x^* - x_n^*)^+ \le \sup_{k\ge n}(x^* - x_k^*)^+ \le x^* \tag{9}$$

almost everywhere on $I$. In consequence, since $\sup_{k\geq n}(x^*-x_n^*)^+\downarrow 0$ almost everywhere and since $x$ is a point of order continuity, by Lemma 2.6 in [10] we obtain

$$\left\|(x^*-x_n^*)^+\right\|_E\to 0.$$

Thus, by the triangle inequality of the norm in $E$, to complete the proof it is enough to show the following condition:

$$\left\|(x_n^*-x^*)^+\right\|_E\to 0. \tag{10}$$

First, by [8, Lemma 3.1] it is clear that $x^{**}(\infty)=0$. Therefore, since $x_n^*\prec x^*$ for all $n\in\mathbb{N}$, by condition (9) it is easy to observe that for any $n\in\mathbb{N}$,

$$\left((x_n^*-x^*)^+\right)^*\leq x_n^*\leq x^{**}\quad\text{and}\quad(x_n^*-x^*)^+\prec x^*, \tag{11}$$

whence, by condition (8) and by property 2.12 in [17] we conclude that

$$\left((x_n^*-x^*)^+\right)^*\to 0 \tag{12}$$

pointwise and also in measure. Furthermore, by condition (11) and by Hardy's lemma (see [1, Proposition 3.6]) for any $y\in E$ and $t>0$, $n\in\mathbb{N}$, we have

$$\int_0^t\left((x_n^*-x^*)^+\right)^*y^*\leq\int_0^t x^*y^*. \tag{13}$$

Define, for any $n,k\in\mathbb{N}$,

$$M_n^k=\left\{t\in I:\left((x_n^*-x^*)^+\right)^*(t)>\frac{1}{k}\right\}.$$

Clearly, by condition (12), for any $k\in\mathbb{N}$ we have $\mu(M_n^k)\to 0$ as $n\to\infty$. Now, letting $y=\chi_{M_n^k}\in E$, by condition (13) and by symmetry of $E$, in view of Corollary 4.7 in [1] we get

$$\left\|\left((x_n^*-x^*)^+\right)^*\chi_{[0,\mu(M_n^k)]}\right\|_E\leq\|x^*\chi_{[0,\mu(M_n^k)]}\|_E$$

for every $n,k\in\mathbb{N}$. Thus, since $x^*\chi_{[0,\mu(M_n^k)]}\leq x^*$ almost everywhere on $I$ for all $n,k\in\mathbb{N}$ and since $x^*$ is a point of order continuity, it follows that, for any $k\in\mathbb{N}$ and $\epsilon>0$, there exists $N\in\mathbb{N}$ such that for any $n\geq N$,

$$\left\|\left((x_n^*-x^*)^+\right)^*\chi_{[0,\mu(M_n^k)]}\right\|_E\leq\frac{\epsilon}{2}.$$

Moreover, by construction of the set $M_n^k$, picking $k\in\mathbb{N}$ such that $\|\chi_I\|_E/k<\epsilon/2$, it is easy to see that

$$\left\|\left((x_n^*-x^*)^+\right)^*\chi_{(\mu(M_n^k),1)}\right\|_E\leq\left\|\frac{1}{k}\chi_{(\mu(M_n^k),1)}\right\|_E\leq\frac{\epsilon}{2}$$

for all $n\in\mathbb{N}$. Finally, by the triangle inequality of the norm in $E$, we prove condition (10) and finish the proof. □

Now, we investigate a similar result as above for a symmetric space $E$ on $[0,\infty)$ under some additional assumptions on $E$.

**Theorem 3.7.** *Let $E$ be a symmetric space on $I=[0,\infty)$, let $\phi$ be the fundamental function of $E$ such that $\phi(t)/t\to 0$ as $t\to\infty$, and let $x\in E\cap L^1$. A point $x$ is an LLUKM point and $\lim_{t\to 0^+}x^*(t)\phi(t)=0$ if and only if $x$ is an LKM point and a point of order continuity.*

*Proof.* Note that proceeding analogously as in the proof of Theorem 3.6 in sufficiency it is enough to show condition (10). First, let us mention that by Lemma 2.5 in [10] and by Lemma 3.1 in [8] and in view of the assumption that $x$ is a point of order continuity, it follows that $x^*(\infty)=x^{**}(\infty)=0$. Let $\epsilon>0$ and $t_\epsilon=d_{x^*}(\epsilon)$. Then it is clear that $t_\epsilon<\infty$, and so by the monotonicity of the decreasing rearrangement $x^*$, we obtain $x^*(t)\leq\epsilon$ for all $t\geq t_\epsilon$. To simplify our notation, let us assume that $y_n=(x_n^*-x^*)^+$ for any $n\in\mathbb{N}$. First, we claim that

$$\|y_n^*\chi_{[0,t_\epsilon)}\|_E\to 0. \tag{14}$$

Define a set

$$A_n=\big\{t\in[0,t_\epsilon]:x^*(t)\leq y_n^*(t)\big\}$$

for every $n\in\mathbb{N}$. Then, by the monotonicity of $x^*$, it is easy to see that $x^*(t)\geq\epsilon$ for any $t\leq t_\epsilon$. Next, in view of condition (12), we observe that

$$\mu(A_n)\leq\mu\big(t\in[0,t_\epsilon]:y_n^*(t)\geq\epsilon\big)\to 0. \tag{15}$$

Moreover, by condition (13) we obtain

$$\int_0^t y_n^*\chi_{[0,\mu(A_n)]}\leq\int_0^t x^*\chi_{[0,\mu(A_n)]}$$

for all $n\in\mathbb{N}$ and $t>0$. Hence, by Proposition 1.1 in [6], for any $t>0$ and $n\in\mathbb{N}$ we get

$$\begin{aligned}(y_n^*\chi_{A_n})^{**}(t)&=\frac{1}{t}\int_0^t(y_n^*\chi_{A_n})^*\leq\frac{1}{t}\int_0^t y_n^*\chi_{[0,\mu(A_n)]}\\&\leq(x^*\chi_{[0,\mu(A_n)]})^{**}(t)\leq x^{**}(t).\end{aligned}$$

Thus, by symmetry of $E$ we conclude that

$$\begin{aligned}\|y_n^*\chi_{[0,t_\epsilon)}\|_E&\leq\|y_n^*\chi_{A_n}\|_E+\|y_n^*\chi_{[0,t_\epsilon)\setminus A_n}\|_E\\&\leq\|x^*\chi_{[0,\mu(A_n)]}\|_E+\|y_n^*\chi_{[0,t_\epsilon)\setminus A_n}\|_E\end{aligned}$$

for each $n\in\mathbb{N}$. Consequently, since $y_n^*\chi_{[0,t_\epsilon)\setminus A_n}\leq x^*$ for any $n\in\mathbb{N}$, by conditions (12) and (15) as well as by the assumption that $x$ is a point of order continuity and in view of Lemma 2.6 in [10], we prove our claim (14). Now, without loss of generality, and passing to a subsequence and relabeling, we may assume that $y_n^*(t_\epsilon)>0$ for all $n\in\mathbb{N}$, because otherwise, in view of claim (14), we finish the proof. Furthermore, by condition (11) and by the assumption that $x\in E\cap L^1$, it is easy to see that

$$\int_{t_\epsilon}^\infty y_n^*\leq\int_0^\infty y_n^*\leq\int_0^\infty x^*<\infty$$

for all $n \in \mathbb{N}$. Denote, for any $n \in \mathbb{N}$,

$$\delta_n = t_\epsilon + \frac{1}{y_n^*(t_\epsilon)} \int_{t_\epsilon}^{\infty} y_n^* \quad \text{and} \quad z_n = y_n^* \chi_{[0,t_\epsilon)} + y_n^*(t_\epsilon)\chi_{[t_\epsilon,\delta_n)}.$$

Now, we prove that

$$\left\| y_n^*(t_\epsilon)\chi_{[t_\epsilon,\delta_n)} \right\|_E \to 0. \tag{16}$$

Assume to the contrary that $a = \inf_{n\in\mathbb{N}} \| y_n^*(t_\epsilon)\chi_{[t_\epsilon,\delta_n)} \|_E > 0$. Then, passing to a subsequence and relabeling if necessary, we obtain

$$\left\| y_n^*(t_\epsilon)\chi_{[t_\epsilon,\delta_n)} \right\|_E \downarrow a.$$

Hence, for any $n \in \mathbb{N}$, we note that

$$\begin{aligned} a &\le \left\| y_n^*(t_\epsilon)\chi_{[t_\epsilon,\delta_n)} \right\|_E = y_n^*(t_\epsilon)\phi(\delta_n - t_\epsilon) \\ &= y_n^*(t_\epsilon)\phi\Big(\frac{1}{y_n^*(t_\epsilon)} \int_{t_\epsilon}^{\infty} y_n^*\Big) \\ &\le y_n^*(t_\epsilon)\phi\Big(\frac{1}{y_n^*(t_\epsilon)} \int_0^{\infty} x^*\Big). \end{aligned}$$

Therefore, letting $s_n = \int_0^\infty x^*/y_n^*(t_\epsilon)$ for all $n \in \mathbb{N}$, we have

$$a \le \frac{\phi(s_n)}{s_n} \int_0^\infty x^*$$

for all $n \in \mathbb{N}$. According to condition (12), we observe that $y_n^*(t_\epsilon) \to 0$ and so $s_n \to \infty$. In consequence, by the assumption that $\phi(t)/t \to 0$ as $t \to \infty$, we get a contradiction which provides condition (16). Now, we show that $y_n \prec z_n$ for all $n \in \mathbb{N}$. Obviously, $y_n^{**} = z_n^{**}$ on $[0, t_\epsilon]$ for each $n \in \mathbb{N}$. Moreover, for any $n \in \mathbb{N}$ and $t \in (t_\epsilon, \delta_n)$, we have

$$\int_0^t z_n^* = \int_0^{t_\epsilon} y_n^* + y_n^*(t_\epsilon)(t - t_\epsilon) \ge \int_0^{t_\epsilon} y_n^* + \int_{t_\epsilon}^{t} y_n^* = \int_0^t y_n^*,$$

and also, for any $t \ge \delta_n$, we have

$$\int_0^t z_n^* = \int_0^{t_\epsilon} y_n^* + y_n^*(t_\epsilon)(\delta_n - t_\epsilon) = \int_0^{t_\epsilon} y_n^* + \int_{t_\epsilon}^{\infty} y_n^* \ge \int_0^t y_n^*.$$

Therefore, by symmetry of $E$ we get $\|z_n\|_E \ge \|y_n\|_E$. Thus, by conditions (14) and (16) and by the triangle inequality of the norm in $E$, we complete the proof. □

Immediately, in view of Remark 3.1 in [7], and by Proposition 3.4 and Theorems 3.6 and 3.7, we obtain the following results.

**Corollary 3.8.** *Let $E$ be a symmetric space on $I = [0, \alpha)$ with $\alpha < \infty$. The space $E$ is LLUKM if and only if $E$ is strictly $K$-monotone and order continuous.*

**Corollary 3.9.** *Let $E$ be a symmetric space on $I = [0, \infty)$ with the fundamental function $\phi$ such that $\phi(t)/t \to 0$ as $t \to \infty$, and let $F \subset E$ be a symmetric sublattice that is embedded in $L^1[0, \infty)$. Then, the space $F$ is LLUKM if and only if $F$ is strictly $K$-monotone and order continuous.*

Now, we investigate a relation between lower local uniform $K$-monotonicity and the Kadec–Klee property for global convergence in measure. First, we show an example of a function in a symmetric space $E$ on $I = [0, \infty)$ that is a point of lower local uniform $K$-monotonicity but is not an $H_g$ point in $E$. We also discuss in this example a symmetric space $E$ on $I = [0, 1)$ that is lower local uniformly $K$-monotone, but does not have the Kadec–Klee property for global convergence in measure. We recall Example 2.8 in [5] and we modify to the case when $I = [0, \alpha)$, where $\alpha \leq \infty$. For the reader's convenience, we present the details of the modified example.

*Example* 3.10. Let $\delta > 0$, and let $\phi_1$, $\phi_2$ be strictly concave functions such that

$$\phi_i(0) = \phi_i(0^+) = 0 \quad \text{and} \quad \phi_i(\infty) = \lim_{t\to\infty} \phi_i(t) = \infty \quad \text{for } i = 1, 2,$$

and also

$$\phi_2(1) > \phi_1(1) + \delta \quad \text{and} \quad \lim_{t\to 0} \frac{\phi_2(t)}{\phi_1(t)} = \lim_{t\to\infty} \frac{\phi_i(t)}{t} = 0 \quad \text{for } i = 1, 2.$$

Consider the space $E = \Lambda_{1,\phi_1'} \cap \Lambda_{1,\phi_2'}$ with a norm given by

$$\|x\|_E = \max\left\{\|x\|_{\Lambda_{1,\phi_1'}}, \|x\|_{\Lambda_{1,\phi_2'}}\right\}$$

for all $x \in E$. Since $\phi_i(\infty) = \infty$ for $i = 1, 2$, it follows that the symmetric space $E$ is order continuous (see [5], [15]). Hence, since $\phi_1$ and $\phi_2$ are strictly concave, by Theorem 2.11 in [5] we get that $E$ is strictly $K$-monotone. Consequently, in the case when $I = [0, 1)$, by Corollary 3.8 we obtain that $E$ is LLUKM. Define

$$x = \chi_{[0,1]} \quad \text{and} \quad x_n = x + \frac{\delta}{\phi_1(\frac{1}{n})} \chi_{[0,\frac{1}{n})}$$

for any $n \in \mathbb{N}$. Obviously, $x_n \to x$ in measure and

$$\|x_n\|_E = \frac{\delta\phi_2(\frac{1}{n})}{\phi_1(\frac{1}{n})} + \phi_2(1) \to \phi_2(1) = \|x\|_E.$$

On the other hand, we observe that $\|x_n - x\|_E \geq \delta$ for any $n \in \mathbb{N}$, from which we infer that $x$ is not an $H_g$ point in $E$, and consequently, $E$ does not have the Kadec–Klee property for global convergence in measure. However, since $x \in L^1[0, \infty)$, by Theorem 3.7 we get that $x$ is an LLUKM point in the space $E$ on $I = [0, \infty)$.

**Theorem 3.11.** *Let $E$ be a symmetric space, let $x, x_n \in E$ with $x^*(\infty) = 0$, and*

(i) *let $x$ be an LKM point and an $H_g$ point;*
(ii) *let $x$ be an LKM point and*

$$x_n^{**} \to x^{**} \quad \text{in measure}, \qquad \|x_n\|_E \to \|x\|_E \quad \Rightarrow \quad \|x_n^* - x^*\|_E \to 0;$$

(iii) *let $x$ be an LLUKM point.*

*Then, (i) $\Rightarrow$ (ii) $\Rightarrow$(iii). If $x$ is an $H_g$ point, then (iii) $\Rightarrow$ (i).*

*Proof.* (i) $\Rightarrow$ (ii) Let $x, x_n \in E$ for any $n \in \mathbb{N}$, $x_n^{**} \to x^{**}$ in measure, and let $\|x_n\|_E \to \|x\|_E$. Now, proceeding analogously as in the proof of Theorem 3.8 in [7], under the assumption that $x$ is an $H_g$ point and $x^*(\infty) = 0$, in view of Theorem 3.3 in [12] we complete the proof.

(ii) $\Rightarrow$ (iii) Let $x, x_n \in E$, $x_n \prec x$ for any $n \in \mathbb{N}$, and let $\|x_n\|_E \to \|x\|_E$. Hence, by Theorem 1 in [9], it follows that $x_n^{**} \to x^{**}$ in measure. Therefore, by condition $(ii)$ we get $\|x_n^* - x^*\|_E \to 0$, which proves that $x$ is an LLUKM point.

(iii) $\Rightarrow$ (i) Let $x$ be an $H_g$ point in $E$. Immediately, by Remark 3.1 in [7], we get that $x$ is an LKM point, and this ends the proof. □

In the next example, we present a symmetric space with the Kadec–Klee property for global convergence in measure which does not have the LLUKM property.

*Example* 3.12. Consider the Lorentz space $\Gamma_{p,w}$ with $0 < p < \infty$, and let $w$ be a nonnegative weight function. If $W(\infty) < \infty$ or $W(t) = \int_0^t w$ is not strictly increasing, then by Proposition 1.4 in [15] or by Theorem 2.10 in [11], respectively, we obtain that the Lorentz space $\Gamma_{p,w}$ is not order continuous or that it is not strictly $K$-monotone, respectively. Moreover, we have $\lim_{t\to 0^+} \|x^*\chi_{[0,t)}\|_{\Gamma_{p,w}} = 0$ (see [15]), whence and by the monotonicity of the decreasing rearrangement $x^*$ we get $\lim_{t\to 0^+} x^*(t)\phi(t) = 0$, where $\phi$ is the fundamental function of $\Gamma_{p,w}$. In consequence, by Remark 3.1 in [7] or by Lemma 3.2, respectively, it follows that $\Gamma_{p,w}$ is not LLUKM. On the other hand, by Theorem 4.1 in [12] we know that the Lorentz space $\Gamma_{p,w}$ has the Kadec–Klee property for global convergence in measure.

Now, we present the full characterization of lower and upper local uniform $K$ monotonicity in a symmetric space $E$ with order continuous norm. Then we establish a correlation between upper local uniform $K$-monotonicity and upper local uniform monotonicity in $E$.

**Theorem 3.13.** *Let $E$ be a symmetric space with order continuous norm. Then, the following conditions are equivalent:*

(i) *$E$ is SKM and for any $(x_n) \subset E$, $x \in E$,*

$$x_n^{**} \to x^{**} \quad \text{in measure and} \quad \|x_n\|_E \to \|x\|_E \quad \Rightarrow \quad \|x_n^* - x^*\|_E \to 0;$$

(ii) *$E$ is LLUKM and has the Kadec–Klee property for global convergence in measure;*

(iii) *$E$ is SKM and has the Kadec–Klee property for global convergence in measure;*

(iv) *$E$ is SKM and has the Kadec–Klee property for local convergence in measure;*

(v) *$E$ is ULUKM.*

*Proof.* It is well known that the equivalences (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) follow directly from Theorem 2.7 in [5]. Immediately, by Theorem 3.8 in [7] and by Theorem 3.5 in [12], we get (i) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (v). Finally, the consequence of Lemma 2.5 in [10] and Theorem 3.11 is the following conclusion (ii) $\Leftrightarrow$ (iii). □

**Theorem 3.14.** *Let $E$ be a symmetric space. If $x \in E$ is a point of order continuity and a ULUKM point, then $|x|$ is a ULUM point and $x$ is an $H_g$ point.*

*Proof.* Let $(x_n) \subset E^+$, $|x| \leq x_n$, and $\|x_n\|_E \to \|x\|_E$. Then, by Proposition 3.2 in [1] we get $x \prec x_n$ for all $n \in \mathbb{N}$, and consequently, by the assumption that $x$ is a ULUKM point, we have $\|x_n^* - x^*\|_E \to 0$. Hence, by the implication (iii) $\Rightarrow$ (ii) in [5, proof of Theorem 3.2], it follows that $x_n$ converges to $|x|$ in measure. Consequently, by the assumption that $x$ is a point of order continuity and by Proposition 2.4 in [13], we have $\|x_n - |x|\|_E \to 0$. Finally, in view of the assumptions, by Theorem 3.8 in [7] and by Theorem 3.5 in [12], we conclude that $x$ is an $H_g$ point in $E$. □

In the next example, we show that if the assumption that $x$ is a point of order continuity of the above theorem is missing, then the implication is not true.

*Example* 3.15. Take $E = L^\infty$ on $I = [0, \infty)$ and $x = \chi_I$. Let $(x_n) \subset E$ be such that $x \prec x_n$ for any $n \in \mathbb{N}$, and let $\|x_n\|_E \to \|x\|_E$. Since $x^* = 1$ on $I$, we claim that $x^* \leq x_n^*$ almost everywhere for all $n \in \mathbb{N}$. Indeed, if it is not true, then there exist $(n_k) \subset \mathbb{N}$ and $(t_k) \subset I$ such that, for any $k \in \mathbb{N}$ and $t \geq t_k$, we have

$$x_{n_k}^*(t) \leq x_{n_k}^*(t_k) < 1.$$

Hence, setting $k \in \mathbb{N}$, we observe that for sufficiently large $t > t_k$,

$$x_{n_k}^{**}(t) < x^{**}(t) = 1.$$

Therefore, by the assumption that $x \prec x_n$ for all $n \in \mathbb{N}$, we get a contradiction which proves our claim. It is easy to see that $x$ is a ULUM point in $E$ (see [10]). Thus, according to the claim and by the assumption that $\|x_n^*\|_E \to \|x^*\|_E$, we obtain

$$\|x_n^* - x^*\|_E \to 0.$$

In consequence, we get that $x$ is a ULUKM point. On the other hand, taking $y_n = \chi_{(\frac{1}{n}, \infty)}$ for any $n \in \mathbb{N}$, it is easy to see that $y_n \to x$ in measure and $\|y_n\|_E = \|x\|_E = 1$, and also that $\|x - y_n\|_E = 1$ for every $n \in \mathbb{N}$. So, it follows that $x$ is not an $H_g$ point in $E$.

Now we discuss a correlation between $K$-order continuity and lower local uniform $K$-monotonicity in symmetric spaces.

**Theorem 3.16.** *Let $E$ be a symmetric space. If $x \in E$ is a point of $K$-order continuity and an LKM point and also $x^*(\infty) = 0$, then $x$ is an LLUKM point.*

*Proof.* Let $(x_n) \subset E$ with $x_n \prec x$ for all $n \in \mathbb{N}$, and let $\|x_n\|_E \to \|x\|_E$. Observe that for each $n \in \mathbb{N}$,

$$(x^* - x_n^*)^+ \leq x^* \quad \text{and} \quad (x_n^* - x^*)^+ \prec x_n^* \prec x^*. \tag{17}$$

Moreover, since $x$ is an LKM point and $x^*(\infty) = 0$, by the assumption that $x_n \prec x$ for any $n \in \mathbb{N}$ and $\|x_n\|_E \to \|x\|_E$ and by Theorem 1 in [9], it follows that $x_n^*$ converges to $x^*$ in measure. Hence, by property 2.11 in [17], we get

$$\left((x_n^* - x^*)^+\right)^* \to 0 \quad \text{and} \quad \left((x^* - x_n^*)^+\right)^* \to 0$$

almost everywhere on $I$. In consequence, by condition (17) and by the assumption that $x$ is a point of $K$-order continuity, we have

$$\left\|\left((x^*-x_n^*)^+\right)^*\right\|_E \to 0 \quad \text{and} \quad \left\|\left((x_n^*-x^*)^+\right)^*\right\|_E \to 0.$$

Thus, by symmetry of $E$ and by the triangle inequality of the norm in $E$, we conclude that $x_n^*$ converges to $x^*$ in norm of $E$. □

We present an example of a symmetric space having upper and lower local uniform $K$-monotonicity but not satisfying $K$-order continuity.

*Remark* 3.17. Let $\psi(t) = t^{1/4}$ for any $t \in I$. Consider the space $E = \Lambda_{1,\psi'} \cap L^1$ on $I$ endowed with the equivalent norm given by $\|x\|_E = \|x\|_{\Lambda_{1,\psi'}} + \|x\|_{L^1}$. We claim that $(E, \|\cdot\|_E)$ is LLUKM and ULUKM, but it is not KOC. First, denote $\phi(t) = \psi(t) + t$ for any $t \in I$. Observe that $E = \Lambda_{1,\phi'}$ and $\phi(t)/t \to 1$ as $t \to \infty$. Define

$$x(t) = \chi_{[0,1)}(t) + \frac{1}{t^2}\chi_{[1,\infty)}(t) \quad \text{and} \quad x_n(t) = \frac{1}{n}\chi_{[0,n)}(t)$$

for any $t > 0$ and $n \in \mathbb{N}$. It is easy to see that $x = x^*$, $x_n = x_n^* \to 0$ almost everywhere. Clearly,

$$x^{**}(t) = \chi_{[0,1)}(t) + \frac{2t-1}{t^2}\chi_{[1,\infty)}(t)$$

and

$$x_n^{**}(t) = \frac{1}{n}\chi_{[0,n)}(t) + \frac{1}{t}\chi_{[n,\infty)}(t)$$

for any $t > 0$ and $n \in \mathbb{N}$, whence $x_n \prec x$ for all $n \in \mathbb{N}$. Note that $x \in E$ and

$$\|x_n\|_E = \|x_n\|_{\Lambda_{1,\psi'}} + \|x_n\|_{L^1} = 1 + \frac{1}{n^{3/4}}$$

for any $n \in \mathbb{N}$. Therefore, $\|x_n\|_E \geq 1$ for every $n \in \mathbb{N}$, from which we infer that $E$ is not KOC. On the other hand, since $\phi(\infty) = \int_0^\infty \phi' = \infty$, by Proposition 1.4 in [15], it follows that the Lorentz space $\Lambda_{1,\phi'}$ is order continuous. Hence, since $\phi$ is strictly concave, by Theorem 2.11 and Proposition 1.7 in [5], we obtain that $\Lambda_{1,\phi'}$ is strictly $K$-monotone and also has the Kadec–Klee property for global convergence in measure. Finally, by Theorem 3.13, we get that $E$ is ULUKM and LLUKM.

According to Theorem 2 in [9] and Remark 3.1 in [7], and also by Lemma 3.2 as well as Theorem 3.16, we conclude with the next theorem.

**Theorem 3.18.** *Let $E$ be a symmetric space on $I = [0,\infty)$, and let $\phi$ be the fundamental function of $E$ and $x \in E$. Then the following conditions are equivalent:*

(i) *$x$ is an LLUKM point and*

$$\lim_{t\to 0^+} \phi(t)x^*(t) = \lim_{s\to\infty} \phi(s)x^{**}(s) = 0;$$

(ii) *$x$ is an LKM point and a point of order continuity, and*

$$\lim_{s\to\infty} \phi(s)x^{**}(s) = 0;$$

(iii) *$x$ is an LKM point and a point of $K$-order continuity, and $x^*(\infty) = 0$.*

**Acknowledgment.** We wish to express our gratitude to the reviewer for many valuable suggestions and remarks.

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