Banach J. Math. Anal. 12 (2018), no. 2, 399–421
https://doi.org/10.1215/17358787-2017-0025
ISSN: 1735-8787 (electronic)
http://projecteuclid.org/bjma

NEW L^p-INEQUALITIES FOR HYPERBOLIC WEIGHTS CONCERNING THE OPERATORS WITH COMPLEX GAUSSIAN KERNELS

BENITO J. GONZÁLEZ* and EMILIO R. NEGRÍN

Communicated by J. A. Ball

ABSTRACT. In this article the authors present a systematic study of several new L^p-boundedness properties and Parseval-type relations concerning the operators with complex Gaussian kernels over the spaces $L^p(\mathbb{R}, \cosh(\alpha x)\, dx)$ and $L^p(\mathbb{R}, \cosh(\alpha x^2)\, dx)$, $1 \le p \le \infty$, $\alpha \in \mathbb{R}$. Relevant connections with various earlier related results are also pointed out.

1. Introduction

In this article we consider the integral operator with complex Gaussian kernel of a suitable complex-valued function f defined on $\mathbb{R}$ by

$$(\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f)(y) = \int_{-\infty}^{+\infty} \exp[-\beta y^2 - \varepsilon x^2 + 2\delta xy + \xi y + \gamma x] f(x)\, dx, \tag{1.1}$$

where $y \in \mathbb{R}$, and $\beta, \varepsilon, \delta, \xi, \gamma \in \mathbb{C}$. This type of integral operator is present in numerous contexts in analysis, probability theory, and mathematical physics. There are several types of examples such as the Fourier transform, the Poisson formula for a solution of the heat equation, and the Mehler formula for the time evolution of a harmonic oscillator (see [3], [4], [10], [14]–[16], among others).

This subject was originally of interest in the context of quantum field theory (see [2], [17]). The complex Gaussian operator (1.1) has an intrinsic interest due

Received Feb. 24, 2017; Accepted Jun. 20, 2017.
First published online Nov. 17, 2017.
*Corresponding author.
2010 *Mathematics Subject Classification*. Primary 47G07; Secondary 46E30.
Keywords. complex Gaussian kernels, L^p-inequalities, hyperbolic weights, Parseval-type relations, Gauss–Weierstrass semigroup.

to the basis role of the extended oscillator semigroup introduced by Howe [11] (see also Folland [5, Chapter 5]).

In his important paper [14], Lieb extends the operator (1.1) in the context of the spaces $L^p(\mathbb{R}^n)$, $1 < p < \infty$. Moreover, he generalizes the results given by Epperson in [4, Section 2] for (1.1). Lieb stated that for the nondegenerate case, that is, $(\Re\delta)^2 < \Re\beta\Re\varepsilon$, the operator $\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}$ has exactly one maximizer which is a centered Gaussian function e^{sy^2}, $s \in \mathbb{C}$. For the degenerate case, that is, $(\Re\delta)^2 = \Re\beta\Re\varepsilon$, the question of the existence of a minimizer is a subtle one. This problem requires a complicated algebraic study, and precise conditions are not given there.

The operator (1.1) is also related to the integral transform studied by Y. J. Lee in [12] and [13].

Our main goal in this article is to establish new L^p-boundedness properties and Parseval-type relations for the operators given by (1.1) over the spaces $L^p(\mathbb{R}, \cosh(\alpha x)\,dx)$ and $L^p(\mathbb{R}, \cosh(\alpha x^2)\,dx)$, where $\alpha \in \mathbb{R}$, and $1 \le p \le \infty$.

Throughout this article we denote the norm of the space $L^p(\mathbb{R}, \cosh(\alpha x)\,dx)$ by

$$\|f\|_{p,\mu_1} = \left(\int_{-\infty}^{+\infty} \big|f(x)\big|^p \cosh(\alpha x)\,dx\right)^{1/p},$$

and we use

$$\|f\|_{p,\mu_2} = \left(\int_{-\infty}^{+\infty} \big|f(x)\big|^p \cosh(\alpha x^2)\,dx\right)^{1/p}$$

to denote the norm of the space $L^p(\mathbb{R}, \cosh(\alpha x^2)\,dx)$.

By using results of [6] and [8], we prove that the operator $\mathfrak{F}_{\varepsilon,\beta,\delta,\gamma,\xi}$ is bounded from the spaces $L^p(\mathbb{R}, \cosh(\alpha x)\,dx)$ into $L^{p'}(\mathbb{R}, \cosh(\alpha x)\,dx)$ and from the spaces $L^p(\mathbb{R}, \cosh(\alpha x^2)\,dx)$ into $L^{p'}(\mathbb{R}, \cosh(\alpha x^2)\,dx)$ under certain conditions of the parameters. Here, p and p' are given by $p + p' = pp'$ if $1 < p < \infty$, $p' = \infty$ if $p = 1$, and $p' = 1$ if $p = \infty$.

Moreover, under these conditions, for $f, g \in L^p(\mathbb{R}, \cosh(\alpha x)\,dx)$ or $f, g \in L^p(\mathbb{R}, \cosh(\alpha x^2)\,dx)$, $1 \le p \le \infty$, and since that the weights are greater than or equal to 1, then we have the following Parseval-type relation:

$$\int_{-\infty}^{+\infty} (\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f)(x) g(x)\,dx = \int_{-\infty}^{\infty} f(x)(\mathfrak{F}_{\varepsilon,\beta,\delta,\gamma,\xi} g)(x)\,dx. \tag{1.2}$$

Let $\mathfrak{F}'_{\varepsilon,\beta,\delta,\gamma,\xi}$ be the adjoint of the operator $\mathfrak{F}_{\varepsilon,\beta,\delta,\gamma,\xi}$; that is,

$$\langle \mathfrak{F}'_{\varepsilon,\beta,\delta,\gamma,\xi} f, g\rangle = \langle f, \mathfrak{F}_{\varepsilon,\beta,\delta,\gamma,\xi} g\rangle.$$

The aforementioned Parseval-type relation (1.2) allows us to obtain an interesting connection between the operator $\mathfrak{F}'_{\varepsilon,\beta,\delta,\gamma,\xi}$ and the operator $\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}$.

We conclude that the operator $\mathfrak{F}'_{\varepsilon,\beta,\delta,\gamma,\xi}$ is the natural extension of the integral operator $\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}$, that is,

$$\mathfrak{F}'_{\varepsilon,\beta,\delta,\gamma,\xi} T_f = T_{\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f},$$

where T_f is given by

$$\langle T_f, g\rangle = \int_{-\infty}^{\infty} f(x) g(x)\,dx.$$

In this article we make use of the well-known facts that

$$(2\pi c)^{(-1/2)} \cdot \int_{-\infty}^{+\infty} \exp\bigl[\nu x - (x^2/2c)\bigr]\, dx = \exp(c\nu^2/2), \quad \nu \in \mathbb{C}, c > 0. \tag{1.3}$$

We also point out how our work relates to various earlier results (see [7], [9]).

Inversion formulae for the operator (1.1) are obtained in [15, Theorem 2.1, Corollary 2.1].

2. The operator $\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}$ over the spaces $L^p(\mathbb{R}, \cosh(\alpha x)\, dx)$ and $L^p(\mathbb{R}, \cosh(\alpha x^2)\, dx)$, $1 < p < \infty$

In this section we study the behavior of the operator $\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}$ on the spaces $L^p(\mathbb{R}, \cosh(\alpha x)\, dx)$ and $L^p(\mathbb{R}, \cosh(\alpha x^2)\, dx)$, $1 < p < \infty$, $\alpha \in \mathbb{R}$.

2.1. **The case of the spaces $L^p(\mathbb{R}, \cosh(\alpha x)\, dx)$.** Indeed, by following [6, Proposition 2.1], we derive Theorem 2.1.

Theorem 2.1. *Assume that $1 < p < \infty$ and that $\alpha \in \mathbb{R}$. We have the following:*

(a) *for $\Re\beta > 0$, $\Re\varepsilon > 0$, $\Re\delta = 0$, and $\xi, \gamma \in \mathbb{C}$, or alternatively for $(\Re\delta)^2 < \Re\beta$, $\Re\varepsilon > 1$, and $\xi, \gamma \in \mathbb{C}$, or alternatively $(\Re\delta)^2 < \Re\varepsilon$, $\Re\beta > 1$, and $\xi, \gamma \in \mathbb{C}$, the operator $\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}$ given by (1.1) is bounded from the spaces $L^p(\mathbb{R}, \cosh(\alpha x)\, dx)$ into $L^q(\mathbb{R}, \cosh(\alpha x)\, dx)$ for any q $(0 < q < \infty)$;*

(b) *for $\Re\beta \geq 0$, $\Re\varepsilon > 0$, $\Re\delta = \Re\xi = 0$, and $\gamma \in \mathbb{C}$, or alternatively $(\Re\delta)^2 \leq \Re\beta$, $\Re\varepsilon > 1$, $\Re\xi = 0$, and $\gamma \in \mathbb{C}$, or alternatively $(\Re\delta)^2 < \Re\varepsilon$, $\Re\beta \geq 1$, $\Re\xi = 0$, and $\gamma \in \mathbb{C}$, the operator $\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}$ is bounded from the spaces $L^p(\mathbb{R}, \cosh(\alpha x)\, dx)$ into $L^\infty(\mathbb{R}, \cosh(\alpha x)\, dx)$.*

Proof. Assuming that $1 < p < \infty$, $p + p' = pp'$ and that $\alpha \in \mathbb{R}$,

(a) from Hölder's inequality, we get

$$\begin{aligned}
&\bigl|(\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f)(y)\bigr| \\
&\quad \leq \int_{-\infty}^{+\infty} \bigl|\exp[-\beta y^2 - \varepsilon x^2 + 2\delta xy + \xi y + \gamma x]\bigr| \bigl|f(x)\bigr|\, dx \\
&\quad = \int_{-\infty}^{+\infty} \bigl|\exp[-\beta y^2 - \varepsilon x^2 + 2\delta xy + \xi y + \gamma x]\bigr| \\
&\qquad \times \bigl|f(x)\bigr| \bigl(\cosh(\alpha x)\bigr)^{1/p} \bigl(\cosh(\alpha x)\bigr)^{-1/p}\, dx \\
&\quad \leq \Bigl(\int_{-\infty}^{+\infty} \bigl|f(x)\bigr|^p \cosh(\alpha x)\, dx\Bigr)^{1/p} \\
&\qquad \times \Bigl(\int_{-\infty}^{+\infty} \bigl|\exp[-\beta y^2 - \varepsilon x^2 + 2\delta xy + \xi y + \gamma x]\bigr|^{p'} \bigl(\cosh(\alpha x)\bigr)^{-p'/p}\, dx\Bigr)^{1/p'} \\
&\quad = \|f\|_{p,\mu_1} \Bigl(\int_{-\infty}^{+\infty} \bigl|\exp[-\beta y^2 - \varepsilon x^2 + 2\delta xy + \xi y + \gamma x]\bigr|^{p'} \bigl(\cosh(\alpha x)\bigr)^{-p'/p}\, dx\Bigr)^{1/p'};
\end{aligned}$$

we therefore have

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \big|(\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f)(y)\big|^q \cosh(\alpha y)\, dy \\
&\quad\le \|f\|_{p,\mu_1}^q \int_{-\infty}^{+\infty} \Big(\int_{-\infty}^{+\infty} \big|\exp[-\beta y^2 - \varepsilon x^2 \\
&\qquad + 2\delta xy + \xi y + \gamma x]\big|^{p'} \big(\cosh(\alpha x)\big)^{-p'/p}\, dx\Big)^{q/p'} \cosh(\alpha y)\, dy;
\end{aligned}$$

hence, for $\Re\delta = 0$, we see that

$$\begin{aligned}
&\big\|(\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f)(y)\big\|_{q,\mu_1} \\
&\quad\le \|f\|_{p,\mu_1} \Big(\int_{-\infty}^{+\infty} \exp\big[(-\Re\varepsilon x^2 + \Re\gamma x)p'\big] \big(\cosh(\alpha x)\big)^{-p'/p}\, dx\Big)^{1/p'} \\
&\quad\times \Big(\int_{-\infty}^{+\infty} \exp\big[(-\Re\beta y^2 + \Re\xi y)q\big] \cosh(\alpha y)\, dy\Big)^{1/q} \qquad (2.1)\\
&\quad= \|f\|_{p,\mu_1} \Big(\int_{-\infty}^{+\infty} \exp\big[(-\Re\varepsilon x^2 + \Re\gamma x)p'\big] \Big(\frac{e^{\alpha x} + e^{-\alpha x}}{2}\Big)^{-p'/p} dx\Big)^{1/p'} \\
&\quad\times \Big(\int_{-\infty}^{+\infty} \exp\big[(-\Re\beta y^2 + \Re\xi y)q\big] \Big(\frac{e^{\alpha y} + e^{-\alpha y}}{2}\Big) dy\Big)^{1/q}.
\end{aligned}$$

In addition, taking into account that $(x - \Re\delta y)^2 \ge 0$, we obtain that $2\Re\delta xy \le x^2 + (\Re\delta)^2 y^2$; thus

$$\begin{aligned}
&\big\|(\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f)(y)\big\|_{q,\mu_1} \\
&\quad\le \|f\|_{p,\mu_1} \Big(\int_{-\infty}^{+\infty} \exp\big[\big(-(\Re\varepsilon - 1)x^2 + \Re\gamma x\big)p'\big] \big(\cosh(\alpha x)\big)^{-p'/p}\, dx\Big)^{1/p'} \\
&\quad\times \Big(\int_{-\infty}^{+\infty} \exp\big[\big(-\big(\Re\beta - (\Re\delta)^2\big)y^2 + \Re\xi y\big)q\big] \cosh(\alpha y)\, dy\Big)^{1/q} \qquad (2.2)\\
&\quad= \|f\|_{p,\mu_1} \Big(\int_{-\infty}^{+\infty} \exp\big[\big(-(\Re\varepsilon - 1)x^2 + \Re\gamma x\big)p'\big] \Big(\frac{e^{\alpha x} + e^{-\alpha x}}{2}\Big)^{-p'/p} dx\Big)^{1/p'} \\
&\quad\times \Big(\int_{-\infty}^{+\infty} \exp\big[\big(-\big(\Re\beta - (\Re\delta)^2\big)y^2 + \Re\xi y\big)q\big] \Big(\frac{e^{\alpha y} + e^{-\alpha y}}{2}\Big) dy\Big)^{1/q}.
\end{aligned}$$

Analogously, taking into account that $(\Re\delta x - y)^2 \ge 0$, we obtain that $2\Re\delta xy \le (\Re\delta)^2 x^2 + y^2$; hence

$$\begin{aligned}
&\big\|(\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f)(y)\big\|_{q,\mu_1} \\
&\quad\le \|f\|_{p,\mu_1} \Big(\int_{-\infty}^{+\infty} \exp\big[\big(-\big(\Re\varepsilon - (\Re\delta)^2\big)x^2 + \Re\gamma x\big)p'\big] \big(\cosh(\alpha x)\big)^{-p'/p}\, dx\Big)^{1/p'} \\
&\quad\times \Big(\int_{-\infty}^{+\infty} \exp\big[\big(-(\Re\beta - 1)y^2 + \Re\xi y\big)q\big] \cosh(\alpha y)\, dy\Big)^{1/q} \qquad (2.3)\\
&\quad= \|f\|_{p,\mu_1} \Big(\int_{-\infty}^{+\infty} \exp\big[\big(-\big(\Re\varepsilon - (\Re\delta)^2\big)x^2 + \Re\gamma x\big)p'\big] \Big(\frac{e^{\alpha x} + e^{-\alpha x}}{2}\Big)^{-p'/p} dx\Big)^{1/p'} \\
&\quad\times \Big(\int_{-\infty}^{+\infty} \exp\big[\big(-(\Re\beta - 1)y^2 + \Re\xi y\big)q\big] \Big(\frac{e^{\alpha y} + e^{-\alpha y}}{2}\Big) dy\Big)^{1/q}.
\end{aligned}$$

Under the hypothesis considered, the integrals involved in (2.1), (2.2), and (2.3) converge; therefore

$$\|\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f\|_{q,\mu_1} \le C\|f\|_{p,\mu_1}$$

for a certain real constant C depending on p and q.

Consequently, the operator $\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}$ is bounded from the spaces $L^p(\mathbb{R}, \cosh(\alpha x)\,dx)$ into $L^q(\mathbb{R}, \cosh(\alpha x)\,dx)$.

(b) Following the same technique, we find by applying Hölder's inequality that

$$\begin{aligned}&\sup_{y\in\mathbb{R}}\bigl\{\bigl|(\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f)(y)\bigr|\bigr\}\\ &\quad\le \|f\|_{p,\mu_1}\sup_{y\in\mathbb{R}}\Bigl\{\Bigl(\int_{-\infty}^{+\infty}\bigl|\exp[-\beta y^2-\varepsilon x^2\\ &\qquad+2\delta xy+\xi y+\gamma x]\bigr|^{p'}\bigl(\cosh(\alpha x)\bigr)^{-p'/p}\,dx\Bigr)^{1/p'}\Bigr\}.\end{aligned}$$

Next we observe that

$$\begin{aligned}&\sup_{y\in\mathbb{R}}\Bigl\{\Bigl(\int_{-\infty}^{+\infty}\bigl|\exp[-\beta y^2-\varepsilon x^2+2\delta xy+\xi y+\gamma x]\bigr|^{p'}\\ &\qquad\times\bigl(\cosh(\alpha x)\bigr)^{-p'/p}\,dx\Bigr)^{1/p'}\Bigr\}\\ &\quad=\sup_{y\in\mathbb{R}}\Bigl\{\Bigl(\int_{-\infty}^{+\infty}\bigl|\exp[-\beta y^2-\varepsilon x^2+2\delta xy+\xi y+\gamma x]\bigr|^{p'}\\ &\qquad\times\Bigl(\frac{e^{\alpha x}+e^{-\alpha x}}{2}\Bigr)^{-p'/p}\,dx\Bigr)^{1/p'}\Bigr\}\\ &\quad=\sup_{y\in\mathbb{R}}\Bigl\{\Bigl(\int_{-\infty}^{+\infty}\exp\bigl[(-\Re\beta y^2-\Re\varepsilon x^2+2\Re\delta xy+\Re\xi y+\Re\gamma x)p'\bigr]\\ &\qquad\times\Bigl(\frac{e^{\alpha x}+e^{-\alpha x}}{2}\Bigr)^{-p'/p}\,dx\Bigr)^{1/p'}\Bigr\};\end{aligned}$$

therefore, for $\Re\delta=0$, we have

$$\begin{aligned}&\sup_{y\in\mathbb{R}}\bigl\{\bigl|(\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f)(y)\bigr|\bigr\}\\ &\quad\le\|f\|_{p,\mu_1}\times\sup_{y\in\mathbb{R}}\bigl\{\exp[-\Re\beta y^2+\Re\xi y]\bigr\}\\ &\qquad\times\Bigl(\int_{-\infty}^{+\infty}\exp\bigl[(-\Re\varepsilon x^2+\Re\gamma x)p'\bigr]\Bigl(\frac{e^{\alpha x}+e^{-\alpha x}}{2}\Bigr)^{-p'/p}\,dx\Bigr)^{1/p'}.\end{aligned}\tag{2.4}$$

Now taking into account that $(x-\Re\delta y)^2\ge 0$, we can see that $2\Re\delta xy\le x^2+(\Re\delta)^2y^2$; consequently,

$$\begin{aligned}&\sup_{y\in\mathbb{R}}\bigl\{\bigl|(\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f)(y)\bigr|\bigr\}\\ &\quad\le\|f\|_{p,\mu_1}\times\sup_{y\in\mathbb{R}}\bigl\{\exp\bigl[-\bigl(\Re\beta-(\Re\delta)^2\bigr)y^2+\Re\xi y\bigr]\bigr\}\\ &\qquad\times\Bigl(\int_{-\infty}^{+\infty}\exp\bigl[\bigl(-(\Re\varepsilon-1)x^2+\Re\gamma x\bigr)p'\bigr]\Bigl(\frac{e^{\alpha x}+e^{-\alpha x}}{2}\Bigr)^{-p'/p}\,dx\Bigr)^{1/p'}.\end{aligned}\tag{2.5}$$

Analogously, taking into account that $(x-\Re\delta y)^2 \geq 0$, we obtain that $2\Re\delta xy \leq x^2+(\Re\delta)^2y^2$; thus

$$\begin{aligned}
&\sup_{y\in\mathbb{R}}\big\{\big|(\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}f)(y)\big|\big\}\\
&\quad\leq \|f\|_{p,\mu_1}\sup_{y\in\mathbb{R}}\big\{\exp\big[-(\Re\beta-1)y^2+\Re\xi y\big]\big\} \qquad (2.6)\\
&\qquad\times\Big(\int_{-\infty}^{+\infty}\exp\big[\big(-\big(\Re\varepsilon-(\Re\delta)^2\big)x^2+\Re\gamma x\big)p'\big]\Big(\frac{e^{\alpha x}+e^{-\alpha x}}{2}\Big)^{-p'/p}dx\Big)^{1/p'}.
\end{aligned}$$

Now the conditions on the parameters show that the integrals involved in (2.4), (2.5), and (2.6) converge; hence

$$\|\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}f\|_\infty \leq C\|f\|_{p,\mu_1}$$

for a certain real constant C depending on p.

Thus the operator $\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}$ is bounded from the spaces $L^p(\mathbb{R},\cosh(\alpha x)\,dx)$ into $L^\infty(\mathbb{R},\cosh(\alpha x)\,dx)$. □

From this result it follows that, if $\Re\beta>0$, $\Re\varepsilon>0$, $\Re\delta=0$, and $\xi,\gamma\in\mathbb{C}$, or alternatively if $(\Re\delta)^2\leq\Re\beta$, $\Re\varepsilon>1$, $\Re\xi=0$, and $\gamma\in\mathbb{C}$, or alternatively if $(\Re\delta)^2<\Re\varepsilon$, $\Re\beta\geq1$, $\Re\xi=0$, and $\gamma\in\mathbb{C}$, then the operator $\mathfrak{F}_{\varepsilon,\beta,\delta,\gamma,\xi}$ is bounded from the spaces $L^p(\mathbb{R},\cosh(\alpha x)\,dx)$ into $L^{p'}(\mathbb{R},\cosh(\alpha x)\,dx)$, $1<p<\infty$, $p+p'=pp'$.

From this theorem and since the weight $\cosh(\alpha x)$ is greater than or equal to 1, Proposition 2.2 in [8] yields the following result.

Theorem 2.2. *For $\alpha\in\mathbb{R}$ the following Parseval-type relation holds:*

$$\int_{-\infty}^{+\infty}(\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}f)(x)g(x)\,dx=\int_{-\infty}^{\infty}f(x)(\mathfrak{F}_{\varepsilon,\beta,\delta,\gamma,\xi}g)(x)\,dx,$$

where $f,g\in L^p(\mathbb{R},\cosh(\alpha x)\,dx)$, and $\Re\beta>0$, $\Re\varepsilon>0$, $\Re\delta=0$, and $\xi,\gamma\in\mathbb{C}$, or alternatively $(\Re\delta)^2<\Re\beta$, $\Re\varepsilon>1$, and $\xi,\gamma\in\mathbb{C}$, or alternatively $(\Re\delta)^2<\Re\varepsilon$, $\Re\beta>1$, and $\xi,\gamma\in\mathbb{C}$.

In [8] we find the following corollary.

Corollary 2.1. *If $f\in L^p(\mathbb{R},\cosh(\alpha x)\,dx)$, and $1<p<\infty$, for $\Re\beta>0$, $\Re\varepsilon>0$, $\Re\delta=0$, and $\xi,\gamma\in\mathbb{C}$, or alternatively for $(\Re\delta)^2<\Re\beta$, $\Re\varepsilon>1$ and $\xi,\gamma\in\mathbb{C}$, or alternatively for $(\Re\delta)^2<\Re\varepsilon$, $\Re\beta>1$, and $\xi,\gamma\in\mathbb{C}$, then*

$$\mathfrak{F}'_{\varepsilon,\beta,\delta,\gamma,\xi}T_f=T_{\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}f}$$

on $(L^p(\mathbb{R},\cosh(\alpha x)\,dx))'$.

2.2. **The case of the spaces $L^p(\mathbb{R},\cosh(\alpha x^2)\,dx)$.** By following [6, Proposition 2.1], we derive Theorem 2.3 below.

Theorem 2.3. *Assume that $1<p<\infty$ and that $\alpha\in\mathbb{R}$. We then see that*

(a) *if $\Re\beta>\frac{|\alpha|}{q}$, $\Re\varepsilon>\frac{|\alpha|}{p}$, $\Re\delta=0$, and $\xi,\gamma\in\mathbb{C}$, or alternatively $(\Re\delta)^2<\Re\beta-\frac{|\alpha|}{q}$, $\Re\varepsilon-\frac{|\alpha|}{p}>1$, and $\xi,\gamma\in\mathbb{C}$, or alternatively $(\Re\delta)^2<\Re\varepsilon-\frac{|\alpha|}{p}$, $\Re\beta-\frac{|\alpha|}{q}>1$, and $\xi,\gamma\in\mathbb{C}$, then the operator $\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}$ given by (1.1) is* *bounded from the spaces* $L^p(\mathbb{R}, \cosh(\alpha x^2)\,dx)$ *into* $L^q(\mathbb{R}, \cosh(\alpha x^2)\,dx)$ *for any* $0 < q < \infty$*;*

(b) *if* $\Re\beta \geq 0$, $\Re\varepsilon > \frac{|\alpha|}{p}$, $\Re\delta = \Re\xi = 0$, *and* $\gamma \in \mathbb{C}$, *or alternatively* $(\Re\delta)^2 \leq \Re\beta$, $\Re\varepsilon - \frac{|\alpha|}{p} > 1$, $\Re\xi = 0$, *and* $\gamma \in \mathbb{C}$, *or alternatively if* $(\Re\delta)^2 < \Re\varepsilon - \frac{|\alpha|}{p}$, $\Re\beta \geq 1$, $\Re\xi = 0$, *and* $\gamma \in \mathbb{C}$, *then the operator* $\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}$ *is bounded from the spaces* $L^p(\mathbb{R}, \cosh(\alpha x^2)\,dx)$ *into* $L^\infty(\mathbb{R}, \cosh(\alpha x^2)\,dx)$.

Proof. Assuming that $1 < p < \infty$, $p + p' = pp'$ and that $\alpha \in \mathbb{R}$, we see the following.

(a) Hölder's inequality shows that

$$\begin{aligned}
&\big|(\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f)(y)\big| \\
&\quad \leq \int_{-\infty}^{+\infty} \big|\exp[-\beta y^2 - \varepsilon x^2 + 2\delta xy + \xi y + \gamma x]\big|\big|f(x)\big|\,dx \\
&\quad = \int_{-\infty}^{+\infty} \big|\exp[-\beta y^2 - \varepsilon x^2 + 2\delta xy + \xi y + \gamma x]\big| \\
&\qquad \times \big|f(x)\big|\big(\cosh(\alpha x^2)\big)^{1/p}\big(\cosh(\alpha x^2)\big)^{-1/p}\,dx \\
&\quad \leq \Big(\int_{-\infty}^{+\infty} \big|f(x)\big|^p \cosh(\alpha x^2)\,dx\Big)^{1/p} \\
&\qquad \times \Big(\int_{-\infty}^{+\infty} \big|\exp[-\beta y^2 - \varepsilon x^2 + 2\delta xy + \xi y + \gamma x]\big|^{p'} \big(\cosh(\alpha x^2)\big)^{-p'/p}\,dx\Big)^{1/p'} \\
&\quad = \|f\|_{p,\mu_2}\Big(\int_{-\infty}^{+\infty} \big|\exp[-\beta y^2 - \varepsilon x^2 + 2\delta xy + \xi y + \gamma x]\big|^{p'} \\
&\qquad \times \big(\cosh(\alpha x^2)\big)^{-p'/p}\,dx\Big)^{1/p'},
\end{aligned}$$

which leads us to the following inequality:

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \big|(\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f)(y)\big|^q \cosh(\alpha y^2)\,dy \\
&\quad \leq \|f\|_{p,\mu_2}^q \int_{-\infty}^{+\infty} \Big(\int_{-\infty}^{+\infty} \big|\exp[-\beta y^2 - \varepsilon x^2 + 2\delta xy + \xi y + \gamma x]\big|^{p'} \\
&\qquad \times \big(\cosh(\alpha x^2)\big)^{-p'/p}\,dx\Big)^{q/p'} \cosh(\alpha y^2)\,dy;
\end{aligned}$$

hence, for $\Re\delta = 0$, we have

$$\begin{aligned}
&\|\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f\|_{q,\mu_2} \\
&\quad \leq \|f\|_{p,\mu_2}\Big(\int_{-\infty}^{+\infty} \exp\big[(-\Re\varepsilon x^2 + \Re\gamma x)p'\big]\big(\cosh(\alpha x^2)\big)^{-p'/p}\,dx\Big)^{1/p'} \\
&\qquad \times \Big(\int_{-\infty}^{+\infty} \exp\big[(-\Re\beta y^2 + \Re\xi y)q\big]\cosh(\alpha y^2)\,dy\Big)^{1/q} \qquad (2.7)
\end{aligned}$$

$$= \|f\|_{p,\mu_2}\Big(\int_{-\infty}^{+\infty} \exp\big[(-\Re\varepsilon x^2 + \Re\gamma x)p'\big]\Big(\frac{e^{\alpha x^2}+e^{-\alpha x^2}}{2}\Big)^{-p'/p}\,dx\Big)^{1/p'}$$
$$\times\Big(\int_{-\infty}^{+\infty} \exp\big[(-\Re\beta y^2 + \Re\xi y)q\big]\Big(\frac{e^{\alpha y^2}+e^{-\alpha y^2}}{2}\Big)\,dy\Big)^{1/q}.$$

Moreover, $2\Re\delta xy \le x^2 + (\Re\delta)^2 y^2$ implies that

$$\begin{aligned}
&\|\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f\|_{q,\mu_2}\\
&\quad\le \|f\|_{p,\mu_2}\Big(\int_{-\infty}^{+\infty} \exp\big[\big(-(\Re\varepsilon - 1)x^2 + \Re\gamma x\big)p'\big]\big(\cosh(\alpha x^2)\big)^{-p'/p}\,dx\Big)^{1/p'}\\
&\qquad\times\Big(\int_{-\infty}^{+\infty} \exp\big[\big(-\big(\Re\beta - (\Re\delta)^2\big)y^2 + \Re\xi y\big)q\big]\cosh(\alpha y^2)\,dy\Big)^{1/q} \qquad (2.8)\\
&\quad= \|f\|_{p,\mu_2}\Big(\int_{-\infty}^{+\infty} \exp\big[\big(-(\Re\varepsilon - 1)x^2 + \Re\gamma x\big)p'\big]\Big(\frac{e^{\alpha x^2}+e^{-\alpha x^2}}{2}\Big)^{-p'/p}\,dx\Big)^{1/p'}\\
&\qquad\times\Big(\int_{-\infty}^{+\infty} \exp\big[\big(-\big(\Re\beta - (\Re\delta)^2\big)y^2 + \Re\xi y\big)q\big]\Big(\frac{e^{\alpha y^2}+e^{-\alpha y^2}}{2}\Big)\,dy\Big)^{1/q}.
\end{aligned}$$

Since $2\Re\delta xy \le (\Re\delta)^2 x^2 + y^2$, one obtains that

$$\begin{aligned}
&\|\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f\|_{q,\mu_2}\\
&\quad\le \|f\|_{p,\mu_2}\Big(\int_{-\infty}^{+\infty} \exp\big[\big(-\big(\Re\varepsilon - (\Re\delta)^2\big)x^2 + \Re\gamma x\big)p'\big]\big(\cosh(\alpha x^2)\big)^{-p'/p}\,dx\Big)^{1/p'}\\
&\qquad\times\Big(\int_{-\infty}^{+\infty} \exp\big[\big(-(\Re\beta - 1)y^2 + \Re\xi y\big)q\big]\cosh(\alpha y^2)\,dy\Big)^{1/q} \qquad (2.9)\\
&\quad= \|f\|_{p,\mu_2}\Big(\int_{-\infty}^{+\infty} \exp\big[\big(-\big(\Re\varepsilon - (\Re\delta)^2\big)x^2 + \Re\gamma x\big)p'\big]\Big(\frac{e^{\alpha x^2}+e^{-\alpha x^2}}{2}\Big)^{-p'/p}\,dx\Big)^{1/p'}\\
&\qquad\times\Big(\int_{-\infty}^{+\infty} \exp\big[\big(-(\Re\beta - 1)y^2 + \Re\xi y\big)q\big]\Big(\frac{e^{\alpha y^2}+e^{-\alpha y^2}}{2}\Big)\,dy\Big)^{1/q}.
\end{aligned}$$

The hypothesis considered guarantees that the integrals involved in (2.7), (2.8), and (2.9) converge; hence

$$\|\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f\|_{q,\mu_2} \le C\|f\|_{p,\mu_2}$$

for a certain real constant C depending on p and q.

Thus the operator $\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}$ is bounded from the spaces $L^p(\mathbb{R}, \cosh(\alpha x^2)\,dx)$ into $L^q(\mathbb{R}, \cosh(\alpha x^2)\,dx)$.

(b) Using Hölder's inequality, it follows that

$$\begin{aligned}
&\sup_{y\in\mathbb{R}}\big\{|\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f|\big\}\\
&\quad\le \|f\|_{p,\mu_2}\sup_{y\in\mathbb{R}}\Big\{\Big(\int_{-\infty}^{+\infty}\big|\exp[-\beta y^2 - \varepsilon x^2 + 2\delta xy + \xi y + \gamma x]\big|^{p'}\\
&\qquad\times\big(\cosh(\alpha x^2)\big)^{-p'/p}\,dx\Big)^{1/p'}\Big\}.
\end{aligned}$$

It also follows that

$$\begin{aligned}
&\sup_{y\in\mathbb{R}}\Big\{\Big(\int_{-\infty}^{+\infty}\big|\exp[-\beta y^2-\varepsilon x^2+2\delta xy+\xi y+\gamma x]\big|^{p'}\big(\cosh(\alpha x^2)\big)^{-p'/p}\,dx\Big)^{1/p'}\Big\}\\
&\quad=\sup_{y\in\mathbb{R}}\Big\{\Big(\int_{-\infty}^{+\infty}\big|\exp[-\beta y^2-\varepsilon x^2+2\delta xy+\xi y+\gamma x]\big|^{p'}\\
&\qquad\times\Big(\frac{e^{\alpha x^2}+e^{-\alpha x^2}}{2}\Big)^{-p'/p}\,dx\Big)^{1/p'}\Big\}\\
&\quad=\sup_{y\in\mathbb{R}}\Big\{\Big(\int_{-\infty}^{+\infty}\exp\big[(-\Re\beta y^2-\Re\varepsilon x^2+2\Re\delta xy+\Re\xi y+\Re\gamma x)p'\big]\\
&\qquad\times\Big(\frac{e^{\alpha x^2}+e^{-\alpha x^2}}{2}\Big)^{-p'/p}\,dx\Big)^{1/p'}\Big\};
\end{aligned}$$

therefore, for $\Re\delta=0$, we see that

$$\begin{aligned}
&\sup_{y\in\mathbb{R}}\big\{\big|(\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}f)(y)\big|\big\}\\
&\quad=\sup_{y\in\mathbb{R}}\big\{\exp[-\Re\beta y^2+\Re\xi y]\big\}\\
&\qquad\times\Big(\int_{-\infty}^{+\infty}\exp\big[(-\Re\varepsilon x^2+\Re\gamma x)p'\big]\Big(\frac{e^{\alpha x^2}+e^{-\alpha x^2}}{2}\Big)^{-p'/p}\,dx\Big)^{1/p'}.
\end{aligned}\tag{2.10}$$

Since $2\Re\delta xy\le x^2+(\Re\delta)^2y^2$, it follows that

$$\begin{aligned}
&\sup_{y\in\mathbb{R}}\big\{\big|(\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}f)(y)\big|\big\}\\
&\quad\le\|f\|_{p,\mu_2}\times\sup_{y\in\mathbb{R}}\big\{\exp\big[-\big(\Re\beta-(\Re\delta)^2\big)y^2+\Re\xi y\big]\big\}\\
&\qquad\times\Big(\int_{-\infty}^{+\infty}\exp\big[\big(-(\Re\varepsilon-1)x^2+\Re\gamma x\big)p'\big]\Big(\frac{e^{\alpha x^2}+e^{-\alpha x^2}}{2}\Big)^{-p'/p}\,dx\Big)^{1/p'}.
\end{aligned}\tag{2.11}$$

Analogously, since $2\Re\delta xy\le x^2+(\Re\delta)^2y^2$, it follows that

$$\begin{aligned}
&\sup_{y\in\mathbb{R}}\big\{\big|(\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}f)(y)\big|\big\}\\
&\quad\le\|f\|_{p,\mu_2}\sup_{y\in\mathbb{R}}\big\{\exp\big[-(\Re\beta-1)y^2+\Re\xi y\big]\big\}\\
&\qquad\times\Big(\int_{-\infty}^{+\infty}\exp\big[\big(-\big(\Re\varepsilon-(\Re\delta)^2\big)x^2+\Re\gamma x\big)p'\big]\Big(\frac{e^{\alpha x^2}+e^{-\alpha x^2}}{2}\Big)^{-p'/p}\,dx\Big)^{1/p'}.
\end{aligned}\tag{2.12}$$

Now, when the conditions in the parameters of the integrals involved in (2.10), (2.11), and (2.12) converge, we see that

$$\|\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}f\|_\infty\le C\|f\|_{p,\mu_2}$$

for a certain real constant C depending on p.

Thus the operator $\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}$ is bounded from the spaces $L^p(\mathbb{R},\cosh(\alpha x^2)\,dx)$ into $L^\infty(\mathbb{R},\cosh(\alpha x^2)\,dx)$. □

In view of this result, for $\alpha \in \mathbb{R}$, it follows that for $\Re\beta > \frac{|\alpha|}{p}$, $\Re\varepsilon > \frac{|\alpha|}{p'}$, $\Re\delta = 0$, and $\xi, \gamma \in \mathbb{C}$, or alternatively $(\Re\delta)^2 < \Re\varepsilon - \frac{|\alpha|}{p'}$, $\Re\beta - \frac{|\alpha|}{p} > 1$, and $\xi, \gamma \in \mathbb{C}$, or alternatively $(\Re\delta)^2 < \Re\beta - \frac{|\alpha|}{p}$, $\Re\varepsilon - \frac{|\alpha|}{p'} > 1$, and $\xi, \gamma \in \mathbb{C}$, the operator $\mathfrak{F}_{\varepsilon,\beta,\delta,\gamma,\xi}$ is bounded from the spaces $L^p(\mathbb{R}, \cosh(\alpha x^2)\,dx)$ into $L^{p'}(\mathbb{R}, \cosh(\alpha x^2)\,dx)$, $1 < p < \infty$, $p + p' = pp'$.

Using this fact and taking into account that the weight $\cosh(\alpha x^2)$ is greater than or equal to 1, Proposition 2.2 in [8] yields the following.

Theorem 2.4. *If we suppose that $1 < p < \infty$, $p + p' = pp'$ and that $\alpha \in \mathbb{R}$, then the following Parseval-type relation holds:*

$$\int_{-\infty}^{+\infty} (\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f)(x) g(x)\,dx = \int_{-\infty}^{\infty} f(x)(\mathfrak{F}_{\varepsilon,\beta,\delta,\gamma,\xi} g)(x)\,dx$$

for $f, g \in L^p(\mathbb{R}, \cosh(\alpha x^2)\,dx)$, and $\Re\beta > \frac{|\alpha|}{p}$, $\Re\varepsilon > \frac{|\alpha|}{p'}$, $\Re\delta = 0$, and $\xi, \gamma \in \mathbb{C}$, or alternatively $(\Re\delta)^2 < \Re\varepsilon - \frac{|\alpha|}{p'}$, $\Re\beta - \frac{|\alpha|}{p} > 1$, and $\xi, \gamma \in \mathbb{C}$, or alternatively $(\Re\delta)^2 < \Re\beta - \frac{|\alpha|}{p}$, $\Re\varepsilon - \frac{|\alpha|}{p'} > 1$, and $\xi, \gamma \in \mathbb{C}$.

In view of Corollary 2.1 in [8], we recognize the following.

Corollary 2.2. *For $f \in L^p(\mathbb{R}, \cosh(\alpha x^2)\,dx)$, with $1 < p < \infty$, $p + p' = pp'$, and $\alpha \in \mathbb{R}$, and if $\Re\beta > \frac{|\alpha|}{p}$, $\Re\varepsilon > \frac{|\alpha|}{p'}$, $\Re\delta = 0$, and $\xi, \gamma \in \mathbb{C}$, or alternatively $(\Re\delta)^2 < \Re\varepsilon - \frac{|\alpha|}{p'}$, $\Re\beta - \frac{|\alpha|}{p} > 1$, and $\xi, \gamma \in \mathbb{C}$, or alternatively $(\Re\delta)^2 < \Re\beta - \frac{|\alpha|}{p}$, $\Re\varepsilon - \frac{|\alpha|}{p'} > 1$, and $\xi, \gamma \in \mathbb{C}$, then we get*

$$\mathfrak{F}'_{\varepsilon,\beta,\delta,\gamma,\xi} T_f = T_{\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f}$$

on $(L^p(\mathbb{R}, \cosh(\alpha x^2)\,dx))'$.

3. The operator $\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}$ over the spaces $L^1(\mathbb{R}, \cosh(\alpha x)\,dx)$ and $L^1(\mathbb{R}, \cosh(\alpha x^2)\,dx)$

We now prove L^p-boundedness results corresponding to the case when $p = 1$.

3.1. The case of the spaces $L^1(\mathbb{R}, \cosh(\alpha x)\,dx)$.

By following [6, Proposition 3.1], we now prove Theorem 3.1.

Theorem 3.1. *If we assume that $\alpha \in \mathbb{R}$, then we have the following.*

(a) *If $\Re\beta > 0$, $\Re\varepsilon \geq 0$, $\Re\delta = \Re\gamma = 0$, and $\xi \in \mathbb{C}$, or alternatively $(\Re\delta)^2 < \Re\beta$, $\Re\varepsilon \geq 1$, $\Re\gamma = 0$, and $\xi, \gamma \in \mathbb{C}$, or alternatively $(\Re\delta)^2 \leq \Re\varepsilon$, $\Re\beta > 1$, $\Re\gamma = 0$, and $\xi \in \mathbb{C}$, then the operator $\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}$ given by (1.1) is bounded from the spaces $L^1(\mathbb{R}, \cosh(\alpha x)\,dx)$ into $L^q(\mathbb{R}, \cosh(\alpha x)\,dx)$ for any q $(0 < q < \infty)$.*

(b) *If $\Re\beta \geq 0$, $\Re\varepsilon \geq 0$, and $\Re\delta = \Re\gamma = \Re\xi = 0$, or alternatively $(\Re\delta)^2 \leq \Re\beta$, $\Re\varepsilon \geq 1$, and $\Re\gamma = \Re\xi = 0$, or alternatively $(\Re\delta)^2 \leq \Re\varepsilon$, $\Re\beta \geq 1$, and $\Re\gamma = \Re\xi = 0$, then the operator $\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}$ is bounded from the spaces $L^1(\mathbb{R}, \cosh(\alpha x)\,dx)$ into $L^\infty(\mathbb{R}, \cosh(\alpha x)\,dx)$.*

Proof. (a) First note that

$$\begin{aligned}
&\big|(\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}f)(y)\big| \\
&\quad\le \int_{-\infty}^{+\infty}\big|\exp[-\beta y^2-\varepsilon x^2+2\delta xy+\xi y+\gamma x]\big|\big|f(x)\big|\,dx \\
&\quad= \int_{-\infty}^{+\infty}\big|\exp[-\beta y^2-\varepsilon x^2+2\delta xy+\xi y+\gamma x]\big| \\
&\qquad\times\big|f(x)\big|\cosh(\alpha x)\big(\cosh(\alpha x)\big)^{-1}\,dx \\
&\quad\le \int_{-\infty}^{+\infty}\sup_{x\in\mathbb{R}}\Big\{\frac{|\exp[-\beta y^2-\varepsilon x^2+2\delta xy+\xi y+\gamma x]|}{\cosh(\alpha x)}\Big\}\big|f(x)\big|\cosh(\alpha x)\,dx \\
&\quad= \int_{-\infty}^{+\infty}\sup_{x\in\mathbb{R}}\Big\{\frac{|\exp[-\Re\beta y^2-\Re\varepsilon x^2+2\Re\delta xy+\Re\xi y+\Re\gamma x]|}{\cosh(\alpha x)}\Big\} \\
&\qquad\times\big|f(x)\big|\cosh(\alpha x)\,dx;
\end{aligned}$$

therefore, for any $0<q<\infty$, we have

$$\begin{aligned}
&\big|(\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}f)(y)\big|^q \\
&\quad\le \|f\|_{1,\mu_1}^q\cdot\Big(\sup_{x\in\mathbb{R}}\Big\{\frac{\exp[-\Re\beta y^2-\Re\varepsilon x^2+2\Re\delta xy+\Re\xi y+\Re\gamma x]}{\cosh(\alpha x)}\Big\}\Big)^q.
\end{aligned}$$

Thus, we clearly find that

$$\begin{aligned}
&\int_{-\infty}^{+\infty}\big|(\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}f)(y)\big|^q\cosh(\alpha y)\,dy \\
&\quad\le \|f\|_{1,\mu_1}^q\cdot\int_{-\infty}^{+\infty}\Big(\sup_{x\in\mathbb{R}}\Big\{\frac{|\exp[-\Re\beta y^2-\Re\varepsilon x^2+2\Re\delta xy+\Re\xi y+\Re\gamma x]|}{\cosh(\alpha x)}\Big\}\Big)^q \\
&\qquad\times\cosh(\alpha y)\,dy,
\end{aligned}$$

and consequently,

$$\begin{aligned}
&\|\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}f\|_{q,\mu_1} \\
&\quad\le \|f\|_{1,\mu_1}\Big(\int_{-\infty}^{+\infty}\Big(\sup_{x\in\mathbb{R}}\Big\{\frac{|\exp[-\Re\beta y^2-\Re\varepsilon x^2+2\Re\delta xy+\Re\xi y+\Re\gamma x]|}{\cosh(\alpha x)}\Big\}\Big)^q \\
&\qquad\times\cosh(\alpha y)\,dy\Big)^{1/q}.
\end{aligned}$$

Hence, for $\Re\delta=0$, we have

$$\begin{aligned}
&\|\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}f\|_{q,\mu_1} \\
&\quad\le \|f\|_{1,\mu_1}\sup_{x\in\mathbb{R}}\Big\{\frac{\exp[-\Re\varepsilon x^2+\Re\gamma x]}{\cosh(\alpha x)}\Big\} \\
&\qquad\times\Big(\int_{-\infty}^{+\infty}\exp\big[(-\Re\beta y^2+\Re\xi y)q\big]\cosh(\alpha y)\,dy\Big)^{1/q} \qquad (3.1)\\
&\quad= \|f\|_{1,\mu_1}\sup_{x\in\mathbb{R}}\Big\{\frac{\exp[-\Re\varepsilon x^2+\Re\gamma x]}{\cosh(\alpha x)}\Big\} \\
&\qquad\times\Big(\int_{-\infty}^{+\infty}\exp\big[(-\Re\beta y^2+\Re\xi y)q\big]\Big(\frac{e^{\alpha y}+e^{-\alpha y}}{2}\Big)\,dy\Big)^{1/q}.
\end{aligned}$$

Given that $2\Re\delta xy \le x^2 + (\Re\delta)^2 y^2$, we have

$$\begin{aligned} &\|\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f\|_{q,\mu_1} \\ &\quad \le \|f\|_{1,\mu_1} \sup_{x\in\mathbb{R}} \Big\{ \frac{\exp[-(\Re\varepsilon - 1)x^2 + \Re\gamma x]}{\cosh(\alpha x)} \Big\} \\ &\qquad \times \Big(\int_{-\infty}^{+\infty} \exp\big[\big(-\big(\Re\beta - (\Re\delta)^2\big) y^2 + \Re\xi y\big) q\big] \cosh(\alpha y)\, dy \Big)^{1/q} \qquad (3.2) \\ &\quad = \|f\|_{1,\mu_1} \sup_{x\in\mathbb{R}} \Big\{ \frac{\exp[-(\Re\varepsilon - 1)x^2 + \Re\gamma x]}{\cosh(\alpha x)} \Big\} \\ &\qquad \times \Big(\int_{-\infty}^{+\infty} \exp\big[\big(-\big(\Re\beta - (\Re\delta)^2\big) y^2 + \Re\xi y\big) q\big] \Big(\frac{e^{\alpha y} + e^{-\alpha y}}{2} \Big)\, dy \Big)^{1/q}. \end{aligned}$$

Similarly, from $2\Re\delta xy \le (\Re\delta)^2 x^2 + y^2$, it follows that

$$\begin{aligned} &\|\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f\|_{q,\mu_1} \\ &\quad \le \|f\|_{1,\mu_1} \sup_{x\in\mathbb{R}} \Big\{ \frac{\exp[-(\Re\varepsilon - 1)x^2 + \Re\gamma x]}{\cosh(\alpha x)} \Big\} \\ &\qquad \times \Big(\int_{-\infty}^{+\infty} \exp\big[\big(-\big(\Re\beta - (\Re\delta)^2\big) y^2 + \Re\xi y\big) q\big] \cosh(\alpha y)\, dy \Big)^{1/q} \qquad (3.3) \\ &\quad = \|f\|_{1,\mu_1} \sup_{x\in\mathbb{R}} \Big\{ \frac{\exp[-(\Re\varepsilon - (\Re\delta)^2)x^2 + \Re\gamma x]}{\cosh(\alpha x)} \Big\} \\ &\qquad \times \Big(\int_{-\infty}^{+\infty} \exp\big[\big(-(\Re\beta - 1) y^2 + \Re\xi y\big) q\big] \Big(\frac{e^{\alpha y} + e^{-\alpha y}}{2} \Big)\, dy \Big)^{1/q}. \end{aligned}$$

Since the integrals involved in (3.1), (3.2), and (3.3) converge, we get

$$\|\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f\|_{q,\mu_1} \le C \|f\|_{1,\mu_1}$$

for a certain real constant C depending on q.

Accordingly, the operator $\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}$ is bounded from the spaces $L^1(\mathbb{R}, \cosh(\alpha x)\, dx)$ into $L^q(\mathbb{R}, \cosh(\alpha x)\, dx)$, $0 < q < \infty$.

(b) Likewise, taking into account that

$$\begin{aligned} &\big|(\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f)(y)\big| \\ &\quad \le \int_{-\infty}^{+\infty} \sup_{x\in\mathbb{R}} \Big\{ \frac{|\exp[-\Re\beta y^2 - \Re\varepsilon x^2 + 2\Re\delta xy + \Re\xi y + \Re\gamma x]|}{\cosh(\alpha x)} \Big\} \\ &\qquad \times \big|f(x)\big| \cosh(\alpha x)\, dx, \end{aligned}$$

we get

$$\begin{aligned} &\|\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f\|_{\infty} \\ &\quad = \sup_{y\in\mathbb{R}} \int_{-\infty}^{+\infty} \sup_{x\in\mathbb{R}} \Big\{ \frac{|\exp[-\Re\beta y^2 - \Re\varepsilon x^2 + 2\Re\delta xy + \Re\xi y + \Re\gamma x]|}{\cosh(\alpha x)} \Big\} \end{aligned}$$

$$\begin{aligned}&\times \bigl|f(x)\bigr| \cosh(\alpha x)\,dx\\ &\le \|f\|_{1,\mu_1} \sup_{y\in\mathbb{R}} \sup_{x\in\mathbb{R}} \Bigl\{\frac{|\exp[-\beta y^2 - \varepsilon x^2 + 2\delta xy + \xi y + \gamma x]|}{\cosh(\alpha x)}\Bigr\};\end{aligned}$$

hence, for $\Re\delta = 0$, we have

$$\begin{aligned}&\|\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f\|_\infty\\ &\quad\le \sup_{x\in\mathbb{R}} \Bigl\{\frac{|\exp[-\Re\varepsilon x^2 + \Re\gamma x]|}{\cosh(\alpha x)}\Bigr\} \sup_{y\in\mathbb{R}} \bigl\{\exp[-\Re\beta y^2 + \Re\xi y]\bigr\}.\end{aligned} \tag{3.4}$$

Moreover, since $2\Re\delta xy \le x^2 + (\Re\delta)^2 y^2$, then we have

$$\begin{aligned}&\|\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f\|_\infty\\ &\quad\le \|f\|_{1,\mu_1} \sup_{y\in\mathbb{R}} \bigl\{\exp\bigl[-\bigl(\Re\beta - (\Re\delta)^2\bigr)y^2 + \Re\xi y\bigr]\bigr\}\\ &\qquad\times \sup_{x\in\mathbb{R}} \Bigl\{\frac{\exp[-(\Re\varepsilon - 1)x^2 + \Re\gamma x]}{\cosh(\alpha x)}\Bigr\}.\end{aligned} \tag{3.5}$$

Again inequality $2\Re\delta xy \le (\Re\delta)^2 x^2 + y^2$ leads to

$$\begin{aligned}&\|\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f\|_\infty\\ &\quad\le \|f\|_{1,\mu_1} \sup_{y\in\mathbb{R}} \bigl\{\exp\bigl[-(\Re\beta - 1)y^2 + \Re\xi y\bigr]\bigr\}\\ &\qquad\times \sup_{x\in\mathbb{R}} \Bigl\{\frac{\exp[-(\Re\varepsilon - (\Re\delta)^2)x^2 + \Re\gamma x]}{\cosh(\alpha x)}\Bigr\}.\end{aligned} \tag{3.6}$$

If the conditions on the parameters are that the integrals involved in (3.4), (3.5), and (3.6) converge, then

$$\|\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f\|_\infty \le C\|f\|_{1,\mu_1}$$

for a certain real constant C.

Thus the operator $\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}$ is bounded from the spaces $L^1(\mathbb{R}, \cosh(\alpha x)\,dx)$ into $L^\infty(\mathbb{R}, \cosh(\alpha x)\,dx)$ for all $\alpha \in \mathbb{R}$, which evidently completes the proof of Theorem 3.1. □

From this result it follows that, for $\Re\beta \ge 0$, $\Re\varepsilon \ge 0$, and $\Re\delta = \Re\gamma = \Re\xi = 0$, or alternatively $(\Re\delta)^2 \le \Re\varepsilon$, $\Re\beta \ge 1$, and $\Re\gamma = \Re\xi = 0$, or alternatively $(\Re\delta)^2 \le \Re\beta$, $\Re\varepsilon \ge 1$, and $\Re\gamma = \Re\xi = 0$, the operator $\mathfrak{F}_{\varepsilon,\beta,\delta,\xi,\gamma}$ is bounded from the spaces $L^1(\mathbb{R}, \cosh(\alpha x)\,dx)$ into $L^\infty(\mathbb{R}, \cosh(\alpha x)\,dx)$.

Using this fact and the fact that the weight $\cosh(\alpha x)$ is greater than or equal to 1, Proposition 3.2 in [8] yields the following.

Theorem 3.2. *For all $\alpha \in \mathbb{R}$, the following Parseval-type relation holds:*

$$\int_{-\infty}^{+\infty} (\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f)(x) g(x)\,dx = \int_{-\infty}^{\infty} f(x)(\mathfrak{F}_{\varepsilon,\beta,\delta,\gamma,\xi} g)(x)\,dx$$

for $f, g \in L^1(\mathbb{R}, \cosh(\alpha x)\,dx)$ with $\Re\beta \ge 0$, $\Re\varepsilon \ge 0$, and $\Re\delta = \Re\gamma = \Re\xi = 0$, or alternatively $(\Re\delta)^2 \le \Re\varepsilon$, $\Re\beta \ge 1$, and $\Re\gamma = \Re\xi = 0$, or alternatively $(\Re\delta)^2 \le \Re\beta$, $\Re\varepsilon \ge 1$, and $\Re\gamma = \Re\xi = 0$.

From Corollary 3.1 in [8] we obtain the following.

Corollary 3.1. *If we assume that $f \in L^1(\mathbb{R}, \cosh(\alpha x)\,dx)$, and $\alpha \in \mathbb{R}$, for $\Re\beta \geq 0$, $\Re\varepsilon \geq 0$, and $\Re\delta = \Re\gamma = \Re\xi = 0$, or alternatively $(\Re\delta)^2 \leq \Re\varepsilon$, $\Re\beta \geq 1$, $\Re\gamma = \Re\xi = 0$, or alternatively $(\Re\delta)^2 \leq \Re\beta$, $\Re\varepsilon \geq 1$, and $\Re\gamma = \Re\xi = 0$, then it follows that*

$$\mathfrak{F}'_{\varepsilon,\beta,\delta,\gamma,\xi} T_f = T_{\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f}$$

on $(L^1(\mathbb{R}, \cosh(\alpha x)\,dx))'$.

3.2. **The case of the spaces $L^1(\mathbb{R}, \cosh(\alpha x^2)\,dx)$.** By following [6, Proposition 3.1], we prove Theorem 3.3 below.

Theorem 3.3. *If we assume that $\alpha \in \mathbb{R}$, then*

(a) *if $\Re\beta > \frac{|\alpha|}{q}$, $\Re\varepsilon \geq 0$, $\Re\delta = \Re\gamma = 0$, and $\xi \in \mathbb{C}$, or alternatively $(\Re\delta)^2 < \Re\beta - \frac{|\alpha|}{q}$, $\Re\varepsilon \geq 1$, $\Re\gamma = 0$, and $\xi \in \mathbb{C}$, or alternatively $(\Re\delta)^2 \leq \Re\varepsilon$, $\Re\beta - \frac{|\alpha|}{q} > 1$, $\Re\gamma = 0$, and $\xi \in \mathbb{C}$, then the operator $\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}$ given by (1.1) is bounded from the spaces $L^1(\mathbb{R}, \cosh(\alpha x^2)\,dx)$ into $L^q(\mathbb{R}, \cosh(\alpha x^2)\,dx)$ for any q $(0 < q < \infty)$;*

(b) *if $\Re\beta \geq 0$, $\Re\varepsilon \geq 0$, and $\Re\delta = \Re\gamma = \Re\xi = 0$, or alternatively $(\Re\delta)^2 \leq \Re\beta$, $\Re\varepsilon \geq 1$, and $\Re\gamma = \Re\xi = 0$, or alternatively $(\Re\delta)^2 \leq \Re\varepsilon$, $\Re\beta \geq 1$, and $\Re\gamma = \Re\xi = 0$, then the operator $\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}$ is bounded from the spaces $L^1(\mathbb{R}, \cosh(\alpha x^2)\,dx)$ into $L^\infty(\mathbb{R}, \cosh(\alpha x^2)\,dx)$.*

Proof. (a) We begin by observing that

$$\begin{aligned}
&\big|(\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f)(y)\big| \\
&\quad\leq \int_{-\infty}^{+\infty} \big|\exp[-\beta y^2 - \varepsilon x^2 + 2\delta xy + \xi y + \gamma x]\big|\big|f(x)\big|\,dx \\
&\quad= \int_{-\infty}^{+\infty} \big|\exp[-\beta y^2 - \varepsilon x^2 + 2\delta xy + \xi y + \gamma x]\big| \\
&\qquad\times \big|f(x)\big| \cosh(\alpha x^2)\big(\cosh(\alpha x^2)\big)^{-1}\,dx \\
&\quad\leq \int_{-\infty}^{+\infty} \sup_{x\in\mathbb{R}} \Big\{ \frac{\exp[-\Re\beta y^2 - \varepsilon x^2 + 2\Re\delta xy + \Re\xi y + \Re\gamma x]}{\cosh(\alpha x^2)} \Big\} \big|f(x)\big| \cosh(\alpha x^2)\,dx \\
&\quad= \|f\|_{1,\mu_2} \cdot \sup_{x\in\mathbb{R}} \Big\{ \frac{|\exp[-\beta y^2 - \varepsilon x^2 + 2\delta xy + \xi y + \gamma x]|}{\cosh(\alpha x^2)} \Big\}.
\end{aligned}$$

Hence, for any $0 < q < \infty$, we have

$$\begin{aligned}
&\big|(\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f)(y)\big|^q \\
&\quad\leq \|f\|_{1,\mu_2}^q \cdot \Big(\sup_{x\in\mathbb{R}} \Big\{ \frac{\exp[-\Re\beta y^2 - \Re\varepsilon x^2 + 2\Re\delta xy + \Re\xi y + \Re\gamma x]}{\cosh(\alpha x^2)} \Big\} \Big)^q.
\end{aligned}$$

Thus, we clearly find that

$$\int_{-\infty}^{+\infty} \left|(\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f)(y)\right|^q \cosh(\alpha y^2)\, dy \\ \leq \|f\|_{1,\mu_2}^q \cdot \int_{-\infty}^{+\infty} \Big(\sup_{x\in\mathbb{R}}\Big\{\frac{\exp[-\Re\beta y^2 - \Re\varepsilon x^2 + 2\Re\delta xy + \Re\xi y + \Re\gamma x]}{\cosh(\alpha x^2)}\Big\}\Big)^q \\ \times \cosh(\alpha y^2)\, dy;$$

therefore,

$$\|\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f\|_{q,\mu_2} \\ \leq \|f\|_{1,\mu_2} \Big(\int_{-\infty}^{+\infty} \Big(\sup_{x\in\mathbb{R}}\Big\{\frac{\exp[-\Re\beta y^2 - \Re\varepsilon x^2 + 2\Re\delta xy + \Re\xi y + \Re\gamma x]}{\cosh(\alpha x^2)}\Big\}\Big)^q \\ \times \cosh(\alpha y^2)\, dy\Big)^{1/q}.$$

Consequently, for $\Re\delta = 0$, we have

$$\begin{aligned} &\|\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f\|_{q,\mu_2} \\ &\quad \leq \|f\|_{1,\mu_2} \sup_{x\in\mathbb{R}}\Big\{\frac{\exp[-\Re\varepsilon x^2 + \Re\gamma x]}{\cosh(\alpha x^2)}\Big\} \\ &\qquad \times \Big(\int_{-\infty}^{+\infty} \exp\big[(-\Re\beta y^2 + \Re\xi y)q\big] \cosh(\alpha y^2)\, dy\Big)^{1/q} \qquad (3.7) \\ &\quad = \|f\|_{1,\mu_2} \sup_{x\in\mathbb{R}}\Big\{\frac{\exp[-\Re\varepsilon x^2 + \Re\gamma x]}{\cosh(\alpha x^2)}\Big\} \\ &\qquad \times \Big(\int_{-\infty}^{+\infty} \exp\big[(-\Re\beta y^2 + \Re\xi y)q\big] \Big(\frac{e^{\alpha y^2} + e^{-\alpha y^2}}{2}\Big)\, dy\Big)^{1/q}. \end{aligned}$$

Applying inequality $2\Re\delta xy \leq x^2 + (\Re\delta)^2 y^2$, we obtain that

$$\begin{aligned} &\|\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f\|_{q,\mu_2} \\ &\quad \leq \|f\|_{1,\mu_2} \sup_{x\in\mathbb{R}}\Big\{\frac{\exp[-(\Re\varepsilon - 1)x^2 + \Re\gamma x]}{\cosh(\alpha x^2)}\Big\} \\ &\qquad \times \Big(\int_{-\infty}^{+\infty} \exp\Big[\Big(-\big(\Re\beta - (\Re\delta)^2\big)y^2 + \Re\xi y\Big)q\Big] \cosh(\alpha y^2)\, dy\Big)^{1/q} \qquad (3.8) \\ &\quad = \|f\|_{1,\mu_2} \sup_{x\in\mathbb{R}}\Big\{\frac{\exp[-(\Re\varepsilon - 1)x^2 + \Re\gamma x]}{\cosh(\alpha x^2)}\Big\} \\ &\qquad \times \Big(\int_{-\infty}^{+\infty} \exp\Big[\Big(-\big(\Re\beta - (\Re\delta)^2\big)y^2 + \Re\xi y\Big)q\Big] \Big(\frac{e^{\alpha y^2} + e^{-\alpha y^2}}{2}\Big)\, dy\Big)^{1/q}; \end{aligned}$$

hence, from $2\Re\delta xy \leq (\Re\delta)^2 x^2 + y^2$, it again follows that

$$\|\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f\|_{q,\mu_2} \\ \leq \|f\|_{1,\mu_2} \sup_{x\in\mathbb{R}}\Big\{\frac{\exp[-(\Re\varepsilon - 1)x^2 + \Re\gamma x]}{\cosh(\alpha x)}\Big\}$$

$$\times\Big(\int_{-\infty}^{+\infty}\exp\big[\big(-\big(\Re\beta-(\Re\delta)^2\big)y^2+\Re\xi y\big)q\big]\cosh(\alpha y)\,dy\Big)^{1/q} \tag{3.9}$$
$$=\|f\|_{1,\mu_2}\sup_{x\in\mathbb{R}}\Big\{\frac{\exp[-(\Re\varepsilon-\Re\delta^2)x^2+\Re\gamma x]}{\cosh(\alpha x)}\Big\}$$
$$\times\Big(\int_{-\infty}^{+\infty}\exp\big[\big(-(\Re\beta-1)y^2+\Re\xi y\big)q\big]\Big(\frac{e^{\alpha y}+e^{-\alpha y}}{2}\Big)\,dy\Big)^{1/q}.$$

The hypothesis considered on the parameters implies that the integrals involved in (3.7), (3.8), and (3.9) converge and that

$$\|\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}f\|_{q,\mu_2}\le C\|f\|_{1,\mu_2}$$

for a certain real constant C depending on q.

Thus, the operator $\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}$ is bounded from the spaces $L^1(\mathbb{R},\cosh(\alpha x^2)\,dx)$ into $L^q(\mathbb{R},\cosh(\alpha x^2)\,dx)$.

(b) Similarly, taking into account

$$\big|(\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}f)(y)\big|$$
$$\le\int_{-\infty}^{+\infty}\sup_{x\in\mathbb{R}}\Big\{\frac{|\exp[-\Re\beta y^2-\Re\varepsilon x^2+2\Re\delta xy+\Re\xi y+\Re\gamma x]|}{\cosh(\alpha x^2)}\Big\}$$
$$\times\big|f(x)\big|\cosh(\alpha x^2)\,dx,$$

we get

$$\|\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}f\|_{\infty}$$
$$=\sup_{y\in\mathbb{R}}\int_{-\infty}^{+\infty}\sup_{x\in\mathbb{R}}\Big\{\frac{|\exp[-\Re\beta y^2-\Re\varepsilon x^2+2\Re\delta xy+\Re\xi y+\Re\gamma x]|}{\cosh(\alpha x^2)}\Big\}$$
$$\times\big|f(x)\big|\cosh(\alpha x)\,dx$$
$$\le\|f\|_{1,\mu_2}\sup_{y\in\mathbb{R}}\sup_{x\in\mathbb{R}}\Big\{\frac{|\exp[-\beta y^2-\varepsilon x^2+2\delta xy+\xi y+\gamma x]|}{\cosh(\alpha x^2)}\Big\};$$

hence, for $\Re\delta=0$, we have

$$\|\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}f\|_{\infty}$$
$$\le\sup_{x\in\mathbb{R}}\Big\{\frac{|\exp[-\Re\varepsilon x^2+\Re\gamma x]|}{\cosh(\alpha x^2)}\Big\}\sup_{y\in\mathbb{R}}\big\{\exp[-\Re\beta y^2+\Re\xi y]\big\}. \tag{3.10}$$

If we again take into account that $2\Re\delta xy\le x^2+(\Re\delta)^2y^2$, then we see that

$$\|\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}f\|_{\infty}$$
$$\le\|f\|_{1,\mu_2}\sup_{y\in\mathbb{R}}\big\{\exp\big[-\big(\Re\beta-(\Re\delta)^2\big)y^2+\Re\xi y\big]\big\} \tag{3.11}$$
$$\times\sup_{x\in\mathbb{R}}\Big\{\frac{\exp[-(\Re\varepsilon-1)x^2+\Re\gamma x]}{\cosh(\alpha x^2)}\Big\}.$$

From inequality $2\Re\delta xy \le (\Re\delta)^2x^2 + y^2$, we have

$$\begin{aligned}&\|\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}f\|_\infty \\ &\quad\le \|f\|_{1,\mu_2} \sup_{y\in\mathbb{R}}\left\{\exp\left[-(\Re\beta-1)y^2+\Re\xi y\right]\right\} \\ &\qquad\times \sup_{x\in\mathbb{R}}\left\{\frac{\exp[-(\Re\varepsilon-(\Re\delta)^2)x^2+\Re\gamma x]}{\cosh(\alpha x^2)}\right\}.\end{aligned} \tag{3.12}$$

From the hypothesis considered, the integrals involved in (3.10), (3.11), and (3.12) converge; thus

$$\|\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}f\|_\infty \le C\|f\|_{1,\mu_2}$$

for a certain real constant C.

Consequently, the operator $\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}$ is bounded from the spaces $L^1(\mathbb{R}, \cosh(\alpha x^2)\,dx)$ into $L^\infty(\mathbb{R}, \cosh(\alpha x^2)\,dx)$ for all $\alpha \in \mathbb{R}$, which completes the proof of Theorem 3.1. □

By virtue of this result, it follows that the operator $\mathfrak{F}_{\varepsilon,\beta,\delta,\gamma,\xi}$ is bounded from the spaces $L^1(\mathbb{R}, \cosh(\alpha x^2)\,dx)$ into $L^\infty(\mathbb{R}, \cosh(\alpha x^2)\,dx)$, for $\Re\varepsilon \ge 0$, $\Re\beta \ge 0$, and $\Re\delta = \Re\gamma = \Re\xi = 0$, or alternatively $(\Re\delta)^2 \le \Re\varepsilon$, $\Re\beta \ge 1$, and $\Re\gamma = \Re\xi = 0$, or alternatively $(\Re\delta)^2 \le \Re\beta$, $\Re\varepsilon \ge 1$, and $\Re\gamma = \Re\xi = 0$.

Using this assertion and the fact that the weight $\cosh(\alpha x^2)$ is greater than or equal to 1, Proposition 3.2 in [8] yields the following.

Theorem 3.4. *For any $\alpha \in \mathbb{R}$, the Parseval-type relation*

$$\int_{-\infty}^{+\infty} (\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}f)(x)g(x)\,dx = \int_{-\infty}^{\infty} f(x)(\mathfrak{F}_{\varepsilon,\beta,\delta,\gamma,\xi}g)(x)\,dx$$

holds for $f, g \in L^1(\mathbb{R}, \cosh(\alpha x^2)\,dx)$ with $\Re\varepsilon \ge 0$, $\Re\beta \ge 0$, and $\Re\delta = \Re\gamma = \Re\xi = 0$, or alternatively $(\Re\delta)^2 \le \Re\varepsilon$, $\Re\beta \ge 1$, and $\Re\gamma = \Re\xi = 0$, or alternatively $(\Re\delta)^2 \le \Re\beta$, $\Re\varepsilon \ge 1$, and $\Re\gamma = \Re\xi = 0$.

Next, from Corollary 3.1 in [8], we obtain the following.

Corollary 3.2. *If we assume that $f \in L^1(\mathbb{R}, \cosh(\alpha x^2)\,dx)$ and $\alpha \in \mathbb{R}$ and that $\Re\varepsilon \ge 0$, $\Re\beta \ge 0$, and $\Re\delta = \Re\gamma = \Re\xi = 0$, or alternatively $(\Re\delta)^2 \le \Re\varepsilon$, $\Re\beta \ge 1$, and $\Re\gamma = \Re\xi = 0$, or alternatively $(\Re\delta)^2 \le \Re\beta$, $\Re\varepsilon \ge 1$, and $\Re\gamma = \Re\xi = 0$, then it follows that*

$$\mathfrak{F}'_{\varepsilon,\beta,\delta,\gamma,\xi}T_f = T_{\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}f}$$

on $(L^1(\mathbb{R}, \cosh(\alpha x^2)\,dx))'$.

4. The operator $\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}$ over the spaces $L^\infty(\mathbb{R}, \cosh(\alpha x)\,dx)$ and $L^\infty(\mathbb{R}, \cosh(\alpha x^2)\,dx)$

In this section, we prove L^p-boundedness results corresponding to the case when $p = \infty$.

4.1. **The case of the spaces $L^\infty(\mathbb{R}, \cosh(\alpha x)\,dx)$.** Indeed, by following [6, Proposition 4.1], we are led to the following result.

Theorem 4.1. *If we assume that $\alpha \in \mathbb{R}$, then*

(a) *for $\Re\varepsilon > 0$, $(\Re\delta)^2 < \Re\varepsilon\Re\beta$, and $\gamma, \xi \in \mathbb{C}$, the operator $\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}$ given by (1.1) is bounded from the spaces $L^\infty(\mathbb{R}, \cosh(\alpha x)\,dx)$ into $L^q(\mathbb{R}, \cosh(\alpha x)\,dx)$ for any q $(0 < q < \infty)$;*

(b) *for $\Re\varepsilon > 0$, $(\Re\delta)^2 \le \Re\varepsilon\Re\beta$, and $\Re\varepsilon\Re\xi + \Re\delta\Re\gamma = 0$, the operator $\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}$ is bounded from the spaces $L^\infty(\mathbb{R}, \cosh(\alpha x)\,dx)$ into $L^\infty(\mathbb{R}, \cosh(\alpha x)\,dx)$.*

Proof. (a) Note that

$$\begin{aligned}
&\left|(\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f)(y)\right| \\
&\quad \le \int_{-\infty}^{+\infty} \left|\exp[-\beta y^2 - \varepsilon x^2 + 2\delta xy + \xi y + \gamma x]\right| \left|f(x)\right| dx \\
&\quad \le \|f\|_\infty \int_{-\infty}^{+\infty} \left|\exp[-\beta y^2 - \varepsilon x^2 + 2\delta xy + \xi y + \gamma x]\right| dx \\
&\quad = \|f\|_\infty \int_{-\infty}^{+\infty} \exp[-\Re\beta y^2 - \Re\epsilon x^2 + 2\Re\delta xy + \Re\xi y + \Re\gamma x]\, dx;
\end{aligned}$$

thus, for any $0 < q < \infty$, we get

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \left|(\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f)(y)\right|^q \cosh(\alpha y)\, dy \\
&\quad \le \|f\|_\infty^q \cdot \int_{-\infty}^{+\infty} \Big(\int_{-\infty}^{+\infty} \exp[-\Re\beta y^2 - \Re\epsilon x^2 + 2\Re\delta xy + \Re\xi y + \Re\gamma x]\, dx\Big)^q \\
&\qquad \times \cosh(\alpha y)\, dy.
\end{aligned}$$

By virtue of (1.3) and because $\Re\varepsilon > 0$ we see that

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \left|(\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f)(y)\right|^q \cosh(\alpha y)\, dy \\
&\quad \le \|f\|_\infty^q \int_{-\infty}^{+\infty} \Big(\frac{\pi}{\Re\varepsilon}\Big)^{q/2} \exp\Big[-q\Re\beta y^2 + \frac{q(\Re\delta)^2}{\Re\varepsilon} y^2 \\
&\qquad + q\Re\xi y + \frac{q\Re\delta\Re\gamma}{\Re\varepsilon} y + \frac{q(\Re\gamma)^2}{4\Re\varepsilon}\Big] \cosh(\alpha y)\, dy
\end{aligned}$$

and therefore that

$$\begin{aligned}
&\|\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f\|_{q,\mu_1} \\
&\quad \le \|f\|_\infty \cdot \Big(\int_{-\infty}^{+\infty} \Big(\frac{\pi}{\Re\varepsilon}\Big)^{q/2} \exp\Big[-q\Re\beta y^2 + \frac{q(\Re\delta)^2}{\Re\varepsilon} y^2 \\
&\qquad + q\Re\xi y + \frac{q\Re\delta\Re\gamma}{\Re\varepsilon} y + \frac{q(\Re\gamma)^2}{4\Re\varepsilon}\Big] \cosh(\alpha y)\, dy\Big)^{1/q},
\end{aligned}$$

which converges under the above conditions.

Hence, we have

$$\|\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f\|_{q,\mu_1} \le C\|f\|_\infty$$

for a certain real constant C depending on q.

Thus, the operator $\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}$ is bounded from the spaces $L^\infty(\mathbb{R}, \cosh(\alpha x)\,dx)$ into $L^q(\mathbb{R}, \cosh(\alpha x)\,dx)$, $0 < q < \infty$.

(b) Also note that

$$\begin{aligned} &\|\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f\|_\infty \\ &\quad \le C\|f\|_\infty \cdot \sup_{y\in\mathbb{R}} \Big\{ \int_{-\infty}^{+\infty} \big|\exp[-\beta y^2 - \varepsilon x^2 + 2\delta xy + \xi y + \gamma x]\big|\,dx \Big\} \\ &\quad = C\|f\|_\infty \cdot \sup_{y\in\mathbb{R}} \Big\{ \int_{-\infty}^{+\infty} \exp[-\Re\beta y^2 - \Re\varepsilon x^2 + 2\Re\delta xy + \Re\xi y + \Re\gamma x]\,dx \Big\}. \end{aligned}$$

By virtue of (1.3) and because $\Re\varepsilon > 0$, this expression is equal to

$$C\|f\|_\infty \cdot \sup_{y\in\mathbb{R}} \Big\{ \Big(\frac{\pi}{\Re\varepsilon}\Big)^{1/2} \exp\Big[-\Re\beta y^2 + \frac{(\Re\delta)^2}{\Re\varepsilon} y^2 + \Re\xi y + \frac{\Re\delta\Re\gamma}{\Re\varepsilon} y + \frac{(\Re\gamma)^2}{4\Re\epsilon}\Big]\Big\},$$

which is bounded under this hypothesis, and therefore the above expression is bounded.

Hence we have

$$\|\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f\|_\infty \le C\|f\|_\infty$$

for a certain real constant C.

In this way, the operator $\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}$ is bounded from the spaces $L^\infty(\mathbb{R}, \cosh(\alpha x)\,dx)$ into $L^\infty(\mathbb{R}, \cosh(\alpha x)\,dx)$. $\square$

From this result, for all $\alpha \in \mathbb{R}$, we see that the operator $\mathfrak{F}_{\varepsilon,\beta,\delta,\gamma,\xi}$ is bounded from the spaces $L^\infty(\mathbb{R}, \cosh(\alpha x)\,dx)$ into $L^1(\mathbb{R}, \cosh(\alpha x)\,dx)$, for $\Re\beta > 0$, $(\Re\delta)^2 < \Re\varepsilon\Re\beta$, and $\gamma, \xi \in \mathbb{C}$.

By Proposition 4.2 in [8], the following can be seen.

Theorem 4.2. *For all $\alpha \in \mathbb{R}$, the following Parseval-type relation*

$$\int_{-\infty}^{+\infty} (\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f)(x) g(x)\,dx = \int_{-\infty}^{+\infty} f(x)(\mathfrak{F}_{\varepsilon,\beta,\delta,\gamma,\xi} g)(x)\,dx$$

holds for $f, g \in L^\infty(\mathbb{R}, \cosh(\alpha x)\,dx)$ with $\Re\beta > 0$, $(\Re\delta)^2 < \Re\varepsilon\Re\beta$, and $\gamma, \xi \in \mathbb{C}$.

In addition, as a consequence of Corollary 4.1 in [8], we obtain the following.

Corollary 4.1. *For $\alpha \in \mathbb{R}$, and $f \in L^\infty(\mathbb{R}, \cosh(\alpha x)\,dx)$, $\Re\beta > 0$, $(\Re\delta)^2 < \Re\varepsilon\Re\beta$, and $\gamma, \xi \in \mathbb{C}$ we get*

$$\mathfrak{F}'_{\varepsilon,\beta,\delta,\gamma,\xi} T_f = T_{\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f}$$

on $(L^\infty(\mathbb{R}, \cosh(\alpha x)\,dx))'$.

4.2. **The case of the spaces $L^\infty(\mathbb{R}, \cosh(\alpha x^2)\,dx)$.** By following [6, Proposition 4.1], we obtain the following result.

Theorem 4.3. *If $\alpha \in \mathbb{R}$, then*

(a) *if $\Re\varepsilon > 0$, $(\Re\delta)^2 < \Re\varepsilon(\Re\beta - \frac{|\alpha|}{q})$, and $\gamma, \xi \in \mathbb{C}$, then the operator $\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}$ given by (1.1) is bounded from $L^\infty(\mathbb{R}, \cosh(\alpha x^2)\,dx)$ into $L^q(\mathbb{R}, \cosh(\alpha x^2)\,dx)$ for any q $(0 < q < \infty)$;*

(b) *if* $\Re\varepsilon > 0$, $(\Re\delta)^2 \le \Re\varepsilon\Re\beta$, *and* $\Re\varepsilon\Re\xi + \Re\delta\Re\gamma = 0$, *then the operator* $\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}$ *is bounded from the spaces* $L^\infty(\mathbb{R}, \cosh(\alpha x^2)\,dx)$ *into* $L^\infty(\mathbb{R}, \cosh(\alpha x^2)\,dx)$.

Proof. (a) Observe that

$$\begin{aligned}
&\left|(\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f)(y)\right| \\
&\quad \le \int_{-\infty}^{+\infty} \left|\exp[-\beta y^2 - \varepsilon x^2 + 2\delta xy + \xi y + \gamma x]\right| \left|f(x)\right| dx \\
&\quad \le \|f\|_\infty \int_{-\infty}^{+\infty} \left|\exp[-\beta y^2 - \varepsilon x^2 + 2\delta xy + \xi y + \gamma x]\right| dx \\
&\quad = \|f\|_\infty \int_{-\infty}^{+\infty} \exp[-\Re\beta y^2 - \Re\epsilon x^2 + 2\Re\delta xy + \Re\xi y + \Re\gamma x]\, dx
\end{aligned}$$

such that, for any $0 < q < \infty$, we get

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \left|(\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f)(y)\right|^q \cosh(\alpha y^2)\, dy \\
&\quad \le \|f\|_\infty^q \cdot \int_{-\infty}^{+\infty} \Big(\int_{-\infty}^{+\infty} \exp[-\Re\beta y^2 - \Re\epsilon x^2 + 2\Re\delta xy + \Re\xi y + \Re\gamma x]\, dx\Big)^q \\
&\quad\quad \times \cosh(\alpha y^2)\, dy.
\end{aligned}$$

By virtue of (1.3) and because $\Re\varepsilon > 0$, we get that

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \left|(\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f)(y)\right|^q \cosh(\alpha y^2)\, dy \\
&\quad \le \|f\|_\infty^q \cdot \int_{-\infty}^{+\infty} \Big(\frac{\pi}{\Re\varepsilon}\Big)^{q/2} \exp\Big[-q\Re\beta y^2 + \frac{q(\Re\delta)^2}{\Re\varepsilon} y^2 + q\Re\xi y \\
&\quad\quad + \frac{q\Re\delta\Re\gamma}{\Re\varepsilon} y + \frac{q(\Re\gamma)^2}{4\Re\varepsilon}\Big] \cosh(\alpha y^2)\, dy
\end{aligned}$$

and therefore that

$$\begin{aligned}
&\|\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f\|_{q,\mu_2} \\
&\quad \le \|f\|_\infty \cdot \Big(\int_{-\infty}^{+\infty} \Big(\frac{\pi}{\Re\varepsilon}\Big)^{q/2} \exp\Big[-q\Re\beta y^2 + \frac{q(\Re\delta)^2}{\Re\varepsilon} y^2 + q\Re\xi y \\
&\quad\quad + \frac{q\Re\delta\Re\gamma}{\Re\varepsilon} y + \frac{q(\Re\gamma)^2}{4\Re\varepsilon}\Big] \cosh(\alpha y^2)\, dy\Big)^{1/q},
\end{aligned}$$

which converges under the above conditions.

Hence, we have

$$\|\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f\|_\infty \le C \|f\|_{q,\mu_2}$$

for a certain real constant C depending on q.

The operator $\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}$ is thus bounded from $L^\infty(\mathbb{R}, \cosh(\alpha x^2)\,dx)$ into $L^q(\mathbb{R}, \cosh(\alpha x^2)\,dx)$, $0 < q < \infty$.

(b) Also observe that

$$\begin{aligned}
&\big\|(\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}f)(y)\big\|_\infty \\
&\quad\le C\|f\|_\infty\cdot\sup_{y\in\mathbb{R}}\Big\{\int_{-\infty}^{+\infty}\big|\exp[-\beta y^2-\varepsilon x^2+2\delta xy+\xi y+\gamma x]\big|\,dx\Big\} \\
&\quad= C\|f\|_\infty\cdot\sup_{y\in\mathbb{R}}\Big\{\int_{-\infty}^{+\infty}\exp[-\Re\beta y^2-\Re\varepsilon x^2+2\Re\delta xy+\Re\xi y+\Re\gamma x]\,dx\Big\}.
\end{aligned}$$

By virtue of (1.3) and because $\Re\varepsilon>0$, this expression is equal to

$$C\|f\|_\infty\cdot\sup_{y\in\mathbb{R}}\Big\{\Big(\frac{\pi}{\Re\varepsilon}\Big)^{1/2}\exp\Big[-\Re\beta y^2+\frac{(\Re\delta)^2}{\Re\varepsilon}y^2+\Re\xi y+\frac{\Re\delta\Re\gamma}{\Re\varepsilon}y+\frac{(\Re\gamma)^2}{4\Re\epsilon}\Big]\Big\},$$

which is bounded under this hypothesis and thus the above integral converges.

Hence, we have

$$\|\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}f\|_\infty\le C\|f\|_\infty$$

for a certain real constant C.

Hence, the operator $\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}$ is bounded from $L^\infty(\mathbb{R},\cosh(\alpha x^2)\,dx)$ into $L^\infty(\mathbb{R},\cosh(\alpha x^2)\,dx)$. □

From this result it follows that, for all $\alpha\in\mathbb{R}$, the operator $\mathfrak{F}_{\varepsilon,\beta,\delta,\gamma,\xi}$ is bounded from $L^\infty(\mathbb{R},\cosh(\alpha x^2)\,dx)$ into $L^1(\mathbb{R},\cosh(\alpha x^2)\,dx)$, for $\Re\beta>0$, $(\Re\delta)^2<\Re\beta(\Re\varepsilon-|\alpha|)$, and $\gamma,\xi\in\mathbb{C}$.

From Proposition 4.2 in [8], the following can be seen.

Theorem 4.4. *For all $\alpha\in\mathbb{R}$ the following Parseval-type relation holds:*

$$\int_{-\infty}^{+\infty}(\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}f)(x)g(x)\,dx=\int_{-\infty}^{+\infty}f(x)(\mathfrak{F}_{\varepsilon,\beta,\delta,\gamma,\xi}g)(x)\,dx,$$

where $f,g\in L^\infty(\mathbb{R},\cosh(\alpha x^2)\,dx)$ with $\Re\beta>0$, $(\Re\delta)^2<\Re\beta(\Re\varepsilon-|\alpha|)$, and $\gamma,\xi\in\mathbb{C}$.

By virtue of Corollary 4.1 in [8], we obtain the following.

Corollary 4.2. *For all $\alpha\in\mathbb{R}$ and $f\in L^\infty(\mathbb{R},\cosh(\alpha x^2)\,dx)$, $\Re\beta>0$, $(\Re\delta)^2<\Re\beta(\Re\varepsilon-|\alpha|)$, and $\gamma,\xi\in\mathbb{C}$, we have*

$$\mathfrak{F}'_{\varepsilon,\beta,\delta,\gamma,\xi}T_f=T_{\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}f}$$

on $(L^\infty(\mathbb{R},\cosh(\alpha x^2)\,dx))'$.

5. An interesting particular case: The Gauss–Weierstrass semigroup

The Gauss–Weierstrass semigroup on $\mathbb{R}$ (see [18, p. 103]) is given by

$$(e^{z\Delta}f)(y)=(4\pi z)^{-1/2}\int_{-\infty}^{\infty}\exp\big[-(y-x)^2/4z\big]f(x)\,dx,$$

where $\Re z\ge 0$ (and $z\neq 0$).

This operator corresponds to the case when $\beta=\varepsilon=\delta=1/4z$, and $\xi=\gamma=0$.

By virtue of Theorem 3.2 and taking into account that $\Re\delta = 0$ corresponds to the line $\Re z = 0$ and that the case $\Re\delta = 1$ corresponds to the circumference $\Re z = 4|z|^2$, we get the following.

Theorem 5.1. *The following Parseval relation holds:*

$$\int_{-\infty}^{+\infty} (e^{z\Delta} f)(x) g(x)\, dx = \int_{-\infty}^{\infty} f(x)(e^{z\Delta} g)(x)\, dx \tag{5.1}$$

for $f, g \in L^1(\mathbb{R}, \cosh(\alpha x)\, dx)$, $\alpha \in \mathbb{R}$, $\Re z = 0$ *(and* $z \neq 0$*), or alternatively,* $\Re z = 4|z|^2$ *(and* $z \neq 0$*).*

Moreover from Corollary 3.1, we have the following.

Corollary 5.1. *For* $f \in L^1(\mathbb{R}, \cosh(\alpha x)\, dx)$, $\alpha \in \mathbb{R}$, $\Re z = 0$ *(and* $z \neq 0$*), or alternatively,* $\Re z = 4|z|^2$ *(and* $z \neq 0$*), it follows that*

$$(e^{z\Delta})' T_f = T_{e^{z\Delta} f} \tag{5.2}$$

on $(L^1(\mathbb{R}, \cosh(\alpha x)\, dx))'$.

By virtue of Theorem 3.4, we obtain the following.

Theorem 5.2. *The following Parseval relation holds that*

$$\int_{-\infty}^{+\infty} (e^{z\Delta} f)(x) g(x)\, dx = \int_{-\infty}^{\infty} f(x)(e^{z\Delta} g)(x)\, dx \tag{5.3}$$

for $f, g \in L^1(\mathbb{R}, \cosh(\alpha x^2)\, dx)$, $\alpha \in \mathbb{R}$, $\Re z = 0$ *(and* $z \neq 0$*), or alternatively,* $\Re z = 4|z|^2$ *(and* $z \neq 0$*).*

Moreover from Corollary 3.2 we have the following.

Corollary 5.2. *For* $f \in L^1(\mathbb{R}, \cosh(\alpha x^2)\, dx)$, $\alpha \in \mathbb{R}$, $\Re z = 0$ *(and* $z \neq 0$*), or alternatively,* $\Re z = 4|z|^2$ *(and* $z \neq 0$*), it follows that*

$$(e^{z\Delta})' T_f = T_{e^{z\Delta} f} \tag{5.4}$$

on $(L^1(\mathbb{R}, \cosh(\alpha x^2)\, dx))'$.

Remark. The Gauss–Weierstrass semigroup appears in [1, p. 521] as well as other publications.

References

1. D. Alpay, P. Jorgensen, and D. Levanony, *A class of Gaussian processes with fractional spectral measures*, J. Funct. Anal. **261** (2011), no. 2, 507–541. Zbl 1229.60042. MR2793121. DOI 10.1016/j.jfa.2011.03.012. 420
2. J. C. Baez, I. Segal, and Z. Zhou, *Introduction to Algebraic and Constructive Quantum Field Theory*, Princeton Series in Physics, Princeton University Press, Princeton, NJ, 1992. Zbl 0760.46061. MR1178936. DOI 10.1515/9781400862504. 399
3. F. Barthe and P. Wolff, *Positivity improvement and Gaussian kernels*, C. R. Math. Acad. Sci. Paris **352** (2014), no. 12, 1017–1021. Zbl 1316.46027. MR3276812. DOI 10.1016/j.crma.2014.09.016. 399
4. J. B. Epperson, *The hypercontractive approach to exactly bounding an operator with complex Gaussian kernel*, J. Funct. Anal. **87** (1989), no. 1, 1–30. Zbl 0696.47028. MR1025881. DOI 10.1016/0022-1236(89)90002-5. 399, 400

5. G. B. Folland, *Harmonic Analysis in Phase Space*, Ann. of Math. Stud. **122**, Princeton University Press, Princeton, NJ, 1989. Zbl 0682.43001. MR0983366. DOI 10.1515/9781400882427. 400
6. B. J. González and E. R. Negrín, *Boundedness properties for a class of integral operators including the index transforms and the operators with complex Gaussian kernels*, J. Math. Anal. Appl. **293** (2004), no. 1, 219–226. Zbl 1055.45006. MR2052542. DOI 10.1016/j.jmaa.2003.12.037. 400, 401, 404, 408, 412, 416, 417
7. B. J. González and E. R. Negrín, *Parseval-Type Relations and L^p-Inequalities for the Operators with Complex Gaussian Kernels*, Complex Anal. Oper. Theory **11** (2017), no. 3, 603–610. Zbl 1364.44003. MR3615521. DOI 10.1007/s11785-016-0612-x. 401
8. N. Hayek, B. J. González, and E. R. Negrín, *On certain integral operators between weighted L^p spaces*, J. Appl. Funct. Anal. **1** (2006), no. 2, 191–201. Zbl 1109.47040. MR2222326. 400, 404, 408, 411, 412, 415, 417, 419
9. N. Hayek, H. M. Srivastava, B. J. González, and E. R. Negrín, *A family of Wiener transforms associated with a pair of operators on Hilbert space*, Integral Transforms Spec. Funct. **24** (2013), no. 1, 1–8. Zbl 1277.44002. MR3011918. DOI 10.1080/10652469.2011.648379. 401
10. T. Hida, *Brownian Motion*, Springer, New York, 1980. Zbl 0432.60002. MR0562914. 399
11. R. Howe, "The oscillator semigroup" in *The Mathematical Heritage of Hermann Weyl (Durham, NC, 1987)*, Proc. Sympos. Pure Math. **48**, Amer. Math. Soc., Providence, 1988, 61–132. Zbl 0687.47034. MR0974332. DOI 10.1090/pspum/048/974332. 400
12. Y. J. Lee, *Integral transforms of analytic functions on abstract Wiener spaces*, J. Funct. Anal. **47** (1982), no. 2, 153–164. Zbl 0487.44006. MR0664334. DOI 10.1016/0022-1236(82)90103-3. 400
13. Y. J. Lee, *Unitary operators on the space of L^2-functions over abstract Wiener spaces*, Soochow J. Math. **13** (1987), no. 2, 165–174. Zbl 0661.46042. MR0960656. 400
14. E. H. Lieb, *Gaussian kernels have only Gaussian maximizers*, Invent. Math. **102** (1990), no. 1, 179–208. Zbl 0726.42005. MR1069246. DOI 10.1007/BF01233426. 399, 400
15. E. R. Negrín, *Complex Gaussian operators in dimension one*, Bull. Inst. Math. Acad. Sin. **23** (1995), no. 1, 37–53. Zbl 0824.47026. MR1319478. 399, 401
16. Y. A. Neretin, *Lectures on Gaussian Integral Operators and Classical Groups*, EMS Ser. Lect. Math., Eur. Math. Soc., Zürich, 2011. Zbl 1211.22001. MR2790054. DOI 10.4171/080. 399
17. I. E. Segal, *Tensor algebras over Hilbert spaces, I*, Trans. Amer. Math. Soc. **81** (1956), 106–134. Zbl 0070.34003. MR0076317. DOI 10.2307/1992855. 399
18. F. B. Weissler, *Two-point inequalities, the Hermite semigroup, and the Gauss–Weierstrass semigroup*, J. Funct. Anal. **32** (1979), no. 1, 102–121. Zbl 0433.47023. MR0533222. DOI 10.1016/0022-1236(79)90080-6. 419

Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de La Laguna (ULL), Campus de Anchieta, ES-38271 La Laguna (Tenerife), España.

E-mail address: bjglez@ull.es; enegrin@ull.es