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ON THE UNIVERSAL FUNCTION FOR WEIGHTED SPACES $L^p_\mu[0,1]$, $p \geq 1$

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Abstract. In this article, we show that there exist a function $g \in L^1[0,1]$ and a weight function $0 < \mu(x) \leq 1$ so that g is universal for each class $L^p_\mu[0,1]$, $p \geq 1$, with respect to signs-subseries of its Fourier–Walsh series.

1. Introduction and preliminaries

Let $|E|$ be the Lebesgue measure of a measurable set $E \subseteq [0,1]$, let $\chi_E(x)$ be its characteristic function, let $L^p(E)$ $(p > 0)$ be the class of all those measurable functions on E that satisfy the condition $\int_E |f(x)|^p\,dx < +\infty$, let $L^p_\mu[0,1]$ (weighted space) be the class of all those measurable functions on $[0,1]$ that satisfy the condition $\int_0^1 |f(x)|^p \mu(x)\,dx < +\infty$, where $0 < \mu(x) \leq 1$ is a weight function (known as the *Muckenhoupt A_p class*; see, e.g., [4], [21]–[23]), and let $\{\varphi_k\}$ be a complete orthonormal system in $L^2[0,1]$.

Definition 1.1. Let $0 < \mu(x) \leq 1$ be a measurable function on the set $[0,1]$. We say that a function $g \in L^1[0,1]$ is *universal* for a class $L^p_\mu[0,1]$ with respect to signs-subseries of its Fourier series by the system $\{\varphi_k\}$, if for each function

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$f \in L^p_\mu[0,1]$ one can choose numbers $\delta_k = \pm 1, 0$ so that the series

$$\sum_{k=0}^{\infty} \delta_k c_k(g)\varphi_k(x) \quad \text{with } c_k(g) = \int_0^1 g(x)\varphi_k(x)\,dx,$$

converges to f in $L^p_\mu[0,1]$ metric; that is,

$$\lim_{m\to\infty} \int_0^1 \Big|\sum_{k=0}^{m} \delta_k c_k(g)\varphi_k(x) - f(x)\Big|^p \mu(x)\,dx = 0.$$

Let us recall the definition of the Walsh orthonormal system $\{W_n(x)\}_{n=0}^{\infty}$. Functions of the Walsh system are defined by means of Rademacher's functions

$$R_n(x) = \operatorname{sign}(\sin 2^n \pi x), \quad x \in [0,1], n = 1, 2, \ldots,$$

in the following way (see [6]): $W_0(x) \equiv 1$ and for $n \geq 1$,

$$W_n(x) = \prod_{i=1}^{p} R_{k_i+1}(x),$$

where $n = 2^{k_1} + 2^{k_2} + \cdots + 2^{k_p}$ $(k_1 > k_2 > \cdots > k_p)$. In the present paper, we prove the following theorem for the Walsh system.

Theorem 1.2. *There exist a function $g \in L^1[0,1]$ and a weight function $0 < \mu(x) \leq 1$ so that g is universal for each class $L^p_\mu[0,1]$, $p \geq 1$, with respect to signs-subseries of its Fourier–Walsh series.*

Moreover, it will be shown that the measure of the set on which $\mu(x) = 1$ can be made arbitrarily close to 1, and the function $g \in L^1[0,1]$ can be chosen to have strictly decreasing Fourier–Walsh coefficients and converging to it by the $L^1[0,1]$ norm Fourier–Walsh series.

Remark 1.3. In the proved theorem, the weight function $\mu(x)$ cannot be made equal to 1 everywhere in $[0,1]$. Moreover, there does not exist a universal function $g \in L^1[0,1]$ (defined above) for any class $L^p[0,1]$, $p \geq 1$.

It can be easily shown that the assumption of existence of such a universal function simply leads to a contradiction. Indeed, if that assumption were true, then for the function $k_0 c_{k_0}(g) W_{k_0}(x)$, where $k_0 > 1$ is any natural number with condition $c_{k_0}(g) \neq 0$, one could find numbers $\delta_k = \pm 1, 0$ so that

$$\lim_{m\to\infty} \int_0^1 \Big|\sum_{k=0}^{m} \delta_k c_k(g) W_k(x) - k_0 c_{k_0}(g) W_{k_0}(x)\Big|^p dx = 0.$$

Hence, we would simply get a contradiction: $\delta_{k_0} = k_0 > 1$.

The existence of functions, which are universal in different senses, have been considered by mathematicians since the beginning of the twentieth century. The first type of universal function was considered by Birkhoff [1] in 1929. He proved that there exists an entire function $g(z)$ which is universal with respect to translations; that is, for every entire function $f(z)$ and for each number $r > 0$, there exists a growing sequence of natural numbers $\{n_k\}_{k=1}^{\infty}$ so that the sequence

$\{g(z+n_k)\}_{k=1}^{\infty}$ uniformly converges to $f(z)$ on $|z| \le r$. In 1952, MacLane [16] proved a similar result for another type of universality, namely, that there exists an entire function $g(z)$ which is universal with respect to derivatives; that is, for every entire function $f(z)$ and for each number $r > 0$, there exists a growing sequence of natural numbers $\{n_k\}_{k=1}^{\infty}$ so that the sequence $\{g^{(n_k)}(z)\}_{k=1}^{\infty}$ uniformly converges to $f(z)$ on $|z| \le r$. Furthermore, in 1975, Voronin [26] proved the universality theorem for the Riemann zeta function $\zeta(s)$, which states that any nonvanishing analytic function can be approximated uniformly by certain purely imaginary shifts of the zeta function in the critical strip; namely, if $0 < r < \frac{1}{4}$ and $g(s)$ is a nonvanishing continuous function on the disk $|s| \le r$, that is, analytic in the interior, then for any $\varepsilon > 0$, there exists such a positive real number τ such that

$$\max_{|s|\le r}\big|g(s) - \zeta(s + 3/4 + i\tau)\big| < \varepsilon.$$

In 1987, Grosse-Erdmann [13] proved the existence of infinitely differentiable functions with universal Taylor expansion, namely, that there exists a function $g \in C^{\infty}(\mathcal{R})$ with $g(0) = 0$ such that, for every function $f \in C(\mathcal{R})$ with $f(0) = 0$ and for each number $r > 0$, there exists a growing sequence of natural numbers $\{n_k\}_{k=1}^{\infty}$ so that the sequence

$$S_{n_k}(g, 0) = \sum_{m=1}^{n_k} \frac{g^{(m)}(0)}{m!} x^m$$

uniformly converges to $f(x)$ on $|x| \le r$.

The first-named author and Sargsyan [11], [12] studied the existence of universal functions for classes $L^p[0,1]$, $p \in (0,1)$, with respect to signs-subseries of Fourier–Walsh series and signs of Fourier–Walsh coefficients. In particular, it was shown in [11] that for each number $p \in (0,1)$ one can construct a function from $L^1[0,1]$ with convergent in $L^1[0,1]$ Fourier–Walsh series having decreasing coefficients, which is universal for the class $L^p[0,1]$ with respect to signs-subseries of Fourier–Walsh series.

Note that the definition of function universality which we gave above could have been framed correspondingly in terms of Fourier series universality. The topic of the existence of universal series (in the common sense, e.g., with respect to rearrangements, partial series, signs of coefficients) in various classical orthogonal systems was also investigated extensively. The most general results were obtained by Menshov [17], Talaljan [24], Ulyanov [25], and their colleagues (see [2], [3], [5], [7]–[10], [14], [15], [19], [20]).

The following questions, which arise in regard to the result of the present article, have yet to be answered.

Question 1.4. Is Theorem 1.2 true for other orthonormal systems (e.g., trigonometric system, Franklin system, etc.)?

Question 1.5. Is it possible to achieve universality with respect to signs of Fourier–Walsh coefficients (i.e., exclude zero values from the sequence δ_k) in Theorem 1.2?

2. Main lemmas

Let us start from known properties of the Walsh system, which will be used during the proofs. It is known (see [6]) that for each natural number m,

$$\sum_{k=0}^{2^m-1} W_k(x) = \begin{cases} 2^m, & \text{when } x \in [0, 2^{-m}), \\ 0, & \text{when } x \in (2^{-m}, 1], \end{cases} \tag{2.1}$$

and, consequently,

$$\sum_{k=2^m}^{2^{m+1}-1} W_k(x) = \begin{cases} 2^m, & \text{when } x \in [0, 2^{-m-1}), \\ -2^m, & \text{when } x \in (2^{-m-1}, 2^{-m}), \\ 0, & \text{when } x \in (2^{-m}, 1]. \end{cases}$$

Thus, for each $p > 0$ we have

$$\int_0^1 \Big| \sum_{k=2^m}^{2^{m+1}-1} W_k(x) \Big|^p dx = 2^{m(p-1)}. \tag{2.2}$$

Let

$$\| \cdot \|_{L^p(E)} = \Big(\int_E | \cdot |^p \, dx \Big)^{\frac{1}{p}} \qquad \text{and} \qquad \| \cdot \|_{L^p_\mu[0,1]} = \Big(\int_0^1 | \cdot |^p \mu(x) \, dx \Big)^{\frac{1}{p}}$$

(where $p \geq 1$, $E \subseteq [0,1]$, and $0 < \mu(x) \leq 1$) be the norms of spaces $L^p(E)$ and $L^p_\mu[0,1]$, respectively. Obviously, for any natural number $M \in [2^m, 2^{m+1})$ and real numbers $\{a_k\}_{k=2^m}^{2^{m+1}-1}$ we have

$$\Big\| \sum_{k=2^m}^{M} a_k W_k \Big\|_{L^1[0,1]} \leq \Big\| \sum_{k=2^m}^{2^{m+1}-1} a_k W_k \Big\|_{L^2[0,1]}. \tag{2.3}$$

Note also that the basicity of the Walsh system in spaces $L^p[0,1]$, $p > 1$, provides the existence of a constant $C_p > 0$, so that for each function $f \in L^p[0,1]$ the following inequality holds:

$$\big\| S_k(f) \big\|_{L^p[0,1]} \leq C_p \|f\|_{L^p[0,1]}, \quad \forall k \in \mathbb{N}, \tag{2.4}$$

where $\{S_k(f)\}$ are partial sums of its expansion by the Walsh system (see [6]).

In this article we use the following lemma, which was proved in [18, Lemma 3].

Lemma 2.1. *For each dyadic interval* $\Delta = [\frac{i}{2^K}, \frac{i+1}{2^K}]$, $0 \leq i < 2^K$, $K \in \mathbb{N}$, *and for every natural number* $M > K$ *such that* $\frac{M-K}{2}$ *is a whole number, there exists a polynomial in the Walsh system*

$$H(x) = \sum_{k=2^M}^{2^{M+1}-1} a_k W_k(x),$$

so that

(1) $|a_k| = 2^{-\frac{M+K}{2}}$, *when* $2^M \leq k < 2^{M+1}$,
(2) $H(x) = -1$, *if* $x \in E_1$, $|E_1| = \frac{1}{2}|\Delta|$,
(3) $H(x) = 1$, *if* $x \in E_2$, $|E_2| = \frac{1}{2}|\Delta|$,

(4) $H(x) = 0$, *if* $x \notin \Delta$,

where E_1 *and* E_2 *are finite unions of dyadic intervals.*

One of the main building blocks in the proof of Theorem 1.2 is Lemma 2.3, which is proved with the help of Lemma 2.2.

Lemma 2.2. *Let* $p > 1$, *let* n_0 *be some natural number, and let* $\Delta \subset [0,1]$ *be a dyadic interval. Then for any numbers* $0 < \varepsilon < 1$, $l \neq 0$, *and natural number* q *there exist a measurable set* $E_q \subset \Delta$ *with measure* $|E_q| = (1 - 2^{-q})|\Delta|$ *and polynomials*

$$P_q(x) = \sum_{k=2^{n_0}}^{2^{n_q}-1} a_k W_k(x) \qquad \text{and} \qquad H_q(x) = \sum_{k=2^{n_0}}^{2^{n_q}-1} \delta_k a_k W_k(x), \quad \delta_k = \pm 1, 0,$$

in the Walsh system, so that $H_q(x) = 0$ *outside* Δ,

$$0 < a_{k+1} \leq a_k < \varepsilon \quad \text{when } k \in [2^{n_0}, 2^{n_q} - 1), \tag{1}$$

$$\|l\chi_\Delta - H_q\|_{L^p(E_q)} = 0, \tag{2}$$

$$\max_{2^{n_0} \leq M < 2^{n_q}} \Big\| \sum_{k=2^{n_0}}^{M} \delta_k a_k W_k \Big\|_{L^p[0,1]} < 2^q C |l| |\Delta|^{\frac{1}{p}}, \tag{3}$$

where C *is a constant defined by the space* $L^p[0,1]$, *and*

$$\max_{2^{n_0} \leq M < 2^{n_q}} \Big\| \sum_{k=2^{n_0}}^{M} a_k W_k \Big\|_{L^1[0,1]} < \varepsilon. \tag{4}$$

Proof. The proof is performed using mathematical induction with respect to the number q. Let $\Delta = [\frac{i}{2^K}, \frac{i+1}{2^K}] \subset [0,1]$, $0 \leq i < 2^K$, $K \in \mathbb{N}$. Choosing a natural number $K_1 > K$ such that

$$|l| 2^{-\frac{K_1+1}{2}} < \frac{\varepsilon}{2}, \tag{2.5}$$

we present the interval Δ in the form of a union of disjoint dyadic intervals

$$\Delta = \bigcup_{i=1}^{N_1} \Delta_i^{(1)}$$

with measure $|\Delta_i^{(1)}| = 2^{-K_1-1}$, $i = \overline{1, N_1}$. Obviously, $N_1 = 2^{K_1-K+1}$.

By denoting $K_0^{(1)} \equiv n_0 - 1$, for each natural number $i \in [1, N_1]$ we choose a natural number $K_i^{(1)} > K_{i-1}^{(1)}$ ($K_1^{(1)} > K_1$) such that the following conditions take place:

(a) $\frac{K_i^{(1)} - K_1 - 1}{2}$ is a whole number,

(b) $(K_i^{(1)} - K_{i-1}^{(1)}) |l| 2^{-\frac{K_i^{(1)} + K_1 + 1}{2}} < \frac{\varepsilon}{4N_1}$,

(c) $2|l| 2^{-\frac{K_i^{(1)}+1}{2}} < \frac{\varepsilon}{2}$.

It immediately follows from (2.5) that

$$|l|2^{-\frac{K_1^{(1)}+K_1+1}{2}} < \varepsilon. \tag{2.6}$$

By successively applying Lemma 2.1 to each dyadic interval $\Delta_i^{(1)}$ $(i=\overline{1,N_1})$ and corresponding number $K_i^{(1)}$, we can find polynomials in the Walsh system

$$\overline{H}_i^{(1)}(x) = \sum_{k=2^{K_i^{(1)}}}^{2^{K_i^{(1)}+1}-1} \bar{a}_k W_k(x), \quad i=\overline{1,N_1} \tag{2.7}$$

such that

$$|\bar{a}_k| = |l|2^{-\frac{K_i^{(1)}+K_1+1}{2}}, \quad \text{when } k\in[2^{K_i^{(1)}}, 2^{K_i^{(1)}+1}), \tag{2.8}$$

$$\overline{H}_i^{(1)}(x) = \begin{cases} -l, & \text{for } x\in \widetilde{E_i}^{(1)} \subset \Delta_i^{(1)}, |\widetilde{E_i}^{(1)}| = \frac{1}{2}|\Delta_i^{(1)}|, \\ l, & \text{for } x\in \widetilde{\widetilde{E_i}}^{(1)} \subset \Delta_i^{(1)}, |\widetilde{\widetilde{E_i}}^{(1)}| = \frac{1}{2}|\Delta_i^{(1)}|, \\ 0, & \text{for } x\notin \Delta_i^{(1)}. \end{cases} \tag{2.9}$$

Hence, by denoting

$$H_1(x) = \sum_{i=1}^{N_1} \overline{H}_i^{(1)}(x), \tag{2.10}$$

we get

$$H_1(x) = \begin{cases} -l, & \text{for } x\in \widetilde{E}_1 \subset \Delta, |\widetilde{E}_1| = \frac{|\Delta|}{2}, \\ l, & \text{for } x\in \Delta\setminus\widetilde{E}_1, \\ 0, & \text{for } x\notin \Delta. \end{cases} \tag{2.11}$$

As the polynomial $\overline{H}_i^{(1)}(x)$ is a linear combination of Walsh functions from the $K_i^{(1)}$ group, it is clear that the set $\widetilde{E}_1$ can be presented as a union of a certain number N_2 of disjoint dyadic intervals

$$\widetilde{E}_1 = \bigcup_{i=1}^{N_2} \Delta_i^{(2)}$$

with measure $|\Delta_i^{(2)}| = 2^{-K_{N_1}^{(1)}-1}$, $i=\overline{1,N_2}$.

By defining

$$E_1 = \Delta\setminus\widetilde{E}_1 \tag{2.12}$$

and

$$\begin{cases} \bar{a}_k = |l|2^{-\frac{K_i^{(1)}+K_1+1}{2}}, & \text{when } k\in[2^{K_{i-1}^{(1)}+1}, 2^{K_i^{(1)}}), i\in[1,N_1], \\ \bar{\delta}_k = \begin{cases} 0, & \text{when } k\in[2^{K_{i-1}^{(1)}+1}, 2^{K_i^{(1)}}), \\ 1, & \text{when } k\in[2^{K_i^{(1)}}, 2^{K_i^{(1)}+1}), \end{cases} & i\in[1,N_1], \\ a_k = |\bar{a}_k|, \qquad \delta_k = \bar{\delta}_k\cdot\frac{\bar{a}_k}{|\bar{a}_k|}, & \text{when } k\in[2^{n_0}, 2^{K_{N_1}^{(1)}+1}), \end{cases} \tag{2.13}$$

let us verify that the set E_1 and polynomials

$$P_1(x)=\sum_{k=2^{n_0}}^{2^{K^{(1)}_{N_1}+1}-1} a_k W_k(x) \qquad \text{and} \qquad H_1(x)=\sum_{k=2^{n_0}}^{2^{K^{(1)}_{N_1}+1}-1} \delta_k a_k W_k(x), \quad \delta_k=\pm 1, 0,$$

satisfy all statements of Lemma 2.2 for $q=1$. Indeed, by using (2.11) and (2.12) we obtain $|E_1|=(1-2^{-1})|\Delta|$. Statement (1) follows from (2.6), (2.8), (2.13), and from the monotonicity of numbers $K_i^{(1)}$ $(i=\overline{1,N_1})$. Statement (2) immediately follows from (2.11) and (2.12). To prove statements (3) and (4), we present the natural number $M\in[2^{n_0},2^{K^{(1)}_{N_1}+1})$ in the form $M=2^{\bar n}+s$, $s\in[0,2^{\bar n})$, where $\bar n\in(K^{(1)}_{m-1},K^{(1)}_m]$ for some $m\in[1,N_1]$. Since intervals $\Delta_i^{(1)}$ $(i=\overline{1,N_1})$ are disjoint, by using (2.4), (2.7), and (2.9)–(2.13) we have

$$\begin{aligned}\Big\|\sum_{k=2^{n_0}}^{M}\delta_k a_k W_k\Big\|_{L^p[0,1]} &\le \Big\|\sum_{i=1}^{m-1}\overline{H}_i^{(1)}\Big\|_{L^p[0,1]}+\Big\|\sum_{k=2^{\bar n}}^{2^{\bar n}+s}\delta_k a_k W_k\Big\|_{L^p[0,1]}\\ &\le \|H_1\|_{L^p[0,1]}+C_p\|\overline{H}_m^{(1)}\|_{L^p[0,1]}\\ &=\big(|l|^p|E_1|+|l|^p|\widetilde{E}_1|\big)^{\frac1p}+C_p|l||\Delta_m^{(1)}|^{\frac1p}<2C|l||\Delta|^{\frac1p},\end{aligned}$$

where $C=C_p+1$.

Furthermore, for each natural number $n\in[n_0,K^{(1)}_{N_1}]$ we denote $b_n=a_k$, $k\in[2^n,2^{n+1})$ (coefficients a_k of Walsh functions from nth group are equal in $H_1(x)$). Taking into account (2.2), (2.3), (2.5), (2.8), (2.13), and condition (b) for numbers $K_i^{(1)}$ $(i=\overline{1,N_1})$, we get

$$\sum_{n=n_0}^{K^{(1)}_{N_1}} b_n=\sum_{i=1}^{N_1}\sum_{n=K^{(1)}_{i-1}+1}^{K_i^{(1)}} b_n=\sum_{i=1}^{N_1}(K_i^{(1)}-K_{i-1}^{(1)})|l|2^{-\frac{K_i^{(1)}+K_1+1}{2}}<\frac{\varepsilon}{4},$$

$$\begin{aligned}\Big\|\sum_{k=2^{n_0}}^{M}a_kW_k\Big\|_{L^1[0,1]} &\le \sum_{n=n_0}^{\bar n-1}b_n+\Big\|\sum_{k=2^{\bar n}}^{2^{\bar n}+s}a_kW_k\Big\|_{L^1[0,1]}\\ &\le \sum_{n=n_0}^{K^{(1)}_{N_1}}b_n+\Big\|\sum_{k=2^{\bar n}}^{2^{\bar n+1}-1}b_{\bar n}W_k\Big\|_{L^2[0,1]}<\frac{\varepsilon}{4}+|l|2^{-\frac{K_m^{(1)}+K_1+1}{2}}2^{\frac{\bar n}{2}}<\varepsilon,\end{aligned}$$

which proves Lemma 2.2(4).

Assume that for $q>1$ the natural numbers

$$K_1^{(1)}<\cdots<K^{(1)}_{N_1}<\cdots<K_1^{(q-1)}<\cdots<K^{(q-1)}_{N_{q-1}},$$

sets

$$\widetilde{E}_{q-1}\subset\Delta \qquad \text{and} \qquad E_{q-1}=\Delta\setminus\widetilde{E}_{q-1}, \tag{2.14}$$

and polynomials

$$P_{q-1}(x)=\sum_{k=2^{n_0}}^{2^{K^{(q-1)}_{N_{q-1}}+1}-1} a_k W_k(x),$$

$$H_{q-1}(x)=\sum_{k=2^{n_0}}^{2^{K^{(q-1)}_{N_{q-1}}+1}-1} \delta_k a_k W_k(x), \quad \delta_k=\pm 1, 0$$

are already chosen to satisfy the conditions

(a′) $\frac{K_i^{(\nu)}-K^{(\nu-1)}_{N_{\nu-1}}-1}{2}$ is a whole number $(K^{(0)}_{N_0}\equiv K_1)$,

(b′) $(K_i^{(\nu)}-K_{i-1}^{(\nu)})2^{(\nu-1)}|l|2^{-\frac{K_i^{(\nu)}+K^{(\nu-1)}_{N_{\nu-1}}+1}{2}}<\frac{\varepsilon}{2^{\nu+1}N_\nu}$,

(c′) $2^\nu|l|2^{-\frac{K_i^{(\nu)}+1}{2}}<\frac{\varepsilon}{2}$,

$$a_k=2^{\nu-1}|l|2^{-\frac{K_i^{(\nu)}+K^{(\nu-1)}_{N_{\nu-1}}+1}{2}} \quad \text{for } k\in[2^{K^{(\nu)}_{i-1}+1},2^{K_i^{(\nu)}+1}), \tag{2.15}$$

$$K_0^{(\nu)}\equiv\begin{cases}K^{(\nu-1)}_{N_{\nu-1}}, & \text{if } \nu>1,\\ n_0-1, & \text{if } \nu=1,\end{cases}$$

for any natural numbers $i\in[1,N_\nu]$ and $\nu\in[1,q-1]$. Besides,

$$\sum_{n=n_0}^{K^{(q-1)}_{N_{q-1}}} b_n<\sum_{k=1}^{q-1}\frac{\varepsilon}{2^{k+1}}, \quad \text{where } b_n\equiv a_k, k\in[2^n,2^{n+1}), \tag{2.16}$$

$$H_{q-1}(x)=\begin{cases}-(2^{q-1}-1)l, & \text{for } x\in\widetilde{E}_{q-1},\\ l, & \text{for } x\in E_{q-1},\\ 0, & \text{for } x\notin\Delta,\end{cases} \tag{2.17}$$

$$|\widetilde{E}_{q-1}|=2^{-q+1}|\Delta| \qquad \text{and} \qquad |E_{q-1}|=(1-2^{-q+1})|\Delta|, \tag{2.18}$$

and the set $\widetilde{E}_{q-1}$ can be presented as a union of a certain number N_q of disjoint dyadic intervals

$$\widetilde{E}_{q-1}=\bigcup_{i=1}^{N_q}\Delta_i^{(q)} \tag{2.19}$$

with measure $|\Delta_i^{(q)}|=2^{-K^{(q-1)}_{N_{q-1}}-1}$, $i=\overline{1,N_q}$.

For each natural number $i\in[1,N_q]$ we choose a natural number $K_i^{(q)}>K_{i-1}^{(q)}$ $(K_0^{(q)}\equiv K^{(q-1)}_{N_{q-1}})$ such that the following conditions hold:

(a″) $\frac{K_i^{(q)}-K^{(q-1)}_{N_{q-1}}-1}{2}$ is a whole number,

(b″) $(K_i^{(q)}-K_{i-1}^{(q)})2^{(q-1)}|l|2^{-\frac{K_i^{(q)}+K^{(q-1)}_{N_{q-1}}+1}{2}}<\frac{\varepsilon}{2^{q+1}N_q}$, (c″) $2^q|l|2^{-\frac{K_i^{(q)}+1}{2}} < \frac{\varepsilon}{2}$.

By successive applications of Lemma 2.1, for each interval $\Delta_i^{(q)} \subset \widetilde{E}_{q-1}$ ($i = \overline{1, N_q}$) and corresponding number $K_i^{(q)}$, we can find polynomials in the Walsh system

$$\overline{H}_i^{(q)}(x) = \sum_{k=2^{K_i^{(q)}}}^{2^{K_i^{(q)}+1}-1} \bar{a}_k W_k(x), \quad i = \overline{1, N_q}, \tag{2.20}$$

such that

$$|\bar{a}_k| = 2^{q-1}|l|2^{-\frac{K_i^{(q)}+K_{N_{q-1}}^{(q-1)}+1}{2}}, \quad \text{when } k \in [2^{K_i^{(q)}}, 2^{K_i^{(q)}+1}), \tag{2.21}$$

$$\overline{H}_i^{(q)}(x) = \begin{cases} -2^{q-1}l, & \text{for } x \in \widetilde{E_i}^{(q)} \subset \Delta_i^{(q)}, |\widetilde{E_i}^{(q)}| = \frac{1}{2}|\Delta_i^{(q)}|, \\ 2^{q-1}l, & \text{for } x \in \widetilde{\widetilde{E_i}}^{(q)} \subset \Delta_i^{(q)}, |\widetilde{\widetilde{E_i}}^{(q)}| = \frac{1}{2}|\Delta_i^{(q)}|, \\ 0, & \text{for } x \notin \Delta_i^{(q)}. \end{cases} \tag{2.22}$$

Hence, by denoting

$$H_q(x) = H_{q-1}(x) + \sum_{i=1}^{N_q} \overline{H}_i^{(q)}(x) \tag{2.23}$$

and taking into account (2.17) and (2.19), we obtain

$$H_q(x) = \begin{cases} -(2^q-1)l, & \text{for } x \in \widetilde{E}_q \subset \widetilde{E}_{q-1}, |\widetilde{E}_q| = \frac{|\Delta|}{2^q}, \\ l, & \text{for } x \in \Delta \setminus \widetilde{E}_q, \\ 0, & \text{for } x \notin \Delta. \end{cases} \tag{2.24}$$

Now, let us define a set

$$E_q = \Delta \setminus \widetilde{E}_q \quad (\text{as in } (2.14)) \tag{2.25}$$

and numbers

$$\begin{cases} \bar{a}_k = 2^{q-1}|l|2^{-\frac{K_i^{(q)}+K_{N_{q-1}}^{(q-1)}+1}{2}}, & \text{when } k \in [2^{K_{i-1}^{(q)}+1}, 2^{K_i^{(q)}}), \\ \bar{\delta}_k = \begin{cases} 0, & \text{when } k \in [2^{K_{i-1}^{(q)}+1}, 2^{K_i^{(q)}}), \\ 1, & \text{when } k \in [2^{K_i^{(q)}}, 2^{K_i^{(q)}+1}), \end{cases} & i \in [1, N_q], \\ a_k = |\bar{a}_k|, \quad \delta_k = \bar{\delta}_k \cdot \frac{\bar{a}_k}{|\bar{a}_k|}, & \text{when } k \in [2^{n_0}, 2^{K_{N_q}^{(q)}+1}), \end{cases} \tag{2.26}$$

and verify that the set E_q and polynomials

$$P_q(x) = \sum_{k=2^{n_0}}^{2^{n_q}-1} a_k W_k(x),$$

$$H_q(x) = \sum_{k=2^{n_0}}^{2^{n_q}-1} \delta_k a_k W_k(x), \quad \delta_k = \pm 1, 0,$$

where $n_q \equiv K^{(q)}_{N_q}+1$, satisfy all statements of Lemma 2.2. Indeed, from (2.24) and (2.25) it follows that $|E_q| = (1-2^{-q})|\Delta|$. Statement (1) follows from (2.6), (2.15), (2.21), (2.26), and from the monotonicity of numbers $K^{(\nu)}_i$, $i \in [1, N_\nu]$, $\nu \in [1, q]$. Statement (2) immediately follows from (2.24) and (2.25). To prove statements (3) and (4), we present the natural number $M \in [2^{n_0}, 2^{n_q})$ in the form $M = 2^{\bar{n}}+s$, $s \in [0, 2^{\bar{n}})$. Let us consider only the case when $\bar{n} \in (K^{(q-1)}_{N_{q-1}}, K^{(q)}_{N_q}]$, since all other cases are under consideration in previous steps of induction. Let $\bar{n} \in (K^{(q)}_{m-1}, K^{(q)}_m]$ for some $m \in [1, N_q]$. From (2.4), (2.17)–(2.20), (2.22), and (2.26) we have

$$\begin{aligned}\Big\|\sum_{k=2^{n_0}}^{M} \delta_k a_k W_k\Big\|_{L^p[0,1]} &\le \Big\|H_{q-1} + \sum_{i=1}^{m-1} \overline{H}^{(q)}_i\Big\|_{L^p[0,1]} + \Big\|\sum_{k=2^{\bar{n}}}^{2^{\bar{n}}+s} \delta_k a_k W_k\Big\|_{L^p[0,1]} \\ &\le \|H_q\|_{L^p[0,1]} + C_p\|\overline{H}^{(q)}_m\|_{L^p[0,1]} \\ &< \big(|l|^p|E_q| + 2^{pq}|l|^p|\widetilde{E}_q|\big)^{\frac{1}{p}} + C_p 2^{q-1}|l||\Delta^{(q)}_m|^{\frac{1}{p}} < 2^q C|l||\Delta|^{\frac{1}{p}}\end{aligned}$$

$(C = C_p + 1)$, which proves statement (3).

Furthermore, for each natural number $n \in [n_0, K^{(q)}_{N_q}]$ as in (2.15), we denote

$$b_n \equiv a_k, \quad \text{when } k \in [2^n, 2^{n+1}).$$

Taking into account (2.2), (2.3), (2.16), (2.21), (2.26), condition (c') for number $K^{(q-1)}_{N_{q-1}}$, and condition (b'') for numbers $K^{(q)}_i$ $(i = \overline{1, N_q})$, we get

$$\begin{aligned}\Big\|\sum_{k=2^{n_0}}^{M} a_k W_k\Big\|_{L^1[0,1]} &\le \sum_{n=n_0}^{\bar{n}-1} b_n + \Big\|\sum_{k=2^{\bar{n}}}^{2^{\bar{n}}+s} a_k W_k\Big\|_{L^1[0,1]} \\ &\le \sum_{n=n_0}^{K^{(q-1)}_{N_{q-1}}} b_n + \sum_{i=1}^{N_q} \sum_{n=K^{(q)}_{i-1}+1}^{K^{(q)}_i} b_n + \Big\|\sum_{k=2^{\bar{n}}}^{2^{\bar{n}+1}-1} b_{\bar{n}} W_k\Big\|_{L^2[0,1]} \\ &\le \sum_{k=1}^{q-1} \frac{\varepsilon}{2^{k+1}} + \sum_{i=1}^{N_q}(K^{(q)}_i - K^{(q)}_{i-1}) 2^{q-1}|l| 2^{-\frac{K^{(q)}_i + K^{(q-1)}_{N_{q-1}}+1}{2}} \\ &\quad + 2^{q-1}|l| 2^{-\frac{K^{(q)}_m + K^{(q-1)}_{N_{q-1}}+1}{2}} 2^{\frac{\bar{n}}{2}} < \varepsilon,\end{aligned}$$

which proves statement (4).

Lemma 2.2 is proved. □

Lemma 2.3. *Let numbers $p_0 > 1$, $n_0 \in \mathbb{N}$, $0 < \varepsilon < 1$, and polynomial $f(x) \not\equiv 0$ in the Walsh system be given. Then one can find a measurable set E_ε with measure $|E_\varepsilon| > 1 - \varepsilon$ and polynomials*

$$P(x) = \sum_{k=2^{n_0}}^{2^n-1} a_k W_k(x) \qquad \text{and} \qquad H(x) = \sum_{k=2^{n_0}}^{2^n-1} \delta_k a_k W_k(x), \quad \delta_k = \pm 1, 0,$$

in the Walsh system which satisfy the following conditions:

(1) $$0 < a_{k+1} < a_k < \varepsilon, \quad k \in [2^{n_0}, 2^n - 1),$$

(2) $$\big\| f(x) - H \big\|_{L^{p_0}(E_\varepsilon)} < \varepsilon,$$

(3) $$\max_{2^{n_0} \le M < 2^n} \Big\| \sum_{k=2^{n_0}}^{M} \delta_k a_k W_k \Big\|_{L^p(F)} < \big\| f \big\|_{L^p(F)} + \varepsilon$$

for any measurable set $F \subseteq E_\varepsilon$ and $p \in [1, p_0]$, and

(4) $$\max_{2^{n_0} \le M < 2^n} \Big\| \sum_{k=2^{n_0}}^{M} a_k W_k \Big\|_{L^1[0,1]} < \varepsilon.$$

Proof. We choose a natural number q so that

$$2^{-q} < \varepsilon, \tag{2.27}$$

and we present the function $f(x)$ in the form

$$f(x) = \sum_{j=1}^{\nu_0} l_j \chi_{\Delta_j}(x),$$

where $l_j \neq 0$, $j = \overline{1, \nu_0}$, and $\{\Delta_j\}_{j=1}^{\nu_0}$ are disjoint dyadic subintervals of the section $[0,1]$. Without loss of generality we can assume that all these intervals have the same length and are small enough to provide the condition

$$\max_{1 \le j \le \nu_0} \left\{ 2^q C |l_j| |\Delta_j|^{\frac{1}{p}} \right\} < \frac{\varepsilon}{2}, \tag{2.28}$$

where C is the positive constant of Lemma 2.2(3).

Applying Lemma 2.2 to each dyadic interval Δ_j, $j = \overline{1, \nu_0}$, and taking into account (2.27) and (2.28), we can find sets $E_q^{(j)} \subset \Delta_j$ with measure

$$|E_q^{(j)}| = (1 - 2^{-q}) |\Delta_j| > (1 - \varepsilon) |\Delta_j| \tag{2.29}$$

and polynomials

$$\bar{P}_q^{(j)}(x) = \sum_{k=2^{n_{j-1}}}^{2^{n_j}-1} \bar{a}_k^{(j)} W_k(x),$$

$$\bar{H}_q^{(j)}(x) = \sum_{k=2^{n_{j-1}}}^{2^{n_j}-1} \delta_k^{(j)} \bar{a}_k^{(j)} W_k(x), \quad \delta_k^{(j)} = \pm 1, 0,$$

in the Walsh system, so that $\bar{H}_q^{(j)}(x) = 0$ outside Δ_j,

$$\begin{cases} 0 < \bar{a}_{k+1}^{(1)} \le \bar{a}_k^{(1)} < \frac{\varepsilon}{2}, & \text{for all } k \in [2^{n_0}, 2^{n_1} - 1), \\ 0 < \bar{a}_{k+1}^{(j)} \le \bar{a}_k^{(j)} < \bar{a}_{2^{n_{j-1}}-1}^{(j-1)}, & \text{for all } k \in [2^{n_{j-1}}, 2^{n_j} - 1), j > 1, \end{cases} \tag{2.30}$$

$$\| l_j \chi_{\Delta_j} - \bar{H}_q^{(j)} \|_{L^{p_0}(E_q^{(j)})} = 0, \tag{2.31}$$

$$\max_{2^{n_{j-1}}\le M<2^{n_j}}\Big\|\sum_{k=2^{n_{j-1}}}^{M}\delta_k^{(j)}\bar a_k^{(j)}W_k\Big\|_{L^{p_0}[0,1]}<2^qC|l_j||\Delta_j|^{\frac{1}{p_0}}<\frac{\varepsilon}{2}, \tag{2.32}$$

$$\max_{2^{n_{j-1}}\le M<2^{n_j}}\Big\|\sum_{k=2^{n_{j-1}}}^{M}\bar a_k^{(j)}W_k\Big\|_{L^1[0,1]}<\frac{\varepsilon}{2^{j+1}}. \tag{2.33}$$

We define a set

$$E_\varepsilon=\bigcup_{j=1}^{\nu_0}E_q^{(j)}\cup\Big([0,1]/\bigcup_{j=1}^{\nu_0}\Delta_j\Big) \tag{2.34}$$

and polynomials

$$\bar P(x)=\sum_{j=1}^{\nu_0}\bar P_q^{(j)}(x)=\sum_{k=2^{n_0}}^{2^{n_{\nu_0}-1}}\bar a_kW_k(x),$$

$$\bar H(x)=\sum_{j=1}^{\nu_0}\bar H_q^{(j)}(x)=\sum_{k=2^{n_0}}^{2^{n_{\nu_0}-1}}\delta_k\bar a_kW_k(x),$$

where $\bar a_k=\bar a_k^{(j)}$ and $\delta_k=\delta_k^{(j)}$ when $k\in[2^{n_{j-1}},2^{n_j})$. Note that $\bar H_q^{(j)}=0$ on the set $[0,1]/\bigcup_{j=1}^{\nu_0}\Delta_j$ (in case it is not empty) for any $j\in[1,\nu_0]$.

From (2.29)–(2.31) and (2.34) it follows that

$$|E_\varepsilon|>1-\varepsilon,$$

$$0<\bar a_{k+1}\le\bar a_k<\frac{\varepsilon}{2},\quad\text{when } k\in[2^{n_0},2^{n_{\nu_0}}-1), \tag{2.35}$$

$$\|f-\bar H\|_{L^{p_0}(E_\varepsilon)}\le\sum_{j=1}^{\nu_0}\|l_j\chi_{\Delta_j}-\bar H_q^{(j)}\|_{L^{p_0}(E_q^{(j)})}=0. \tag{2.36}$$

Furthermore, let M be a natural number from $[2^{n_0},2^{n_{\nu_0}})$. Then $M\in[2^{n_{m-1}},2^{n_m})$ for some $m\in[1,\nu_0]$. Taking into account (2.31), (2.32), and (2.34), for any measurable set $F\subseteq E_\varepsilon$ and $p\in[1,p_0]$ we have

$$\begin{aligned}
&\Big\|\sum_{k=2^{n_0}}^{M}\delta_k\bar a_kW_k\Big\|_{L^p(F)}\\
&\quad\le\Big\|\sum_{j=1}^{m-1}\bar H_q^{(j)}\Big\|_{L^p(F)}+\Big\|\sum_{k=2^{n_{m-1}}}^{M}\delta_k^{(m)}a_k^{(m)}W_k\Big\|_{L^p(F)}\\
&\quad\le\sum_{j=1}^{m-1}\|l_j\chi_{\Delta_j}-\bar H_q^{(j)}\|_{L^{p_0}(E_q^{(j)})}+\Big\|\sum_{j=1}^{m-1}l_j\chi_{\Delta_j}\Big\|_{L^p(F)}\\
&\qquad+\Big\|\sum_{k=2^{n_{m-1}}}^{M}\delta_k^{(m)}a_k^{(m)}W_k\Big\|_{L^{p_0}[0,1]}<\|f\|_{L^p(F)}+\frac{\varepsilon}{2}
\end{aligned} \tag{2.37}$$

and, by using (2.33), we obtain

$$\Big\|\sum_{k=2^{n_0}}^{M}\bar a_k W_k\Big\|_{L^1[0,1]}\le\sum_{j=1}^{\nu_0}\max_{2^{n_{j-1}}\le N<2^{n_j}}\Big\|\sum_{k=2^{n_{j-1}}}^{N}\bar a_k^{(j)}W_k\Big\|_{L^1[0,1]}<\frac{\varepsilon}{2}. \tag{2.38}$$

Hence, polynomials $\bar P(x)$ and $\bar H(x)$ satisfy all statements of Lemma 2.3 except for (1). To have strict inequalities between coefficients, we choose such a natural number N_0 that

$$2^{-N_0}<\frac{\varepsilon}{2} \tag{2.39}$$

and define polynomials

$$P(x)=\sum_{k=2^{n_0}}^{2^{n_{\nu_0}}-1}a_kW_k(x)\qquad\text{and}\qquad H(x)=\sum_{k=2^{n_0}}^{2^{n_{\nu_0}}-1}\delta_k a_kW_k(x),$$

where

$$a_k=\bar a_k+2^{-(N_0+k)}. \tag{2.40}$$

It is not hard to verify that polynomials $P(x)$ and $H(x)$ satisfy all statements of Lemma 2.3. Indeed, statement (1) immediately follows from (2.35), (2.39), and (2.40). Furthermore, considering (2.36)–(2.40) for each natural number $M\in[2^{n_0},2^{n_{\nu_0}})$, measurable set $F\subseteq E_\varepsilon$, and $p\in[1,p_0]$ we get

$$\begin{aligned}\|f-H\|_{L^{p_0}(E_\varepsilon)}&\le\|f-\bar H\|_{L^{p_0}(E_\varepsilon)}+\Big\|\sum_{k=2^{n_0}}^{2^{n_{\nu_0}}-1}\delta_k2^{-(N_0+k)}W_k\Big\|_{L^{p_0}[0,1]}\\&\le\sum_{k=2^{n_0}}^{2^{n_{\nu_0}}-1}\|\delta_k2^{-(N_0+k)}W_k\|_{L^{p_0}[0,1]}<2^{-N_0}<\varepsilon,\end{aligned}$$

$$\begin{aligned}\Big\|\sum_{k=2^{n_0}}^{M}\delta_ka_kW_k\Big\|_{L^p(F)}&\le\Big\|\sum_{k=2^{n_0}}^{M}\delta_k\bar a_kW_k\Big\|_{L^p(F)}+\Big\|\sum_{k=2^{n_0}}^{M}\delta_k2^{-(N_0+k)}W_k\Big\|_{L^p[0,1]}\\&\le\|f\|_{L^p(F)}+\frac{\varepsilon}{2}+\sum_{k=2^{n_0}}^{M}\|\delta_k2^{-(N_0+k)}W_k\|_{L^p[0,1]}\\&<\|f\|_{L^p(F)}+\frac{\varepsilon}{2}+2^{-N_0}<\|f\|_{L^p(F)}+\varepsilon\end{aligned}$$

and

$$\begin{aligned}\Big\|\sum_{k=2^{n_0}}^{M}a_kW_k\Big\|_{L^1[0,1]}&\le\Big\|\sum_{k=2^{n_0}}^{M}\bar a_kW_k\Big\|_{L^1[0,1]}+\sum_{k=2^{n_0}}^{M}\|2^{-(N_0+k)}\|_{L^1[0,1]}\\&<\frac{\varepsilon}{2}+2^{-N_0}<\varepsilon.\end{aligned}$$

Lemma 2.3 is proved. □

Now with the help of Lemma 2.3 we will prove the main lemma of the article, which will be used in the proof of the main theorem.

Lemma 2.4. *For any $\delta \in (0,1)$ there exists a weight function $0 < \mu(x) \leq 1$, with $|\{x \in [0,1]; \mu(x) = 1\}| > 1-\delta$, so that for any numbers $p_0 > 1$, $n_0 \in \mathbb{N}$, $\varepsilon \in (0,1)$, and polynomial $f(x) \not\equiv 0$ in the Walsh system, one can find polynomials in the Walsh system*

$$P(x) = \sum_{k=2^{n_0}}^{2^n-1} a_k W_k(x) \qquad \textit{and} \qquad H(x) = \sum_{k=2^{n_0}}^{2^n-1} \delta_k a_k W_k(x), \quad \delta_k = \pm 1, 0,$$

satisfying the following conditions:

(1) $$0 < a_{k+1} < a_k < \varepsilon, \quad k \in [2^{n_0}, 2^n - 1),$$

(2) $$\|f - H\|_{L^{p_0}_{\mu}[0,1]} < \varepsilon,$$

(3) $$\max_{2^{n_0} \leq M < 2^n} \Big\| \sum_{k=2^{n_0}}^{M} \delta_k a_k W_k \Big\|_{L^p_{\mu}[0,1]} < 2\|f\|_{L^p_{\mu}[0,1]} + \varepsilon, \quad \forall p \in [1, p_0],$$

(4) $$\max_{2^{n_0} \leq M < 2^n} \Big\| \sum_{k=2^{n_0}}^{M} a_k W_k \Big\|_{L^1[0,1]} < \varepsilon.$$

Proof. Let $p_m \nearrow +\infty$, $\delta \in (0,1)$, and $N_0 = 1$, and let $\{f_m(x)\}_{m=1}^{\infty}$, $x \in [0,1]$, be a sequence of all polynomials in the Walsh system with rational coefficients. By successive applications of Lemma 2.3, it is possible to find sets $E_m \subset [0,1]$ and polynomials in the Walsh system of the form

$$P_m(x) = \sum_{k=2^{N_{m-1}}}^{2^{N_m}-1} a_k^{(m)} W_k(x), \tag{2.41}$$

$$H_m(x) = \sum_{k=2^{N_{m-1}}}^{2^{N_m}-1} \delta_k^{(m)} a_k^{(m)} W_k(x), \quad \delta_k^{(m)} = \pm 1, 0, \tag{2.42}$$

which satisfy the following conditions for any natural number m:

$$|E_m| > 1 - \frac{1}{2^{m+1}}, \tag{2.43}$$

$$0 < a_{k+1}^{(m)} < a_k^{(m)} < \frac{1}{4^{N_{m-1}}}, \quad k \in [2^{N_{m-1}}, 2^{N_m} - 1), \tag{2.44}$$

$$\|f_m - H_m\|_{L^{p_m}(E_m)} < \frac{1}{2^{m+2}}, \quad \text{and} \tag{2.45}$$

for any measurable set $F \subseteq E_m$ and $p \in [1, p_m]$ we have

$$\max_{2^{N_{m-1}} \leq M < 2^{N_m}} \Big\| \sum_{k=2^{N_{m-1}}}^{M} \delta_k^{(m)} a_k^{(m)} W_k \Big\|_{L^p(F)} < \|f_m\|_{L^p(F)} + \frac{1}{2^{m+2}}, \tag{2.46}$$

and

$$\max_{2^{N_{m-1}} \leq M < 2^{N_m}} \Big\| \sum_{k=2^{N_{m-1}}}^{M} a_k^{(m)} W_k \Big\|_{L^1[0,1]} < \frac{1}{2^{m+2}}. \tag{2.47}$$

We set

$$\begin{cases} \Omega_n = \bigcap_{m=n}^{+\infty} E_m, \quad n \in \mathbb{N}, \\ E = \Omega_{\widetilde{n}} = \bigcap_{m=\widetilde{n}}^{+\infty} E_m, \quad \widetilde{n} = [\log_{1/2} \delta] + 1, \\ B = \Omega_{\widetilde{n}} \cup (\bigcup_{n=\widetilde{n}+1}^{+\infty} \Omega_n \setminus \Omega_{n-1}). \end{cases} \tag{2.48}$$

It is clear (see (2.43) and (2.48)) that

$$|B| = 1, \qquad |E| > 1 - \delta.$$

We define a function $\mu(x)$ in the following way:

$$\mu(x) = \begin{cases} 1, & x \in E \cup ([0,1] \setminus B), \\ \mu_n, & x \in \Omega_n \setminus \Omega_{n-1}, n \geq \widetilde{n} + 1, \end{cases} \tag{2.49}$$

where

$$\mu_n = \frac{1}{2^{p_n(n+2)}} \cdot \Big[\prod_{m=1}^{n} h_m\Big]^{-1}, \tag{2.50}$$

$$h_m = \max_{1 \leq p \leq p_m} \Big\{1 + \int_0^1 \big|f_m(x)\big|^p \, dx + \max_{2^{N_{m-1}} \leq M < 2^{N_m}} \int_0^1 \Big| \sum_{k=2^{N_{m-1}}}^{M} \delta_k^{(m)} a_k^{(m)} W_k(x) \Big|^p \, dx \Big\}.$$

It follows from (2.48)–(2.50) that for all $m \geq \widetilde{n}$,

$$\begin{aligned} \int_{[0,1]\setminus\Omega_m} \big|H_m(x)\big|^{p_m} \mu(x) \, dx &= \sum_{n=m+1}^{+\infty} \Big(\int_{\Omega_n \setminus \Omega_{n-1}} \big|H_m(x)\big|^{p_m} \mu_n \, dx \Big) \\ &< \sum_{n=m+1}^{\infty} \frac{1}{2^{p_n(n+2)} h_m} \Big(\int_0^1 \big|H_m(x)\big|^{p_m} \, dx \Big) \\ &< \frac{1}{2^{p_m(m+2)}}. \end{aligned} \tag{2.51}$$

In a similar way, for all $m \geq \widetilde{n}$, $M \in [2^{N_{m-1}}, 2^{N_m})$, and $p \in [1, p_m]$ we have

$$\int_{[0,1]\setminus\Omega_m} \big|f_m(x)\big|^{p_m} \mu(x) \, dx < \frac{1}{2^{p_m(m+2)}} \tag{2.52}$$

and

$$\int_{[0,1]\setminus\Omega_m} \Big| \sum_{k=2^{N_{m-1}}}^{M} \delta_k^{(m)} a_k^{(m)} W_k(x) \Big|^p \mu(x) \, dx < \frac{1}{2^{p(m+2)}}. \tag{2.53}$$

Since $\Omega_m \subset E_m$, by using conditions (2.45), (2.48)–(2.52), and Jensen's inequality, for all $m \geq \widetilde{n}$ we obtain

$$\begin{aligned} &\int_0^1 \big|f_m(x) - H_m(x)\big|^{p_m} \mu(x) \, dx \\ &\quad = \int_{\Omega_m} \big|f_m(x) - H_m(x)\big|^{p_m} \mu(x) \, dx \end{aligned}$$

$$+\int_{[0,1]\setminus\Omega_m}\bigl|f_m(x)-H_m(x)\bigr|^{p_m}\mu(x)\,dx$$
$$<\frac{1}{2^{p_m(m+2)}}+2\cdot 2^{p_m}\frac{1}{2^{p_m(m+2)}}<\frac{1}{2^{p_m(m-1)}}$$

or

$$\|f_m-H_m\|_{L^{p_m}_\mu[0,1]}<\frac{1}{2^{m-1}}. \tag{2.54}$$

Furthermore, taking relations (2.46), (2.48)–(2.50), (2.53), and Jensen's inequality into account for all $M\in[2^{N_{m-1}},2^{N_m})$, $p\in[1,p_m]$, and $m\geq\widetilde{n}+1$ we get

$$\int_0^1\Bigl|\sum_{k=2^{N_{m-1}}}^{M}\delta_k^{(m)}a_k^{(m)}W_k(x)\Bigr|^p\mu(x)\,dx$$
$$=\int_{\Omega_m}\Bigl|\sum_{k=2^{N_{m-1}}}^{M}\delta_k^{(m)}a_k^{(m)}W_k(x)\Bigr|^p\mu(x)\,dx$$
$$+\int_{[0,1]\setminus\Omega_m}\Bigl|\sum_{k=2^{N_{m-1}}}^{M}\delta_k^{(m)}a_k^{(m)}W_k(x)\Bigr|^p\mu(x)\,dx$$
$$<\int_{\Omega_{\widetilde{n}}}\Bigl|\sum_{k=2^{N_{m-1}}}^{M}\delta_k^{(m)}a_k^{(m)}W_k(x)\Bigr|^p\mu(x)\,dx$$
$$+\sum_{n=\widetilde{n}+1}^{m}\mu_n\cdot\int_{\Omega_n\setminus\Omega_{n-1}}\Bigl|\sum_{k=2^{N_{m-1}}}^{M}\delta_k^{(m)}a_k^{(m)}W_k(x)\Bigr|^p dx+\frac{1}{2^{p(m+2)}}$$
$$<\Bigl(\|f_m\|_{L^p(\Omega_{\widetilde{n}})}+\frac{1}{2^{m+2}}\Bigr)^p+\sum_{n=\widetilde{n}+1}^{m}\mu_n\Bigl(\|f_m\|_{L^p(\Omega_n\setminus\Omega_{n-1})}+\frac{1}{2^{m+2}}\Bigr)^p+\frac{1}{2^{p(m+2)}}$$
$$\leq 2^p\Bigl(\int_{\Omega_{\widetilde{n}}}\bigl|f_m(x)\bigr|^p dx+\sum_{n=\widetilde{n}+1}^{m}\int_{\Omega_n\setminus\Omega_{n-1}}\bigl|f_m(x)\bigr|^p\cdot\mu_n\,dx\Bigr)$$
$$+\frac{1}{2^{p(m+2)}}\Bigl(2^p+2^p\cdot\sum_{n=\widetilde{n}+1}^{m}\mu_n+1\Bigr)<2^p\|f_m\|^p_{L^p_\mu[0,1]}+\frac{1}{2^{p(m-1)}}$$

or

$$\Bigl\|\sum_{k=2^{N_{m-1}}}^{M}\delta_k^{(m)}a_k^{(m)}W_k\Bigr\|_{L^p_\mu[0,1]}<2\|f_m\|_{L^p_\mu[0,1]}+\frac{1}{2^{m-1}}. \tag{2.55}$$

Let $n_0\in\mathbb{N}$ and $\varepsilon\in(0,1)$ be arbitrarily given. From the sequence $\{f_m(x)\}_{m=1}^{\infty}$ we choose such a function $f_{m_0}(x)$ that

$$m_0>\max\Bigl\{\tilde{n},\log_2\frac{8}{\varepsilon}\Bigr\},\quad p_{m_0}>p_0,2^{N_{m_0-1}}>2^{n_0}, \tag{2.56}$$
$$\|f-f_{m_0}\|_{L^{p_0}[0,1]}<\frac{\epsilon}{4}, \tag{2.57}$$

and for $k \in [2^{n_0}, 2^{N_{m_0}})$ set

$$a_k = \begin{cases} a^{(m_0)}_{2^{N_{m_0-1}}} + \frac{1}{2^{k+m_0}}, & \text{when } k \in [2^{n_0}, 2^{N_{m_0-1}}), \\ a^{(m_0)}_k, & \text{when } k \in [2^{N_{m_0-1}}, 2^{N_{m_0}}), \end{cases} \tag{2.58}$$

$$\delta_k = \begin{cases} 0, & \text{when } k \in [2^{n_0}, 2^{N_{m_0-1}}), \\ \delta^{(m_0)}_k = \pm 1, 0, & \text{when } k \in [2^{N_{m_0-1}}, 2^{N_{m_0}}), \end{cases} \tag{2.59}$$

and

$$P(x) = \sum_{k=2^{n_0}}^{2^{N_{m_0}}-1} a_k W_k(x) = \sum_{k=2^{n_0}}^{2^{N_{m_0-1}}-1} a_k W_k(x) + P_{m_0}(x),$$

$$H(x) = \sum_{k=2^{n_0}}^{2^{N_{m_0}}-1} \delta_k a_k W_k(x) = H_{m_0}(x).$$

Now it is not hard to verify that the function $\mu(x)$ and polynomials $P(x)$ and $H(x)$ satisfy all the requirements of Lemma 2.4. Indeed, statements (1)–(3) immediately follow from (2.44) and (2.54)–(2.59). Furthermore, by using (2.47) and (2.56)–(2.58) we obtain

$$\begin{aligned} \max_{2^{n_0} \le M < 2^{N_{m_0}}} \Big\| \sum_{k=2^{n_0}}^{M} a_k W_k \Big\|_{L^1[0,1]} &\le \max_{2^{n_0} \le M_1 < 2^{N_{m_0-1}}} \Big\| \sum_{k=2^{n_0}}^{M_1} a_k W_k \Big\|_{L^1[0,1]} \\ &\quad + \max_{2^{N_{m_0-1}} \le M_2 < 2^{N_{m_0}}} \Big\| \sum_{k=2^{N_{m_0-1}}}^{M_2} a^{(m_0)}_k W_k \Big\|_{L^1[0,1]} \\ &< \max_{2^{n_0} \le M_1 < 2^{N_{m_0-1}}} \Big\| \sum_{k=2^{n_0}}^{M_1} a_k W_k \Big\|_{L^1[0,1]} + \frac{\varepsilon}{2}. \end{aligned}$$

Let M_1 be an arbitrary natural number in $[2^{n_0}, 2^{N_{m_0-1}})$. Then $M_1 \in [2^{n_1}, 2^{n_1+1})$ for some $n_1 \in [n_0, N_{m_0-1})$ and, considering (2.1), we have

$$\begin{aligned} \Big\| \sum_{k=2^{n_0}}^{M_1} a_k W_k \Big\|_{L^1[0,1]} &< a^{(m_0)}_{2^{N_{m_0-1}}} \cdot \Big\| \sum_{k=2^{n_0}}^{2^{n_1}-1} W_k \Big\|_{L^1[0,1]} + a^{(m_0)}_{2^{N_{m_0-1}}} \cdot 2^{n_1} \\ &\quad + \sum_{k=2^{n_0}}^{M_1} \frac{1}{2^{k+m_0}} < \frac{\varepsilon}{2}, \end{aligned}$$

which proves statement (4).

Lemma 2.4 is proved. □

3. Proof of theorem 1.2

Let $\delta \in (0,1)$, $p_m \nearrow +\infty$, and let $\{f_m(x)\}_{m=1}^{\infty}$, $x \in [0,1]$, be a sequence of all polynomials in the Walsh system with rational coefficients. By virtue of

Lemma 2.4, there exist a weight function $0 < \mu(x) \le 1$ with $|\{x \in [0,1], \mu(x) = 1\}| > 1 - \delta$ and polynomials

$$P_m(x) = \sum_{k=N_{m-1}}^{N_m - 1} a_k^{(m)} W_k(x), \tag{3.1}$$

$$H_m(x) = \sum_{k=N_{m-1}}^{N_m - 1} \delta_k^{(m)} a_k^{(m)} W_k(x), \quad \delta_k^{(m)} = \pm 1, 0, \tag{3.2}$$

in the Walsh system which satisfy the following conditions for any natural number m:

$$\begin{cases} 0 < a_{k+1}^{(1)} < a_k^{(1)}, \\ 0 < a_{k+1}^{(m)} < a_k^{(m)} < \min\{2^{-m}, a_{N_{m-1}-1}^{(m-1)}\} \quad \text{for } m > 1, \end{cases} \tag{3.3}$$

when $k \in [N_{m-1}, N_m - 1)$,

$$\|f_m - H_m\|_{L_\mu^{p_m}[0,1]} < 2^{-m-1}, \tag{3.4}$$

$$\max_{N_{m-1} \le M < N_m} \Big\| \sum_{k=N_{m-1}}^{M} \delta_k^{(m)} a_k^{(m)} W_k \Big\|_{L_\mu^p[0,1]} < 2\|f_m\|_{L_\mu^p[0,1]} + 2^{-m}, \tag{3.5}$$

for any $p \in [1, p_m]$, and

$$\max_{N_{m-1} \le M < N_m} \Big\| \sum_{k=N_{m-1}}^{M} a_k^{(m)} W_k \Big\|_{L^1[0,1]} < 2^{-m-1}. \tag{3.6}$$

From (3.1) and (3.6) it immediately follows that

$$\Big\| \sum_{m=1}^{\infty} P_m \Big\|_{L^1[0,1]} \le \sum_{m=1}^{\infty} \|P_m\|_{L^1[0,1]} < +\infty. \tag{3.7}$$

By denoting

$$P_0(x) = \sum_{k=0}^{N_0 - 1} a_k W_k(x), \tag{3.8}$$

where coefficients a_k, $k \in [0, N_0)$, are arbitrary monotonically decreasing positive numbers with $a_{N_0-1} > a_{N_0}^{(1)}$, we define a function $g(x)$ and a series $\sum_{k=0}^{\infty} a_k W_k(x)$ as follows:

$$g(x) = \sum_{m=0}^{\infty} P_m(x), \tag{3.9}$$

$$a_k = a_k^{(m)}, \quad \text{when } k \in [N_{m-1}, N_m), m \in \mathbb{N}, \tag{3.10}$$

and a_k are coefficients in $P_0(x)$ (see (3.8)), when $k \in [0, N_0)$. By using (3.3) and (3.6)–(3.10) we conclude that the series $\sum_{k=0}^{\infty} a_k W_k(x)$ converges to $g \in L^1[0,1]$ in $L^1[0,1]$ metric, and $a_k = \int_0^1 g(t) W_k(t)\, dt \searrow 0$.

Let $p \geq 1$, and let $f \in L^p_\mu(0,1)$. We choose such a polynomial $f_{\nu_1}(x)$ from the sequence $\{f_m(x)\}_{m=1}^\infty$ such that

$$\|f - f_{\nu_1}\|_{L^p_\mu[0,1]} < 2^{-2} \qquad \text{and} \qquad p_{\nu_1} > p. \tag{3.11}$$

By denoting

$$\delta_k = \begin{cases} \delta_k^{(\nu_1)} = \pm 1, 0, & \text{when } k \in [N_{\nu_1 - 1}, N_{\nu_1}), \\ 0, & \text{when } k \in [0, N_{\nu_1 - 1}), \end{cases}$$

and taking into account (3.2), (3.4), (3.5), and (3.11), we have

$$\begin{aligned} &\Big\| f - \sum_{k=0}^{N_{\nu_1}-1} \delta_k a_k W_k \Big\|_{L^p_\mu[0,1]} \\ &\quad \leq \|f - f_{\nu_1}\|_{L^p_\mu[0,1]} + \|f_{\nu_1} - H_{\nu_1}\|_{L^{p_{\nu_1}}_\mu[0,1]} \\ &\quad < 2^{-2} + 2^{-\nu_1 - 1} < 2^{-1} \end{aligned}$$

and

$$\max_{N_{\nu_1 - 1} \leq M < N_{\nu_1}} \Big\| \sum_{k=N_{\nu_1 - 1}}^{M} \delta_k a_k W_k \Big\|_{L^p_\mu[0,1]} < 2\|f_{\nu_1}\|_{L^p_\mu[0,1]} + 2^{-\nu_1}.$$

Assume that for $q > 1$ numbers $\nu_1 < \nu_2 < \cdots < \nu_{q-1}$ and $\{\delta_k = \pm 1, 0\}_{k=0}^{N_{\nu_{q-1}}-1}$ are already chosen, so that for each natural number $j \in [1, q-1]$ the following conditions hold:

$$\begin{gathered} \delta_k = \begin{cases} \delta_k^{(\nu_j)} = \pm 1, 0, & \text{when } k \in [N_{\nu_j - 1}, N_{\nu_j}), \\ 0, & \text{when } k \notin \bigcup_{j=1}^{q-1} [N_{\nu_j - 1}, N_{\nu_j}), \end{cases} \\ \Big\| f - \sum_{k=0}^{N_{\nu_j}-1} \delta_k a_k W_k \Big\|_{L^p_\mu[0,1]} < 2^{-j}, \\ \max_{N_{\nu_j - 1} \leq M < N_{\nu_j}} \Big\| \sum_{k=N_{\nu_j - 1}}^{M} \delta_k a_k W_k \Big\|_{L^p_\mu[0,1]} < 2\|f_{\nu_j}\|_{L^p_\mu[0,1]} + 2^{-\nu_j}. \end{gathered} \tag{3.12}$$

We choose a function $f_{\nu_q}(x)$ from the sequence $\{f_m(x)\}_{m=1}^\infty$ with $\nu_q > \nu_{q-1}$ so that

$$\Big\| f - \sum_{k=0}^{N_{\nu_{q-1}}-1} \delta_k a_k W_k(x) - f_{\nu_q} \Big\|_{L^p_\mu[0,1]} < 2^{-q-1}, \tag{3.13}$$

and we define

$$\delta_k = \begin{cases} \delta_k^{(\nu_q)} = \pm 1, 0, & \text{when } k \in [N_{\nu_q - 1}, N_{\nu_q}), \\ 0, & \text{when } k \notin \bigcup_{j=1}^{q} [N_{\nu_j - 1}, N_{\nu_j}). \end{cases} \tag{3.14}$$

Taking into account (3.2), (3.4), (3.13), and (3.14), we get

$$\begin{aligned}\Big\| f - \sum_{k=0}^{N_{\nu_q}-1} \delta_k a_k W_k \Big\|_{L^p_\mu[0,1]} & \\ \leq \Big\| f - \sum_{k=0}^{N_{\nu_{q-1}}-1} \delta_k a_k W_k - f_{\nu_q} \Big\|_{L^p_\mu[0,1]} &+ \|f_{\nu_q} - H_{\nu_q}\|_{L^{p_{\nu_q}}_\mu[0,1]} \\ < 2^{-q-1} + 2^{-\nu_q-1} < 2^{-q}. & \qquad (3.15)\end{aligned}$$

Furthermore, from (3.12) and (3.13) we have

$$\begin{aligned}\|f_{\nu_q}\|_{L^p_\mu[0,1]} < \Big\| f - \sum_{k=0}^{N_{\nu_{q-1}}-1} \delta_k a_k W_k - f_{\nu_q} \Big\|_{L^p_\mu[0,1]} & \\ + \Big\| f - \sum_{k=0}^{N_{\nu_{q-1}}-1} \delta_k a_k W_k \Big\|_{L^p_\mu[0,1]} &< 2^{-q-1} + 2^{-q+1} < 2^{-q+2}.\end{aligned}$$

Thus, from (3.5) and (3.14) it follows that for each natural number $M \in [N_{\nu_q-1}, N_{\nu_q})$,

$$\Big\| \sum_{k=N_{\nu_{q-1}}}^{M} \delta_k a_k W_k \Big\|_{L^p_\mu[0,1]} < 2\|f_{\nu_q}\|_{L^p_\mu[0,1]} + 2^{-\nu_q} < 2^{-q+4}. \qquad (3.16)$$

By induction, we easily determine a growing sequence of indexes $\{\nu_q\}_{q=1}^{+\infty}$ and numbers $\{\delta_k = \pm 1, 0\}_{k=0}^{+\infty}$ so that conditions (3.14)–(3.16) hold for any $q \in \mathbb{N}$. Hence, we obtain a series

$$\sum_{k=0}^{+\infty} \delta_k a_k W_k(x), \quad \delta_k = \pm 1, 0, \qquad (3.17)$$

which converges to f in $L^p_\mu[0,1]$ metric. Indeed, from (3.15) it follows that the subsequence $\{S_{N_{\nu_q}}(x)\}_{q=1}^{+\infty}$ of its partial sums

$$S_N(x) \equiv \sum_{k=0}^{N-1} \delta_k a_k W_k(x), \quad N = 1, 2, \ldots,$$

converges to f in $L^p_\mu[0,1]$ metric, and (3.16) provides the convergence of the whole sequence $S_N(x)$.

Theorem 1.2 is proved. □

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