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# TENT SPACES AT ENDPOINTS

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Abstract. In 1985, Coifman, Meyer, and Stein gave the duality of the tent spaces; that is, $(T_q^p(\mathbb{R}_+^{n+1}))^* = T_{q'}^{p'}(\mathbb{R}_+^{n+1})$ for $1 < p, q < \infty$, and $(T_\infty^1(\mathbb{R}_+^{n+1}))^* = \mathscr{C}(\mathbb{R}_+^{n+1})$, $(T_q^1(\mathbb{R}_+^{n+1}))^* = T_{q'}^{\infty}(\mathbb{R}_+^{n+1})$ for $1 < q < \infty$, where $\mathscr{C}(\mathbb{R}_+^{n+1})$ denotes the Carleson measure space on $\mathbb{R}_+^{n+1}$. We prove that $(\mathscr{C}_v(\mathbb{R}_+^{n+1}))^* = T_\infty^1(\mathbb{R}_+^{n+1})$, which we introduced recently, where $\mathscr{C}_v(\mathbb{R}_+^{n+1})$ is the vanishing Carleson measure space on $\mathbb{R}_+^{n+1}$. We also give the characterizations of $T_q^\infty(\mathbb{R}_+^{n+1})$ by the boundedness of the Poisson integral. As application, we give the boundedness and compactness on $L^q(\mathbb{R}^n)$ of the paraproduct $\pi_F$ associated with the tent space $T_q^\infty(\mathbb{R}_+^{n+1})$, and we extend partially an interesting result given by Coifman, Meyer, and Stein, which establishes a close connection between the tent spaces $T_2^p(\mathbb{R}_+^{n+1})$ $(1 \le p \le \infty)$ and $L^p(\mathbb{R}^n)$, $H^p(\mathbb{R}^n)$ and $BMO(\mathbb{R}^n)$ spaces.

## 1. Introduction

In 1985, Coifman, Meyer, and Stein [9] developed a theory of tent spaces. Because tent spaces are closely related to many important concepts in harmonic analysis, such as the Carleson measure, square operators, nontangential maximal function, Hardy space, and $BMO$ space, they have many interesting applications in harmonic analysis and PDE (see, for example, [2], [8], [20], [23]). Recently, P. Auscher et al. considered the boundedness of some operators in tent spaces (see [3]–[5]).

Let us recall the definition of tent spaces given in [9].

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**Definition 1.1** (Tent space)**.** For $0 < q \le \infty$ and a measurable function $f$ on $\mathbb{R}^{n+1}_+$, let

$$A_q(f)(x) = \begin{cases} (\int_{\Gamma(x)} |f(y,t)|^q \frac{dy\,dt}{t^{n+1}})^{\frac{1}{q}}, & 0 < q < \infty, \\ \sup_{(y,t)\in\Gamma(x)} |f(y,t)|, & q = \infty, \end{cases}$$

where in the following, $\Gamma_\alpha(x) = \{(y,t) \in \mathbb{R}^{n+1}_+ : |y-x| < \alpha t\}$, and denote $\Gamma(x) = \Gamma_1(x)$ briefly. For $0 < p < \infty$ and $0 < q < \infty$, the tent space $T^p_q(\mathbb{R}^{n+1}_+)$ is defined by

$$T^p_q(\mathbb{R}^{n+1}_+) = \big\{f : \|f\|_{T^p_q} = \big\|A_q(f)\big\|_{L^p} < \infty\big\}.$$

For $q = \infty$ and $0 < p < \infty$, the tent space $T^p_\infty(\mathbb{R}^{n+1}_+)$ is defined by

$$T^p_\infty(\mathbb{R}^{n+1}_+) = \big\{f \in \mathbf{C}(\mathbb{R}^{n+1}_+) : \|f\|_{T^p_\infty} = \big\|A_\infty(f)\big\|_{L^p} < \infty \text{ and } \lim_{\epsilon\to 0} \|f - f_\epsilon\|_{T^p_\infty} = 0\big\},$$

where $\mathbf{C}(\mathbb{R}^{n+1}_+)$ denotes all continuous functions on $\mathbb{R}^{n+1}_+$, and $f_\epsilon(x,t) = f(x,t+\epsilon)$. For $p = \infty$ and $1 < q < \infty$, the tent space $T^\infty_q(\mathbb{R}^{n+1}_+)$ is defined by

$$T^\infty_q(\mathbb{R}^{n+1}_+) = \big\{f : \|f\|_{T^\infty_q} = \sup_{a>0} M_a(f,q) < \infty\big\}$$

with

$$M(f,Q,q) = \Big(\frac{1}{|Q|}\int_{\hat{Q}} \big|f(y,t)\big|^q \frac{dy\,dt}{t}\Big)^{\frac{1}{q}} \qquad \text{and} \qquad M_a(f,q) = \sup_{|Q|=a} M(f,Q,q),$$

where $a > 0$, and $\hat{Q} = \{(x,t) \in \mathbb{R}^{n+1}_+ : x \in Q, 0 < t < \ell(Q)\}$ is the tent over $Q$.

It was shown in [9] that $T^p_\infty(\mathbb{R}^{n+1}_+)$ consists of exactly those $f$ which are continuous in $\mathbb{R}^{n+1}_+$, such that $A_\infty(f) \in L^p(\mathbb{R}^n)$, and for which $f(x,t)$ has nontangential limits at the boundary almost everywhere. Coifman, Meyer, and Stein also observe that all tent spaces $T^p_q(\mathbb{R}^{n+1}_+)$ $(1 \le p, q \le \infty)$ are Banach spaces with the norm $\|\cdot\|_{T^p_q}$. Note that the tent spaces are closely related to the Carleson measure; thus we need to give the definition of the Carleson measure.

**Definition 1.2** (Carleson measure)**.** Suppose that $\mu$ is a positive measure on $\mathbb{R}^{n+1}_+$. For any cube $Q \subset \mathbb{R}^n$ and $a > 0$, let

$$\mathscr{N}(\mu,Q) = \frac{\mu(\hat{Q})}{|Q|}, \qquad \text{and let} \qquad \mathscr{N}_a(\mu) = \sup_{|Q|=a} \mathscr{N}(\mu,Q).$$

A positive measure $\mu$ on $\mathbb{R}^{n+1}_+$ is called a *Carleson measure* written by $\mu \in \mathscr{C}(\mathbb{R}^{n+1}_+)$ if there exists a constant $C > 0$ such that

$$\|\mu\|_{\mathscr{C}} := \sup_{a>0} \mathscr{N}_a(\mu) \le C,$$

where $\|\mu\|_{\mathscr{C}}$ is called the Carleson constant of $\mu$. It is well known that $\|\cdot\|_{\mathscr{C}}$ is a norm and that $\mathscr{C}(\mathbb{R}^{n+1}_+)$ is a Banach space in the norm $\|\cdot\|_{\mathscr{C}}$.

Coifman, Meyer, and Stein gave the duality between the tent spaces and Carleson measure.

**Theorem A** (See [9])**.** *The tent spaces have the following duality:*

(1) *for* $1 < q, p < \infty$, $(T_q^p)^* = T_{q'}^{p'}$;
(2) *for* $1 < q < \infty$, $(T_q^1)^* = T_{q'}^{\infty}$; *and*
(3) $(T_\infty^1)^* = \mathscr{C}(\mathbb{R}_+^{n+1})$.

The conclusion (1) in Theorem A shows that the tent spaces $T_q^p(\mathbb{R}_+^{n+1})$ are self-adjoint if $1 < q$, $p < \infty$. The conclusions (2) and (3) can be seen as the endpoint cases of the conclusion (1). Obviously, $T_q^1(\mathbb{R}_+^{n+1})$ $(1 < q \le \infty)$ are not self-adjoint.

In 1985, Wang [29] considered predual spaces of the tent spaces $T_\infty^1(\mathbb{R}_+^{n+1})$. He introduced the subclass of $\mathscr{C}(\mathbb{R}_+^{n+1})$

$$VCM(\mathbb{R}_+^{n+1}) = \big\{\mu \in \mathscr{C}(\mathbb{R}_+^{n+1}) : \lim_{a\to 0} \mathscr{N}_a(\mu) = 0\big\},$$

and, using the method in [10], he argued (without proof) that the dual of $VCM(\mathbb{R}_+^{n+1})$ is the tent space $T_\infty^1(\mathbb{R}_+^{n+1})$. The examples we give in Remark 2.2 below show, however, that it is impossible for the space $VCM(\mathbb{R}_+^{n+1})$ to become a predual of the tent space $T_\infty^1(\mathbb{R}_+^{n+1})$.

In this paper, we introduce a subclass $\mathscr{C}_c(\mathbb{R}_+^{n+1})$ of the Carleson measure space $\mathscr{C}(\mathbb{R}_+^{n+1})$ (see Section 2.1 for its definition). Then we show that

$$\overline{\mathscr{C}_c(\mathbb{R}_+^{n+1})}^{\|\cdot\|_{\mathscr{C}}} = \mathscr{C}_v(\mathbb{R}_+^{n+1}),$$

where $\mathscr{C}_v(\mathbb{R}_+^{n+1})$ is the vanishing Carleson measure space, which was introduced in [14] (see Definition 2.1 below). Then we prove that $(\mathscr{C}_v(\mathbb{R}_+^{n+1}))^* = T_\infty^1(\mathbb{R}_+^{n+1})$ (see Theorem 2.1). An important fact is that, by their definitions, $\mathscr{C}_v(\mathbb{R}_+^{n+1}) \subsetneqq VCM(\mathbb{R}_+^{n+1})$ (see also [12]).

Another aim of this paper is to give a characterization of the tent space $T_q^\infty(\mathbb{R}_+^{n+1})$ $(1 < q < \infty)$ and its subspace $T_{q,v}^\infty(\mathbb{R}_+^{n+1})$ (see (2.14) for the definition) by Poisson integral. By Carleson's famous works ([6], [7]), it is well known that the Carleson measure space $\mathscr{C}(\mathbb{R}_+^{n+1})$ can be characterized by the boundedness of the Poisson integral from $L^p(\mathbb{R}^n; dx)$ to $L^p(\mathbb{R}_+^{n+1}; d\mu)$. Recently, in [14] we proved that the vanishing Carleson measure space $\mathscr{C}_v(\mathbb{R}_+^{n+1})$ can be characterized by the compactness of the Poisson integral from $L^p(\mathbb{R}^n; dx)$ to $L^p(\mathbb{R}_+^{n+1}; d\mu)$. In Section 3, we show simply that $T_q^\infty(\mathbb{R}_+^{n+1})$ $(1 < q < \infty)$ and that $T_{q,v}^\infty(\mathbb{R}_+^{n+1})$ can also be characterized by Poisson integral, which is simply a direct application of the characterizations of the Carleson measure and the vanishing Carleson measure mentioned above.

In the last section, we give some applications of tent spaces $T_q^\infty(\mathbb{R}_+^{n+1})$ and $T_{q,v}^\infty(\mathbb{R}_+^{n+1})$. To be precise, we define a paraproduct $\pi_F$ associated with $F \in T_q^\infty(\mathbb{R}_+^{n+1})$, and we establish the boundedness and compactness of $\pi_F$ on $L^q(\mathbb{R}^n)$ for $1 < q \le 2$. Note that in [9], the authors established a connection between tent spaces $T_2^p(\mathbb{R}_+^{n+1})$ $(1 \le p \le \infty)$ and $L^p(\mathbb{R}^n)$, $H^p(\mathbb{R}^n)$ and $BMO(\mathbb{R}^n)$ spaces. In this section, we discuss the connection between $T_q^p(\mathbb{R}_+^{n+1})$ $(1 < q \le 2, 1 \le p \le \infty)$ and $L^p(\mathbb{R}^n)$, $H^p(\mathbb{R}^n)$, and we thereby partially extend the interesting result in [9] mentioned above.

In this paper, $C$ will denote a positive constant that may change its value on each statement without special instruction.

## 2. Predual of tent space $T^1_\infty(\mathbb{R}^{n+1}_+)$

In our recent paper [14], we introduced a subclass of $\mathscr{C}(\mathbb{R}^{n+1}_+)$, the vanishing Carleson measure space $\mathscr{C}_v(\mathbb{R}^{n+1}_+)$, and we gave a characterization of $\mathscr{C}_v(\mathbb{R}^{n+1}_+)$ by the compactness of Poisson integral (see [14, Corollary 2.2]).

**Definition 2.1** (vanishing Carleson measure)**.** A positive measure $\mu$ on $\mathbb{R}^{n+1}_+$ is said to be a *vanishing Carleson measure* if $\mu \in \mathscr{C}(\mathbb{R}^{n+1}_+)$, and it satisfies

(1) $\lim_{a\to 0} \mathscr{N}_a(\mu) = 0$;
(2) $\lim_{a\to\infty} \mathscr{N}_a(\mu) = 0$; and
(3) $\lim_{|x|\to\infty} \mathscr{N}(\mu, Q + x) = 0$, for any cube $Q \subset \mathbb{R}^n$.

The set of all vanishing Carleson measures on $\mathbb{R}^{n+1}_+$ is denoted by $\mathscr{C}_v(\mathbb{R}^{n+1}_+)$.

In this section we will prove that the vanishing Carleson measure space $\mathscr{C}_v(\mathbb{R}^{n+1}_+)$ is just the predual of the tent space $T^1_\infty$.

**Theorem 2.1.** *The dual space of $\mathscr{C}_v(\mathbb{R}^{n+1}_+)$ is the tent space $T^1_\infty(\mathbb{R}^{n+1}_+)$, that is, $(\mathscr{C}_v)^* = T^1_\infty$. More precisely, the pairing $\langle f, d\mu\rangle = \int_{\mathbb{R}^{n+1}_+} f(x,t)\,d\mu(x,t)$ realizes the duality of $\mathscr{C}_v(\mathbb{R}^{n+1}_+)$ with $T^1_\infty(\mathbb{R}^{n+1}_+)$.*

**Remark 2.2.** Here we give three examples to show that each condition in Definition 2.1 is not removable for the desired duality result in Theorem 2.1.

Suppose that $k \in \mathbb{Z}^+$ and that $E_k = I_k \times J_k$, where $I_k$ and $J_k$ are some intervals in $\mathbb{R}$ and $(0,\infty)$, respectively. Choose $\varphi_k \in C^\infty_c(\mathbb{R}^2_+)$ supported in $E_k$ with $0 \le \varphi_k \le 1$, which has nontangential limits at the boundary almost everywhere.

(i) Set $E_k = [0, \frac{1}{k}] \times (0, \frac{1}{k})$, and set $\{f^1_k\}_{k\in\mathbb{Z}^+} = \{\frac{1}{3}k\varphi_k\}_{k\in\mathbb{Z}^+}$. Note that

$$\int_{\mathbb{R}} \sup_{|x-y|<t} \big|f^1_k(y,t)\big|\,dx \le \frac{k}{3}\int_{-\frac{1}{k}}^{\frac{2}{k}} \sup_{|x-y|<t} \big|\varphi_k(y,t)\big|\,dx \le 1;$$

hence $\{f^1_k\}_{k\in\mathbb{Z}^+} \subset B_1(T^1_\infty(\mathbb{R}^2_+))$, the closed unit ball of the dual space $T^1_\infty(\mathbb{R}^2_+)$ of $\mathscr{C}_v(\mathbb{R}^2_+)$. Since the closed unit ball of a dual space is weak* compact by the Banach–Alaoglu theorem (see [22]), then there exists a weak* convergent subsequence of $\{f^1_k\}_{k\in\mathbb{Z}^+}$, which is denoted still by $\{f^1_k\}$. In fact, $\{f^1_k\}_{k\in\mathbb{Z}^+}$ converges to zero in weak*-topology. It is easy to see that, for any $\mu \in \mathscr{C}_v(\mathbb{R}^2_+)$,

$$\begin{aligned}\Big|\int_{\mathbb{R}^2_+} f^1_k(x,t)\,d\mu(x,t)\Big| &\le \frac{k}{3}\mu\Big(\Big[0,\frac{1}{k}\Big]\times\Big(0,\frac{1}{k}\Big)\Big)\\ &= \frac{\mu(\widehat{[0,\frac{1}{k}]})}{3|[0,\frac{1}{k}]|} \to 0 \quad \text{as } k\to\infty, \end{aligned} \tag{2.1}$$

since $\mu$ satisfies the condition (i) in Definition 2.1. Obviously, if $\mu \in \mathscr{C}(\mathbb{R}^2_+)$, but $\mu$ does not satisfy the condition (i), then the convergence in (2.1) does not hold for any subsequence of $\{f^1_k\}_{k\in\mathbb{Z}^+}$.

(ii) Set $E_k = [0,k] \times (0,k)$, and set $\{f_k^2\}_{k\in\mathbb{Z}^+} = \{\frac{1}{3k}\varphi_k\}_{k\in\mathbb{Z}^+}$. Noting that

$$\int_{\mathbb{R}} \sup_{|x-y|<t} \left|f_k^2(t,y)\right| dx \le \frac{1}{3k}\int_{-k}^{2k} \sup_{|x-y|<t} \left|\varphi_k(y,t)\right| dx \le 1,$$

we see that $\{f_k^2\}_{k\in\mathbb{Z}^+} \subset B_1(T^1_\infty(\mathbb{R}^2_+))$ also. Thus for any $\mu \in \mathscr{C}_v(\mathbb{R}^2_+)$,

$$\left|\int_{\mathbb{R}^2_+} f_k^2(x,t)\, d\mu(x,t)\right| \le \frac{1}{3k}\mu\big([0,k]\times(0,k)\big) = \frac{\mu(\widehat{[0,k]})}{3|[0,k]|} \to 0 \quad \text{as } k\to\infty, \quad (2.2)$$

since $\mu$ satisfies the condition (ii) in Definition 2.1. If, however, $\mu \in \mathscr{C}(\mathbb{R}^2_+)$, but $\mu$ does not satisfy the condition (ii), then the convergence in (2.2) does not hold for any subsequence of $\{f_k^2\}_{k\in\mathbb{Z}^+}$.

(iii) Set $E_k = [k-1,k+1] \times (0,2)$, and set $\{f_k^3\}_{k\in\mathbb{Z}^+} = \{\frac{1}{6}\varphi_k\}_{k\in\mathbb{Z}^+}$. Then

$$\int_{\mathbb{R}} \sup_{|x-y|<t} \left|f_k^3(y,t)\right| dx \le \frac{1}{6}\int_{k-3}^{k+3} \sup_{|x-y|<t} \left|\varphi_k(y,t)\right| dx \le 1,$$

which shows that $\{f_k^3\}_{k\in\mathbb{Z}^+} \subset B_1(T^1_\infty(\mathbb{R}^2_+))$. Note that

$$\left|\int_{\mathbb{R}^2_+} f_k^3(x,t)\, d\mu(x,t)\right| \le \frac{1}{6}\mu\big([k-1,k+1]\times(0,2)\big) = \frac{\mu(\widehat{[k-1,k+1]})}{3|[k-1,k+1]|}. \quad (2.3)$$

Using the similar discussion above, it is easy to see that, for any $\mu \in \mathscr{C}(\mathbb{R}^2_+)$ which does not satisfy the condition (iii), the right-hand side of (2.3) does not go to zero as $k\to\infty$. This leads to a contradiction.

To prove Theorem 2.1, we need some elementary properties of $\mathscr{C}_v(\mathbb{R}^{n+1}_+)$ and $T^1_\infty(\mathbb{R}^{n+1}_+)$, which are given in Sections 2.1 and 2.2, respectively.

2.1. **A dense subset of $\mathscr{C}_v(\mathbb{R}^{n+1}_+)$.** In this section, we introduce a subclass $\mathscr{C}_c(\mathbb{R}^{n+1}_+)$ of $\mathscr{C}(\mathbb{R}^{n+1}_+)$, and we prove that $\mathscr{C}_c(\mathbb{R}^{n+1}_+)$ is a dense subset of $\mathscr{C}_v(\mathbb{R}^{n+1}_+)$ in the norm $\|\cdot\|_{\mathscr{C}}$. We begin with two lemmas; since the first one is obvious, we omit its proof.

**Lemma 2.3.**

(1) $\lim_{a\to 0} \mathscr{N}_a(\mu) = 0 \Longleftrightarrow \lim_{a\to 0} \sup_{|Q|\le a} \mathscr{N}(\mu, Q) = 0$;
(2) $\lim_{a\to \infty} \mathscr{N}_a(\mu) = 0 \Longleftrightarrow \lim_{a\to \infty} \sup_{|Q|\ge a} \mathscr{N}(\mu, Q) = 0$;
(3) $\lim_{|x|\to \infty} \mathscr{N}(\mu, Q+x) = 0 \Longleftrightarrow \lim_{a\to \infty} \sup_{|x|>a} \mathscr{N}(\mu, Q+x) = 0$, *where* $Q$ *is any cube in* $\mathbb{R}^n$.

**Lemma 2.4.** $\mathscr{C}_v(\mathbb{R}^{n+1}_+)$ *is a Banach space equipped with the norm* $\|\cdot\|_{\mathscr{C}}$.

*Proof.* If we suppose that $\{\mu_k\}$ is a Cauchy sequence in $\mathscr{C}_v(\mathbb{R}^{n+1}_+)$, then $\{\mu_k\}$ is also the Cauchy sequence in $\mathscr{C}(\mathbb{R}^{n+1}_+)$ since $\mathscr{C}_v(\mathbb{R}^{n+1}_+) \subset \mathscr{C}(\mathbb{R}^{n+1}_+)$. Applying the completeness of $\mathscr{C}(\mathbb{R}^{n+1}_+)$, there exists a measure $\mu \in \mathscr{C}(\mathbb{R}^{n+1}_+)$ such that $\|\mu - \mu_k\|_{\mathscr{C}} \to 0$ as $k\to\infty$. It remains to show that $\mu \in \mathscr{C}_v(\mathbb{R}^{n+1}_+)$.

For any $a>0$, note that

$$\mathscr{N}_a(\mu) \le \mathscr{N}_a(\mu_k) + \mathscr{N}_a(\mu-\mu_k) \le \mathscr{N}_a(\mu_k) + \|\mu-\mu_k\|_{\mathscr{C}}.$$

Thus for both cases, $s = 0$ and $s = \infty$, we have

$$\lim_{a\to s} \mathscr{N}_a(\mu) \le \|\mu - \mu_k\|_{\mathscr{C}} \to 0 \quad \text{as } k \to \infty.$$

We also notice that for any cube $Q$ in $\mathbb{R}^n$,

$$\mathscr{N}(\mu, Q) \le \mathscr{N}(\mu_k, Q) + \mathscr{N}(\mu - \mu_k, Q) \le \mathscr{N}(\mu_k, Q) + \|\mu - \mu_k\|_{\mathscr{C}}.$$

Hence

$$\lim_{|x|\to\infty} \mathscr{N}(\mu, Q + x) \le \|\mu - \mu_k\|_{\mathscr{C}} \to 0 \quad \text{as } k \to \infty.$$

That is, $\mu \in \mathscr{C}_v$, and $\mathscr{C}_v(\mathbb{R}^{n+1}_+)$ is a Banach space. □

Now we introduce a subclass of $\mathscr{C}(\mathbb{R}^{n+1}_+)$ as follows.

$$\mathscr{C}_c(\mathbb{R}^{n+1}_+) = \big\{\mu \in \mathscr{C} : \text{there exists a compact set } K \text{ in } \mathbb{R}^{n+1}_+ \text{ such that for any } \mu \\ - \text{measurable set } E \text{ in } \mathbb{R}^{n+1}_+, \mu(E) = \mu(E \cap K)\big\}.$$

We claim that $\mathscr{C}_c(\mathbb{R}^{n+1}_+)$ is dense in $\mathscr{C}_v(\mathbb{R}^{n+1}_+)$ in the norm $\|\cdot\|_{\mathscr{C}}$.

**Lemma 2.5.** *$\mathscr{C}_c(\mathbb{R}^{n+1}_+)$ is dense in $\mathscr{C}_v(\mathbb{R}^{n+1}_+)$ in the norm $\|\cdot\|_{\mathscr{C}}(\mathbb{R}^{n+1}_+)$. That is,*

$$\overline{\mathscr{C}_c(\mathbb{R}^{n+1}_+)}^{\|\cdot\|_{\mathscr{C}}} = \mathscr{C}_v(\mathbb{R}^{n+1}_+).$$

*Proof.* We first prove that $\mathscr{C}_c \subset \mathscr{C}_v$. In fact, for any $\mu \in \mathscr{C}_c$, there exists a compact set $K \subset \mathbb{R}^{n+1}_+$ such that for any $\mu$-measurable set $E \subset \mathbb{R}^{n+1}_+$, $\mu(E) = \mu(E \cap K)$. For $a > 0$ and any cube $Q \subset \mathbb{R}^n$ with $|Q| = a$, then we have the following facts:

(i) If $a$ is small enough, then $\hat{Q} \cap K = \emptyset$; thus $\frac{\mu(\hat{Q})}{|Q|} = \frac{\mu(\hat{Q}\cap K)}{|Q|} = 0$.

(ii) If $a$ is large enough, such that $\hat{Q} \cap K \ne \emptyset$, then $\frac{\mu(\hat{Q})}{|Q|} \le a^{-1}\mu(K)$; thus $\mathscr{N}_a(\mu) \le a^{-1}\mu(K)$ and $\lim_{a\to\infty} \mathscr{N}_a(\mu) = 0$.

(iii) If $|x| \to \infty$, then $\widehat{Q + x} \cap K = \emptyset$; thus $\mu(\widehat{Q + x}) = 0$, and $\lim_{|x|\to\infty} \frac{\mu(\widehat{Q+x})}{|Q|} = 0$. Hence $\mathscr{C}_c \subset \mathscr{C}_v$, and $\overline{\mathscr{C}_c}^{\|\cdot\|_{\mathscr{C}}} \subset \mathscr{C}_v$ by Lemma 2.4.

Below we verify $\mathscr{C}_v \subset \overline{\mathscr{C}_c}^{\|\cdot\|_{\mathscr{C}}}$. Let $E_k = \{(y, t) \in \mathbb{R}^{n+1}_+ : |y| \le k, \frac{1}{k} \le t \le k\}$ for $k \in \mathbb{N}$. For $\mu \in \mathscr{C}_v$, denote $\mu_k(E) = \mu(E \cap E_k)$ for any $\mu$-measurable set $E$ in $\mathbb{R}^{n+1}_+$. It is then easy to see that $\mu_k \in \mathscr{C}_c$; thus to finish the proof of Lemma 2.5 it only remains to show that

$$\lim_{k\to\infty} \|\mu - \mu_k\|_{\mathscr{C}} = 0. \tag{2.4}$$

Let

$$F^1_k = \big\{(y, t) \in \mathbb{R}^{n+1}_+ : t > k\big\},$$
$$F^2_k = \Big\{(y, t) \in \mathbb{R}^{n+1}_+ : 0 < t < \frac{1}{k}\Big\},$$
$$F^3_k = \Big\{(y, t) \in \mathbb{R}^{n+1}_+ : |y| > k, \frac{1}{k} \le t \le k\Big\}.$$

Then it is easy to see that $\mathbb{R}^{n+1}_+ = E_k \cup F_k^1 \cup F_k^2 \cup F_k^3$ for any $k \in \mathbb{N}$ and that, for any cube $Q$ in $\mathbb{R}^n$ with center $x_Q$ and sidelength $\ell(Q)$,

$$\frac{(\mu-\mu_k)(\hat{Q})}{|Q|} \le \frac{\mu(\hat{Q}\cap F_k^1)}{|Q|} + \frac{\mu(\hat{Q}\cap F_k^2)}{|Q|} + \frac{\mu(\hat{Q}\cap F_k^3)}{|Q|} =: I_1 + I_2 + I_3.$$

Thus to get (2.4) we need only show that

$$\lim_{k\to\infty} \sup_{Q\subset\mathbb{R}^n} I_i = 0, \quad \text{for } i = 1,2,3. \tag{2.5}$$

*Case* $i = 1$. If $\ell(Q) \le k/2$, then $\hat{Q}\cap F_k^1 = \emptyset$, and we have $\sup_{|Q|\le(\frac{k}{2})^n} I_1 = 0$. If $\ell(Q) > k/2$, then $|Q| \ge (\frac{k}{2})^n \to \infty$ as $k \to \infty$. Thus $\lim_{k\to\infty} \sup_{|Q|\ge(\frac{k}{2})^n} I_1 = 0$ by $\mu \in \mathscr{C}_v$ and Lemma 2.3. Hence (2.5) holds for $i = 1$.

*Case* $i = 2$. If $\ell(Q) \le \frac{2}{k}$, since $\mu \in \mathscr{C}_v$ and applying Lemma 2.3, we have

$$\sup_{\ell(Q)\le\frac{2}{k}} I_2 \le \sup_{\ell(Q)\le\frac{2}{k}} \frac{\mu(\hat{Q})}{|Q|} \to 0 \quad \text{as } k \to \infty. \tag{2.6}$$

If $\ell(Q) > \frac{2}{k}$, applying a Besicovitch covering lemma (see [19, p. 39]), then there exists a sequence of cubes $\{Q_j\}$ and $c_n$ only depending on the dimension $n$ such that:

(i) $\ell(Q_j) \in (\frac{1}{k}, \frac{2}{k})$;
(ii) $Q \subset \bigcup_j Q_j$; and
(iii) $\sum_j \chi_{Q_j}(x) \le c_n$, for each $x \in \mathbb{R}^n$.

Then it is easy to see that $(\hat{Q}\cap F_k^2) \subset \bigcup_j \hat{Q}_j$. Thus by (2.6) we have

$$I_2 \le \sum_j \frac{\mu(\hat{Q}_j)}{|Q|} \le \sum_j \frac{|Q_j|}{|Q|} \sup_{\ell(Q)\le\frac{2}{k}} \frac{\mu(\hat{Q})}{|Q|} \le c_n \sup_{\ell(Q)\le\frac{2}{k}} \frac{\mu(\hat{Q})}{|Q|} \to 0 \quad \text{as } k\to\infty.$$

From this we see that (2.5) holds for $i = 2$.

*Case* $i = 3$. Obviously we need only consider the limit of $\sup_{\hat{Q}\cap F_k^3\neq\emptyset} I_3$ as $k \to \infty$. By $\mu \in \mathscr{C}_v$ and Lemma 2.3, we have

$$\begin{aligned}
\sup_{\hat{Q}\cap F_k^3\neq\emptyset} I_3 &\le \sup_{|x_Q|\ge k} I_3 + \sup_{\substack{|x_Q|<k \\ \frac{1}{2}\ell(Q)>k-|x_Q|}} I_3 \\
&\le \sup_{|x_Q|\ge k} I_3 + \sup_{\substack{|x_Q|<k/2 \\ \frac{1}{2}\ell(Q)>k-|x_Q|}} I_3 + \sup_{\substack{k/2\le|x_Q|<k \\ \frac{1}{2}\ell(Q)>k-|x_Q|}} I_3 \\
&\le \sup_{|x_Q|\ge k} I_3 + \sup_{\ell(Q)>k} I_3 + \sup_{k/2\le|x_Q|<k} I_3 \\
&\le 2\sup_{|x_Q|\ge k/2} \frac{\mu(\hat{Q})}{|Q|} + \sup_{\ell(Q)>k} \frac{\mu(\hat{Q})}{|Q|} \to 0 \quad \text{as } k\to\infty.
\end{aligned}$$

Thus (2.5) still holds in this case. We therefore show that $\mu \in \overline{\mathscr{C}_c}^{\|\cdot\|_{\mathscr{C}}}$ and we complete the proof of Lemma 2.5. □

2.2. **Some facts on the tent space $T^1_\infty(\mathbb{R}^{n+1}_+)$.** In this subsection, we give some facts on the tent space $T^1_\infty(\mathbb{R}^{n+1}_+)$, which will be used in the proof of Theorem 2.1.

**Lemma 2.6.** *The norm of $T^1_\infty(\mathbb{R}^{n+1}_+)$ can be characterized via $\mathscr{C}_c(\mathbb{R}^{n+1}_+)$; more precisely,*

$$\|f\|_{T^1_\infty} = \sup_{\substack{\mu\in\mathscr{C}_c \\ \|\mu\|_{\mathscr{C}}\le 1}} \Big|\int_{\mathbb{R}^{n+1}_+} f(y,t)\, d\mu(y,t)\Big|.$$

*Proof.* Since $(T^1_\infty(\mathbb{R}^{n+1}_+))^* = \mathscr{C}(\mathbb{R}^{n+1}_+)$, and $\mathscr{C}_c(\mathbb{R}^{n+1}_+) \subset \mathscr{C}(\mathbb{R}^{n+1}_+)$, we have

$$\|f\|_{T^1_\infty} = \sup_{\substack{\mu\in\mathscr{C} \\ \|\mu\|_{\mathscr{C}}\le 1}} \Big|\int_{\mathbb{R}^{n+1}_+} f(y,t)\, d\mu(y,t)\Big| \ge \sup_{\substack{\mu\in\mathscr{C}_c \\ \|\mu\|_{\mathscr{C}}\le 1}} \Big|\int_{\mathbb{R}^{n+1}_+} f(y,t)\, d\mu(y,t)\Big|. \tag{2.7}$$

It remains to prove that for any $\epsilon > 0$, there exists $\mu_0 \in \mathscr{C}_c(\mathbb{R}^{n+1}_+)$ with $\|\mu_0\|_{\mathscr{C}} \le 1$ such that

$$\Big|\int_{\mathbb{R}^{n+1}_+} f(y,t)\, d\mu_0(y,t)\Big| \ge \|f\|_{T^1_\infty} - \epsilon.$$

In fact, from (2.7), there exists a measure $\mu \in \mathscr{C}(\mathbb{R}^{n+1}_+)$ with $\|\mu\|_{\mathscr{C}} \le 1$ such that

$$\Big|\int_{\mathbb{R}^{n+1}_+} f(y,t)\, d\mu(y,t)\Big| \ge \|f\|_{T^1_\infty} - \frac{\epsilon}{2}.$$

For $k \in \mathbb{N}$, let $\chi_k = \chi_{\{(y,t)\in\mathbb{R}^{n+1}_+ : |y|\le k, \frac{1}{k}\le t\le k\}}$. Using the Lebesgue dominated convergent theorem, it is easy to see that

$$\Big|\int_{\mathbb{R}^{n+1}_+} f(y,t)\big(1-\chi_k(y,t)\big)\, d\mu(y,t)\Big| \to 0 \quad \text{as } k\to\infty.$$

Thus there exists $k_0 > 0$, such that

$$\Big|\int_{\mathbb{R}^{n+1}_+} f(y,t)\big(1-\chi_{k_0}(y,t)\big)\, d\mu(y,t)\Big| < \frac{\epsilon}{2}.$$

Hence

$$\Big|\int_{\mathbb{R}^{n+1}_+} f(y,t)\chi_{k_0}(y,t)\, d\mu(y,t)\Big| \ge \|f\|_{T^1_\infty} - \epsilon.$$

If we denote $d\mu_0 := \chi_{k_0}\, d\mu$, then it is easy to see that$\mu_0 \in \mathscr{C}_c(\mathbb{R}^{n+1}_+)$ and that $\|\mu_0\|_{\mathscr{C}} \le 1$. Hence we prove Lemma 2.6. □

In [9], Coifman, Meyer, and Stein gave the atom decomposition of the tent space $T^1_q$ with $1 < q \le \infty$. A function $a$ on $\mathbb{R}^{n+1}_+$ is said to be a $T^1_q$ *atom* if:

(i) $a$ is supported in $\hat{Q}$ (for some cube $Q \subset \mathbb{R}^n$); and
(ii) $\|a\|_{L^q(\mathbb{R}^{n+1}_+; \frac{dy\,dt}{t})} \le |Q|^{-1/q'}$.

**Lemma 2.7** (See [9])**.** *Suppose that $f \in T^1_q$ $(1 < q \le \infty)$. Then $f = \sum_{i=1}^\infty \lambda_i a_i$, where each $a_i$ is a $T^1_q$ atom, $\lambda_i \in \mathbb{C}$, and $\sum_{i=1}^\infty |\lambda_i| \le C\|f\|_{T^1_q}$, where the constant $C$ is independent of $\{\lambda_i\}$ and $f$.*

Notice that in the atom decomposition given above, the relationship of the support set of each atom is not clear. In [29], Wang gave a more delicate atom decomposition of $T^1_\infty(\mathbb{R}^{n+1}_+)$.

**Lemma 2.8** (See [29]). *For every fixed $k \in \mathbb{Z}$, there is a sequence $\{Q_{jk}\}_j$ of cubes in $\mathbb{R}^n$ which satisfies*

(1) $|Q_{jk}| = \beta^k, \beta = 3^n$, *for any* $j = 1, 2, \dots$;
(2) $\bigcup_{j=1}^\infty Q_{jk} = \mathbb{R}^n$;
(3) $\sum_{j=1}^\infty \chi_{Q_{jk}}(x) \le \beta$ *for any* $x \in \mathbb{R}^n$; *and*
(4) *for each* $f \in T^1_\infty(\mathbb{R}^{n+1}_+)$, *we have* $f = \sum_{k\in\mathbb{Z}} \sum_{j=1}^\infty \lambda_{jk} a_{jk}$, *where* $a_{jk}$ *is the* $T^1_\infty(\mathbb{R}^{n+1}_+)$ *atom supported in* $\hat{Q}_{jk}$, *and* $\sum_{k\in\mathbb{Z}}\sum_{j=1}^\infty |\lambda_{jk}| \le C\|f\|_{T^1_\infty}$, *and the constant* $C$ *is independent of the sequence* $\{\lambda_{jk}\}$ *and* $f$.

In order to prove Theorem 2.1, we also need the following lemma which is given by Coifman and Weiss in [10].

**Lemma 2.9** (See [10]). *If we suppose that* $\lambda_{jk} \ge 0,\ j, k = 1, 2, \dots$ *satisfies* $\sum_{j=1}^\infty \lambda_{jk} \le 1$ *for each* $k = 1, 2, \dots$, *then there exists an increasing sequence of natural numbers,* $k_1 < k_2 < \cdots < k_l < \cdots$ *such that* $\lim_{l\to\infty} \lambda_{jk_l} = \lambda_j$ *for each* $j$, *and* $\sum_{j=1}^\infty \lambda_j \le 1$.

The following result plays a key role in the proof of Theorem 2.1.

**Lemma 2.10.** *Suppose that* $\{f_l\}_{l\in\mathbb{N}} \subset T^1_\infty(\mathbb{R}^{n+1}_+)$ *with* $\|f_l\|_{T^1_\infty} \le D$, *where* $D > 0$ *is independent of* $l = 1, 2, \dots$. *Then there exists a function* $f \in T^1_\infty(\mathbb{R}^{n+1}_+)$ *and a subsequence* $\{f_{l_s}\}_s$ *such that*

$$\lim_{s\to\infty} \int_{\mathbb{R}^{n+1}_+} f_{l_s}(y,t)\, d\mu(y,t) = \int_{\mathbb{R}^{n+1}_+} f(y,t)\, d\mu(y,t) \quad \text{for any } \mu \in \mathscr{C}_c(\mathbb{R}^{n+1}_+). \tag{2.8}$$

*Proof.* Applying Lemma 2.8, $f_l = \sum_{k\in\mathbb{Z}}\sum_{j=1}^\infty \lambda_{l,jk} a_{l,jk}$ with $\sum_{k\in\mathbb{Z}}\sum_{j=1}^\infty |\lambda_{l,jk}| \le C\|f_l\|_{T^1_\infty}$, where $C$ is independent of $\{\lambda_{l,jk}\}$ and $f_l$. For any fixed $k \in \mathbb{Z}$ and $j \in \mathbb{N}$, by Lemma 2.9, there exist subsequences $\{\lambda_{l_s,jk}\}_s$ and $\lambda_{jk}$ such that

$$\lim_{s\to\infty} |\lambda_{l_s,jk}| = |\lambda_{jk}|, \qquad \text{and} \qquad \sum_{k\in\mathbb{Z}}\sum_{j=1}^\infty |\lambda_{jk}| \le C\|f_l\|_{T^1_\infty} \le CD.$$

Notice that all $a_{l,jk}$ are $T^1_\infty(\mathbb{R}^{n+1}_+)$ atoms supported in $\hat{Q}_{jk}$ with $Q_{jk}$ satisfying (1), (2), (3) in Lemma 2.8. From the proof of Lemma 2.8, it is easy to see that the cube sequence $\{Q_{jk}\}$ is independent of $l_s$. As to $(j,k)$ fixed,

$$\|a_{l_s,jk}\|_{L^\infty(\hat{Q}_{jk};\frac{dy\,dt}{t})} \le |Q_{jk}|^{-1} = \beta^{-k} < \infty \tag{2.9}$$

holds uniformly with the bound independent of $l_s$. Hence there exist a subsequence, which still is denoted by $\{a_{l_s,jk}\}_s$, and a function $a_{jk} \in L^\infty(\hat{Q}_{jk};\frac{dy\,dt}{t})$ such that for $g \in L^1(\hat{Q}_{jk};\frac{dy\,dt}{t})$,

$$\lim_{s\to\infty} \int_{\hat{Q}_{jk}} a_{l_s,jk}(y,t) g(y,t) \frac{dy\,dt}{t} = \int_{\hat{Q}_{jk}} a_{jk}(y,t) g(y,t) \frac{dy\,dt}{t}. \tag{2.10}$$

Thus, by (2.10), it is easy to see that

$$\begin{aligned}\|a_{jk}\|_{L^\infty(\hat{Q}_{jk};\frac{dy\,dt}{t})} &= \sup_{\|g\|_{L^1(\hat{Q}_{jk};\frac{dy\,dt}{t})}\le 1}\Big|\int_{\hat{Q}_{jk}} a_{jk}(y,t)g(y,t)\frac{dy\,dt}{t}\Big|\\ &= \sup_{\|g\|_{L^1(\hat{Q}_{jk};\frac{dy\,dt}{t})}\le 1}\lim_{s\to\infty}\Big|\int_{\hat{Q}_{jk}} a_{l_s,jk}(y,t)g(y,t)\frac{dy\,dt}{t}\Big|\\ &\le \sup_{\|g\|_{L^1(\hat{Q}_{jk};\frac{dy\,dt}{t})}\le 1}|Q_{jk}|^{-1}\|g\|_{L^1(\hat{Q}_{jk};\frac{dy\,dt}{t})}\\ &\le |Q_{jk}|^{-1}.\end{aligned}$$

Hence $a_{jk}$ is a $T^1_\infty(\mathbb{R}^{n+1}_+)$ atom. If we let $f=\sum_{k\in\mathbb{Z}}\sum_{j=1}^\infty \lambda_{jk}a_{jk}$, then $f\in T^1_\infty(\mathbb{R}^{n+1}_+)$. Below we prove that $f$ satisfies (2.8).

Assume that $\mu\in\mathscr{C}_c(\mathbb{R}^{n+1}_+)$ and that there exists a compact set $K\subset\mathbb{R}^{n+1}_+$ such that for any $\mu$-measurable set $E\subset\mathbb{R}^{n+1}_+$, $\mu(E)=\mu(E\cap K)$, and then

$$\begin{aligned}\int_{\mathbb{R}^{n+1}_+} f_{l_s}(y,t)\,d\mu(y,t) &= \int_{\mathbb{R}^{n+1}_+}\Big(\sum_{-N\le k\le N}+\sum_{k<-N}+\sum_{k>N}\Big)\sum_{j=1}^\infty \lambda_{l_s,jk}a_{l_s,jk}(y,t)\,d\mu(y,t)\\ &=: II_1+II_2+II_3.\end{aligned}$$

Since $K$ is a compact set in $\mathbb{R}^{n+1}_+$, then there is a $t_0>0$ such that $K\subset\{(x,t)\in\mathbb{R}^{n+1}_+ : t>t_0\}$. We now make $N$ large enough so that $\beta^{-\frac{N}{n}}<t_0$.

*Estimate of* $II_2$. Note that $\ell(Q_{jk})=\beta^{\frac{k}{n}}<\beta^{-\frac{N}{n}}<t_0$ for all $j\in\mathbb{N}$. Thus

$$\Big(\bigcup_{k<-N}\bigcup_{j=1}^\infty \hat{Q}_{jk}\Big)\cap K=\emptyset.$$

Hence $II_2=0$.

*Estimate of* $II_3$. By (2.9), it is easy to see that

$$\begin{aligned}|II_3| &\le \sum_{k>N}\sum_{j=1}^\infty |\lambda_{l_s,jk}|\|a_{l_s,jk}\|_{L^\infty(\hat{Q}_{jk};\frac{dy\,dt}{t})}\mu(K)\\ &\le C_K\|\mu\|_{\mathscr{C}}\sum_{k>N}\sum_{j=1}^\infty |\lambda_{l_s,jk}||Q_{jk}|^{-1}\\ &\le CDC_K\|\mu\|_{\mathscr{C}}\beta^{-N}\to 0 \quad \text{as } N\to\infty.\end{aligned}$$

*Estimate of* $II_1$. Notice that, for any $k\in[-N,N]$ fixed, the set $\{j\in\mathbb{N} : \hat{Q}_{jk}\cap K\neq\emptyset\}$ is a finite set; that is, there exists an integer $m>0$ such that

$$\begin{aligned}II_1 &= \int_{\mathbb{R}^{n+1}_+}\sum_{-N\le k\le N}\sum_{j=1}^m \lambda_{l_s,jk}a_{l_s,jk}(y,t)\,d\mu(y,t)\\ &= \sum_{-N\le k\le N}\sum_{j=1}^m \lambda_{l_s,jk}\int_{\mathbb{R}^{n+1}_+} a_{l_s,jk}(y,t)\,d\mu(y,t).\end{aligned}$$

Notice that $td\mu \in L^1(\hat{Q}_{jk}; \frac{dy\,dt}{t})$; then from (2.10), notice that

$$\lim_{s\to\infty} II_1 = \sum_{-N\le k\le N} \sum_{j=1}^{m} \lambda_{jk} \int_{\mathbb{R}^{n+1}_+} a_{jk}(y,t)\, d\mu(y,t);$$

hence

$$\begin{aligned}\lim_{s\to\infty}\lim_{N\to\infty} II_1 &= \lim_{N\to\infty} \sum_{-N\le k\le N} \sum_{j=1}^{m} \lambda_{jk} \int_{\mathbb{R}^{n+1}_+} a_{jk}(y,t)\, d\mu(y,t)\\ &= \lim_{N\to\infty} \int_{\mathbb{R}^{n+1}_+} \sum_{-N\le k\le N} \sum_{j=1}^{\infty} \lambda_{jk} a_{jk}(y,t)\, d\mu(y,t)\\ &= \int_{\mathbb{R}^{n+1}_+} f(y,t)\, d\mu(y,t).\end{aligned}$$

We thus complete the proof of Lemma 2.10. □

2.3. **Proof of Theorem 2.1.** The proof of Theorem 2.1 needs to use a general result in functional analysis. Let us give the definition of the total set, which can be found in [16, p. 58].

**Definition 2.2** (total set)**.** A set $W$ of maps which map a vector space $X$ into another vector space $Y$ is called a total set if $x = 0$ is the only vector for which $\phi(x) = 0$ for all $\phi \in W$.

**Lemma 2.11** ([16, p. 439])**.** *Let $X$ be a locally convex linear topological space, and let $W$ be a linear subspace of $X^*$. Then $W$ is $X$-dense in $X^*$ if and only if $W$ is a total set of functionals on $X$.*

*Proof of Theorem 2.1.* Note that $(T^1_\infty(\mathbb{R}^{n+1}_+))^* = \mathscr{C}(\mathbb{R}^{n+1}_+) \supset \mathscr{C}_v(\mathbb{R}^{n+1}_+)$ (see the conclusion (3) in Theorem A), so $T^1_\infty \subset (\mathscr{C}_v)^*$.

On the other hand, if there exists a $\mu \in \mathscr{C}_v(\mathbb{R}^{n+1}_+)$ such that

$$\int_{\mathbb{R}^{n+1}_+} f(y,t)\, d\mu(y,t) = 0 \quad \text{for all } f \in T^1_\infty(\mathbb{R}^{n+1}_+), \tag{2.11}$$

then by $(T^1_\infty)^* = \mathscr{C}(\mathbb{R}^{n+1}_+)$, we see that

$$\|\mu\|_{\mathscr{C}} = \sup_{\|f\|_{T^1_\infty}\le 1} \left| \int_{\mathbb{R}^{n+1}_+} f(y,t)\, d\mu(y,t) \right| = 0.$$

Thus $\mu = 0$. In particular, since $\mathscr{C}_v(\mathbb{R}^{n+1}_+)$ is a Banach space, then it is obvious that $\mu$ is the only measure in $\mathscr{C}_v(\mathbb{R}^{n+1}_+)$ such that (2.11) holds. Thus $T^1_\infty(\mathbb{R}^{n+1}_+)$ is a total set on $\mathscr{C}_v(\mathbb{R}^{n+1}_+)$ by Definition 2.2. Clearly, $\mathscr{C}_v(\mathbb{R}^{n+1}_+)$ is a locally convex linear topological space by Lemma 2.4. Applying Lemma 2.11, $T^1_\infty(\mathbb{R}^{n+1}_+)$ is weak* dense in $(\mathscr{C}_v)^*$. Hence, for any $\ell \in (\mathscr{C}_v)^*$, there exists a sequence of functions $\{f_k\} \subset T^1_\infty(\mathbb{R}^{n+1}_+)$ such that

$$\ell(\mu) = \lim_{k\to\infty} \langle f_k, \mu\rangle = \lim_{k\to\infty} \int_{\mathbb{R}^{n+1}_+} f_k(y,t)\, d\mu(y,t) \quad \text{for all } \mu \in \mathscr{C}_v(\mathbb{R}^{n+1}_+).$$

From this we see that, for each fixed $\mu \in \mathscr{C}_v(\mathbb{R}^{n+1}_+)$,

$$\sup_k \left| \int_{\mathbb{R}^{n+1}_+} f_k(y,t)\, d\mu(y,t) \right| < \infty. \tag{2.12}$$

The uniform boundedness principle (Banach–Steinhaus theorem) and (2.12) imply that

$$\sup_k \left| \int_{\mathbb{R}^{n+1}_+} f_k(y,t)\, d\mu(y,t) \right| \leq C \|\mu\|_{\mathscr{C}} \quad \text{for any } \mu \in \mathscr{C}_v(\mathbb{R}^{n+1}_+);$$

thus, by Lemma 2.6, we get $\|f_k\|_{T^1_\infty} \leq C$ where $C$ is independent of $k$. Now, applying Lemma 2.10, we can obtain a subsequence $\{f_{k_j}\}_j$ and an $f \in T^1_\infty(\mathbb{R}^{n+1}_+)$ such that

$$\lim_{j\to\infty} \int_{\mathbb{R}^{n+1}_+} f_{k_j}(y,t)\, d\mu(y,t) = \int_{\mathbb{R}^{n+1}_+} f(y,t)\, d\mu(y,t) \quad \text{for any } \mu \in \mathscr{C}_c(\mathbb{R}^{n+1}_+);$$

hence

$$\ell(\mu) = \int_{\mathbb{R}^{n+1}_+} f(y,t)\, d\mu(y,t) \quad \text{for any } \mu \in \mathscr{C}_c(\mathbb{R}^{n+1}_+). \tag{2.13}$$

Finally, by the density of $\mathscr{C}_c(\mathbb{R}^{n+1}_+)$ in $\mathscr{C}_v(\mathbb{R}^{n+1}_+)$ (i.e., Lemma 2.5), we can check (2.13), which still holds for all $\mu \in \mathscr{C}_v(\mathbb{R}^{n+1}_+)$ and which shows that the linear functional $\ell$ on $\mathscr{C}_v(\mathbb{R}^{n+1}_+)$ can be represented by a function $f$ in $T^1_\infty(\mathbb{R}^{n+1}_+)$. We therefore prove that $(\mathscr{C}_v)^* \subset T^1_\infty$, and we complete the proof of Theorem 2.1. □

**Remark 2.12.** Using the idea of proving Theorem 2.1, we also can consider a predual of the tent space $T^1_q(\mathbb{R}^{n+1}_+)$ $(1 < q < \infty)$. Similar to the definition of the vanishing Carleson measure, we can introduce a subclass of $T^\infty_q(\mathbb{R}^{n+1}_+)$ $(1 < q < \infty)$, the vanishing tent space $T^\infty_{q,v}(\mathbb{R}^{n+1}_+)$ $(1 < q < \infty)$, which is defined by

$$\begin{aligned} T^\infty_{q,v}(\mathbb{R}^{n+1}_+) = \Big\{ f \in T^\infty_q : \lim_{a\to 0} M_a(f,q) = 0,\ \lim_{a\to\infty} M_a(f,q) = 0, \\ \text{and for any cube } Q \subset \mathbb{R}^n,\ \lim_{|x|\to\infty} M(f, Q+x, q) = 0 \Big\}, \end{aligned} \tag{2.14}$$

where $M_a(f,q)$ and $M(f,Q+x,q)$ are defined in Definition 1.1. This is completely similar to the proof of Theorem 2.1; thus we can obtain $(T^\infty_{q',v})^* = T^1_q$ $(1 < q < \infty)$.

## 3. Characterizations of tent space $T^\infty_q(\mathbb{R}^{n+1}_+)$ and its subspace $T^\infty_{q,v}(\mathbb{R}^{n+1}_+)$

In this section, we give the characterizations of tent space $T^\infty_q(\mathbb{R}^{n+1}_+)$ and subspace $T^\infty_{q,v}(\mathbb{R}^{n+1}_+)$, respectively. Let us first recall the characterizations of the Carleson measure space $\mathscr{C}(\mathbb{R}^{n+1}_+)$ via the boundedness of the Poisson integral. For $f \in L^p(\mathbb{R}^n)$ $(1 \leq p \leq \infty)$, the Poisson integral of $f$ is defined by $u(x,t) := p_t * f(x)$ $(t > 0)$, where $p_t(x) = c_n \frac{t}{(|x|^2+t^2)^{(n+1)/2}}$ with $c_n = \frac{\Gamma(\frac{n+1}{2})}{\pi^{(n+1)/2}}$ is the Poisson kernel on $\mathbb{R}^{n+1}_+$.

**Theorem B** (see [6], [7])**.** *The following are equivalent:*

(1) $\mu \in \mathscr{C}(\mathbb{R}^{n+1}_+)$;
(2) *the Poisson integral is bounded from* $L^p(\mathbb{R}^n; dx)$ *to* $L^p(\mathbb{R}^{n+1}_+; d\mu)$ *for all* $1 < p < \infty$*; and*
(3) *the Poisson integral is bounded from* $L^p(\mathbb{R}^n; dx)$ *to* $L^p(\mathbb{R}^{n+1}_+; d\mu)$ *for some* $1 < p < \infty$.

The following Theorem C is an extension of Theorem B. Suppose that the function $\varphi$ on $\mathbb{R}^n$ satisfies

$$\big|\varphi(x)\big| \le C\big(1 + |x|\big)^{-n-\theta} \quad \text{for some } C, \theta > 0 \text{ and that all } x \in \mathbb{R}^n. \tag{3.1}$$

The convolution operator associated with $\varphi$ is denoted by

$$\mathscr{L}_\varphi(f) : f \mapsto \varphi_t * f, \tag{3.2}$$

where $\varphi_t(x) = t^{-n}\varphi(x/t)$ for $t > 0$.

**Theorem C** (see [18, p. 177])**.** *Suppose that $\varphi$ satisfies (3.1).*

(1) *If the measure $\mu$ on $\mathbb{R}^{n+1}_+$ is a Carleson measure, then for every $1 < p < \infty$, the operator $\mathscr{L}_\varphi$ defined in (3.2) is bounded from $L^p(\mathbb{R}^n; dx)$ to $L^p(\mathbb{R}^{n+1}_+; d\mu)$ with the norm $\|\mathscr{L}_\varphi\|_{L^p(dx)\to L^p(d\mu)} \le C_{n,p}\|\mu\|_{\mathscr{C}}^{1/p}$.*
(2) *If $\varphi \ge 0$ and $\int_{|x|\le 1} \varphi(x)\, dx > 0$ yet, and a measure $\mu$ is defined on $\mathbb{R}^{n+1}_+$ such that the operator $\mathscr{L}_\varphi$ is bounded from $L^p(\mathbb{R}^n; dx)$ to $L^p(\mathbb{R}^{n+1}_+; d\mu)$ for some $1 < p < \infty$, then $\mu$ is a Carleson measure, and $\|\mu\|_{\mathscr{C}} \le \|\mathscr{L}_\varphi\|^p_{L^p(dx)\to L^p(d\mu)}$.*

**Remark 3.1.** Theorem C still holds if the condition (3.1) assumed on $\varphi$ is replaced by $\phi \in L^1 \cap L^\infty$ with $\phi(x) := \operatorname{ess\,sup}_{|x|\le|y|} |\varphi(y)|$ (see [17]).

### 3.1. A characterization of the tent space $T^\infty_q(\mathbb{R}^{n+1}_+)$.

Applying Theorem C, we can show simply that the tent space $T^\infty_q(\mathbb{R}^{n+1}_+)$ can be characterized via the boundedness of the operator $\mathscr{L}_\varphi$.

**Theorem 3.2.** *Suppose that $\varphi$ satisfies (3.1) and that $1 < q < \infty$.*

(1) *If $g \in T^\infty_q(\mathbb{R}^{n+1}_+)$, then for every $1 < p < \infty$, the operator $\mathscr{L}_\varphi$ is bounded from $L^p(\mathbb{R}^n; dx)$ to $L^p(\mathbb{R}^{n+1}_+; |g(y,t)|^q \frac{dy\,dt}{t})$ with the norm $\|\mathscr{L}_\varphi\| \le C\|g\|^{\frac{q}{p}}_{T^\infty_q}$.*
(2) *If $\varphi \ge 0$ and $\int_{|x|\le 1} \varphi(x)\, dx > 0$ yet, and $g$ is defined on $\mathbb{R}^{n+1}_+$ such that the operator $\mathscr{L}_\varphi$ is bounded from $L^p(\mathbb{R}^n; dx)$ to $L^p(\mathbb{R}^{n+1}_+; |g(y,t)|^q \frac{dy\,dt}{t})$ for some $1 < p < \infty$, then $g \in T^\infty_q(\mathbb{R}^{n+1}_+)$, and $\|g\|_{T^\infty_q} \le C\|\mathscr{L}_\varphi\|^{\frac{p}{q}}$.*

*Proof.* (1) For any $g \in T^\infty_q(\mathbb{R}^{n+1}_+)$, let $d\nu(y,t) = |g(y,t)|^q \frac{dy\,dt}{t}$, then $\nu$ is a Carleson measure on $\mathbb{R}^{n+1}_+$ by the definition of $T^\infty_q(\mathbb{R}^{n+1}_+)$. Applying the conclusion (1) of Theorem C, the operator $\mathscr{L}_\varphi$ is bounded from $L^p(\mathbb{R}^n; dx)$ to $L^p(\mathbb{R}^{n+1}_+; d\nu) = L^p(\mathbb{R}^{n+1}_+; |g(y,t)|^q \frac{dy\,dt}{t})$ for all $1 < p < \infty$, and $\|\mathscr{L}_\varphi\| \le C_{n,p}\|\nu\|_{\mathscr{C}}^{1/p} = C_{n,p}\|g\|^{\frac{q}{p}}_{T^\infty_q}$.

(2) If $d\nu(y,t) = |g(y,t)|^q \frac{dy\,dt}{t}$, then $\nu$ is a positive measure on $\mathbb{R}^{n+1}_+$, and the condition of the conclusion (2) in Theorem 3.2 implies that the operator $\mathscr{L}_\varphi$ is bounded from $L^p(\mathbb{R}^n;dx)$ to $L^p(\mathbb{R}^{n+1}_+;d\nu)$ for some $1 < p < \infty$. Hence $\nu$ is a Carleson measure on $\mathbb{R}^{n+1}_+$ by Theorem C(2). Consequently, there exists a constant $C > 0$ such that for any cube $Q$ in $\mathbb{R}^n$,

$$\frac{\nu(\hat{Q})}{|Q|} = \frac{1}{|Q|}\int_{\hat{Q}} \big|g(y,t)\big|^q \frac{dy\,dt}{t} \le C,$$

which shows that $g \in T^\infty_q(\mathbb{R}^{n+1}_+)$ and that $\|g\|_{T^\infty_q} = \|\nu\|^{1/q}_{\mathscr{C}} \le C\|\mathscr{L}_\varphi\|^{\frac{p}{q}}$. □

**Remark 3.3.** Theorem 3.2 still holds if the condition (3.1) assumed on $\varphi$ is replaced by $\phi \in L^1 \cap L^\infty$ with $\phi(x) := \operatorname{ess\,sup}_{|x|\le|y|} |\varphi(y)|$.

The following corollary is a direct result of Theorem 3.2.

**Corollary 3.4.** *The following are equivalent:*

(1) $g \in T^\infty_q(\mathbb{R}^{n+1}_+)$ $(1 < q < \infty)$;
(2) *the Poisson integral is bounded from* $L^p(\mathbb{R}^n;dx)$ *to* $L^p(\mathbb{R}^{n+1}_+;|g(y,t)|^q\frac{dy\,dt}{t})$ *for all* $1 < p < \infty$*; and*
(3) *the Poisson integral is bounded from* $L^p(\mathbb{R}^n;dx)$ *to* $L^p(\mathbb{R}^{n+1}_+;|g(y,t)|^q\frac{dy\,dt}{t})$ *for some* $1 < p < \infty$.

3.2. **A characterization of the vanishing tent space $T^\infty_{q,v}(\mathbb{R}^{n+1}_+)$.** The characterization of $T^\infty_{q,v}(\mathbb{R}^{n+1}_+)$ $(1 < q < \infty)$ defined in (2.14) which will be given below is closely related to the characterization of the vanishing Carleson measure space $\mathscr{C}_v(\mathbb{R}^{n+1}_+)$. The latter has been given by the authors in [14] recently.

**Theorem D** ([14, Theorem 2.1]). *Suppose that $\varphi$ satisfies (3.1).*

(1) *If the measure $\mu$ on $\mathbb{R}^{n+1}_+$ is a vanishing Carleson measure, then for every $1 < p < \infty$, the operator $\mathscr{L}_\varphi$ is compact from $L^p(\mathbb{R}^n;dx)$ to $L^p(\mathbb{R}^{n+1}_+;d\mu)$.*
(2) *If $\varphi \ge 0$ and $\int_{|x|\le 1}\varphi(x)\,dx > 0$ yet, and a measure $\mu$ is defined on $\mathbb{R}^{n+1}_+$ such that the operator $\mathscr{L}_\varphi$ is compact from $L^p(\mathbb{R}^n;dx)$ to $L^p(\mathbb{R}^{n+1}_+;d\mu)$ for some $1 < p < \infty$, then $\mu$ is a vanishing Carleson measure.*

**Corollary E** ([14, Corollary 2.2]). *The following are equivalent:*

(1) $\mu \in \mathscr{C}_v(\mathbb{R}^{n+1}_+)$;
(2) *the Poisson integral is compact from* $L^p(\mathbb{R}^n;dx)$ *to* $L^p(\mathbb{R}^{n+1}_+;d\mu)$ *for all* $1 < p < \infty$*; and*
(3) *the Poisson integral is compact from* $L^p(\mathbb{R}^n;dx)$ *to* $L^p(\mathbb{R}^{n+1}_+;d\mu)$ *for some* $1 < p < \infty$.

By Theorem D, it is easy to obtain the characterization of the vanishing tent space $T^\infty_{q,v}(\mathbb{R}^{n+1}_+)$.

**Theorem 3.5.** *Suppose that $\varphi$ satisfies (3.1) and that $1 < q < \infty$.*

(1) *If $g \in T^\infty_{q,v}(\mathbb{R}^{n+1}_+)$, then for every $1 < p < \infty$, the operator $\mathscr{L}_\varphi$ is compact from $L^p(\mathbb{R}^n;dx)$ to $L^p(\mathbb{R}^{n+1}_+;|g(y,t)|^q\frac{dy\,dt}{t})$.*

(2) *If* $\varphi \geq 0$ *with* $\int_{|x|\leq 1}\varphi(x)\,dx > 0$ *yet, and* $g$ *is defined on* $\mathbb{R}^{n+1}_+$ *such that the operator* $\mathscr{L}_\varphi$ *is compact from* $L^p(\mathbb{R}^n;dx)$ *to* $L^p(\mathbb{R}^{n+1}_+;|g(y,t)|^q\frac{dy\,dt}{t})$ *for some* $1<p<\infty$*, then* $g\in T^\infty_{q,v}(\mathbb{R}^{n+1}_+)$.

*Proof.* (1) For any $g\in T^\infty_{q,v}(\mathbb{R}^{n+1}_+)$, let $d\nu(y,t)=|g(y,t)|^q\frac{dy\,dt}{t}$, and it is easy to check that $\nu\in\mathscr{C}_v(\mathbb{R}^{n+1}_+)$. Applying the conclusion (1) of Theorem D, $\mathscr{L}_\varphi$ is a compact operator from $L^p(\mathbb{R}^n;dx)$ to $L^p(\mathbb{R}^{n+1}_+;d\nu)=L^p(\mathbb{R}^{n+1}_+;|g(y,t)|^q\frac{dy\,dt}{t})$ for all $1<p<\infty$.

(2) If $d\nu(y,t)=|g(y,t)|^q\frac{dy\,dt}{t}$, then $\nu$ is a positive measure on $\mathbb{R}^{n+1}_+$, and the condition of Theorem 3.5(2) shows that the operator $\mathscr{L}_\varphi$ is compact from $L^p(\mathbb{R}^n;dx)$ to $L^p(\mathbb{R}^{n+1}_+;d\nu)$ for some $1<p<\infty$. Thus, by the conclusion (2) of Theorem D, the measure $\nu\in\mathscr{C}_v(\mathbb{R}^{n+1}_+)\subset\mathscr{C}(\mathbb{R}^{n+1}_+)$. Hence it is obvious that $g\in T^\infty_q(\mathbb{R}^{n+1}_+)$. In particular, since $\nu$ satisfies (1)∼(3) in the Definition 2.1, then $g\in T^\infty_{q,v}(\mathbb{R}^{n+1}_+)$. □

**Remark 3.6.** Theorem 3.5 still holds if the condition (3.1) assumed on $\varphi$ is replaced by $\phi\in L^1\cap L^\infty$ with $\phi(x):=\operatorname{ess\,sup}_{|x|\leq|y|}|\varphi(y)|$.

The following consequence of Theorem 3.5 is obvious.

**Corollary 3.7.** *The following are equivalent:*

(1) $g\in T^\infty_{q,v}(\mathbb{R}^{n+1}_+)$ $(1<q<\infty)$*;*

(2) *Poisson integral is compact from* $L^p(\mathbb{R}^n;dx)$ *to* $L^p(\mathbb{R}^{n+1}_+;|g(y,t)|^q\frac{dy\,dt}{t})$ *for all* $1<p<\infty$*; and*

(3) *Poisson integral is compact from* $L^p(\mathbb{R}^n;dx)$ *to* $L^p(\mathbb{R}^{n+1}_+;|g(y,t)|^q\frac{dy\,dt}{t})$ *for some* $1<p<\infty$.

## 4. Applications

In the last part of this paper, we first introduce a paraproduct $\pi_F$ associated with the tent spaces $T^\infty_q(\mathbb{R}^{n+1}_+)$ $(1<q<\infty)$, and we then give the boundedness and compactness of $\pi_F$ with smooth kernel and rough kernel, respectively. We shall also extend partially an interesting result in [9], which gives the relation between the tent spaces $T^p_2(\mathbb{R}^{n+1}_+)$ $(1\leq p\leq\infty)$ and $L^p(\mathbb{R}^n)$, $H^p(\mathbb{R}^n)$ and $BMO(\mathbb{R}^n)$ spaces.

4.1. **Boundedness and compactness of paraproducts with smooth kernel.** Let us begin by giving the definition of a general Littlewood–Paley $g$-function, which plays an important role in the study of paraproducts.

**Definition 4.1** (The general Littlewood–Paley $g$-function)**.** Suppose that $1<q<\infty$ and that $\eta\in\mathscr{S}(\mathbb{R}^n)$ (the Schwartz class on $\mathbb{R}^n$), which satisfies

$$\int_{\mathbb{R}^n}\eta(x)\,dx=0. \tag{4.1}$$

For $f\in L^p(\mathbb{R}^n)$ $(1<p<\infty)$, the general Littlewood–Paley $g$-function $g_{\eta,q}$ is defined by

$$g_{\eta,q}(f)(x)=\Big(\int_0^\infty\big|f*\eta_t(x)\big|^q\frac{dt}{t}\Big)^{\frac{1}{q}}.$$

It is well known that if $q = 2$, then $g_{\eta,2}$ is just the classical Littlewood–Paley $g$-function, which is a bounded operator on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ (see, for example, [28, p. 159] or [18, p. 356]). For $2 < q < \infty$, the following conclusion is obvious:

**Lemma 4.1.** *For $2 < q < \infty$, $g_{\eta,q}$ is a bounded operator on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.*

In fact, it is clear that for $2 < q < \infty$,

$$g_{\eta,q}(f)(x) \le C g_{\eta,2}(f)(x)^{\frac{2}{q}} Mf(x)^{\frac{q-2}{q}},$$

where $M$ is the classical Hardy–Littlewood maximal operator defined by

$$Mf(x) = \sup_{r>0} \frac{1}{r^n} \int_{|x-y|\le r} \big|f(y)\big|\, dy.$$

Thus for any fixed $1 < p < \infty$, Hölder's inequality, the $L^p$ $(1 < p < \infty)$-boundedness of $g_{\eta,2}$, and $M$ (see [27]) imply that

$$\Big(\int_{\mathbb{R}^n} \big|g_{\eta,q}(f)(x)\big|^p\, dx\Big)^{\frac{1}{p}} \le C\big\|g_{\eta,2}(f)\big\|_p^{\frac{2}{q}} \|Mf\|_p^{\frac{q-2}{q}} \le C\|f\|_p. \tag{4.2}$$

Now we introduce a paraproduct associated with the tent space.

**Definition 4.2** (Paraproduct)**.** For a fixed $F \in T_q^\infty(\mathbb{R}_+^{n+1})$ $(1 < q < \infty)$, we define a paraproduct $\pi_F$ by

$$\pi_F(f)(x) = \int_0^\infty \eta_t * \big((f * \varphi_t)(\cdot)F(\cdot,t)\big)(x)\frac{dt}{t}, \tag{4.3}$$

where $\varphi$ satisfies (3.1) and where $\eta \in \mathscr{S}(\mathbb{R}^n)$ satisfies (4.1).

We can obtain the $L^q$ $(1 < q \le 2)$ boundedness of the paraproduct $\pi_F(f)$ if $F \in T_q^\infty(\mathbb{R}_+^{n+1})$.

**Theorem 4.2.** *Denote $\phi(x) := \operatorname{ess\,sup}_{|x|\le|y|} |\varphi(y)|$. If $\phi \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, and $F \in T_q^\infty(\mathbb{R}_+^{n+1})$ $(1 < q \le 2)$, then $\pi_F$ defined in (4.3) is a bounded operator on $L^q(\mathbb{R}^n)$.*

*Proof.* Applying Remark 3.3 and $g \in T_q^\infty(\mathbb{R}_+^{n+1})$, we know that

$$\Big(\int_{\mathbb{R}_+^{n+1}} \big|f * \varphi_t(x)\big|^q \big|F(x,t)\big|^q \frac{dx\,dt}{t}\Big)^{\frac{1}{q}} \le C\|F\|_{T_q^\infty} \|f\|_{L^q}.$$

For any $h \in L^{q'}(\mathbb{R}^n)$ with $\|h\|_{L^{q'}} \le 1$, by (4.2) we have

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^n} \pi_F(f)(x)h(x)\,dx\Big| \\
&\quad = \Big|\int_{\mathbb{R}^n}\int_0^\infty \eta_t * \big((f*\varphi_t)(\cdot)F(\cdot,t)\big)(x)h(x)\frac{dt}{t}\,dx\Big| \\
&\quad = \Big|\int_{\mathbb{R}^n}\int_0^\infty (f*\varphi_t)(x)F(x,t)(h*\widetilde{\eta}_t)(x)\frac{dt}{t}\,dx\Big|
\end{aligned}$$

$$
\begin{aligned}
&\le \Big(\int_{\mathbb{R}^{n+1}_+} \big|h*\widetilde{\eta}_t(x)\big|^{q'} \frac{dx\,dt}{t}\Big)^{\frac{1}{q'}} \Big(\int_{\mathbb{R}^{n+1}_+} \big|f*\varphi_t(x)\big|^{q} \big|F(x,t)\big|^{q} \frac{dx\,dt}{t}\Big)^{\frac{1}{q}} \\
&\le C\big\|g_{\widetilde{\eta},q'}(h)\big\|_{q'} \|F\|_{T^\infty_q} \|f\|_{L^q} \\
&\le C\|F\|_{T^\infty_q} \|f\|_{L^q}, \qquad (4.4)
\end{aligned}
$$

where in the following, $\widetilde{\eta}(x) = \eta(-x)$. We thus complete the proof of Theorem 4.2. □

Below we show that $\pi_g$ is a compact operator on $L^q$ if $F \in T^\infty_{q,v}(\mathbb{R}^{n+1}_+)$ $(1 < q \le 2)$.

**Theorem 4.3.** *Denote $\phi(x) := \operatorname{ess\,sup}_{|x|\le|y|} |\varphi(y)|$. If $\phi \in L^1 \cap L^\infty$, and $F \in T^\infty_{q,v}(\mathbb{R}^{n+1}_+)$ $(1 < q \le 2)$, then $\pi_F$ is a compact operator on $L^q(\mathbb{R}^n)$.*

*Proof.* Since $F \in T^\infty_{q,v}(\mathbb{R}^{n+1}_+)$, applying Remark 3.6, the operator $\mathscr{L}_\varphi$ is a compact operator from $L^p(\mathbb{R}^n; dx)$ to $L^p(\mathbb{R}^{n+1}_+; |F(x,t)|^q \frac{dx\,dt}{t})$ for all $1 < p < \infty$. Hence, for any sequence $\{f_k\}$ in $L^p(\mathbb{R}^n)$ which converges weakly to zero, we have

$$
\lim_{k\to\infty} \int_{\mathbb{R}^{n+1}_+} \big|f_k*\varphi_t(x)\big|^p \big|F(x,t)\big|^q \frac{dx\,dt}{t} = 0. \qquad (4.5)
$$

Notice that $L^q(\mathbb{R}^n)$ $(1 < q \le 2)$ is a reflexive space. When combining that with the fact in [25, p. 113, exe. 18] to prove that $\pi_F$ is a compact operator on $L^q(\mathbb{R}^n)$, it suffices to verify that, for any sequence $\{f_k\}$ in $L^q(\mathbb{R}^n)$ which converges weakly to zero, $\{\pi_F(f_k)\}$ converges to zero in $L^q$ norm. Equivalently, we need only show that

$$
\lim_{k\to\infty} \sup_{\|h\|_{q'}\le 1} \Big|\int_{\mathbb{R}^n} \pi_F(f_k)(x)h(x)\,dx\Big| = 0. \qquad (4.6)
$$

In fact, for any $h \in L^{q'}(\mathbb{R}^n)$ with $\|h\|_{q'} \le 1$, by (4.4) with $f$ instead by $f_k$ and by applying (4.2), we get

$$
\begin{aligned}
&\sup_{\|h\|_{q'}\le 1} \Big|\int_{\mathbb{R}^n} \pi_F(f_k)(x)h(x)\,dx\Big| \\
&\quad\le \sup_{\|h\|_{q'}\le 1} \Big(\int_{\mathbb{R}^{n+1}_+} \big|h*\widetilde{\eta}_t(x)\big|^{q'} \frac{dx\,dt}{t}\Big)^{\frac{1}{q'}} \Big(\int_{\mathbb{R}^{n+1}_+} \big|f*\varphi_t(x)\big|^{q} \big|F(x,t)\big|^{q} \frac{dx\,dt}{t}\Big)^{\frac{1}{q}} \\
&\quad\le C\Big(\int_{\mathbb{R}^{n+1}_+} \big|f_k*\varphi_t(x)\big|^{q} \big|F(x,t)\big|^{q} \frac{dx\,dt}{t}\Big)^{\frac{1}{q}}.
\end{aligned}
$$

Hence (4.6) holds from the above estimate and from (4.5). We therefore finish the proof of Theorem 4.3. □

4.2. **Boundedness and compactness of paraproducts with rough kernel.** From the proofs of Theorem 4.2 and Theorem 4.3, it can be seen that the $L^q$ boundedness of $g_{\eta,q}$ $(2 \le q < \infty)$ plays a very important role. Now we point out that, after removing the smoothness condition assumed on $\eta$ in Definition 4.2, we still may get the $L^q$ $(1 < q \le 2)$ boundedness and compactness of $\pi_F$ by using some known results.

Suppose that the function $\Omega$ on $\mathbb{R}^n \setminus \{0\}$ satisfies the following conditions:

(A) $\Omega(\lambda x) = \Omega(x)$, for any $\lambda > 0$ and $x \in \mathbb{R}^n \setminus \{0\}$;
(B) $\int_{\mathbb{S}^{n-1}} \Omega(x')\, d\sigma(x') = 0$; and
(C) $\int_{\mathbb{S}^{n-1}} |\Omega(x')|\, d\sigma(x') < \infty$.

Let $\eta(x) = \Omega(x)|x|^{1-n}\chi_{\{|x|\le 1\}}(x)$; then, for $1 < q < \infty$, the general Littlewood–Paley $g$-function $g_{\eta,q}$ is defined by

$$g_{\eta,q}(f)(x) := \Big(\int_0^\infty \big|f * \eta_t(x)\big|^q \frac{dt}{t}\Big)^{\frac{1}{q}} = \Big(\int_0^\infty \Big|\int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)\, dy\Big|^q \frac{dt}{t^{1+q}}\Big)^{\frac{1}{q}}.$$

Note that $g_{\eta,2}$ is just the Marcinkiewicz integral $g_\Omega$, which was first introduced by Stein [26]. It is easy to check that, for $2 < q < \infty$,

$$g_{\eta,q}(f)(x) \le C g_\Omega(f)(x)^{\frac{2}{q}} M_\Omega f(x)^{\frac{q-2}{q}}, \tag{4.7}$$

where $M_\Omega$ denotes the rough maximal operator defined by

$$M_\Omega f(x) = \sup_{r>0} \frac{1}{r^n} \int_{|x-y|\le r} \big|\Omega(x-y)\big|\big|f(y)\big|\, dy.$$

Thus the $L^p$ boundedness of $M_\Omega$ and the Marcinkiewicz integral $g_\Omega$ imply the $L^p$ boundedness of $g_{\eta,q}$ by (4.7). Now let us recall some known results on the $L^p$ boundedness of $M_\Omega$ and $g_\Omega$.

**Theorem F** (see [28])**.** *If we suppose that $\Omega$ satisfies the conditions (A) and (C), then $M_\Omega$ is bounded on $L^p$ for $1 < p \le \infty$.*

**Theorem G.** *Suppose that $\Omega$ satisfies the conditions (A) and (B).*

(1) *If $\Omega \in Lip_\alpha(\mathbb{S}^{n-1})$ $(0 < \alpha \le 1)$, then $g_\Omega$ is bounded on $L^p$ for $1 < p \le 2$ (see [26]) ;*
(2) *If $\Omega \in H^1(\mathbb{S}^{n-1})$, then $g_\Omega$ is bounded on $L^p$ for $1 < p < \infty$(see [13]);*
(3) *If $\Omega \in L(\log^+ L)^{\frac{1}{2}}(\mathbb{S}^{n-1})$, then $g_\Omega$ is bounded on $L^p$ for $1 < p < \infty$,*

*where $H^1(\mathbb{S}^{n-1})$ denotes the Hardy space on $\mathbb{S}^{n-1}$ (see [1]). The definition and some facts on $H^1(\mathbb{S}^{n-1})$ can be found in [11], [21] and [24].*

**Remark 4.4.** Notice the following well-known containing relation between some function spaces on $\mathbb{S}^{n-1}$:

$$L^\infty(\mathbb{S}^{n-1}) \subsetneq L^r(\mathbb{S}^{n-1})\ (1 < r < \infty) \subsetneq L\log^+ L(\mathbb{S}^{n-1}) \subsetneq H^1(\mathbb{S}^{n-1}) \subsetneq L^1(\mathbb{S}^{n-1}).$$

Moreover, the spaces $H^1(\mathbb{S}^{n-1})$ and $L(\log^+ L)^{\frac{1}{2}}(\mathbb{S}^{n-1})$ do not contain each other.

Thus, applying Theorem F and Theorem G, we get

**Lemma 4.5.** *Suppose that $\eta(x) = \Omega(x)|x|^{1-n}\chi_{\{|x|\le 1\}}(x)$ with $\Omega$ satisfying the conditions (A) and (B). Then for $2 < q < \infty$, $g_{\eta,q}$ is a bounded operator on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ if $\Omega \in H^1(\mathbb{S}^{n-1})$ or $\Omega \in L(\log^+ L)^{\frac{1}{2}}(\mathbb{S}^{n-1})$.*

Thus, by Lemma 4.5 and by using the same methods for proving Theorem 4.2 and Theorem 4.3, we can obtain the $L^q(\mathbb{R}^n)$ boundedness and compactness of the rough paraproduct $\pi_F$ for $1 < q \leq 2$.

**Theorem 4.6.** *Suppose that the paraproduct $\pi_F$ is defined by (4.3), where*

(1) $F \in T_q^\infty(\mathbb{R}_+^{n+1})$ $(1 < q \leq 2)$;
(2) $\eta(x) = \Omega(x)|x|^{1-n}\chi_{\{|x|\leq 1\}}(x)$ *with* $\Omega$ *satisfying the conditions (A) and (B); and*
(3) $\varphi$ *satisfies (3.1) (or* $\phi \in L^1 \cap L^\infty$ *with* $\phi(x) := \operatorname{ess\,sup}_{|x|\leq|y|} |\varphi(y)|$).

*If* $\Omega \in H^1(\mathbb{S}^{n-1})$ *or* $\Omega \in L(\log^+ L)^{\frac{1}{2}}(\mathbb{S}^{n-1})$, *then* $\pi_F$ *is a bounded operator on* $L^q(\mathbb{R}^n)$.

**Theorem 4.7.** *Suppose that the paraproduct $\pi_F$ is defined by (4.3), where*

(1) $F \in T_{q,v}^\infty(\mathbb{R}_+^{n+1})$ $(1 < q \leq 2)$;
(2) $\eta(x) = \Omega(x)|x|^{1-n}\chi_{\{|x|\leq 1\}}(x)$ *with* $\Omega$ *satisfying the conditions (A) and (B); and*
(3) $\varphi$ *satisfies (3.1) and* $\phi \in L^1 \cap L^\infty$ *with* $\phi(x) := \operatorname{ess\,sup}_{|x|\leq|y|} |\varphi(y)|$.

*If* $\Omega \in H^1(\mathbb{S}^{n-1})$ *or* $\Omega \in L(\log^+ L)^{\frac{1}{2}}(\mathbb{S}^{n-1})$, *then* $\pi_F$ *is a compact operator on* $L^q(\mathbb{R}^n)$.

4.3. **The map from tent spaces to $L^p$ and Hardy space.** In [9], the authors established a close connection between the tent space $T_2^p(\mathbb{R}_+^{n+1})$ and $L^p(\mathbb{R}^n)$, the Hardy space $H^p(\mathbb{R}^n)$ and $BMO(\mathbb{R}^n)$ space. Suppose that the function $\Phi$ satisfies the following conditions:

(i) $\operatorname{supp}(\Phi) \subset \{x \in \mathbb{R}^n : |x| < 1\}$;
(ii) there exists a constant $B > 0$, such that $|\Phi(x)| \leq B, |\Phi(x+h) - \Phi(x)| \leq B(|h|/|x|)^\epsilon$ for some $\epsilon > 0$;
(iii) $\int \Phi(x)\, dx = 0$; and
($\text{iii}_N$) $\int x^\gamma \Phi(x)\, dx = 0$, for all $|\gamma| \leq N$.

The operator $\Pi_\Phi$ is defined by

$$\Pi_\Phi(F)(x) = \int_0^\infty \big(F(\cdot,t) * \Phi_t\big)(x)\frac{dt}{t}, \tag{4.8}$$

where $\Phi_t(x) = t^{-n}\Phi(x/t)$.

**Theorem H** ([9, p. 328])**.** *If $\Phi$ satisfies (i), (ii), (iii), then the linear operator $\Pi_\Phi$ defined in (4.8) can be extended to a bounded operator:*

(1) *from* $T_2^p(\mathbb{R}_+^{n+1})$ *to* $L^p(\mathbb{R}^n)$, *if* $1 < p < \infty$;
(2) *from* $T_2^1(\mathbb{R}_+^{n+1})$ *to* $H^1(\mathbb{R}^n)$; *and*
(3) *from* $T_2^\infty(\mathbb{R}_+^{n+1})$ *to* $BMO(\mathbb{R}^n)$.
(4) *If* ($\text{iii}_N$) *is satisfied with* $N \geq n[1/p - 1]$ ($[x]$ *indicates the integer part of* $x$), *then* $\Pi_\Phi$ *can be extended to a bounded operator from* $T_2^p(\mathbb{R}_+^{n+1})$ *to* $H^p(\mathbb{R}^n)$ *for* $0 < p \leq 1$.

**Remark 4.8.** In [9], the authors pointed out that Theorem H still holds if the conditions (i) and (ii) assumed on $\Phi$ are replaced by the following condition:

(i′) there exists a constant $B > 0$, such that $|\Phi(x)| + |\nabla\Phi(x)| \leq B(1+|x|)^{-n-1}$.

Below we show that the conclusions (1), (2), and (4) of Theorem H still hold if replacing the space $T^p_2(\mathbb{R}^{n+1}_+)$ by $T^p_q(\mathbb{R}^{n+1}_+)$ $(1<q<2)$.

**Theorem 4.9.** *If $\Phi$ satisfies (i)–(iii) (or (i′), (iii)), and $1<q<2$, then the operator $\Pi_\Phi$ can be extended to a bounded operator:*

(1) *from $T^p_q(\mathbb{R}^{n+1}_+)$ to $L^p(\mathbb{R}^n)$, if $1<p<\infty$; and*
(2) *from $T^1_q(\mathbb{R}^{n+1}_+)$ to $H^1(\mathbb{R}^n)$.*
(3) *If ($iii_N$) is satisfied with $N\ge n[1/p-1]$ ($[x]$ indicates the integer part of $x$), then $\Pi_\Phi$ can be extended to a bounded operator from $T^p_q(\mathbb{R}^{n+1}_+)$ to $H^p(\mathbb{R}^n)$ for $0<p\le 1$.*

The key to verifying Theorem 4.9 is to apply the $L^p$ boundedness of the following general area integral operator, which is defined for $1<q<\infty$ by

$$S_{\Phi,q}(f)(x)=\Big(\int_{\Gamma(x)}\big|f*\Phi_t(y)\big|^q\frac{dy\,dt}{t^{n+1}}\Big)^{\frac1q}\quad\text{for any } f\in L^p\ (1<p<\infty).$$

It is easy to see that $S_{\Phi,2}(f)$ is the classical square function. Thus $S_{\Phi,2}$ is bounded on $L^p(\mathbb{R}^n)$ for $1<p<\infty$. For $2<q<\infty$, it is easy to check that

$$S_{\Phi,q}(f)(x)\le CS_{\Phi,2}(f)(x)^{\frac2q}Mf(x)^{\frac{q-2}{q}}.$$

For any $1<p<\infty$, as with (4.2), using Hölder's inequality with the indexes $\frac q2$ and $\frac{q}{q-2}$, we have

$$\Big(\int_{\mathbb{R}^n}\big|S_{\Phi,q}f(x)\big|^p\,dx\Big)^{\frac1p}\le C\big\|S_{\Phi,2}(f)\big\|_p^{\frac2q}\|Mf\|_p^{\frac{q-2}{q}}\le C\|f\|_p. \tag{4.9}$$

*Proof of Theorem 4.9.* First we consider the conclusion (a). For any $h\in L^{p'}(\mathbb{R}^n)$ with $\|h\|_{p'}\le 1$, for $1<q<2$, applying an estimate [9, (5.1)] and (4.9), we have

$$\begin{aligned}\Big|\int_{\mathbb{R}^n}\Pi_\Phi(F)(x)h(x)\,dx\Big|&=\Big|\int_{\mathbb{R}^{n+1}_+}\big(F(\cdot,t)*\Phi_t\big)(x)h(x)\frac{dx\,dt}{t}\Big|\\&=\Big|\int_{\mathbb{R}^{n+1}_+}F(x,t)(h*\tilde\Phi_t)(x)\frac{dx\,dt}{t}\Big|\\&\le C\int_{\mathbb{R}^n}A_q(F)(x)A_{q'}(h*\tilde\Phi_t)(x)\,dx\\&\le C\|F\|_{T^p_q}\big\|S_{\tilde\Phi,q'}(h)\big\|_{p'}\\&\le C\|F\|_{T^p_q},\end{aligned}$$

where $\tilde\Phi(x)=\Phi(-x)$. Thus we show the conclusion (a). Using the conclusion (a) and the idea of proving the conclusion (b) of Theorem H, we may get Theorem 4.9(b). The proof of (c) is similar. Thus we finish the proof of Theorem 4.9.

**Remark 4.10.** We wonder whether the operator $S_{\Phi,q}$ is the bounded Hardy space $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ when $q>2$. Thus it is also not clear whether the operator $\Pi_\Phi$ can be extended to a bounded operator from $T^\infty_q(\mathbb{R}^{n+1}_+)$ to $BMO(\mathbb{R}^n)$ for $1<q<2$.

**Remark 4.11.** Notice that, for any $F \in T_2^p(\mathbb{R}_+^{n+1})$ $(1 < p < \infty)$ (or $F \in T_2^1(\mathbb{R}_+^{n+1})$) and any $h \in L^{p'}(\mathbb{R}^n)$ (or $h \in BMO(\mathbb{R}^n)$), by Theorem H we see that

$$\begin{aligned}\infty > \langle \Pi_\Phi(F), h\rangle &= \int_{\mathbb{R}_+^{n+1}} \big(F(\cdot,t) * \Phi_t\big)(x)h(x)\frac{dx\,dt}{t}\\ &= \int_{\mathbb{R}_+^{n+1}} F(x,t)(h * \tilde{\Phi}_t)(x)\frac{dx\,dt}{t}\\ &= \int_{\mathbb{R}_+^{n+1}} F(x,t)\mathscr{L}_{\tilde{\Phi}}(h)(x)\frac{dx\,dt}{t}. \end{aligned}\tag{4.10}$$

Thus (4.10) and Theorem A show that the adjoint operator of $\Pi_\Phi$ is $\mathscr{L}_{\tilde{\Phi}}$ in some sense; the latter is defined in (3.2). In other words, the operator $\mathscr{L}_{\tilde{\Phi}}$ maps $L^{p'}(\mathbb{R}^n)$ to $T_2^{p'}(\mathbb{R}_+^{n+1})$ (or maps $BMO$ to $T_2^\infty(\mathbb{R}_+^{n+1})$). Morever, it is easy to verify that the operator $\mathscr{L}_{\tilde{\Phi}}$ is actually a bounded operator from $L^{p'}(\mathbb{R}^n)$ to $T_2^{p'}(\mathbb{R}_+^{n+1})$ and from $BMO(\mathbb{R}^n)$ to $T_2^\infty(\mathbb{R}_+^{n+1})$.

If $\Phi$ satisfies (i)–(iii) (or (i′), (iii)), and $h \in BMO(\mathbb{R}^n)$, define the paraproduct $\pi$ as follows:

$$\pi(f)(x) = \int_0^\infty \eta_t * \big((f * \varphi_t)(\cdot)(\tilde{\Phi}_t * h)(\cdot)\big)(x)\frac{dt}{t},\tag{4.11}$$

where $\varphi$ and $\eta$ are as Definition 4.2. The $L^2$ boundedness and $L^2$ compactness of the paraproduct $\pi$ were studied in [15] and [14], respectively.

Obviously, the paraproduct defined in (4.11) is only a particular case of the paraproduct $\pi_F$ defined in (4.3) with $F := (\tilde{\Phi}_t * h)(x) = \mathscr{L}_{\tilde{\Phi}}(h) \in T_2^\infty(\mathbb{R}_+^{n+1})$.

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