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# TOEPLITZ OPERATORS ON THE SPACE OF REAL ANALYTIC FUNCTIONS: THE FREDHOLM PROPERTY

PAWEL DOMAŃSKI† and MICHAL JASICZAK*

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Abstract. We completely characterize those continuous operators on the space of real analytic functions on the real line for which the associated matrix is Toeplitz (that is, we describe Toeplitz operators on this space). We also prove a necessary and sufficient condition for such operators to be Fredholm operators. While the space of real analytic functions is neither Banach space nor has a basis which makes available methods completely different from classical cases of Hardy spaces or Bergman spaces, nevertheless the results themselves show surprisingly strong similarity to the classical Hardy-space theory.

## 1. Introduction

An (infinite) Toeplitz matrix is an infinite matrix of the form

$$M = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & \dots \\ a_1 & a_0 & a_{-1} & \dots \\ a_2 & a_1 & a_0 & \dots \\ \dots & \dots & \dots & \ddots \end{pmatrix}, \tag{1.1}$$

with $\dots, a_{-1}, a_0, a_1, \dots \in \mathbb{C}$. Toeplitz operators as operators for which an associated matrix is Toeplitz have been considered on various function and sequence


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†Professor Domański passed away on August 4, 2016.
*Corresponding author.
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spaces like $\ell^2(\mathbb{N})$, Hardy spaces $H^2(\mathbb{T})$ or $H^p(\mathbb{T})$, Bergman spaces $A^2(\mathbb{D})$ (see for instance [1], [28], [29], [22]), and Fock spaces (see for example [3], [6] and the monograph [30]), among others. Such theories are now well established—an excellent reference is the monograph [10]. Moreover, recently some generalizations have been considered (see, e.g., [26]). Toeplitz operators constitute one of the most important classes of operators. Its theory is a beautiful interplay between operator theory and function theory. There are connections between the theory of Toeplitz operators and probability theory, information theory, and control theory. It is also still a very active field of study.

In the present article, we develop the analogous theory for one of the most prominent classes of functions: the space $\mathcal{A}(\mathbb{R})$ of real analytic functions on the real line. The latter space is not a Banach space (or not even a metrizable space) and, as shown by Domański and Vogt [18], it has no Schauder basis. Also, $\mathcal{A}(\mathbb{R})$ has a natural locally convex topology giving the natural sequence convergence, and every linear operator $A$ on $\mathcal{A}(\mathbb{R})$ has an associated matrix uniquely determining $A$, that is, a matrix $(a_{mn})_{m,n\in\mathbb{N}}$ such that

$$A(x^n)(\xi) = \sum_{m=0}^{\infty} a_{mn}\xi^m \tag{1.2}$$

around 0. We should emphasize that the theory we have discovered is parallel in its results but completely different in its methods in comparison to the classical theory, which makes the results surprising. The reason for the difference is the fact that $\mathcal{A}(\mathbb{R})$ is neither Banach nor metrizable but rather that elements of $\mathcal{A}(\mathbb{R})$ are in fact germs of holomorphic functions, so they are not all defined on a fixed open complex set.

First, we make precise the concept of Toeplitz operator on the space of real analytic functions on the real line.

*Definition* 1.1. We consider a continuous linear operator $T : \mathcal{A}(\mathbb{R}) \to \mathcal{A}(\mathbb{R})$ a Toeplitz operator if there exist complex numbers $\ldots, a_{-1}, a_0, a_1, \ldots \in \mathbb{C}$ such that, for each $n \in \mathbb{N}_0$ locally near 0,

$$T(x^n)(\xi) = a_{-n} + a_{-n+1}\xi + a_{-n+2}\xi^2 + \cdots . \tag{1.3}$$

Since $T(x^n)$ is real analytic, there exists $r > 0$ such that the series (1.3) converges in the ball $B(0, r)$ of radius $r$ around 0. Notice that it is a consequence of condition (1.3) that the radius $r$ is the same for each $n \in \mathbb{N}_0$. This follows readily from Hadamard's theorem.

Already for Toeplitz operators on $\ell^2(\mathbb{N})$ it is an intriguing question for which matrices (1.1) the associated Toeplitz operator is well defined (i.e., continuous and defined from $\ell^2(\mathbb{N})$ to $\ell^2(\mathbb{N})$). While it is easy to define a Toeplitz operator by means of condition (1.1), the answer to the question raised above requires considerably more sophisticated tools. One needs to realize $l^2(\mathbb{N})$ as the Hardy space $H^2(\mathbb{T})$. This space consists of all functions holomorphic in the unit disk $\mathbb{D}$ whose boundary (nontangential) values belong to $L^2(\mathbb{T})$—the space of functions square-integrable with respect to the Lebesgue measure on the unit circle $\mathbb{T}$. It is a fundamental theorem of Brown and Halmos [11, Theorem 4] that $T$ defined by the Toeplitz matrix $M$ (see (1.1)) maps $l^2(\mathbb{N})$ boundedly into itself if and only if $a_j, j \in \mathbb{Z}$, are the Fourier coefficients of an $L^\infty(\mathbb{T})$-function $\phi$. Then

$$T = PM_\phi, \tag{1.4}$$

where $P$ is the Szegö projection—the orthogonal projection of $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$ and $M_\phi : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is the operator of multiplication by $\phi$, $M_\phi : f \mapsto \phi f$. The function $\phi$ is called the *symbol* of the corresponding Toeplitz operator.

The main idea in the whole theory is to relate properties of the symbol with properties of the operator. It turns out that a candidate for the symbol space for Toeplitz operators on the space of real analytic functions is

$$\mathcal{X} := \mathcal{X}(\mathbb{R}) := \bigcup_{K \subset U} H(U \setminus K). \tag{1.5}$$

The sets $K$ in (1.5) run through all compact subsets of the real line and $U$ through all open complex neighborhoods of $\mathbb{R}$. We intend to assign to any $F \in \mathcal{X}(\mathbb{R})$ a continuous operator $T_F$ on $\mathcal{A}(\mathbb{R})$ which is a Toeplitz operator in the sense of Definition 1.1. First, however, we need to discuss certain properties of the space $\mathcal{X}(\mathbb{R})$.

We will show in Section 3 that every element $F \in \mathcal{X}(\mathbb{R})$ splits uniquely into the sum $F_+ + F_-$, where $F_+$ is real analytic (so holomorphic in some open complex neighborhood of $\mathbb{R}$) and $F_-$ is holomorphic on the Riemann sphere $\mathbb{C}_\infty$ except some compact subset of $\mathbb{R}$. In fact, topologically

$$\mathcal{X}(\mathbb{R}) = \mathcal{A}(\mathbb{R}) \oplus H_0(\mathbb{C}_\infty \setminus \mathbb{R}),$$

where

$$H_0(\mathbb{C}_\infty \setminus \mathbb{R}) = \operatorname{ind}_n H_0\big(\mathbb{C}_\infty \setminus [-n, n]\big)$$

and the subscript 0 means that functions vanish at infinity. By the Köthe–Grothendieck duality (see Section 2), the space $H_0(\mathbb{C}_\infty \setminus \mathbb{R})$ can be isomorphically identified with the dual space $\mathcal{A}(\mathbb{R})'$. Hence,

$$\mathcal{X}(\mathbb{R}) \cong \mathcal{A}(\mathbb{R}) \oplus \mathcal{A}(\mathbb{R})'.$$

We now define the projection $\mathcal{C} : \mathcal{X}(\mathbb{R}) \to \mathcal{X}(\mathbb{R})$ onto $\mathcal{A}(\mathbb{R})$. Assume that $F \in \mathcal{X}(\mathbb{R})$. Thus there exist a compact set $K \subset \mathbb{R}$ and an open set $U \supset \mathbb{R}$ such that $F \in H(U \setminus K)$. Let $\gamma : \mathbb{T} \to U \setminus K$ be a $C^\infty$ diffeomorphic map defined on the unit circle $\mathbb{T}$ such that $\operatorname{Ind}_\gamma(\zeta) = 1$ for $\zeta \in K$ (we may assume that $K$ is connected). For a point $z$ which belongs to the domain bounded by $\gamma$, we set

$$(\mathcal{C}_\gamma F)(z) = \frac{1}{2\pi i} \int_\gamma \frac{F(\zeta)}{\zeta - z}\, d\zeta. \tag{1.6}$$

The function $\mathcal{C}_\gamma F$ is holomorphic on the domain bounded by $\gamma$. It is immediate that the definition of $\mathcal{C}_\gamma F$ does not depend on the particular choice of $\gamma$ as long as $z$ belongs to the domain bounded by $\gamma$. It is also clear that for any $z \in U$ we can find such a closed curve with $z$ belonging to the domain bounded by $\gamma$. Thus, for $z \in U$ we define $(\mathcal{C}F)(z) := (\mathcal{C}_\gamma F)(z)$ with appropriately chosen closed curve $\gamma$. We will call $\mathcal{C}$ the *Cauchy projection*. Let us introduce here the following convention: whenever we write *a closed curve in* $U \setminus K$, we always mean a $C^\infty$

diffeomorphic map $\gamma : \mathbb{T} \to U \setminus K$ such that $\mathrm{Ind}_\gamma(\zeta) = 1$ for $\zeta \in K$—without loss of generality, we may assume that $K$ is nonempty.

Now, the first main theorem of the paper is the following justification of the definition of Toeplitz operators on $\mathcal{A}(\mathbb{R})$ and the analogue of the Brown and Halmos theorem.

**Theorem 1.** *The following assertions are equivalent.*

(i) $T : \mathcal{A}(\mathbb{R}) \to \mathcal{A}(\mathbb{R})$ *is a Toeplitz operator; that is, $T$ is a continuous linear operator such that, locally near* $0$,

$$T(x^n)(\xi) = a_{-n} + a_{-n+1}\xi + a_{-n+2}\xi^2 + \cdots \tag{1.7}$$

*for some complex numbers $a_n, n \in \mathbb{Z}$.*

(ii) *There exists a function $F \in \mathcal{X}(\mathbb{R})$ such that*

$$T = \mathcal{C}M_F, \tag{1.8}$$

*where $M_F : \mathcal{X}(\mathbb{R}) \to \mathcal{X}(\mathbb{R})$ is the multiplication operator $M_F : f \mapsto Ff$ and $\mathcal{C}$ is the Cauchy projection defined above*

$$\mathcal{C} : \mathcal{X}(\mathbb{R}) \to \mathcal{A}(\mathbb{R}) \subset \mathcal{X}(\mathbb{R}).$$

*Then (1.7) holds with*

$$a_n = \frac{1}{2\pi i}\int_\gamma F(\zeta)\zeta^{-n-1}\,d\zeta,$$

*where $\gamma$ is a closed curve in $U \setminus K$ and $F \in H(U \setminus K)$.*

(iii) *There exist $G \in \mathcal{A}(\mathbb{R})$ and $\Phi \in \mathcal{A}(\mathbb{R})'$ such that*

$$(Tf)(z) = G(z)f(z) + \left\langle \frac{f(z) - f(\cdot)}{z - \cdot}, \Phi \right\rangle.$$

*Then close to $0$ we have*

$$G(z) = \sum_{n=0}^{\infty} c_n z^n$$

*and (1.7) holds with $a_n = c_n, n \in \mathbb{N}_0$, and $a_{-n}, n \in \mathbb{N}$, the sequence of moments of $\Phi$, that is,*

$$a_{-n-1} = \langle z^n, \Phi\rangle, \quad n = 0, 1, 2, \ldots.$$

Observe that formula (1.8) says that Toeplitz operators on $\mathcal{A}(\mathbb{R})$ are of the form very similar to (1.4). They are compositions of the multiplication $f \mapsto Ff$ which maps $\mathcal{A}(\mathbb{R})$ into $\mathcal{X}(\mathbb{R})$ and the projection $\mathcal{C} : \mathcal{X}(\mathbb{R}) \to \mathcal{X}(\mathbb{R})$ onto $\mathcal{A}(\mathbb{R})$.

As we have already stated, the theory of Toeplitz operators is a beautiful interplay between function theory and operator theory. A very important example of the interplay is the *Gohberg–Krein index formula* [19] (see also [9]), which says that if $\phi \in C(\mathbb{T})$ and $\phi(z) \neq 0$ for all $z \in \mathbb{T}$, then the operator $T := PM_\phi : H^2(\mathbb{T}) \to H^2(\mathbb{T})$ is a Fredholm operator and

$$\operatorname{index} T = -\operatorname{Ind}_\phi(0).$$

Recall that an operator $T$ between Banach spaces is considered a Fredholm operator if it has a finite-dimensional kernel and a finite-dimensional cokernel (i.e., its image is of finite codimension). The index of a Fredholm operator $T$ is then defined as

$$\operatorname{index} T := \dim\ker T - \dim\operatorname{coker} T.$$

We adopt the same definitions for operators acting on $\mathcal{A}(\mathbb{R})$. We remark here that a Fredholm operator has necessarily closed range. For Banach spaces this is classical—for a nonBanach space $\mathcal{A}(\mathbb{R})$ this will be shown in Section 5. Our next main result is the complete characterization of Fredholm Toeplitz operators on $\mathcal{A}(\mathbb{R})$.

**Theorem 2.** *A Toeplitz operator $T_F : \mathcal{A}(\mathbb{R}) \to \mathcal{A}(\mathbb{R})$ is a Fredholm operator if and only if there exist an open complex set $U \supset \mathbb{R}$ and a compact set $K \subset \mathbb{R}$ such that $F \in H(U \setminus K)$ and $F(z) \neq 0$ for $z \in U \setminus K$. Furthermore,*

$$\operatorname{index} T_F = -\operatorname{winding} F.$$

While it is clear what $\operatorname{Ind}_\phi(0)$ is for $\phi \in C(\mathbb{T})$, the definition of winding $F$ for $F \in \mathcal{X}$ requires some explanation. This is provided in Section 5.

It is important to realize that many arguments available for Fredholm operators on Banach spaces are not available on locally convex spaces, in particular for $\mathcal{A}(\mathbb{R})$. For instance, the index in the Hilbert space case is known by Dieudonné's result to be locally constant and, as a result, homotopy-invariant. It does not seem that such arguments are available in our case. The proof of Theorem 2 must therefore be completely different comparing with the proof of the classical Gohberg–Krein formula. It does make use of the $H^2(\mathbb{T})$ case in a very strong sense. We believe that Theorem 2 alone justifies our interest in Toeplitz operators on $\mathcal{A}(\mathbb{R})$.

The key element in the proof of Theorem 1 is the Köthe–Grothendieck duality. In order to prove Theorem 2, we consider Hardy spaces $H^2(\gamma)$ on closed curves $\gamma$ in $U \setminus K$, where $U$ is an open neighborhood of $\mathbb{R}$ and where $K$ is a compact subset of $\mathbb{R}$. Each $F \in H(U \setminus K)$ determines not only the Toeplitz operator on $\mathcal{A}(\mathbb{R})$ but also the whole family of Toeplitz operators $T_{F,\gamma}$ on spaces $H^2(\gamma)$ (this will be explained in Section 2 and Section 5). We prove that the Fredholm index of these operators is constant when $F \neq 0$. Furthermore, we show that both kernels and cokernels of Toeplitz operators on spaces $H^2(\gamma)$ are globally generated. More precisely, we show that under the assumption that $F(z) \neq 0$ for $z \in U \setminus K$, there exist functions $f_1, \dots, f_m$ holomorphic on $U$ such that

$$\ker\left(T_{F,\gamma} : H^2(\gamma) \to H^2(\gamma)\right) = \operatorname{span}\{f_1, \dots, f_m\}$$

for any closed curve $\gamma$ (Theorem 5.6 below). Similarly, we show that there exist functions $g_1, \dots, g_n$ holomorphic in some open neighborhood $\tilde{U}$ of $\mathbb{R}$ such that classes

$$g_1 + \operatorname{im} T_{F,\gamma}, \dots, g_n + \operatorname{im} T_{F,\gamma}$$

span $H^2(\gamma)/\operatorname{im} T_{F,\gamma}$ for any closed curve $\gamma$ in $\tilde{U} \setminus K$ (Theorem 5.10 below). These results imply readily that $T_F$ is a Fredholm operator on $\mathcal{A}(\mathbb{R})$ when $F \neq 0$. Both these results require rather delicate knowledge concerning the Cauchy transform. In the proof of the second result we solve an additive Cousin problem. Similar arguments, also based on properties of the Cauchy transform on the Hardy spaces, show that $F(z) \neq 0$ for $z \in U \setminus K$ is the necessary condition for $T_F$ to be a Fredholm operators on $\mathcal{A}(\mathbb{R})$. Both Theorem 5.6 and Theorem 5.10, which were briefly presented above, seem to be of independent interest. Importantly, when $\gamma$ is fixed, the operator $\mathcal{C}_\gamma$ is the well-known Cauchy transform. Not surprisingly, results concerning the Cauchy transform are crucial to our study and will be recalled in Section 2. We want to emphasize that the Cauchy transform is defined on different spaces, for instance the $L^2$ space on the curve $\gamma$. We will need this extension of $\mathcal{C}_\gamma$ in our study of the Fredholm property.

This paper is addressed both to specialists in locally convex spaces, in particular the space of real analytic functions, and to specialists in Toeplitz operators working in different spaces such as Hardy space, Bergman space, and Fock space. This justifies its style. We recall basic facts concerning locally convex spaces in Section 2 and provide rather detailed arguments concerning the structure of the space $\mathcal{X}(\mathbb{R})$. Similarly, we provide detailed background on Hardy spaces and their Toeplitz operators. While theory of Hardy spaces on the unit disk is considered basic and elementary in mathematical analysis, its counterpart on different domains is not.

Not much is known about continuous operators on natural locally convex spaces, like spaces of all smooth or all holomorphic functions on a given domain, probably with a prominent exception of differential operators, convolution operators and, maybe, composition operators. This concerns in particular the space of real analytic functions. There is, however, an interesting class of operators on $\mathcal{A}(\mathbb{R})$ which has been studied recently and whose study motivated our research. These are the so-called *Hadamard multipliers*, that is, continuous operators on $\mathcal{A}(\mathbb{R})$ such that the matrix $(a_{mn})$ in (1.2) is diagonal. These operators were studied by Domański and Langenbruch in a series of papers (see [14], [13], [15], [16]); by Domański, Langenbruch, and Vogt in [17]; and by Vogt in [27]. There is a broad literature on Hadamard-type operators on holomorphic functions (see the literature of [17]). Operators with matrices like (1.1) seem to be natural in the study of operators on $\mathcal{A}(\mathbb{R})$ comparing with $(a_{mn})$ being a diagonal matrix.

This paper is divided into five sections. First, we recall basic information concerning the space of real analytic functions as a locally convex space and continuous operators on this space (Section 2.1). We also provide background on Hardy spaces on curves and the corresponding Cauchy transform (Section 2.2). Then, in Section 3, we analyze the symbol space $\mathcal{X}(\mathbb{R})$ and we present basic information concerning Toeplitz operators on $\mathcal{A}(\mathbb{R})$. In Section 4, we prove Theorem 1. In the last long Section 5, we provide arguments which prove Theorem 2.

## 2. Preliminaries

2.1. **The space of real analytic functions on $\mathbb{R}$.** We refer the reader to [21] for a nice summary on real analytic functions and to [12] for an instructive survey on the space of real analytic functions. Here we only present facts which we will need later on. (For information on locally convex spaces, in particular projective and inductive topologies, see [24, Chapter 24].)

Recall that the symbol $\mathcal{A}(\mathbb{R})$ stands for the *space of real analytic functions on* $\mathbb{R}$, that is, those functions which can be locally developed into a Taylor series convergent (locally) to the function itself. There are two ways of introducing topologies on this space, but both of them define the same *natural* topology. Let $V$ be an open complex neighborhood of $\mathbb{R}$ and let $K$ be a compact subset of $\mathbb{R}$. Let $H(K)$ stand for the space of germs of holomorphic functions on $K$. We have

$$H(K) = \bigcup_{U \supset K} H(U), \tag{2.1}$$

where $U$ are arbitrary open complex neighborhoods of $K$. Note that $H(U)$ is the (Fréchet) space of all functions holomorphic in $U$ with the topology induced by seminorms

$$|\cdot|_L : f \mapsto |f|_L := \sup_{z \in L} \big|f(z)\big|, \tag{2.2}$$

where $L$ runs over compact subsets $L$ of $U$. Thus $H(K)$ carries in a natural way the (locally convex) inductive topology of the system $(H(U) \hookrightarrow H(K))_{U \supset K}$ (i.e., the strongest locally convex topology where all the mentioned embeddings are continuous).

As for the space of real analytic functions $\mathcal{A}(\mathbb{R})$, we have two types of restriction maps

$$R : H(V) \to \mathcal{A}(\mathbb{R}), \qquad r : \mathcal{A}(\mathbb{R}) \to H(K),$$

the families of which are used to introduce topologies on $\mathcal{A}(\mathbb{R})$. Here $V$ is an open set containing $\mathbb{R}$. It holds algebraically that

$$\mathcal{A}(\mathbb{R}) = \bigcap_{K \subset\subset \mathbb{R}} H(K).$$

The system $(\mathcal{A}(\mathbb{R}) \to H(K))_{K \subset\subset \mathbb{R}}$ is a projective system. Thus $\mathcal{A}(\mathbb{R})$ can be equipped with the projective topology induced by this system. This topology is the weakest topology such that the restrictions $r : \mathcal{A}(\mathbb{R}) \to H(K)$ are continuous for every compact set $K \subset \mathbb{R}$. It is easy to observe that such a topology exists. It also holds algebraically that

$$\mathcal{A}(\mathbb{R}) = \bigcup_{\substack{V \supset \mathbb{R} \\ V \text{ open in } \mathbb{C}}} H(V).$$

The system $(H(U) \to \mathcal{A}(\mathbb{R}))_{U \supset \mathbb{R}}$ is an inductive system. The space $\mathcal{A}(\mathbb{R})$ can be equipped with the locally convex inductive topology induced by this system. Such topology exists and is the strongest locally convex topology which makes all the restrictions $R : H(V) \to \mathcal{A}(\mathbb{R})$ continuous. Since $H(V)$ is a Fréchet space, it is therefore ultrabornological (see Remark 24.15 in [24]). Recall that this means that $H(V)$ has the topology of some inductive system of Banach spaces. Thus $\mathcal{A}(\mathbb{R})$ equipped with the inductive topology is ultrabornological (see Proposition 24.16 in [24]). We remark here that we always assume that the inductive topology is Hausdorff. This is not assumed in all textbooks on locally convex spaces.

There are therefore two natural topologies on $\mathcal{A}(\mathbb{R})$.

**Theorem 2.1** (Martineau [23, Théorème 1.2]). *Both defined above topologies on $\mathcal{A}(\mathbb{R})$ coincide (i.e., the projective topology is equal to the inductive topology).*

In particular, it follows (see [12]) that the standard tools of functional analysis work on $\mathcal{A}(\mathbb{R})$ equipped with this natural topology: the Hahn–Banach theorem, the uniform boundedness principle, the open mapping theorem for surjective operators, and the closed graph theorem are all true on $\mathcal{A}(\mathbb{R})$. Even more, a linear operator $T : \mathcal{A}(\mathbb{R}) \to \mathcal{A}(\mathbb{R})$ is continuous if and only if it is sequentially continuous. A sequence $(f_n) \subset \mathcal{A}(\mathbb{R})$ is convergent to $f \in \mathcal{A}(\mathbb{R})$ if and only if there is a complex neighborhood $U$ of $\mathbb{R}$ such that $f_n, f$ all extend holomorphically to $U$ and $f_n \to f$ uniformly on compact subsets of $U$.

We need to understand when an operator $T : \mathcal{A}(\mathbb{R}) \to \mathcal{A}(\mathbb{R})$ is continuous. We start with the inductive topology. It is a property of an inductive topology that an operator $T : \mathcal{A}(\mathbb{R}) \to \mathcal{A}(\mathbb{R})$ is continuous if $T : H(V) \to \mathcal{A}(\mathbb{R})$ is continuous for every $V$—open complex neighborhood of $\mathbb{R}$. By the defintion of the inductive topology the inclusions $H(V) \hookrightarrow \mathcal{A}(\mathbb{R})$ are continuous. Thus we have the following continuity criteria.

**Theorem 2.2** (Continuity criteria I). *If for every $V_1 \supset \mathbb{R}$ there exists $V_2 \supset \mathbb{R}$, $V_1, V_2$ open in $\mathbb{C}$, such that*

$$T : H(V_1) \to H(V_2)$$

*is continuous, then $T$ is continuous as an operator on $\mathcal{A}(\mathbb{R})$.*

Recall that topology of $H(V_i), i = 1, 2$ is induced by seminorms (2.2).

Now we turn our attention to the projective picture. First we need some simplification. Choose compact connected sets $K_n \subset \mathbb{R}, n \in \mathbb{N}$ such that

(i)
$$K_n \subset \operatorname{int} K_{n+1},$$

(ii)
$$\mathbb{R} = \bigcup_{n=1}^{\infty} K_n.$$

The system $(\mathcal{A}(\mathbb{R}) \to H(K_n))_{n\in\mathbb{N}}$ induces the topology of $\mathcal{A}(\mathbb{R})$. In particular, the induced topology does not depend on the choice of sets $K_n$ as long as (i) and (ii) are satisfied.

**Theorem 2.3** (Continuity criteria II; [12, Theorem 1.35]). *A linear operator $T : \mathcal{A}(\mathbb{R}) \to \mathcal{A}(\mathbb{R})$ is continuous if and only if for every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $T$ extends to a continuous map $T : H(K_m) \to H(K_n)$.*

There is one more step we should make as far as understanding of continuity of operators on $\mathcal{A}(\mathbb{R})$ is concerned. Namely, for the given sets $K_n$ choose open sets $U_{n,\nu}, n, \nu \in \mathbb{N}$ such that

(i)
$$\bar{U}_{n,\nu+1} \subset\subset U_{n,\nu},$$

(ii)

$$K_n = \bigcap_{\nu=1}^{\infty} U_{n,\nu}.$$

Then algebraically and topologically,

$$\mathcal{A}(\mathbb{R}) = \bigcap_{n=1}^{\infty} H(K_n) = \bigcap_{n=1} \bigcup_{\nu=1}^{\infty} H(U_{n,\nu}). \tag{2.3}$$

It is important to realize that instead of the spaces $H(U_{n,\nu})$, one can take $H^\infty(U_{n,\nu})$ or even for instance Bergman spaces over $U_{n,\nu}$. Then the linking maps turn out to be compact and, as a result, $\mathcal{A}(\mathbb{R})$ is a PLS space (i.e., the projective limit of a sequence of DFS spaces; we refer the reader to [12] for explanation of these terms).

We are, however, interested in conditions which guarantee continuity of a linear map $T$ acting on $\mathcal{A}(\mathbb{R})$. Theorem 2.3, the fact that $H(K_n)$ carries an inductive topology by (2.1), and Grothendieck's factorization theorem (Theorem 24.33 in [24]) yield the following fact.

**Theorem 2.4** (Continuity criteria IIA)**.** *A linear map $T : \mathcal{A}(\mathbb{R}) \to \mathcal{A}(\mathbb{R})$ is continuous if and only if for every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that for every $\mu \in \mathbb{N}$ there is $\nu \in \mathbb{N}$ such that $T$ extends to a continuous map*

$$T : H(U_{m,\mu}) \to H(U_{n,\nu}).$$

Although monomials do not form a basis of $\mathcal{A}(\mathbb{R})$ by the result of Domański and Vogt [18], polynomials are dense in $\mathcal{A}(\mathbb{R})$.

**Theorem 2.5** ([12, p. 12])**.** *Polynomials are dense in $\mathcal{A}(\mathbb{R})$.*

Every continuous linear functional $\xi$ on $H(K)$, $K \subset \mathbb{C}$, corresponds to a holomorphic function $f_\xi \in H_0(\mathbb{C}_\infty \setminus K)$ (cf. Theorem 1.3.5 in [8]). $H_0(\mathbb{C}_\infty \setminus K)$ stands for the space of all holomorphic functions on $\mathbb{C}_\infty \setminus K$ which vanish at infinity. Naturally, $\mathbb{C}_\infty$ is the Riemann sphere. The (Köthe–Grothendieck) duality between $H(K)$ and $H_0(\mathbb{C}_\infty \setminus K)$ is given by

$$H(K) \times H_0(\mathbb{C}_\infty \setminus K) \ni (g, f) \mapsto \langle g, f \rangle = \frac{1}{2\pi i} \int_\gamma g(z) f(z)\, dz,$$

where $\gamma$ is a finite union of closed curves contained in $U \setminus K$ if $g \in H(U)$, $U$ an open neighborhood of $K$, such that $\mathrm{Ind}_\gamma(z) = 1$ for any $z \in K$.

2.2. **Hardy spaces on curves and the Cauchy transform.** We present background on Hardy spaces on curves and the Cauchy transform. We follow the book by Bell [7], which has greatly influenced the authors' understanding of the subject and the approach taken in this paper.

Let $\Omega$ be a bounded domain in $\mathbb{C}$ with $C^\infty$ smooth boundary. We assume that $\Omega$ is simply connected. In this paper, $\Omega$ is always a domain bounded by a $C^\infty$ diffeomorphic image of the unit circle.

Let $b\Omega$ denote the boundary of $\Omega$. There is a $C^\infty$ complex-valued function $z(t), t \in [0, 1]$, which parameterizes the boundary curve. This means that $z(t)$

and all its derivatives agree at the endpoints $t = 0$ and $t = 1$, $z'(t)$ does not vanish, and $z(t)$ traces out the boundary exactly once. We assume that $-iz'(t)$ is a complex number representing the direction of the outward pointing normal vector to the boundary at the point $z(t)$.

If $k$ is a positive integer, $C^k(\bar{\Omega})$ denotes the space of continuous complex-valued functions on $\bar{\Omega}$ whose partial derivatives up to and including order $k$ exist and are continuous on $\Omega$ and extend continuously to $\bar{\Omega}$. The space $C^\infty(\bar{\Omega})$ is the set of functions in $C^k(\bar{\Omega})$ for all $k$, $A^\infty(\Omega)$ denotes the space of holomorphic functions on $\Omega$ that are in $C^\infty(\bar{\Omega})$. The symbol $A^\infty(b\Omega)$ stands for the set of functions on $b\Omega$ which are the boundary values of functions in $A^\infty(\Omega)$.

Let $ds$ denote the differential element of arc length on the boundary of $\Omega$. For $u$ and $v$ in $C^\infty(b\Omega)$, the $L^2$ inner product on $b\Omega$ of $u$ and $v$ is defined via $\langle u, v\rangle_b = \int_{b\Omega} u\bar{v}\, ds$. The space $L^2(b\Omega)$ is defined to be the Hilbert space obtained by completing the space $C^\infty(b\Omega)$ with respect to this inner product. The space $L^2(b\Omega)$ is equal to the set of complex valued functions on $u$ on $b\Omega$ such that $u(z(t))$ is a measurable function of $t$ and $\|u\|^2 = \int_0^1 |u(z(t))|^2 |z'(t)|\, dt$ is finite. This definition is independent of the choice of the parametrization of the boundary.

The Hardy space $H^2(b\Omega)$ is defined to be the closure in $L^2(b\Omega)$ of $A^\infty(b\Omega)$. Classically the Hardy space was defined in a different way. We will recall this definition now. Let $z(t)$ denote a parametrization of the boundary of $\Omega$ as descibed above. Let $\mathcal{T}(z), z \in b\Omega$ be the complex number which represents a unit tangent vector at $z \in b\Omega$ (i.e., $\mathcal{T}(z(t)) = \frac{z'(t)}{|z'(t)|}$). The function $z_\varepsilon(t) = z(t) + i\varepsilon\mathcal{T}(z(t))$ parameterizes the curve obtained by allowing a point at a distance $\varepsilon$ along the inward pointing normal to $z(t) \in b\Omega$ to trace out a curve as $z(t)$ ranges over the boundary. Classically the Hardy space was defined to be the space of holomorphic functions $H$ on $\Omega$ such that

$$\sup_{0<\varepsilon<\delta} \left( \int \big|H\big(z_\varepsilon(t)\big)\big|^2 \big|{z_\varepsilon}'(t)\big|\, dt \right)^{1/2} \tag{2.4}$$

is finite. Naturally we need to know that our definition of the Hardy space and the classical one are equivalent. For this we need the Cauchy transform. Let $u$ be a $C^\infty$ function defined on $b\Omega$. The Cauchy transform of $u$ is a holomorphic function $\mathcal{C}_{b\Omega}u$ on $\Omega$ given by

$$(\mathcal{C}_{b\Omega}u)(z) = \frac{1}{2\pi i} \int_{b\Omega} \frac{u(\zeta)}{\zeta - z}\, d\zeta. \tag{2.5}$$

Observe that the Cauchy transform of a $C^\infty(b\Omega)$ function is an extension of the projection defined in (1.6). This justifies the fact that we use the same symbol to denote both operators.

Following [7], we now gather fundamental properties of the Cauchy transform.

**Theorem 2.6** ([7, Theorem 3.1]). *The Cauchy transform maps $C^\infty(b\Omega)$ into $A^\infty(\Omega)$.*

This theorem allows us to treat the Cauchy transform as an operator which maps the space $C^\infty(b\Omega)$ into $C^\infty(\bar{\Omega})$, or even as an operator from $C^\infty(b\Omega)$ into itself. Notice, however, that then $\mathcal{C}_{b\Omega}$ is not given by the integral representation (2.5) due to the singularity in the denominator.

**Theorem 2.7** ([7, Theorem 4.1]). *The Cauchy transform extends to a bounded operator from $L^2(b\Omega)$ into $H^2(b\Omega)$.*

We will denote the extension of the Cauchy transform to the operator from $L^2(b\Omega)$ to $H^2(b\Omega)$ by the same symbol $\mathcal{C}_{b\Omega}$. Lastly we can formulate the aforementioned equivalence of the definitions of the Hardy space.

**Theorem 2.8** ([7, Theorem 6.1]). *Functions that satisfy the classical Hardy condition (2.4) are Cauchy integrals of functions in $H^2(b\Omega)$.*

**Theorem 2.9** ([7, Theorem 6.2]). *Cauchy integrals of functions in $H^2(b\Omega)$ are holomorphic functions on $\Omega$ that satisfy the classical Hardy condition (2.4).*

In the proof of Theorem 2.9 one shows for $H \in A^\infty(\Omega)$ the following inequality

$$\int \left|H\left(z_\varepsilon(t)\right)\right|^2 \left|z_\varepsilon'(t)\right| dt \leq C\|H\|_{H^2(b\Omega)}.$$

Thus the Cauchy integral is actually an isomorphism between $H^2(b\Omega)$ and the classical Hardy space. Given $h \in H^2(b\Omega)$, let $H(z) = (\mathcal{C}_{b\Omega}h)(z)$ be the holomorphic function on $\Omega$ given by the Cauchy integral of $h$ and let $u_\varepsilon(z(t)) := H(z_\varepsilon(t))$, $\varepsilon > 0$.

**Theorem 2.10** ([7, Theorem 6.3]). *If $h \in H^2(b\Omega)$, then $u_\varepsilon \to h$ in $L^2(b\Omega)$ as $\varepsilon \to 0$.*

The map which associates to a classical Hardy space function $H$ its $L^2$ boundary values sets a one-to-one correspondence between the classical Hardy space and $H^2(b\Omega)$. We use therefore the same symbol to denote a function in $H^2(b\Omega)$ and the holomorphic function on $\Omega$ which is its Cauchy integral.

Crucial tool in our study is the following theorem.

**Theorem 2.11** ([7, Theorem 3.4]). *Suppose that $u \in C^\infty(b\Omega)$. If $M$ is a positive integer, then there is a function $\Psi \in C^\infty(\bar{\Omega})$ which vanishes to order $M$ on the boundary such that the boundary values of $\mathcal{C}_{b\Omega}u$ are expressed via*

$$(\mathcal{C}_{b\Omega}u)(z) = u(z) - \frac{1}{2\pi i}\int\int_\Omega \frac{\Psi(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta}$$

*for $z \in b\Omega$.*

The function $\Psi$ can be viewed as a function in $C^M(\mathbb{C})$ via extension by 0.

Let $\gamma$ be a closed curve in $U \setminus K$. The curve $\gamma$ surrounds a simply connected region $\Omega$ whose boundary is $\gamma$. We will denote the closure of $\Omega$ by $\hat{\gamma}$—naturally this is the polynomially convex hull of $\gamma$. We will write $H^2(\gamma)$ to denote $H^2(b\Omega)$ and $\mathcal{C}_\gamma$ to denote the corresponding Cauchy transform.

## 3. Symbol space

Let us define

$$\mathcal{X} := \mathcal{X}(\mathbb{R}) := \bigcup_{K \subset U} H(U \setminus K), \tag{3.1}$$

where $K \subset \mathbb{R}$ runs through all compact subsets of the real line and $U$ through all open subsets of $\mathbb{C}$ which contain $\mathbb{R}$. Observe that if $U_1 \subset U_2$ and $K_1 \subset K_2$ then we can naturally treat elements of $H(U_2 \setminus K_1)$ as elements of $H(U_1 \setminus K_2)$. This is tacitly assumed in (3.1). Also, any open neighborhood of $\mathbb{R}$ contains a connected, simply connected neighborhood of $\mathbb{R}$. Any compact subset $K \subset \mathbb{R}$ is contained in a connected compact subset of $\mathbb{R}$ which contains 0. We may therefore assume that the union in (3.1) is over connected and simply connected open sets $U \supset \mathbb{R}$ and connected compact sets $K \subset \mathbb{R}$ which contain 0.

Let $F \in \mathcal{X}$. Thus there exist an open in $\mathbb{C}$, connected, and simply connected set $U \supset \mathbb{R}$ and a compact, connected subset $K \subset \mathbb{R}, 0 \in K$, such that $F \in H(U \setminus K)$. The meaning of the symbols $K$ and $U$ will be fixed throughout the paper; that is, we always assume that $F \in H(U \setminus K)$.

Let $z \in U$. There exists a closed curve in $U \setminus K$ such that $z$ belongs to the domain bounded by $\gamma$. As was stated in Section 1, by "a closed curve in $U \setminus K$," we mean a $C^\infty$ diffeomorphic map $\gamma : \mathbb{T} \to U \setminus K$ such that $\operatorname{Ind}_\gamma(z) = 1$ for any $z \in K$. The Cauchy transform $\mathcal{C}_\gamma F$ is well defined by the formula

$$(\mathcal{C}_\gamma F)(z) = \frac{1}{2\pi i} \int_\gamma \frac{F(\zeta)}{\zeta - z}\, d\zeta.$$

For any two closed curves $\gamma_1, \gamma_2$ in $U \setminus K$ such that $\operatorname{Ind}_{\gamma_i}(z) = 1, i = 1, 2$, it holds that

$$(\mathcal{C}_{\gamma_1} F)(z) = (\mathcal{C}_{\gamma_2} F)(z).$$

Hence for $z \in U$ we define

$$(\mathcal{C} F)(z) := (\mathcal{C}_\gamma F)(z),$$

where $\gamma$ is any closed curve in $U \setminus K$ such that $\operatorname{Ind}_\gamma(z) = 1$. Naturally, $\mathcal{C}F$ is a holomorphic function in $U$. Furthermore, if $F \in \mathcal{A}(\mathbb{R})$, that is if $F \in H(U)$ for some open subset $U \supset \mathbb{R}$, then

$$\mathcal{C}F = F. \tag{3.2}$$

We claim that $\mathcal{C} : \mathcal{X}(\mathbb{R}) \to \mathcal{X}(\mathbb{R})$ is a continuous projection onto $\mathcal{A}(\mathbb{R})$. To make this statement meaningful, we need to introduce a topology in $\mathcal{X}(\mathbb{R})$. Naturally this is going to be the locally convex inductive topology given by the system

$$\big(H(U \setminus K) \hookrightarrow \mathcal{X}(\mathbb{R})\big)_{K,U}.$$

We show that such a topology exists. There is an obvious repetition of some arguments below. This is justified by the fact that we wanted to have a clear definition of the map $\mathcal{C}$ which is crucial in the whole study.

Let as before $F \in \mathcal{X}$. Let $\gamma_1, \gamma_2$ be closed curves in $U \setminus K$ such that $\gamma_1 \subset \operatorname{int} \hat{\gamma}_2$. Let $\gamma_0$ be a path which joins $\gamma_1$ with $\gamma_2$. Let $\Gamma := (-\gamma_1) \cup \gamma_0 \cup \gamma_2 \cup (-\gamma_0)$. Then for $z \in (\operatorname{int} \hat{\gamma}_2) \setminus (\hat{\gamma}_1 \cup \gamma_0)$, we have

$$F(z) = \frac{1}{2\pi i} \int_\Gamma \frac{F(\zeta)}{\zeta - z} \, d\zeta$$

and

$$\frac{1}{2\pi i} \int_\Gamma \frac{F(\zeta)}{\zeta - z} \, d\zeta = \frac{1}{2\pi} \int_{\gamma_2} \frac{F(\zeta)}{\zeta - z} \, d\zeta + \frac{1}{2\pi i} \int_{-\gamma_1} \frac{F(\zeta)}{\zeta - z} \, d\zeta,$$

since integrals over $\gamma_0$ and $(-\gamma_0)$ cancel out. For $z \in (\operatorname{int} \hat{\gamma}_2) \setminus \hat{\gamma}_1$, define

$$F_+(z) := \frac{1}{2\pi} \int_{\gamma_2} \frac{F(\zeta)}{\zeta - z} \, d\zeta,$$
$$F_-(z) := \frac{1}{2\pi i} \int_{-\gamma_1} \frac{F(\zeta)}{\zeta - z} \, d\zeta.$$

Then $F(z) = F_+(z) + F_-(z)$. Obviously, $F_+(z) = (\mathcal{C}F)(z)$. Furthermore, $F_+$ is holomorphic in $\operatorname{int} \hat{\gamma}_2$, while $F_-$ is holomorphic in ${}^c\hat{\gamma}_1$ and vanishes at infinity. It is clear that we can repeat the whole construction for any $z \in U \setminus K$ and the outcome does not depend on $\gamma_0, \gamma_1, \gamma_2$. We have that any $F \in H(U \setminus K)$ decomposes into

$$F = F_+ + F_-,$$

with $F_+ \in H(U)$ and $F_- \in H_0(\mathbb{C}_\infty \setminus K)$. The symbol $H_0(\mathbb{C}_\infty \setminus K)$ stands for the space of all functions holomorphic in $\mathbb{C}_\infty \setminus K$, and $\mathbb{C}_\infty$ is the Riemann sphere, which vanish at infinity. The decomposition is unique by Liouville's theorem. This implies the following well-known fact, which we will refer to later on.

**Proposition 3.1.** *For any open connected and simply connected neighborhood $U$ of the real line and a compact connected subset $K$ of $\mathbb{R}$ the space $H(U \setminus K)$ is isomorphic to*

$$H(U) \oplus H_0(\mathbb{C}_\infty \setminus K).$$

*Proof.* Naturally, the map

$$H(U \setminus K) \ni F \mapsto (F_+, F_-) \in H(U) \oplus H_0(\mathbb{C}_\infty \setminus K)$$

is continuous when each of the spaces is equipped with the compact-open topology. By uniqueness of the decomposition, it has a continuous inverse

$$H(U) \oplus H_0(\mathbb{C}_\infty \setminus K) \ni (F_+, F_-) \mapsto F_+ + F_- \in H(U \setminus K). \qquad \square$$

Proposition 3.1 means that algebraically,

$$\mathcal{X}(\mathbb{R}) = \bigcup_{\substack{K \subset\subset \mathbb{R} \\ \mathbb{R} \subset U}} H(U \setminus K) = \bigcup_{\mathbb{R} \subset U} H(U) \oplus \bigcup_{K \subset\subset \mathbb{R}} H_0(\mathbb{C}_\infty \setminus K). \tag{3.3}$$

Relation (3.3) can be used to equip the space $\mathcal{X}(\mathbb{R})$ with a separated locally convex inductive topology of the system $(H(U \setminus K) \hookrightarrow \mathcal{X}(\mathbb{R}))_{K \subset\subset U}$. Point evaluations applied to $F_+$ and Laurent coefficients of $F_-$ provide linear forms which are continuous on every $H(U \setminus K)$ and separate functions in $\mathcal{X}(\mathbb{R})$. This shows that such a topology exists (see [24, Lemma 24.6]).

Observe that the similar arguments show that there exist inductive topologies on

$$\bigcup_{\substack{U \supset \mathbb{R} \\ U \text{ open}}} H(U)$$

and

$$\bigcup_{K \subset\subset \mathbb{R}} H_0(\mathbb{C}_\infty \setminus K).$$

Naturally, this is well known. We have now all the necessary elements to formulate the following theorem.

**Theorem 3.2.** *The map $\mathcal{C} : \mathcal{X}(\mathbb{R}) \to \mathcal{X}(\mathbb{R})$ is a continuous projection onto $\mathcal{A}(\mathbb{R})$.*

*Proof.* For any $F \in H(U \setminus K)$ the function $\mathcal{C}F$ is holomorphic in $U$. Also, if $F \in H(U)$ then $\mathcal{C}F = F$. This means that $\mathcal{C}F \in \mathcal{A}(\mathbb{R})$ for any $F \in \mathcal{X}(\mathbb{R})$ and $\mathcal{C}F = F$ when $F \in \mathcal{A}(\mathbb{R})$.

Thus only continuity of $\mathcal{C}$ requires some comment. It is an elementary property of the Cauchy transform that $\mathcal{C}$ maps continuously $H(U \setminus K)$ into $H(U) \hookrightarrow H(U \setminus K)$. Since $\mathcal{X}(\mathbb{R})$ carries the inductive topology, the claim follows from this immediately. Indeed, in order to prove that $\mathcal{C}$ is a continuous operator on $\mathcal{X}(\mathbb{R})$ it suffices to show that for any $U, K$

$$\mathcal{C} : H(U \setminus K) \to \mathcal{X}(\mathbb{R})$$

is continuous. But $\mathcal{C}$ maps continuously $H(U \setminus K)$ into $H(U)$ and the latter embeds continuously into $H(U \setminus K)$, which in turn embeds continuously into $\mathcal{X}(\mathbb{R})$. $\square$

Notice that we have actually proved the following fact.

**Theorem 3.3.** *Let*

$$H_0(\mathbb{C}_\infty \setminus \mathbb{R}) := \operatorname{ind}_n H_0\big(\mathbb{C}_\infty \setminus [-n, n]\big).$$

*Then*

$$\mathcal{X}(\mathbb{R}) \cong \mathcal{A}(\mathbb{R}) \oplus H_0(\mathbb{C}_\infty \setminus \mathbb{R})$$

*and, as a result,*

$$\mathcal{X}(\mathbb{R}) \cong \mathcal{A}(\mathbb{R}) \oplus \mathcal{A}(\mathbb{R})'.$$

*Proof.* The proof follows from above given arguments and Köthe–Grothendieck duality. Indeed, as we know by Proposition 3.1,

$$H(U \setminus K) \cong H(U) \oplus H_0(\mathbb{C}_\infty \setminus K).$$

This, however, implies that

$$\begin{aligned} \operatorname{ind}_{K \subset\subset U} H(U \setminus K) &\cong \operatorname{ind}_{K \subset\subset U} H(U) \oplus H_0(\mathbb{C}_\infty \setminus K) \\ &\cong \operatorname{ind}_{U \supset \mathbb{R}} H(U) \oplus \operatorname{ind}_{K \subset\subset \mathbb{R}} H_0(\mathbb{C}_\infty \setminus K). \end{aligned}$$

Also,

$$\operatorname{ind}_{U\supset\mathbb{R}} H(U) = \mathcal{A}(\mathbb{R})$$

and, by Köthe–Grothendieck duality,

$$\mathcal{A}(\mathbb{R})' \cong \operatorname{ind}_{K\subset\subset\mathbb{R}} H(K)' \cong \operatorname{ind} H_0(\mathbb{C}_\infty \setminus K).$$

Obviously, it is enough to take compact sets of the form $K = [-n, n], n \in \mathbb{N}$ above. $\square$

Observe that $\mathcal{X}(\mathbb{R})$ is an algebra and for a fixed function $F \in \mathcal{X}(\mathbb{R})$ the map

$$M_F : \mathcal{X} \ni f \mapsto Ff \in \mathcal{X}$$

is continuous. Indeed, assume that $f \in H(V \setminus L)$ for some open $V \supset \mathbb{R}$ and $L \subset\subset \mathbb{R}$. Then $Ff \in H(U \cap V \setminus (L \cup K))$. The map

$$H(V \setminus L) \ni f \mapsto Ff \in H\big(U \cap V \setminus (K \cup L)\big)$$

is clearly continuous. The statement follows now from properties of the inductive topology on $\mathcal{X}(\mathbb{R})$ by the argument analogous to the one which appeared in the proof of Theorem 3.2. It seems natural to call the operator $M_F : \mathcal{X}(\mathbb{R}) \to \mathcal{X}(\mathbb{R})$ a Laurent operator.

For any $F \in \mathcal{X}(\mathbb{R})$, the map

$$\mathcal{A}(\mathbb{R}) \ni f \mapsto Ff \in \mathcal{X}(\mathbb{R}),$$

is continuous, since $\mathcal{A}(\mathbb{R}) \hookrightarrow \mathcal{X}(\mathbb{R})$ continuously, We will denote this map by the same symbol $M_F$.

## 4. Toeplitz operators

Observe that by results of the previous section operators of the form

$$T_F := \mathcal{C}M_F$$

map continuously $\mathcal{A}(\mathbb{R})$ into itself. We intend to show that these operators are Toeplitz operators in the sense of Definition 1.1 and that each Toeplitz operator on $\mathcal{A}(\mathbb{R})$ is an operator $T_F$ for some function $F \in \mathcal{X}(\mathbb{R})$.

Our first main result is the following theorem.

**Theorem 1.** *The following assertions are equivalent.*

(i) $T : \mathcal{A}(\mathbb{R}) \to \mathcal{A}(\mathbb{R})$ *is a Toeplitz operator, that is,* $T$ *is a continuous linear operator such that, locally near* $0$,

$$T(x^n)(\xi) = a_{-n} + a_{-n+1}\xi + a_{-n+2}\xi^2 + \cdots \tag{4.1}$$

*for some complex numbers* $a_n, n \in \mathbb{Z}$.

(ii) *There exists a function* $F \in \mathcal{X}(\mathbb{R})$ *such that*

$$T = T_F = \mathcal{C}M_F, \tag{4.2}$$

*where* $M_F : \mathcal{X}(\mathbb{R}) \to \mathcal{X}(\mathbb{R})$ *is the multiplication operator* $M_F : f \mapsto Ff$ *and* $\mathcal{C}$ *is the projection*

$$\mathcal{C} : \mathcal{X}(\mathbb{R}) \to \mathcal{A}(\mathbb{R}) \subset \mathcal{X}(\mathbb{R}).$$

*Then (4.1) holds with*

$$a_n = \frac{1}{2\pi i}\int_\gamma F(\zeta)\zeta^{-n-1}\,d\zeta, \tag{4.3}$$

*where $\gamma$ is a closed curve in $U \setminus K$ and $F \in H(U \setminus K)$.*

(iii) *There exist $G \in \mathcal{A}(\mathbb{R})$ and $\Phi \in \mathcal{A}(\mathbb{R})'$ such that*

$$(Tf)(z) = G(z)f(z) + \Big\langle \frac{f(z)-f(\cdot)}{z-\cdot}, \Phi \Big\rangle.$$

*Then close to* $0$,

$$G(z) = \sum_{n=0}^{\infty} c_n z^n$$

*and (4.1) holds with $a_n = c_n, n \in \mathbb{N}_0$ and $a_{-n}, n \in \mathbb{N}$ the sequence of moments of $\Phi$; that is,*

$$a_{-n-1} = \langle z^n, \Phi\rangle, \quad n \in \mathbb{N}_0. \tag{4.4}$$

Before we prove Theorem 1 we single out the following reasoning, which appears in the proof of Theorem 1 twice.

**Proposition 4.1.** *Assume that $F \in \mathcal{X}$. There exists $r > 0$ such that for $\xi \in B(0,r)$ and any $n \in \mathbb{N}_0$, it holds that*

$$T_F(x^n)(\xi) = F_{-n} + F_{-n+1}\xi + \cdots + F_{-1}\xi^{n-1} + F_0\xi^n + \cdots,$$

*where*

$$F_n = \frac{1}{2\pi i}\int_\gamma F(\zeta)\zeta^{-n-1}\,d\zeta \tag{4.5}$$

*and $\gamma$ is a closed curve in $U \setminus K$.*

*Proof.* Decompose $F = F_+ + F_-$, where $F_+ \in H(U)$ and $F_- \in H_0(\mathbb{C}_\infty \setminus K)$, as in Proposition 3.1. Let $F_n, n \in \mathbb{Z}$ be given by formula (4.5). There is an $r > 0$ such that, in $B(0,r)$,

$$F_+(z) = \sum_{n=0}^{\infty} F_n z^n,$$

and there is also $R > 0$ such that, for $|z| > R$,

$$F_-(z) = \sum_{n=1}^{\infty} \frac{F_{-n}}{z^n}.$$

Let $\xi \in B(0,r)$. We have

$$\begin{aligned} T_F(x^n)(\xi) &= T_F(z^n)(\xi) = T_{F_+ + F_-}(z^n)(\xi) = T_{F_+}(z^n)(\xi) + T_{F_-}(z^n)(\xi) \\ &= \xi^n\Big(\sum_{k=0}^{\infty} F_k\xi^k\Big) + T_{F_-}(x^n)(\xi), \end{aligned}$$

since $F_+z^n$ is holomorphic in $U$. We need to compute $T_{F_-}(x^n)(\xi)$. Let $\gamma$ be a closed curve in $\mathbb{C}_\infty \setminus K$ such that $\xi \in \operatorname{int}\hat{\gamma}$. We have

$$\begin{aligned} T_{F_-}(x^n)(\xi) &= \frac{1}{2\pi i}\int_\gamma \frac{F_-(\zeta)\zeta^n}{\zeta-\xi}\,d\zeta \\ &= \frac{1}{2\pi i}\int_{|\zeta|=2R} \frac{F_-(\zeta)\zeta^n}{\zeta-\xi}\,d\zeta \\ &= \sum_{k=1}^\infty\sum_{m=0}^\infty F_{-k}\Big(\frac{1}{2\pi i}\int_{|\zeta|=2R}\zeta^n\cdot\frac{1}{\zeta^{m+k+1}}\,d\zeta\Big)\cdot\xi^m \\ &= F_{-n}+F_{-(n-1)}\xi+\cdots+F_{-1}\xi^{n-1}. \end{aligned}$$

Altogether, we obtain

$$T_F(x^n)(\xi) = F_{-n}+F_{-n+1}\xi+\cdots+F_{-1}\xi^{n-1}+F_0\xi^n+\cdots. \tag{4.6}$$

Observe that the series (4.6) converges by the choice of $\xi$. The "minus" part of $F$ does not have any influence on convergence. ☐

*Proof of Theorem 1.* As we have already observed for any $F \in \mathcal{X}(\mathbb{R})$ the operator $T_F = \mathcal{C}M_F$ is a continous linear operator on $\mathcal{A}(\mathbb{R})$. This together with Proposition 4.1 proves that (ii) implies (i).

We show that (i) implies (ii). Set $F_+ := T1$. Naturally, $F_+ \in \mathcal{A}(\mathbb{R})$, since $T$ maps $\mathcal{A}(\mathbb{R})$ into itself, and near 0 we have

$$F_+(z) = \sum_{n=0}^\infty a_n z^n.$$

We define $\varphi \in \mathcal{A}(\mathbb{R})'$ by $\varphi(f) := T(f)(0)$, then $\varphi(x_n) = a_{-n}$ for all $n \in \mathbb{N}_0$. Since $\varphi \in \mathcal{A}(\mathbb{R})'$, there is $G_- \in H_0(\mathbb{C}_\infty\setminus K)$ for some $K \subset\subset \mathbb{R}$ such that $\varphi(f) = \langle f, G_-\rangle$ for all $f \in \mathcal{A}(\mathbb{R})$. For $|z| > R$, $R$ large enough, there is an expansion

$$G_-(z) = \sum_{m=1}^\infty \frac{G_{-m}}{z^m}$$

and we obtain

$$a_{-n} = \varphi(x^n) = \langle x^n, G_-\rangle = G_{-(n+1)}.$$

We set

$$F(z) = F_+(z) + zG_-(z) - a_0.$$

Then $F \in \mathcal{X}(\mathbb{R})$ and Proposition 4.1 yields the result.

We show now that (ii) implies (iii). Assume that $F \in \mathcal{X}(\mathbb{R})$ and consider the operator $T_F$. According to Proposition 3.1,

$$F = F_+ + F_-$$

with $F_+ \in H(U)$ and $F_- \in H_0(\mathbb{C}_\infty \setminus K)$. Thus for $f \in \mathcal{A}(\mathbb{R})$,

$$T_F f = T_{F_++F_-}f = T_{F_+}f + T_{F_-}f = F_+f + T_{F_-}f. \tag{4.7}$$

Since $F_- \in H_0(\mathbb{C}_\infty \setminus K)$, we have

$$\frac{1}{2\pi i}\int_\gamma \frac{F_-(\zeta)}{\zeta - z}\,d\zeta = 0 \tag{4.8}$$

for any closed curve in $\mathbb{C}_\infty \setminus K$ and $z \in \operatorname{int}\hat{\gamma}$. Assume that $f \in H(V)$, where $V$ is an open neighborhood of $\mathbb{R}$. Let $z \in \mathbb{R}$ and let $\gamma$ be a closed curve in $V \setminus K$ such that $\operatorname{Ind}_\gamma(z) = 1$. Then

$$\begin{aligned}(T_{F_-}f)(z) = \mathcal{C}_\gamma(F_-f)(z) &= \frac{1}{2\pi i}\int_\gamma \frac{(F_-f)(\zeta)}{\zeta - z}\,d\zeta \\ &= \frac{1}{2\pi i}\int_\gamma \frac{(F_-f)(\zeta)}{\zeta - z}\,d\zeta - f(z)\frac{1}{2\pi i}\int_\gamma \frac{F_-(\zeta)}{\zeta - z}\,d\zeta \\ &= \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta) - f(z)}{\zeta - z}F_-(\zeta)\,d\zeta.\end{aligned}$$

Naturally, if $f \in H(V)$, then the function

$$g(z,\zeta) := \begin{cases} \frac{f(\zeta)-f(z)}{\zeta - z}, & \zeta \neq z \\ f'(z), & \zeta = z, \end{cases}$$

belongs for each fixed $z \in V$ to $H(V)$. This means that

$$(T_{F_-}f)(z) = \Big\langle \frac{f(z) - f(\cdot)}{z - \cdot}, \Phi \Big\rangle,$$

where the functional $\Phi \in \mathcal{A}(\mathbb{R})'$ is determined by the function $F_- \in H_0(\mathbb{C}_\infty \setminus K)$ (see the information on duality at the end of Section 2.1). This, in view of (4.7), proves that (ii) implies (iii).

Now any continuous functional on $\mathcal{A}(\mathbb{R})$ corresponds to a function $F_- \in H_0(\mathbb{C}_\infty \setminus K)$ for some $K \subset\subset \mathbb{R}$ by Köthe–Grothendieck duality. Let now $f \in H(V)$ and let $\gamma$ be a closed curve in $V \setminus K$. The value of the action of $F_-$ on $g(z,\cdot)$ does not depend on $\gamma$, as long as $\gamma$ is a closed curve in $V \setminus K$. We may therefore assume that $z \in \operatorname{int}\hat{\gamma}$. Thus, by (4.8),

$$\begin{aligned}\Big\langle \frac{f(z) - f(\cdot)}{z - \cdot}, \Phi \Big\rangle &= \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta) - f(z)}{\zeta - z}F_-(\zeta)\,d\zeta \\ &= \frac{1}{2\pi i}\int_\gamma \frac{(F_-f)(\zeta)}{\zeta - z}\,d\zeta = \mathcal{C}(F_-f)(z).\end{aligned}$$

This proves that (iii) implies (ii). Obviously, we have

$$\frac{z^n - \zeta^n}{z - \zeta} = z^{n-1} + z^{n-2}\zeta + \cdots + \zeta^{n-1}.$$

Thus

$$\Big\langle \frac{z^n - \zeta^n}{z - \zeta}, \Phi_\zeta \Big\rangle = \langle 1, \Phi_\zeta\rangle z^{n-1} + \langle \zeta, \Phi_\zeta\rangle z^{n-2} + \cdots + \langle \zeta^{n-1}, \Phi_\zeta\rangle.$$

This shows (4.4). $\square$

Theorem 1 is a source of examples of Toeplitz operators on $\mathcal{A}(\mathbb{R})$. We provide now an easy and interesting example of such an operator. Namely, for $k \in \mathbb{N}_0$ and $a \in \mathbb{R}$ let us put

$$(D_a^k f)(\zeta) := \begin{cases} \frac{d^k}{dz^k}\left(\frac{f(z)-f(\zeta)}{z-\zeta}\right)\Big|_{z=a}, & \zeta \neq a, \\ f^{(k)}(a), & \zeta = a. \end{cases}$$

It is a consequence of Theorem 1 that operator

$$f \mapsto D_a^k f$$

is a Toeplitz operator on $\mathcal{A}(\mathbb{R})$. Hence, Theorem 2 provides information on solvability of equations of the form

$$\sum_{k=1}^{N} A_k D_{a_k}^{n_k} f = F,$$

since operators

$$\sum_{k=1}^{N} A_k D_{a_k}^{n_k}$$

are Fredholm operators.

In Theorem 1 we have shown that in full analogy to the Hardy space case, a Toeplitz operator on the space of real analytic functions $\mathcal{A}(\mathbb{R})$ is the composition of the multiplication $f \mapsto Ff$ which maps $\mathcal{A}(\mathbb{R})$ into $\mathcal{X}(\mathbb{R})$ and the projection from $\mathcal{X}(\mathbb{R})$ onto $\mathcal{A}(\mathbb{R})$. The first step in studying Toeplitz operators on $H^2(\mathbb{T})$ is investigating Laurent operators $M_\phi$ on $L^2(\mathbb{T})$. The idea in many proofs in the $H^2(\mathbb{T})$ case is to read properties of a Toeplitz operator from the properties of the corresponding Laurent operator. This was the idea in the seminal paper [11]. Our approach above was different. Nonetheless, it seems important to pursue this analogy further also in the case of spaces $\mathcal{A}(\mathbb{R})$ and $\mathcal{X}(\mathbb{R})$. We include the simple results on Laurent operators on $\mathcal{X}$.

Naturally $Wf := zf$ acts on $\mathcal{X}$. Furthermore, $W$ is invertible.

**Theorem 4.2.** *A necessary and sufficient condition that an operator on $\mathcal{X}$ be a Laurent operator is that it commutes with $W$.*

*Proof.* Assume that $A : \mathcal{X} \to \mathcal{X}$ commutes with $W$. Let $\varphi := A1$. Then $\varphi \in \mathcal{X}$. We have

$$Az^n = AW^n 1 = W^n A1 = z^n \varphi = \varphi \cdot z^n.$$

Also, since $AW = WA$

$$AW^{-1} = W^{-1}A.$$

Therefore, we have

$$Aq = \varphi \cdot q$$

for any rational function. Since such functions are dense by Runge's theorem in $\mathcal{X}$ we have $A = M_\varphi$. □

Observe that

$$\begin{aligned} a_{i+1,j+1} &= \langle Az^{i+1}, z^{-(j+1)-1}\rangle = \langle \varphi z^{i+1}, z^{-(j+1)-1}\rangle = \langle \varphi z^{i}, z^{-(j+1)}\rangle \\ &= \langle Az^{i}, z^{-(j+1)}\rangle = a_{ij}. \end{aligned}$$

**Theorem 4.3.** *A necessary and sufficient condition that an operator on $\mathcal{X}$ be a Laurent operator is that its matrix satisfies $a_{i+1,j+1} = a_{i,j}$ for any $i, j \in \mathbb{Z}$.*

*Proof.* It suffices to prove that $A$ commutes with $W$. We have

$$\begin{aligned} \langle AWz^{i}, z^{-(j+1)}\rangle &= \langle Az^{i+1}, z^{-(j+1)}\rangle = a_{i+1,j} = a_{i,j-1} = \langle Az^{i}, z^{-j}\rangle \\ &= \langle Az^{i}, Wz^{-(j+1)}\rangle = \langle WAz^{i}, z^{-(j+1)}\rangle. \end{aligned}$$ □

## 5. The Fredholm property

A continuous operator $T : \mathcal{A}(\mathbb{R}) \to \mathcal{A}(\mathbb{R})$ is a Fredholm operator if

(i) $\dim \ker T < \infty$,
(ii) $\dim \operatorname{coker} T < \infty$.

When $T$ is a Fredholm operator we define

$$\operatorname{index} T := \dim \ker T - \dim \operatorname{coker} T.$$

A Fredholm operator on a Banach space has closed range. The same is true in case of $\mathcal{A}(\mathbb{R})$.

**Proposition 5.1.** *Assume that $T : \mathcal{A}(\mathbb{R}) \to \mathcal{A}(\mathbb{R})$ is a continuous operator such that $\operatorname{im} T$ is of finite codimension. Then $\operatorname{im} T$ is closed.*

*Proof.* In order to prove the proposition we will use the de Wilde *open mapping theorem* (Theorem 24.30 in [24]). Let $Z$ be a finite-dimensional subspace of $\mathcal{A}(\mathbb{R})$ such that $\operatorname{im} T \oplus Z = \mathcal{A}(\mathbb{R})$. Obviously, $Z$ is closed. The space $\mathcal{A}(\mathbb{R})$ has a web as a PLS space. As a result, the quotient $\mathcal{A}(\mathbb{R})/\ker T$ has a web. Since $Z$ is of finite dimension $\mathcal{A}(\mathbb{R})/\ker T \oplus Z$ has a web. Consider the operator

$$S : \mathcal{A}(\mathbb{R})/\ker T \oplus Z \ni (f + \ker T) \oplus z \mapsto Tf + z \in \mathcal{A}(\mathbb{R}).$$

The operator $S$ is a continuous, surjective linear operator. Since $\mathcal{A}(\mathbb{R})$ is ultrabornological, by the open mapping theorem, $S$ is open. Hence, $S$ is invertible, since $S$ is one-to-one. Under the topological isomorphism $S$ the space $\operatorname{im} T$ corresponds to the closed subspace $\mathcal{A}(\mathbb{R})/\ker T \oplus \{0\}$ of $\mathcal{A}(\mathbb{R})/\ker T \oplus Z$. Hence $\operatorname{im} T$ is closed. □

We intend to prove our second main result.

**Theorem 2.** *A Toeplitz operator $T_F : \mathcal{A}(\mathbb{R}) \to \mathcal{A}(\mathbb{R})$ with $F \in \mathcal{X}(\mathbb{R})$ is a Fredholm operator if and only if there exist an open complex set $U \supset \mathbb{R}$ and a compact set $K \subset \mathbb{R}$ such that $F(z) \neq 0$ for $z \in U \setminus K$.*

*When $F(z) \neq 0$ for $z \in U \setminus K$ then*

$$\operatorname{index} T_F = -\operatorname{winding} F.$$

We will explain the meaning of symbol winding $F$. Let $F \in H(U \setminus K)$ be such that $F \neq 0$ in $U \setminus K$. We may assume that $U$ is connected and simply connected, $K$ is compact connected and contains 0. Let $\gamma : \mathbb{T} \to U \setminus K$ be a diffeomorphic map such that $\operatorname{Ind}_\gamma(0) = \deg \gamma = 1$. We set

$$\operatorname{winding} F := \operatorname{Ind}_{F\circ\gamma}(0). \tag{5.1}$$

This definition is correct (i.e., it does not depend on $\gamma$).

**Proposition 5.2.** *Assume that $F \in H(U \setminus K)$ and $F \neq 0$. For any $\gamma_0, \gamma_1 : S^1 \to U \setminus K$ such that* $\operatorname{Ind}_{\gamma_0}(0) = \operatorname{Ind}_{\gamma_1}(0)$

$$\operatorname{Ind}_{F\circ\gamma_0}(0) = \operatorname{Ind}_{F\circ\gamma_1}(0).$$

*Proof.* The proof follows at once from Cauchy's theorem since $\gamma_0 - \gamma_1$ is homologous to 0 in $U \setminus K$. $\square$

We intend now to prove that if $F \in H(U \setminus K)$ and $F \neq 0$, then the operator $T_F : \mathcal{A}(\mathbb{R}) \to \mathcal{A}(\mathbb{R})$ is a Fredholm operator. Our idea to prove this fact is to derive it from the case of the Hardy spaces on curves. This is why we need some information concerning operators on $H^2(\gamma)$.

Let $\gamma \subset U \setminus K$ be a closed curve. Recall that the Cauchy transform (on $\gamma$) is defined for functions $f \in C^\infty(\gamma)$ by the formula

$$\mathcal{C}_\gamma f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta \tag{5.2}$$

for $z \in \operatorname{int}\hat{\gamma}$. According to Theorem 2.6, the Cauchy transform maps $C^\infty(\gamma)$ into $A^\infty(\Omega) \subset C^\infty(\gamma)$, where $\Omega = \operatorname{int}\hat{\gamma}$. By Theorem 2.7 the Cauchy transform extends to an operator from $L^2(\gamma)$ into $H^2(\gamma)$—still denoted by the symbol $\mathcal{C}_\gamma$. Let $\phi \in C(\gamma)$, in particular $\phi = F|_\gamma$ when $F \in H(U \setminus K)$. We consider operators

$$T_{\phi,\gamma} : H^2(\gamma) \ni f \mapsto \mathcal{C}_\gamma(\phi f) \in H^2(\gamma)$$

and call them also Toeplitz operators. One remark here is in order. The usual definition of a Toeplitz operator on $H^2(\gamma)$ involves the Szegö projection rather than the Cauchy transform. It is, however, a consequence of Kerzman–Stein formula ([7, p. 11]) that the Cauchy transform and the Szegö projection differ on $H^2(\gamma)$ by a compact operator. This justifies the fact that we call operators $T_{\phi,\gamma}$ also Toeplitz operators.

Recall that we defined $H^2(\gamma)$ to be the closure in $L^2(\gamma)$ of $A^\infty(\Omega)$, $\Omega = \operatorname{int}\hat{\gamma}$. Formally $\mathcal{C}_\gamma(\phi f)$ is an element of $H^2(\gamma)$ and "does not exist" in $\operatorname{int}\hat{\gamma}$. As we know, Cauchy integral gives the extension of a function in $H^2(\gamma)$ to a holomorphic function in $\operatorname{int}\hat{\gamma}$ which satisfies the classical definition of the Hardy space (2.4). We want to treat elements of $H^2(\gamma)$, in particular functions of the form $T_{\phi,\gamma} f$ for $f \in H^2(\gamma)$, as holomorphic functions, not their traces. This means that we work with functions of the form

$$\frac{1}{2\pi i} \int_\gamma \frac{\mathcal{C}_\gamma(Ff)(\zeta)}{\zeta - z}\, d\zeta, \quad z \in \operatorname{int}\hat{\gamma}.$$

Importantly, this expression can be simplified.

**Proposition 5.3.** *For any $\phi \in C(\gamma)$, $f \in H^2(\gamma)$, and $z \in \operatorname{int}\hat{\gamma}$,*

$$\frac{1}{2\pi i}\int_\gamma \frac{\mathcal{C}_\gamma(\phi\cdot f)(\zeta)}{\zeta - z}\,d\zeta = \frac{1}{2\pi i}\int_\gamma \frac{(\phi\cdot f)(\zeta)}{\zeta - z}\,d\zeta. \tag{5.3}$$

Proposition 5.3 means that we may also treat the Toeplitz operator as an operator on $H^2(\gamma)$ with values in $H^2(\gamma)$ understood in the classical sense and defined by

$$(T_{\phi,\gamma}f)(z) = \frac{1}{2\pi i}\int_\gamma \frac{(\phi\cdot f)(\zeta)}{\zeta - z}\,d\zeta.$$

When $\phi = F|_\gamma$ the right hand side of (5.3) is the extension of $\mathcal{C}_\gamma(Ff)$, considered before for $f \in \mathcal{A}(\mathbb{R})$, to functions $f \in H^2(\gamma)$. Thus a Toeplitz operator on $H^2(\gamma)$ with a symbol $F \in \mathcal{X}(\mathbb{R})$ is the extension of the corresponding Toeplitz operator on $\mathcal{A}(\mathbb{R})$. We want to emphasize that here $\gamma$ is fixed. This is in contrary to the case of $\mathcal{A}(\mathbb{R})$ where it is of utmost importance that $\gamma$ can vary.

*Proof.* Assume that $u \in C^\infty(\gamma)$. We prove that

$$\frac{1}{2\pi i}\int_\gamma \frac{\mathcal{C}_\gamma(u)(\zeta)}{\zeta - z}\,d\zeta = \frac{1}{2\pi i}\int_\gamma \frac{u(\zeta)}{\zeta - z}\,d\zeta \tag{5.4}$$

when $z \in \operatorname{int}\hat{\gamma}$. Once (5.4) is proved, we choose $u_n \in C^\infty(\gamma)$ such that $u_n \to \phi f$ in $L^2(\gamma)$. From the definition of $\mathcal{C}_\gamma$ on $L^2(\gamma)$ we have $\mathcal{C}_\gamma(u_n) \to \mathcal{C}_\gamma(\phi\cdot f)$ in $L^2(\gamma)$. Since $z \in \operatorname{int}\hat{\gamma}$ the function $\frac{\gamma'(t)}{\gamma(t)-z}$ is bounded. Hence

$$\int_\gamma \frac{\mathcal{C}_\gamma(u_n)(\zeta)\,d\zeta}{\zeta - z} \to \int_\gamma \frac{\mathcal{C}_\gamma(\phi\cdot f)(\zeta)}{\zeta - z}\,d\zeta,$$
$$\int_\gamma \frac{u_n(\zeta)\,d\zeta}{\zeta - z} \to \int_\gamma \frac{(\phi\cdot f)(\zeta)}{\zeta - z}\,d\zeta,$$

as $n \to \infty$, which completes the proof.

It remains therefore to prove (5.4) for $u \in C^\infty(\gamma)$. To accomplish this task we invoke Theorem 2.11. There exists a function $\Psi \in C^M(\hat{\gamma})$ which vanishes to order $M \in \mathbb{N}$ on $\gamma$ such that

$$\mathcal{C}_\gamma(u)(z) = u(z) - \frac{1}{2\pi i}\int\int_{\operatorname{int}\hat{\gamma}} \frac{\Psi(\zeta)}{\zeta - z}\,d\zeta\wedge d\bar{\zeta}$$

for $z \in \gamma$. We need to show that

$$\int_\gamma \frac{\Phi(z)}{z - w}\,dz = 0,$$

where

$$\Phi(z) := \frac{1}{2\pi i}\int\int_{\operatorname{int}\hat{\gamma}} \frac{\Psi(\zeta)}{\zeta - z}\,d\zeta\wedge d\bar{\zeta}$$

and $w \in \operatorname{int}\hat{\gamma}$. The function $\Phi$ belongs to $C^M(\mathbb{C}) \cap H_0(\mathbb{C}_\infty \setminus \hat{\gamma})$. Consider curves $\gamma_\varepsilon(t) := \gamma(t) - i\varepsilon\mathcal{T}(\gamma(t))$ for small $\varepsilon > 0$. Since $\mathcal{T}(\gamma(t))$ is the complex number which represents the unit tangent vector at the point $\gamma(t)$, $\gamma_\varepsilon(t)$ is "the outside dilation" of $\gamma$. Define functions $\Phi_\varepsilon(\gamma(t)) := \Phi(\gamma_\varepsilon(t))$ and write

$$\begin{aligned}\int_0^1 \Phi\big(\gamma_\varepsilon(t)\big)\frac{\gamma_\varepsilon{}'(t)}{\gamma_\varepsilon(t)-w}\,dt - \int_0^1 \Phi\big(\gamma(t)\big)\frac{\gamma'(t)}{\gamma(t)-w}\,dt \\ = \int_0^1 \big(\Phi_\varepsilon\big(\gamma(t)\big) - \Phi\big(\gamma(t)\big)\big)\frac{\gamma'(t)}{\gamma(t)-w}\,dt \\ + \int_0^1 \Phi\big(\gamma_\varepsilon(t)\big)\Big(\frac{\gamma_\varepsilon{}'(t)}{\gamma_\varepsilon(t)-w} - \frac{\gamma'(t)}{\gamma(t)-w}\Big)\,dt.\end{aligned} \tag{5.5}$$

Since $\Phi \in C^M(\mathbb{C})$, the functions $\Phi_\varepsilon$ tend to $\Phi$ uniformly on $\gamma$. This implies that the first integral on the right-hand side of (5.5) tends to 0. Note that the functions $\Phi_\varepsilon$ are uniformly bounded and that

$$\frac{\gamma_\varepsilon{}'(t)}{\gamma_\varepsilon(t)-w} - \frac{\gamma'(t)}{\gamma(t)-w}$$

tends uniformly to 0. We have shown that

$$\int_{\gamma_\varepsilon} \frac{\Phi(z)}{z-w}\,dz \to \int_\gamma \frac{\Phi(z)}{z-w}\,dz, \tag{5.6}$$

as $\varepsilon \to 0$. By Cauchy's theorem, for any $\varepsilon > 0$,

$$\int_{\gamma_\varepsilon} \frac{\Phi(z)}{z-w}\,dz = \int_{|z|=R} \frac{\Phi(z)}{z-w}\,dz$$

for any $R$ large enough. But

$$\int_{|z|=R} \frac{\Phi(z)}{z-w}\,dz = \int_0^1 \Phi(Re^{2\pi it})\frac{2\pi i Re^{2\pi it}}{Re^{2\pi it}-w}\,dt \to 0,$$

as $R \to \infty$ by Lebesgue's dominated convergence theorem, since $\Phi$ vanishes at infinity. This implies that for any $\varepsilon > 0$,

$$\int_{\gamma_\varepsilon} \frac{\Phi(z)}{z-w}\,dz = 0,$$

which, in view of (5.6), completes the proof. □

The next result is essentially due to Mitrea et al. [25]. There is a minor difference comparing with [25]. Our Toeplitz operators act on $H^2(\gamma)$ rather than $L^2(\gamma)$. Nonetheless the argument is essentially the same. Therefore we omit the proof.

**Theorem 5.4.** *Assume that $\gamma$ is a closed curve in $U \setminus K$. For any $\phi \in C(\gamma)$, $\phi \neq 0$ the operator*

$$T_{\phi,\gamma} : H^2(\gamma) \to H^2(\gamma)$$

*is a Fredholm operator and*

$$\operatorname{index} T_{\phi,\gamma} = -\operatorname{Ind}_{\phi\circ\gamma}(0).$$

Observe the following fact follows at once from Theorem 5.4.

**Corollary 5.5.** *Assume that $F \in H(U \setminus K)$, $F \neq 0$. Then*

$$\operatorname{index}\big(T_{F,\gamma} : H^2(\gamma) \to H^2(\gamma)\big) = -\operatorname{winding} F \equiv \textit{constant}$$

*for any closed curve $\gamma$ in $U \setminus K$.*

We claim that $T_F : \mathcal{A}(\mathbb{R}) \to \mathcal{A}(\mathbb{R})$ has kernel and cokernel of finite dimension. We will prove that not only the index but also the dimension of the kernel and cokernel of $T_{F,\gamma} : H^2(\gamma) \to H^2(\gamma)$ does not depend on $\gamma$. Actually, we prove even more.

Recall our convention: $U$ is a connected, simply connected open neighborhood of $\mathbb{R}$, $K$ is a connected, compact subset of $\mathbb{R}$, $0 \in K$ and a closed curve in $U \setminus K$ is a diffeomorphism from $\mathbb{T}$ to $U \setminus K$ such that $\operatorname{Ind}_\gamma(0) = 1$.

**Theorem 5.6.** *Assume that $F \in H(U \setminus K)$ and $F(z) \neq 0$ for $z \in U \setminus K$. Then, either* $\dim\ker T_{F,\gamma} = 0$ *for any closed curve $\gamma \subset U \setminus K$ or there exist $m \in \mathbb{N}$ and linearly independent functions $f_1, \dots, f_m \in H(U)$ such that, for any closed curve $\gamma \subset U \setminus K$,*

$$\ker\big(T_{F,\gamma} : H^2(\gamma) \to H^2(\gamma)\big) = \operatorname{span}\{f_1, \dots, f_m\}.$$

*As a result, there exists a number $m \in \mathbb{N}_0$ such that, for any closed curve $\gamma \subset U \setminus K$,*

$$\dim\ker T_{F,\gamma} = m.$$

This implies the following theorem.

**Theorem 5.7.** *Assume that $F \in H(U \setminus K)$ and $F(z) \neq 0$ for $z \in U \setminus K$. There exists a number $m \in \mathbb{N}_0$ such that*

$$\dim\ker\big(T_F : \mathcal{A}(\mathbb{R}) \to \mathcal{A}(\mathbb{R})\big) = m.$$

*Furthermore, for any closed curve $\gamma$ in $U \setminus K$*

$$\dim\ker\big(T_F : \mathcal{A}(\mathbb{R}) \to \mathcal{A}(\mathbb{R})\big) = \dim\ker\big(T_{F,\gamma} : H^2(\gamma) \to H^2(\gamma)\big).$$

*Proof.* Assume that for any $\gamma \subset U \setminus K$, $\ker T_{F,\gamma} = \{0\}$. Then $\ker T_F = \{0\}$. Indeed, if there exists $f \in \mathcal{A}(\mathbb{R})$, $f \neq 0$, such that $f \in \ker T_F$, then for some $\gamma$ we must have $T_{F,\gamma} f = 0$ and $f \in H^2(\gamma)$, which contradicts our assumption.

Let $m > 0$ and $f_1, \dots, f_m$ be functions from Theorem 5.6. Functions $f_1, \dots, f_m$ are linearly independent and belong to $H(U)$. Naturally, $f_1, \dots, f_m \in \mathcal{A}(\mathbb{R})$ and $f_1, \dots, f_m \in \ker T_F$. Let $f$ be any real analytic function such that $T_F f = 0$. Thus $f \in H(V)$ for some open neighborhood $V$ of $\mathbb{R}$—we may assume that $U \cap V$ is connected and simply connected. Choose $z \in U \cap V$ and take a closed curve $\gamma \subset U \cap V \setminus K$ such that $z \in \operatorname{int}\hat{\gamma}$. Then, by definition of $T_F$,

$$(T_F f)(z) = (T_{F,\gamma} f)(z)$$

for any $z \in \operatorname{int}\hat{\gamma}$. Since $T_F f = 0$ we have in particular $T_{F,\gamma} f = 0$. As a result, by Theorem 5.6, $f$ belongs to $\operatorname{span}\{f_1, \dots, f_m\}$. □

*Proof of Theorem 5.6.* The main tool in the proof is the following lemma.

**Lemma 5.8.** *Assume that $F \in H(U \setminus K)$ and that $F(z) \neq 0$ for $z \in U \setminus K$. Let $\gamma \subset U \setminus K$ be a closed curve and let $f \in H^2(\gamma)$. If $T_{F,\gamma} f = 0$, then $f \in H(U)$.*

*Proof.* Assume that $T_{F,\gamma}f = 0$. Let $\gamma_{\varepsilon_0}$ be a smaller closed curve (i.e., $\gamma_{\varepsilon_0}(t) = \gamma(t) + i\varepsilon_0\mathcal{T}(\gamma(t))$ with $\varepsilon_0 > 0$ small enough to guarantee that $\gamma_{\varepsilon_0} \subset U \setminus K$).

**Claim.** *For $z \in \operatorname{int}\hat{\gamma}_{\varepsilon_0}$, we ahve*

$$(T_{F,\gamma_{\varepsilon_0}} f)(z) = 0.$$

*Proof.* Assume that $z \in \operatorname{int}\hat{\gamma}_{\varepsilon_0}$. We have $T_{F,\gamma}f(z) = 0$. For any $f \in H^2(\gamma)$,

$$\mathcal{C}_\gamma(Ff)(z) = \mathcal{C}_{\gamma_\varepsilon}(Ff)(z)$$

for $z \in \operatorname{int}\hat{\gamma}_{\varepsilon_0}$ and $0 < \varepsilon \le \varepsilon_0$. Indeed, first of all

$$\mathcal{C}_{\gamma_\varepsilon}(Ff)(z) \to \mathcal{C}_\gamma(Ff)(z) \tag{5.7}$$

as $\varepsilon \to 0$ for any $z \in \operatorname{int}\hat{\gamma}_{\varepsilon_0}$. Although this is a standard fact, we sketch the argument. Define functions $u_\varepsilon(\gamma(t)) := Ff(\gamma_\varepsilon(t))$ for small $\varepsilon > 0$. Then $u_\varepsilon \to Ff$ in $L^2(\gamma)$ (see Theorem 2.10). We have therefore

$$\begin{aligned}\big|\mathcal{C}_\gamma(Ff)(z) - \mathcal{C}_{\gamma_\varepsilon}(Ff)(z)\big| \le& \int \big|Ff\big(\gamma(t)\big) - u_\varepsilon\big(\gamma(t)\big)\big|\left|\frac{\gamma'(t)}{\gamma(t)-z}\right| dt \\ &+ \int \big|Ff\big(\gamma_\varepsilon(t)\big)\big|\left|\frac{\gamma(t)}{\gamma(t)-z} - \frac{\gamma_\varepsilon{}'(t)}{\gamma_\varepsilon(t)-z}\right| dt \\ \to& \; 0,\end{aligned}$$

since $Ff(\gamma_\varepsilon(t))$ is bounded in $L^2$ and the other term in the second integral tends uniformly to 0.

From Cauchy's theorem, when $z \in \operatorname{int}\hat{\gamma}_{\varepsilon_0}$,

$$(C_{\gamma_{\varepsilon_1}}Ff)(z) = (C_{\gamma_{\epsilon_2}}Ff)(z) \tag{5.8}$$

for any $\varepsilon_1, \varepsilon_2 > 0, \varepsilon_i \le \varepsilon_0, i = 1, 2$, since $Ff$ is holomorphic in $\operatorname{int}\hat{\gamma} \setminus K$. This proves the claim. □

Observe that we have proved that $\mathcal{C}_{\gamma_{\varepsilon_0}}(Ff) = 0$ when $\mathcal{C}_{\gamma_{\varepsilon_0}}(Ff)$ is considered as a function on $\gamma_{\varepsilon_0}$. Indeed, $Ff \in C^\infty(\gamma_{\varepsilon_0})$, thus $\mathcal{C}_{\gamma_{\varepsilon_0}}(Ff) \in C^\infty(\hat{\gamma}_{\varepsilon_0})$ by Theorem 2.6. For $z \in \operatorname{int}\hat{\gamma}_{\varepsilon_0}$ we have $\mathcal{C}_{\gamma_{\varepsilon_0}}(Ff)(z) = 0$. It must also be that

$$\mathcal{C}_{\gamma_{\varepsilon_0}}(Ff)(z) = 0$$

when $z \in \gamma_{\varepsilon_0}$.

It follows from Theorem 2.11 that for $z \in \gamma_{\varepsilon_0}$,

$$0 = \mathcal{C}_{\gamma_{\varepsilon_0}}(Ff)(z) = Ff(z) - \frac{1}{2\pi i}\int\int_{\operatorname{int}\hat{\gamma}_{\varepsilon_0}} \frac{\Psi(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta}$$

for some function $\Psi \in C^\infty(\hat{\gamma}_{\varepsilon_0})$, which vanishes to order $M$ on the boundary of $\hat{\gamma}_{\varepsilon_0}$. Hence, for $z \in \gamma_{\varepsilon_0}$,

$$f(z) = \frac{1}{2\pi i}\frac{1}{F(z)}\int\int_{\operatorname{int}\hat{\gamma}_{\varepsilon_0}} \frac{\Psi(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta} := g(z), \tag{5.9}$$

since $F$ does not vanish in $U \setminus K$, in particular in $\gamma_{\varepsilon_0}$.

The function

$$\int\int_{\operatorname{int}\hat{\gamma}_{\varepsilon_0}} \frac{\Psi(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta}$$

belongs to $C^1(\mathbb{C})$. This is a standard fact. Since the suppport of $\Psi$ is contained in $\hat{\gamma}_{\varepsilon_0}$, the integral can be considered as an integral over $\mathbb{C}$. In order to show that the integral is $C^1$, one first changes variables $\zeta \mapsto \zeta + z$ and then differentiate, which is possible since the singularity $\frac{1}{\zeta}$ is integrable. Also, $g$ is holomorphic in $U \cap {}^c\hat{\gamma}_{\varepsilon_0}$ (i.e., outside the support of $\Psi$, where $F$ is defined and nonzero). Functions $f$ and $g$ are equal on $\gamma_{\varepsilon_0}$. Thus they are equal on the open and connected set $\operatorname{int}\hat{\gamma} \cap {}^c\hat{\gamma}_{\varepsilon_0}$. Indeed, let $W$ be an open connected and simply connected subset of $\operatorname{int}\hat{\gamma} \cap {}^c\hat{\gamma}_{\varepsilon_0}$ bounded by a $C^\infty$ smooth curve $bW$ which is the sum of curves $\beta_1 : [0,1] \to \mathbb{C}$ and $\beta_2 : (0,1) \to \mathbb{C}$. $\beta_1$ is contained in $\gamma_{\varepsilon_0}$. The interior of the arc of $\gamma_{\varepsilon_0}$ determined by $\beta_1$ is nonempty. $\beta_2$ is contained in $\operatorname{int}\hat{\gamma} \cap {}^c\hat{\gamma}_{\varepsilon_0}$. Let $\Phi$ be the Riemann map between $W$ and $\mathbb{D}$. $\Phi$ extends to diffeomorphisms of closures of $W$ and $\mathbb{D}$. Consider the functions $f \circ \Phi^{-1}$ and $g \circ \Phi^{-1}$. Both belong to $H(\mathbb{D}) \cap C^1(\bar{\mathbb{D}})$ and are equal on a piece of boundary with nonempty interior in the boundary. Thus they are equal on $\mathbb{D}$. Hence $f$ and $g$ are equal on $W$ as claimed.

This means that $f$ extends to a holomorphic function on $U$ and completes the proof of the lemma. □

We continue with the proof of Theorem 5.6. There are two possibilities: either for any closed curve $\gamma \subset U \setminus K$ we have $\dim\ker T_{F,\gamma} = 0$ and the proof is completed or there exists a closed curve $\gamma \subset U \setminus K$ such that $\ker T_{F,\gamma} \neq \{0\}$. Since $T_{F,\gamma}$ is a Fredholm operator there exists $m \in \mathbb{N}$ such that

$$\dim\ker T_{F,\gamma} = m.$$

This means that there exist linearly independent functions $f_1, \ldots, f_m \in H^2(\gamma)$ such that $T_{F,\gamma} f_i = 0, i = 1, \ldots, m$ which span $\ker T_{F,\gamma}$. It follows from Lemma 5.8 that $f_1, \ldots, f_m$ extend to functions in $H(U)$. Let now $\gamma_1 \subset U \setminus K$ be any closed curve. Cauchy's theorem implies now that we must also have $\mathcal{C}_{\gamma_1}(F f_i) = 0$, $i = 1, \ldots, m$. Naturally, functions $f_1, \ldots, f_m$ are linearly independent as elements of $H^2(\gamma_1)$. Indeed, assume that $\alpha_1 f_1 + \cdots + \alpha_m f_m = 0$ in $H^2(\gamma_1)$. Then $\alpha_1 f_1(z) + \cdots + \alpha_m f_m(z) = 0$ for $z \in \operatorname{int}\hat{\gamma}_1$. But this means that $\alpha_1 f_1(z) + \cdots + \alpha_m f_m(z)$ for $z \in U$, in particular also in $\operatorname{int}\hat{\gamma}$. Hence, we have $\alpha_1 = \cdots = \alpha_m = 0$, since $f_1, \ldots, f_m$ were chosen to be linearly independent elements of $H^2(\gamma)$.

Thus $\dim\ker T_{F,\gamma_1} \geq m$. Let now $f \in \ker T_{F,\gamma_1}$. It follows from Lemma 5.8 that $f \in H(U)$ and so $f \in \ker T_{F,\gamma}$. Thus $f \in \operatorname{span}\{f_1, \ldots, f_m\}$ which completes the proof. We have shown that

$$\dim\ker\big(T_{F,\gamma} : H^2(\gamma) \to H^2(\gamma)\big) = m$$

for any closed curve $\gamma \subset U \setminus K$ and

$$\ker\big(T_{F,\gamma} : H^2(\gamma) \to H^2(\gamma)\big) = \operatorname{span}\{f_1, \ldots, f_m\}. \qquad \square$$

Importantly, the reasoning which was used to prove Lemma 5.8 can be generalized.

**Lemma 5.9.** *Assume that $F \in H(U \setminus K)$ and that $F(z) \neq 0$ for $z \in U \setminus K$. Assume that $V \supset K$ is an open set such that $U \cap V$ is connected and simply connected. Let $\gamma \subset U \cap V \setminus K$ be a closed curve. Assume that*

(i) $f \in H(V)$,
(ii) $h \in H^2(\gamma)$,
(iii) $f(z) = \mathcal{C}_\gamma(Fh)(z)$ *for* $z \in \operatorname{int}\hat{\gamma}$.

*Then $h \in H(V \cap U)$.*

*Proof.* We essentially repeat the arguments which prove Lemma 5.8.

Take a smaller closed curve $\gamma_{\varepsilon_0}$ contained in $(U \cap V) \setminus K$. Then for $z \in \operatorname{int}\hat{\gamma}_{\varepsilon_0}$,

$$f(z) = \mathcal{C}_{\gamma_{\varepsilon_0}}(Fh)(z).$$

This follows from Cauchy's theorem and the limit argument in $H^2(\gamma)$. Both $f$ and $\mathcal{C}_{\gamma_{\varepsilon_0}}(Fh)(z)$ belong to $C^\infty(\hat{\gamma}_{\varepsilon_0})$. Thus

$$f(z) = \mathcal{C}_{\gamma_{\varepsilon_0}}(Fh)(z)$$

for $z \in \gamma_{\varepsilon_0}$. It follows from Theorem 2.11 that, for a $C^M(\mathbb{C})$ function $\Psi$ with the support contained in $\hat{\gamma}_{\varepsilon_0}$,

$$f(z) = \mathcal{C}_{\gamma_{\varepsilon_0}}(Fh)(z) = Fh(z) - \frac{1}{2\pi i}\int\int_{\operatorname{int}\gamma_{\varepsilon_0}} \frac{\Psi(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta}$$

for $z \in \gamma_{\varepsilon_0}$.

Thus for $z \in \gamma_{\varepsilon_0}$ we have

$$h(z) = \frac{1}{F(z)}\Big(f(z) + \frac{1}{2\pi i}\int\int_{\operatorname{int}\hat{\gamma}_{\varepsilon_0}} \frac{\Psi(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta}\Big) := H(z).$$

The function $H$ is holomorphic in $V \cap U \setminus \hat{\gamma}_{\varepsilon_0}$. As in the proof of Lemma 5.8 we show that $H$ is the holomorphic extension of $h$. Thus $h$ is holomorphic in $U \cap V$. $\square$

We have proved so far the following.

(i) Index $T_{F,\gamma} : H^2(\gamma) \to H^2(\gamma)$ does not depend on $\gamma \subset U \setminus K$.
(ii) The dimension of the kernel $\ker(T_{F,\gamma} : H^2(\gamma) \to H^2(\gamma))$ is constant for each $\gamma$.

Thus for any closed curve $\gamma \subset U \setminus K$,

$$\dim\operatorname{coker}\big(T_{F,\gamma} : H^2(\gamma) \to H^2(\gamma)\big) = n \in \mathbb{N}_0$$

is constant and does not depend on $\gamma$. We will prove that the cokernels are also globally generated.

**Theorem 5.10.** *Assume that $F \in H(U \setminus K)$ and that $F \neq 0$. Either $T_{F,\gamma}$ is surjective for any closed curve $\gamma \subset U \setminus K$ or there exist $n \in \mathbb{N}$, an open set $\tilde{U} \subset U$, $\tilde{U} \supset \mathbb{R}$, and functions $f_1, \dots, f_n \in H(\tilde{U})$ such that classes $[f_1], \dots, [f_n]$ generate $H^2(\gamma)/\operatorname{im} T_{F,\gamma}$ and are linearly independent in $H^2(\gamma)/\operatorname{im} T_{F,\gamma}$ for any closed curve $\gamma \subset \tilde{U} \setminus K$.*

This theorem implies the following fact.

**Theorem 5.11.** *Assume that $F \in H(U\setminus K)$ and that $F \neq 0$. There exists $n \in \mathbb{N}_0$ such that*

$$\dim \mathcal{A}(\mathbb{R})/\operatorname{im} T_F = n.$$

*Furthermore, for any closed curve $\gamma$ in $\tilde{U} \setminus K$*

$$\dim \operatorname{coker} \big(T_F : \mathcal{A}(\mathbb{R}) \to \mathcal{A}(\mathbb{R})\big) = \dim \operatorname{coker} \big(T_{F,\gamma} : H^2(\gamma) \to H^2(\gamma)\big).$$

*The set $\tilde{U}$ is the open neighborhood of $\mathbb{R}$, the existence of which is proved in Theorem 5.10.*

*Proof.* Assume that $T_{F,\gamma}$ is surjective for any curve $\gamma \subset U \setminus K$. We claim that $T_F : \mathcal{A}(\mathbb{R}) \to \mathcal{A}(\mathbb{R})$ is surjective. Let $f \in \mathcal{A}(\mathbb{R})$. Thus $f \in H(V)$ for some open neighborhood $V \supset \mathbb{R}$. We may naturally assume that $U \cap V$ is connected and simply connected. Let $\gamma \subset U \cap V \setminus K$ be a closed curve. Naturally, $f \in H^2(\gamma)$. Since $T_{F,\gamma}$ is surjective, there exists $h \in H^2(\gamma)$ such that $f = T_{F,\gamma}h$. It follows from Lemma 5.9 that $h \in H(U\cap V)$. Obviously, $f = T_F h$ from Cauchy's theorem.

Assume now that $\dim \operatorname{coker} T_{F,\gamma} = n$ for some $n \in \mathbb{N}$ and any $\gamma \subset U\setminus K$. Assume that $g_1, \dots, g_{n+1} \in \mathcal{A}(\mathbb{R})$ are given. There exists a complex neighborhood $V$ of the real line $\mathbb{R}$ such that $g_1, \dots, g_{n+1} \in H(V)$, as usually we may assume that $U \cap V$ is connected and simply connected. Let $\gamma$ be a closed curve in $U \cap V \setminus K$. Functions $g_1, \dots, g_{n+1}$ belong to $H^2(\gamma)$. Therefore, there exist scalars $\alpha_1, \dots, \alpha_{n+1}$ not every equal to 0 such that $\alpha_1[g_1] + \dots + \alpha_{n+1}[g_{n+1}] = 0$ in $H^2(\gamma)/\operatorname{im} T_{F,\gamma}$. In other words, for these scalars there exists a function $h \in H^2(\gamma)$ such that

$$\alpha_1 g_1 + \dots + \alpha_{n+1} g_{n+1} = T_{F,\gamma} h.$$

It follows from Lemma 5.9 that $h \in H(V \cap U)$ and also $h \in \mathcal{A}(\mathbb{R})$. Thus classes $g_1, \dots, g_{n+1}$ are linearly dependent in $\mathcal{A}(\mathbb{R})/\operatorname{im} T_F$. This means that $\dim \operatorname{coker} T_F \leq n$.

We need to show that $\dim \operatorname{coker} T_F \geq n$. It is enough to show that classes of $f_1, \dots, f_n$ from Theorem 5.10 are linearly independent in $\mathcal{A}(\mathbb{R})/\operatorname{im} T_F$. Assume that there exist scalars $\alpha_1, \dots, \alpha_n$ such that

$$\alpha_1 f_1 + \dots + \alpha_n f_n = T_F h$$

for some $h \in \mathcal{A}(\mathbb{R})$. There exists an open set $V \supset \mathbb{R}$ such that $f_1, \dots, f_n$, $h \in H(V)$. Take any closed curve $\gamma \subset \tilde{U} \cap V \setminus K$ then

$$\alpha_1 f_1 + \dots + \alpha_n f_n = T_{F,\gamma} h$$

and $h \in H^2(\gamma)$. This, in view of Theorem 5.10, is possible only if $\alpha_1 = \dots = \alpha_n = 0$. □

*Proof of Theorem 5.10.* We know that there exists $n \in \mathbb{N}_0$ such that, for any closed curve $\gamma \subset U \setminus K$,

$$\dim H^2(\gamma)/\operatorname{im} T_{F,\gamma} = n.$$

We may assume that $n \neq 0$ since otherwise the theorem is proved.

Choose open sets $U_n, n \in \mathbb{N}$, bounded by closed curves $\gamma_n \subset U \setminus K$ such that

$$\tilde{U} := \bigcup_{n=1}^{\infty} U_n$$

is an open neighborhood of $\mathbb{R}$ in the complex plane. Also, we assume that sets $U_n$ are convex. We will construct functions $f_1, \dots, f_n \in H(\tilde{U})$ such that classes $[f_1], \dots, [f_n]$ span $H^2(\gamma)/\operatorname{im} T_{F,\gamma}$ and are linearly independent in $H^2(\gamma)/\operatorname{im} T_{F,\gamma}$ for any closed curve $\gamma \subset \tilde{U} \setminus K$.

Consider the space $H^2(\gamma_1)$ and the operator $T_{F,\gamma_1}$. We know that there exist functions

$$f_1^1, \dots, f_n^1 \in H^2(\gamma_1)$$

which span $H^2(\gamma_1)/\operatorname{im} T_{F,\gamma_1}$. There are also functions

$$f_1^2, \dots, f_n^2 \in H^2(\gamma_2)$$

(the same $n$!) which span $H^2(\gamma_2)/\operatorname{im} T_{F,\gamma_2}$. The intersection $U_1 \cap U_2$ is convex and nonempty and contains $K$. Let $\gamma_{12}$ be a closed curve in $U_1 \cap U_2 \setminus K$. Naturally, both $f_1^1, \dots, f_n^1$ and $f_1^2, \dots, f_n^2$ belong to $H^2(\gamma_{12})$.

**Claim two.** *Classes of functions $f_1^1, \dots, f_n^1$ are linearly independent in $H^2(\gamma_{12})/\operatorname{im} T_{F,\gamma_{12}}$. As a result, they span $H^2(\gamma_{12})/\operatorname{im} T_{F,\gamma_{12}}$.*

*Proof.* Assume that there exists a nontrivial combination

$$\alpha_1 f_1^1 + \cdots + \alpha_n f_n^1 = T_{F,\gamma_{12}} h$$

for some $h \in H^2(\gamma_{12})$.

*Remark* 5.12. It follows from Lemma 5.9 that $h \in H(U_1 \cap U)$. However, it is easy to observe that $h \in H^2(\gamma_1)$.

Indeed, this follows from the fact that the extension of $h$ is equal near $\gamma_1$ to

$$\frac{1}{F(z)}\Big(\underbrace{\alpha_1 f_1^1 + \cdots + \alpha_n f_n^1}_{\text{in } H^2(\gamma_1)} + \frac{1}{2\pi i}\int\int_{\operatorname{int}\hat{\gamma}_{120}} \frac{\Psi(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta}\Big),$$

where $\gamma_{120}$ is a smaller closed curve. The function

$$\frac{1}{2\pi i}\int\int_{\operatorname{int}\hat{\gamma}_{120}} \frac{\Psi(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta}$$

belongs to $C^1(\mathbb{C}) \cap H({}^c\hat{\gamma}_{120})$. This completes the proof of Remark 5.12. □

Thus, $h \in H^2(\gamma_1)$, and by Cauchy's theorem and the limit argument,

$$\alpha_1 f_1^1 + \cdots + \alpha_n f_n^1 = T_{F,\gamma_1} h.$$

Hence classes of $f_1^1, \dots, f_n^1$ are linearly dependent in $H^2(\gamma_1)/\operatorname{im} T_{F,\gamma_1}$, which contradicts our assumption. □

It follows from the Claim that classes of $f_1^2, \dots, f_n^2$ span $H^2(\gamma_{12})/\operatorname{im} T_{F,\gamma_{12}}$. As a result, there exist scalars $\beta_1, \dots, \beta_n$ such that

$$f_1^1 - \beta_1 f_1^2 - \dots - \beta_n f_n^2 = T_{F,\gamma_{12}} h$$

for some function $h \in H^2(\gamma_{12})$. We rename now functions in $H^2(\gamma_2)$. Let the new $f_1^2$ be equal to

$$\beta_1 f_1^2 + \dots + \beta_n f_n^2.$$

We repeat this procedure for other functions $f_2^1, \dots, f_n^1$. In such a way we have functions $f_1^2, \dots, f_n^2 \in H^2(\gamma_2)$ such that

$$f_i^1 - f_i^2 = T_{F,\gamma_{12}} h_i^{12},$$

where $h_i^{12} \in H^2(\gamma_{12}), i = 1, \dots, n$.

We now fix $i$ and supress denoting dependence on $i$. Take the set $U_3$. This set has a nonempty intersection with $U_1$. In the same manner as above we produce function $f^3 \in H^2(\gamma_3)$ and $h^{13} \in H^2(\gamma_{13})$ such that

$$f^1 - f^3 = T_{F,\gamma_{13}} h^{13}$$

with $h^{13} \in H^2(\gamma_{13})$, where $\gamma_{13}$ is a closed curve in $U_1 \cap U_3 \setminus K$. We repeat the procedure for $j > 3$. As a result, for any $j \in \mathbb{N}$ we have

$$f^1 - f^j = T_{F,\gamma_{1j}} h^{1j}$$

with $\gamma_{1j} \subset U_1 \cap U_j \setminus K$. If $j = 1$, then we set $h^{1j} = 0$, which matters for Cousin data. It follows from Lemma 5.9 that $h^{1j} \in H(U_1 \cap U_j)$.

We also set $h^{j1} = -h^{1j}$ for $j \in \mathbb{N}$. We intend to produce consistent Cousin data now. Let $k$ and $l$ be given. Then $f^k$ exists in $H^2(\gamma_k)$ and $f^l$ exists in $H^2(\gamma_l)$. Observe that $U_1 \cap U_k \cap U_l \neq \emptyset$. Thus it is meaningful to consider

$$f^1 - f^k - (f^1 - f^l) = f^l - f^k.$$

This is equal to

$$T_{F,\gamma_{1k}} h^{1k} - T_{F,\gamma_{1l}} h^{1l}.$$

More precisely, for $z \in \operatorname{int} \hat{\gamma}_{1k}$,

$$f^1(z) - f^k(z) = T_{F,\gamma_{1k}}(h^{1k})(z).$$

For $z \in \operatorname{int} \hat{\gamma}_{1l}$

$$f^1(z) - f^l(z) = T_{F,\gamma_{1l}}(h^{1l})(z).$$

The function $h^{1k}$ exists in $H^2(\gamma_{1k})$, and the function $h^{1l}$ exists in $H^2(\gamma_{1l})$. Choose now a smaller closed curve $\gamma_{kl} \subset \operatorname{int} \hat{\gamma}_{1k} \cap \operatorname{int} \hat{\gamma}_{1l} \setminus K$. Then from Cauchy's theorem we have

$$f^l(z) - f^k(z) = T_{F,\gamma_{kl}}(h^{1k} - h^{1l})(z)$$

for $z \in \operatorname{int} \hat{\gamma}_{kl}$. But from Lemma 5.9 we know that $h^{1k} - h^{1l}$ exists where $f^l - f^k$ exists (i.e., in $U_k \cap U_l$). Importantly, it does not mean that $h^{1k}$ or $h^{1l}$ extend to $U_k \cap U_l$ whatsoever, but the difference $h^{1k} - h^{1l}$ does.

Set $H^{1j} = h^{1j}, H^{j1} = h^{j1}, j \in \mathbb{N}$ and $H^{lk} = h^{1k} - h^{1l}$ if both $k$ and $l$ are not equal to 1. We have

$$H^{kl} \in H(U_k \cap U_l)$$

and

$$f^l - f^k = T_{F,\gamma_{lk}} H^{lk}.$$

**Claim.** *$H^{lk}$ are Cousin data.*

*Proof.* It follows from the definition that $H^{1j} = -H^{j1}$. Also, $H^{lk} = -H^{kl}$. Indeed, $H^{lk}$ is the extension to $U_k \cap U_l$ of $h^{1k} - h^{1l}$, while $H^{kl}$ is the extension of $h^{1l} - h^{1k}$. On an open set we have $H^{kl} = -H^{lk}$, which completes the argument.

Take now $j, k, l \in \mathbb{N}$. Assume that $j = k = 1$. Then

$$H^{11} + H^{1l} + H^{l1} = 0.$$

Assume now that $j = 1$ and that $k, l \neq 1$. We have, in $U_j \cap U_k \cap U_l$,

$$H^{1k} + H^{kl} + H^{l1} = h^{1k} + h^{1l} - h^{1k} + h^{l1} = 0.$$

Assume now that $j, k, l \neq 1$ and consider $U_1 \cap U_j \cap U_k \cap U_l$, which is nonempty and open. We have

$$H^{jk} + H^{kl} + H^{lj} = h^{1k} - h^{1j} + h^{1l} - h^{1k} + h^{1j} - h^{1l} = 0.$$

Thus $H^{jk} + H^{kl} + H^{lj} = 0$ in $U_j \cap U_k \cap U_l$. □

Recall the following.

**Theorem 5.13** ([20, Theorem 1.4.5])**.** *Let $\Omega = \bigcup_1^\infty \Omega_j$ and let $H^{jk} \in H(\Omega_j \cap \Omega_k)$, $j, k = 1, 2, \ldots$ satisfy the conditions*

$$H^{jk} = -H^{kj}, \qquad H^{jk} + H^{kl} + H^{lj} = 0 \text{ in } \Omega_j \cap \Omega_k \cap \Omega_k, \quad j, k, l \in \mathbb{N}.$$

*Then one can find $H^j \in H(\Omega_j)$ so that*

$$H^{jk} = H^k - H^j \text{ in } \Omega_j \cap \Omega_k \quad \text{for all } j \text{ and } k.$$

There exist functions $H_j$ holomorphic in $U_j, j \in \mathbb{N}$ such that $H^{jk} = H^k - H^j$ in $U_j \cap U_k$. We have therefore

$$f^j - f^k = T_{F,\gamma_{jk}} H^{jk} = T_{F,\gamma_{jk}} H^k - T_{F,\gamma_{jk}} H^j.$$

Thus

$$f^j + T_{F,\gamma_{jk}} H^j = f^k + T_{F,\gamma_{jk}} H^k$$

in $\operatorname{int} \hat{\gamma}_{jk}$. Now $\gamma_{jk} \subset U_j \cap U_k \setminus K$ and for any closed curve $\gamma \subset U_j \setminus K$ we have $T_{F,\gamma_{jk}} H^j = T_{F,\gamma} H^j$ in $\operatorname{int} \hat{\gamma}_{jk} \cap \operatorname{int} \hat{\gamma}$. This means that $F_j := T_{F,\gamma_{jk}} H^j$ is a function in $U_j$. Thus we have

$$f^j + F_j = f^k + F_k \text{ in } U_j \cap U_k.$$

We may glue the functions $f^j + F_j$ in order to obtain a function $\tilde{f} \in H(\tilde{U})$.

So far $i$ was fixed. The same procedure can be applied to other among functions $f_1^1, \ldots, f_n^1$. We have therefore constructed functions $\tilde{f}_1, \ldots, \tilde{f}_n \in H(\tilde{U})$. We claim that for any $\gamma \subset \tilde{U} \setminus K$ functions so obtained are linearly independent in $H^2(\gamma)/\operatorname{im} T_{F,\gamma}$.

**Claim.** *Let $\gamma$ be a closed curve contained in $\operatorname{int} \hat{\gamma}_1 \setminus K$. Classes of functions $f_1^1, \ldots, f_n^1$ are linearly independent in $H^2(\gamma)/\operatorname{im} T_{F,\gamma}$.*

We have functions $\tilde{f}_1, \dots, \tilde{f}_n$ which exist and are holomorphic in $\tilde{U} = \bigcup U_n$. On $U_1$ these functions are of the form

$$f_1^1 + T_{F,\gamma_0} H^1, \dots, f_n^1 + T_{F,\gamma_0} H^n$$

for functions $H^1, \dots, H^n \in H(U_1)$ and appropriately chosen closed curve $\gamma_0$ in $U_1 \setminus K$. Assume that

$$\alpha_1 \tilde{f}_1 + \dots + \alpha_n \tilde{f}_n = T_{F,\gamma} h$$

for some function $h \in H^2(\gamma)$ and a curve $\gamma \subset \tilde{U} \setminus K$.

Take a closed curve $\tilde{\gamma} \subset \operatorname{int} \hat{\gamma} \cap \operatorname{int} \hat{\gamma}_1 \setminus K$. We have

$$\alpha_1 \tilde{f}_1 + \dots + \alpha_n \tilde{f}_n = T_{F,\tilde{\gamma}} h$$

in $\operatorname{int} \widehat{\tilde{\gamma}}$. Naturally, we also have $h \in H^2(\tilde{\gamma})$. In other words,

$$\alpha_1 (f_1^1 + T_{F,\gamma_0} H^1) + \dots + \alpha_n (f_n^1 + T_{F,\gamma_0} H^n) = T_{F,\tilde{\gamma}} h$$

for $z \in \operatorname{int} \widehat{\tilde{\gamma}} \cap \operatorname{int} \hat{\gamma}_0$ with $H^1, \dots, H^n \in H(U_1)$. This means that

$$\alpha_1 (f_1^1 + T_{F,\tilde{\gamma}} H^1) + \dots + \alpha_n (f_n^1 + T_{F,\tilde{\gamma}} H^n) = T_{F,\tilde{\gamma}} h.$$

Thus

$$\alpha_1 f_1^1 + \dots + \alpha_n f_n^1 = T_{F,\tilde{\gamma}} (h - \alpha_1 H^1 - \dots - \alpha_n H^n).$$

This means, however, that classes of $f_1^1, \dots, f_n^1$ are linearly dependent in $H^2(\tilde{\gamma}) / \operatorname{im} T_{F,\tilde{\gamma}}$. This is impossible in view of the previous claim. □

Theorem 5.7 and 5.11 show that if $F \in H(U \setminus K)$, $F \neq 0$ then $T_F : \mathcal{A}(\mathbb{R}) \to \mathcal{A}(\mathbb{R})$ is a Fredholm operator; that is, both kernel of $T_F$ and cokernel are finite-dimensional. We now show that if $T_F$ is a Fredholm operator then $F \in H(U \setminus K)$ and $F \neq 0$ in $U \setminus K$. We start with the following easy observation.

**Proposition 5.14.** *Assume that $F \in H(U \setminus K)$. For any open neighborhood of $\tilde{U} \subset U$ of $\mathbb{R}$ and a compact set $\tilde{K} \subset \mathbb{R}, K \subset \tilde{K}$ there exists $z \in \tilde{U} \setminus \tilde{K}$ such that $F(z) = 0$ if and only if either*

(i) *there exists $z_n \in \mathbb{R}$, $z_n \to \pm\infty$, such that $F(z_n) = 0$,*
*or*
(ii) *there exists $z_n \to z \in K$, $z_n \in U \setminus \mathbb{R}$, such that $F(z_n) = 0$.*

*Proof.* It is obvious that if (i) or (ii) is satisfied, then for any $\tilde{U}$ and $\tilde{K}$ there is $z \in \tilde{U} \setminus \tilde{K}$ such that $F(z) = 0$. Assume that $F$ is not identically equal to 0, otherwise the claim is trivial. Assume that (ii) is not satisfied. For any $N \in \mathbb{N}$ there exists $\varepsilon(N) > 0$ such that in $[-N, N] \times [-\varepsilon(N), \varepsilon(N)]$ there is only finitely many points $z$ such that $F(z) = 0$. Otherwise such points accumulate at a point in $\mathbb{R} \setminus K$, which implies that $F \equiv 0$. There are two possibilities: either there exists a sequence $z_n \in \mathbb{R}, z_n \to \pm\infty$, such that $F(z_n) = 0$, which completes the argument, or for any $N$ finitely many zeros of $F$ in $[-N, N] \times [-\varepsilon(N), \varepsilon(N)]$ are not real. In this case it is easy to find an open neighborhood of $\mathbb{R}$ which contains no 0 of $F$. □

**Theorem 5.15.** *Assume that $F \in H(U \setminus K)$ and that there exists a sequence $(z_n) \subset U \setminus \mathbb{R}$, $z_n \to z \in K$, such that $F(z_n) = 0$ and $F(z) \neq 0$ in $\mathbb{R} \setminus K$. Then*

$$\dim \ker T_F = \infty.$$

*In particular, $T_F : \mathcal{A}(\mathbb{R}) \to \mathcal{A}(\mathbb{R})$ is not a Fredholm operator.*

*Proof.* Assume that $F$ is not identically 0. Otherwise the theorem is proved. We will show that there exist infinitely many linearly independent functions $f_n \in \mathcal{A}(\mathbb{R})$ such that $T_F f_n = 0$. Strictly speaking, we show that for any $\nu \in \mathbb{N}$ there are linearly independent functions $f_1, \ldots, f_\nu \in \mathcal{A}(\mathbb{R})$ which belong to the kernel of $T_F$.

Let $\gamma_{\text{outer}}$ be a closed curve in $U \setminus K$ such that $z_n \in \operatorname{int} \hat{\gamma}_{\text{outer}}, n \in \mathbb{N}$. For simplicity we assume that $z_n$ are simple zeros of $F$. Let $\gamma_n$ be a closed curve in $U \setminus K$ such that $z_1, \ldots, z_n \notin \hat{\gamma}_n$ and $z_{n+1}, \ldots \in \operatorname{int} \hat{\gamma}_n$. Also let $\gamma_{\text{join}}$ be any path which joins $\gamma_{\text{outer}}$ with $\gamma_n$ and which does not contain any 0 of $F$.

The number of zeros inside $\Gamma := \gamma_{\text{outer}} \cup \gamma_{\text{join}} \cup (-\gamma_{\text{join}}) \cup (-\gamma_n)$ is at least $n$. Hence,

$$\begin{aligned} n &\leq \frac{1}{2\pi i} \int_\Gamma \frac{F'(z)}{F(z)}\, dz \\ &= \frac{1}{2\pi i} \int_{\gamma_{\text{outer}}} \frac{F'(z)}{F(z)}\, dz - \frac{1}{2\pi i} \int_{\gamma_n} \frac{F'(z)}{F(z)}\, dz. \end{aligned}$$

We have therefore that

$$\frac{1}{2\pi i} \int_{\gamma_n} \frac{F'(z)}{F(z)}\, dz \to -\infty.$$

Also, by Theorem 5.4

$$\begin{aligned} \operatorname{index}\left(T_{F,\gamma_n} : H^2(\gamma_n) \to H^2(\gamma_n)\right) &= -\operatorname{Ind}_{F \circ \gamma_n}(0) \\ &= -\frac{1}{2\pi i} \int_{\gamma_n} \frac{F'(z)}{F(z)}\, dz. \end{aligned}$$

This means that

$$\operatorname{index}\left(T_{F,\gamma_n} : H^2(\gamma_n) \to H^2(\gamma_n)\right) \to \infty.$$

This, however, implies that

$$\dim \ker \left(T_{F,\gamma_n} : H^2(\gamma_n) \to H^2(\gamma_n)\right) \to \infty. \tag{5.10}$$

Observe that when not all $z_n$ are simple, then the conclusion is the same: there exist closed curves $\gamma_n$ in $U \setminus K$ such that (5.10) holds true.

For any $\nu \in \mathbb{N}$ we find therefore $n \in \mathbb{N}$ such that

$$\dim \ker T_{F,\gamma_n} \geq \nu.$$

We claim that a function $f \in \ker T_{F,\gamma_n}$ extends to a function in $H(\tilde{U})$ for some open $\tilde{U} \supset \mathbb{R}$. This follows from the method of the proof of Lemma 5.8. Indeed, we again take a smaller closed curve $\gamma_{n0}$ such that $z_{n+1}, \dots$ lie inside $\gamma_{n0}$. Then $0 = \mathcal{C}_{\gamma_{n0}}(Ff)(z)$ for $z \in \operatorname{int} \hat{\gamma}_{n0}$ and, as a result,

$$f(z) = \frac{1}{F(z)} \frac{1}{2\pi i} \int\int_{\operatorname{int} \hat{\gamma}_{n0}} \frac{\Psi(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta} := h(z)$$

for $z \in \gamma_{n0}$ and appropriate function $\Psi$, the existence of which is guaranteed by Theorem 2.11. The function $h$ is the holomorphic extension of $f$ on some neighborhood of $\mathbb{R}$, since there is no 0 of $F$ in $\mathbb{R} \setminus K$.

This means that we have $\nu$ linearly independent functions $f_1, \dots, f_\nu \in \mathcal{A}(\mathbb{R})$ which belong to the kernel of $T_F$ by Cauchy's theorem and the limiting argument in $H^2(\gamma_n)$. □

The well-known theorem of Coburn on Toeplitz operators $T$ on $H^2$ says that either $\ker T = \{0\}$ or $\ker T^* = \{0\}$. Consequently, either $T$ is injective or it has dense image. We expect a similar result for Toeplitz operators on the space of real analytic functions. The next theorem should be viewed in this light.

**Theorem 5.16.** *Assume that $F \in H(U \setminus K)$. If there exists a sequence $(z_n) \subset \mathbb{R}$, $z_n \to \infty$, such that $F(z_n) = 0$, then $T_F$ is injective.*

*Proof.* Assume that $f \in \mathcal{A}(\mathbb{R})$ belongs to $\ker T_F$. This means that there exists an open set $V \supset \mathbb{R}$, as usually we assume that $U \cap V$ is connected and simply connected, such that $f \in H(V)$ and for any closed curve $\gamma \subset U \cap V \setminus K$ it holds that

$$T_{F,\gamma} f = 0.$$

As before, $Ff \in C^\infty(\gamma)$ and $T_{F,\gamma} f = \mathcal{C}_\gamma(Ff) \in A^\infty(\hat{\gamma})$. This means that for $z \in \gamma$ we have $T_{F,\gamma} f(z) = 0$. Thus for $z \in \gamma$,

$$0 = Ff(z) - \int\int_{\operatorname{int} \hat{\gamma}} \frac{\Psi(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta}$$

for some function $\Psi \in C^\infty(\hat{\gamma})$ which vanishes to order $M \in \mathbb{N}$ on $\gamma$ by Theorem 2.11. This means that

$$Ff(z) = \int\int_{\operatorname{int} \hat{\gamma}} \frac{\Psi(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta}$$

for $z \in \gamma$. The function

$$H(z) := \int\int_{\operatorname{int} \hat{\gamma}} \frac{\Psi(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta}$$

belongs to $H_0(\mathbb{C}_\infty \setminus \hat{\gamma})$. For $z$ outside some ball we have $H(z) \neq 0$ since otherwise $H \equiv 0$. We may assume therefore that $H(z_n) \neq 0$. This means that $Ff(z)$ extends holomorphically on the set $\mathbb{C}_\infty \setminus \hat{\gamma}$ and $Ff(z_n) \neq 0$. We have $F(z_n) = 0$ therefore $Ff(z_n) = 0$ since $f$ is holomorphic on $V$. This is a contradiction. Thus it must be that $H \equiv 0$ and $Ff \equiv 0$, and hence that $f = 0$. □

**Theorem 5.17.** *If there exist a sequence $(z_n) \subset \mathbb{R}$ such that $z_n \to \infty$ and $F(z_n) = 0$, then $T_F : \mathcal{A}(\mathbb{R}) \to \mathcal{A}(\mathbb{R})$ is not a Fredholm operator.*

*Proof.* We will prove that the cokernel of $T_F$ is of infinite dimension. Assume that $H \in \operatorname{im} T_F$. Then there exists a function $f \in \mathcal{A}(\mathbb{R})$ such that $H = T_F f$. We may assume that both $H$ and $f$ are defined on an open set $V \supset \mathbb{R}$, which is connected, simply connected and contained in $U$. Let $\gamma \subset V \setminus K$ be a closed curve. Then for $z \in \operatorname{int} \hat{\gamma}$,

$$H(z) = T_{F,\gamma} f(z) = \mathcal{C}_\gamma(Ff)(z).$$

Both $H$ and $\mathcal{C}_\gamma(Ff)(z)$ belong to $C^\infty(\hat{\gamma})$. Thus

$$H(z) = \mathcal{C}_\gamma(Ff)(z)$$

also for $z \in \gamma$. According to Theorem 2.11 there exists a function $\Psi \in C^M(\hat{\gamma})$ which vanishes to order $M$ on $\gamma$ such that

$$\mathcal{C}_\gamma(Ff)(z) = Ff(z) - \frac{1}{2\pi i} \int\int_{\operatorname{int} \hat{\gamma}} \frac{\Psi(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta}$$

for $z \in \gamma$. Hence for $z \in \gamma$,

$$H(z) = Ff(z) - \frac{1}{2\pi i} \int\int_{\operatorname{int} \hat{\gamma}} \frac{\Psi(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta} := G(z).$$

Both $H$ and $G$ are holomorphic in $V \setminus \hat{\gamma}$ and are $C^1(V)$. Since they are equal on $\gamma$, the argument used in the proof of Lemma 5.8 shows that $H(z) = G(z)$ for $z \in V \setminus \hat{\gamma}$. Since $F(z_n) = 0$ we have

$$H(z_n) = -\frac{1}{2\pi i} \int\int_{\operatorname{int} \hat{\gamma}} \frac{\Psi(\zeta)}{\zeta - z_n}\, d\zeta \wedge d\bar{\zeta}.$$

Also,

$$\lim_{n\to\infty} \frac{1}{2\pi i} \int\int_{\operatorname{int} \hat{\gamma}} \frac{\Psi(\zeta)}{\zeta - z_n}\, d\zeta \wedge d\bar{\zeta} = 0.$$

Hence, for any $H \in \operatorname{im} T_F$, we have

$$\lim_{n\to\infty} H(z_n) = 0.$$

We will show that classes of functions $1, z, z^2, \ldots, z^N$ are linearly independent in $\mathcal{A}(\mathbb{R}) / \operatorname{im} T_F$. Assume that

$$\alpha_0 + \alpha_1 z + \cdots + \alpha_N z^N = T_F f$$

for some $f \in \mathcal{A}(\mathbb{R})$. Then

$$\lim_{n\to\infty} \alpha_0 + \alpha_1 z_n + \cdots + \alpha_N z_n^N = 0.$$

This is only possible if $\alpha_0 = \alpha_1 = \cdots = \alpha_N = 0$. This implies that $\mathcal{A}(\mathbb{R}) / \operatorname{im} T_F$ cannot be of finite dimension. □

Theorem 5.7, Theorem 5.11, Theorem 5.15, and Theorem 5.17 in view of Proposition 5.14 complete the proof of Theorem 2. Indeed, observe that we have also shown that

$$\operatorname{index} T_F = -\operatorname{winding} F.$$

It follows from Theorem 5.7 and Theorem 5.11 that

$$\begin{aligned}\operatorname{index} T_F &= \dim\ker T_F - \dim\operatorname{coker} T_F\\ &= \dim\ker T_{F,\gamma} - \dim\operatorname{coker} T_{F,\gamma} = \operatorname{index} T_{F,\gamma}\end{aligned}$$

for any closed curve $\gamma$ in $\tilde{U}\setminus K$—the set $\tilde{U}$ is the set from Theorem 5.10. We have also proved in Theorem 5.4 that

$$\operatorname{index} T_{F,\gamma} = -\operatorname{Ind}_{F\circ\gamma}(0) = -\operatorname{winding} F.$$

This completes the proof of Theorem 2.

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## References

1. S. Axler and Ž. Čučković, *Commuting Toeplitz operators with harmonic symbols*, Integral Equations Operator Theory **14** (1991), no. 1, 1–11. Zbl 0733.47027. MR1079815. DOI 10.1007/BF01194925. 32
2. W. Bauer, *Berezin–Toeplitz quantization and composition formulas*, J. Funct. Anal. **256** (2009), no. 10, 3107–3142. Zbl 1169.47017. MR2504520. DOI 10.1016/j.jfa.2008.10.002.
3. W. Bauer, B. R. Choe, and H. Koo, *Commuting Toeplitz operators with pluriharmonic symbols on the Fock space*, J. Funct. Anal. **268** (2015), no. 10, 3017–3060. Zbl 1327.47023. MR3331792. DOI 10.1016/j.jfa.2015.03.003. 32
4. W. Bauer and H. Issa, *Commuting Toeplitz operators with quasi-homogeneous symbols on the Segal–Bargmann space*, J. Math. Anal. Appl. **386** (2012), no. 1, 213–235. Zbl 1228.47030. MR2834879. DOI 10.1016/j.jmaa.2011.07.058.
5. W. Bauer and T. Le, *Algebraic properties and the finite rank problem for Toeplitz operators on the Segal–Bargmann space*, J. Funct. Anal. **261** (2011), no. 9, 2617–2640. Zbl 1339.47035. MR2826408. DOI 10.1016/j.jfa.2011.07.006.
6. W. Bauer and Y. J. Lee, *Commuting Toeplitz operators on the Segal–Bargmann space*, J. Funct. Anal. **260** (2011), no. 2, 460–489. Zbl 1307.47026. MR2737409. DOI 10.1016/j.jfa.2010.09.007. 32
7. S. R. Bell, *The Cauchy Transform, Potential Theory, and Conformal Mapping*, 2nd ed., CRC Press, Boca Raton, FL, 2016. Zbl 1359.30001. MR1228442. 39, 40, 41, 51
8. C. A. Berenstein and R. Gay, *Complex Analysis and Special Topics in Harmonic Analysis*, Springer, Berlin, 1995. Zbl 0837.30001. MR1344448. DOI 10.1007/978-1-4613-8445-8. 39
9. B. Booss and D. D. Bleecker, *Topology and Analysis: The Atiyah–Singer Index Formula and Gauge-Theoretic Physics*, Universitext, Springer, New York, 1985. Zbl 0551.58031. MR0771117. DOI 10.1007/978-1-4684-0627-6. 34
10. A. Böttcher and B. Silbermann, *Analysis of Toeplitz Operators*, Springer, Berlin, 1990. Zbl 0732.47029. MR1071374. DOI 10.1007/978-3-662-02652-6. 32
11. A. Brown and P. R. Halmos, *Algebraic properties of Toeplitz operators*, J. Reine Angew. Math. **213** (1963), 89–102. Zbl 0116.32501. MR0160136. DOI 10.1007/978-1-4613-8208-9_19. 32, 49
12. P. Domański, "Notes on real analytic functions and classical operators" in *Topics in Complex Analysis and Operator Theory (Valencia, 2010)*, Amer. Math. Soc., Providence, 2012, 3–47. MR2905538. DOI 10.1090/conm/561/11108. 36, 38, 39

13. P. Domański and M. Langenbruch, *Algebra of multipliers on the space of real analytic functions of one variable*, Studia Math. **212** (2012), no. 2, 155–171. Zbl 1268.46021. MR3008439. DOI 10.4064/sm212-2-4. 36
14. P. Domański and M. Langenbruch, *Representation of multipliers of real analytic functions*, Analysis **31** (2012), 1001–1026. Zbl 1267.46036. MR3043719. DOI 10.1524/anly.2012.1150. 36
15. P. Domański and M. Langenbruch, *Hadamard multipliers on spaces of real analytic functions*, Adv. Math. **240** (2013), 575–612. Zbl 1288.46019. MR3046319. DOI 10.1016/j.aim.2013.01.015. 36
16. P. Domański and M. Langenbruch, *Interpolation of holomorphic functions and surjectivity of Taylor coefficient multiplieres*, Adv. Math. **293** (2016), 782–855. Zbl 1336.35115. MR3474335. DOI 10.1016/j.aim.2016.02.028. 36
17. P. Domański, M. Langenbruch, and D. Vogt, *Hadamard type operators on spaces of real analytic functions in several variables*, J. Funct. Anal. **269** (2015), no. 12, 3868–3913. Zbl 1341.46016. MR3418073. DOI 10.1016/j.jfa.2015.09.011. 36
18. P. Domański and D. Vogt, *The space of real analytic functions has no basis*, Studia Math. **142** (2000), no. 2, 187–200. Zbl 0990.46015. MR1792604. 32, 39
19. I. C. Gohberg and M. G. Kreĭn, *The basic propositions on defect numbers, root numbers, and indices of linear operators*, Amer. Math. Soc. Transl. (2) **13** (1960), 185–264. Zbl 0089.32201. MR0113146. 34
20. L. Hörmander, *An Introduction to Complex Analysis in Several Variables*, 3rd. ed., North Holland, Amsterdam 1990. Zbl 0685.32001. MR1045639. 61
21. S. G. Krantz and H. R. Parks, *A Primer of Real Analytic Functions*, 2nd ed., Birkhäuser, Boston, 2002. Zbl 1015.26030. MR1916029. DOI 10.1007/978-0-8176-8134-0. 36
22. T. Le, *The commutants of certain Toeplitz operators on weighted Bergman spaces*, J. Math. Anal. Appl. **348** (2008), no. 1, 1–11. Zbl 1168.47025. MR2449321. DOI 10.1016/j.jmaa.2008.07.005. 32
23. A. Martineau, *Sur la topologie des espaces de fonctions holomorphes*, Math. Ann. **163** (1966), 62–88. Zbl 0138.38101. MR0190697. DOI 10.1007/BF02052485. 38
24. R. Meise and D. Vogt, *Introduction to Functional Analysis*, Oxf. Grad. Texts Math. **2**, Clarendon Press, Oxford, 1997. Zbl 0924.46002. MR1483073. 37, 39, 44, 50
25. I. Mitrea, M. Mitrea, and M. Taylor, *Cauchy integrals, Calderón projectors, Toeplitz operators on uniformly rectifiable domains*, Adv. Math. **208** (2015), 666–757. Zbl 1305.31003. MR3276607. DOI 10.1016/j.aim.2014.09.020. 53
26. G. Rozenblum and N. Vasilevski, *Toeplitz operators defined by sesquilinear forms: Fock space case*, J. Funct. Anal. **267** (2014), no. 11, 4399–4430. Zbl 1342.47044. MR3269881. DOI 10.1016/j.jfa.2014.10.001. 32
27. D. Vogt, *Operators of Hadamard type on spaces of smooth functions*, Math. Nach. **288** (2015), no. 2–3, 353–361. Zbl 1314.47119. MR3310518. DOI 10.1002/mana.201300269. 36
28. D. Zheng, *Semi-commutators of Toeplitz operators on the Bergman space*, Integral Equations Operator Theory **25** (1996), no. 3, 347–372. Zbl 0864.47015. MR1395711. DOI 10.1007/BF01262299. 32
29. D. Zheng, *Commuting Toeplitz operators with pluriharmonic symbols*, Trans. Amer. Math. Soc. **350** (1998), no. 4, 1595–1618. Zbl 0893.47015. MR1443898. DOI 10.1090/S0002-9947-98-02051-0. 32
30. K. Zhu, *Analysis on Fock Spaces*, Grad. Texts in Math. **263**, Springer, New York 2012. Zbl 1262.30003. MR2934601. DOI 10.1007/978-1-4419-8801-0. 32



Faculty of Mathematics and Computer Science, Adam Mickiewicz University, ul. Umultowska 87, 61-614 Poznań, Poland.

*E-mail address*: mjk@amu.edu.pl