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LIMIT DYNAMICAL SYSTEMS AND C^*-ALGEBRAS FROM SELF-SIMILAR GRAPH ACTIONS

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Abstract. In this article, we study dynamical and C^*-algebraic properties of self-similar group actions on finite directed graphs. We investigate the structure of limit dynamical systems induced from group actions on graphs, and we deduce conditions of group actions and graphs for the groupoid C^*-algebras defined by limit dynamical systems to be simple, separable, purely infinite, nuclear, and satisfying the universal coefficient theorem.

1. Introduction

Since Nekrashevych's book [15] and article [16], self-similar groups have become an important topic in the study of geometric group theory, dynamical systems, and C^*-algebras. In their recent article [7], Exel and Pardo introduced a generalization of self-similar groups: the self-similar graph action.

A *self-similar graph action* (G, E) consists of a finitely generated group G and a directed graph E, as well as the cocycle $\varphi : G \times E^1 \to G$ with a "self-similar" G-action on E^* (the set of finite path space of E). The self-similar groups of Nekrashevych are special cases of self-similar graph actions where E is an n-bouquet X and the G-action on X is faithful so that the cocycle of $g \in G$ at $e \in X$ is given as $g|_e$, the restriction of g at e. Whereas Nekrashevych constructed a Cuntz–Pimsner algebra $\mathcal{O}_G$ and a groupoid algebra $\mathcal{O}_f$ from a self-similar group (G, X) and its corresponding limit dynamical system, Exel and Pardo built a Cuntz–Pimsner

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algebra $\mathcal{O}_{G,E}$ from a self-similar graph action (G, E), and they showed that $\mathcal{O}_{G,E}$ is isomorphic to the C^*-algebra of the groupoid $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$. The purpose of this article is to study limit dynamical systems of self-similar graph actions and their associated groupoid C^*-algebras.

Because Exel and Pardo's argument covers a very large class of group actions on graphs, we need to give some restrictions on it; our graph E is a finite, connected, and directed graph with no sink or source, and our group G is a finitely generated countable group. Under these restrictions, we imitate [15] and [16] to construct the limit space $J_{(G,E)}$ as the quotient of the one-sided infinite path space over the graph E by an asymptotic equivalence relation and to define the shift map σ on $J_{(G,E)}$ so that the limit dynamical system $(J_{(G,E)}, \sigma)$ is obtained.

Even under these strong restrictions, the limit space and the limit dynamical system of (G, E) are quite different from the case of self-similar groups. While the limit space of a self-similar group is connected and locally connected with *level-transitive* and *recurrent* conditions, $J_{(G,E)}$ is neither connected nor locally connected in general. The shift map on the limit space of a self-similar group is a covering map with *regular* condition, and it is unclear whether or not the shift map σ on $J_{(G,E)}$ is an open map. The topological and dynamical properties of self-similar graph actions are much more complex than those of self-similar groups, and this makes the structure of the groupoid C^*-algebra obtained from $(J_{(G,E)}, \sigma)$ very complicated. Moreover, as Thomsen observed in [21], the groupoid of Anantharaman-Delaroche, Deaconu, and Renault (see [1], [2], [5], [19]) naturally associated to a dynamical system may not be an étale groupoid if the map of the dynamical system is not open. Our limit dynamical system $(J_{(G,E)}, \sigma)$ is one of those cases where its groupoid $\Gamma_{(G,E)}$ may not be étale in the general situation. Hence it is important to determine the conditions on (G, E) that will make $\Gamma_{(G,E)}$ an étale groupoid.

As in the case of $\mathcal{O}_G$ and $\mathcal{O}_f$ from self-similar groups (see [16]), it is reasonable to guess that the properties of $\Gamma_{(G,E)}$ are strongly related to those of $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$. Following Exel and Pardo's observation that their groupoid $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ is remarkably similar to the groupoid defined by the shift map on the infinite path space of the graph described in [13], we are able to explain the limit dynamical system $(J_{(G,E)}, \sigma)$ in the terminology of Exel and Pardo. Then we find conditions for $\Gamma_{(G,E)}$ to become an étale and amenable groupoid. Although our conditions are very restrictive, they induce the groupoid algebra $C^*(\Gamma_{(G,E)})$, which is a Kirchberg algebra with some extra conditions introduced by Exel and Pardo.

This article is organized as follows. In Section 2, we introduce our notation and review facts for later use. We modify the definition of the self-similar graph action by Exel and Pardo to group actions on finite graphs. In Section 3, we check topological properties of the limit space $J_{(G,E)}$ of a self-similar graph action (G, E). We show that $J_{(G,E)}$ is a compact Hausdorff metrizable space that is not connected or locally connected in general. In Section 4, we build the limit dynamical system $(J_{(G,E)}, \sigma)$ from (G, E), and we show that G-transitivity and pseudofree conditions of Exel and Pardo imply topological transitivity and the topologically free property, respectively, of $(J_{(G,E)}, \sigma)$. In the last section, we prove our main result: if (G, E) is a contracting, regular, and pseudofree self-similar action and the graph E satisfies Condition (L) and the G-transitive condition, then the groupoid algebra $C^*(\Gamma_{(G,E)})$ is a simple, separable, nuclear, and purely infinite C^*-algebra satisfying the universal coefficient theorem (UCT).

2. Self-similar graph actions

We modify the definition of self-similar actions of groups on directed graphs by Exel and Pardo [7] to group actions on finite graphs. For general references on the notion of graphs, we refer the reader to [10], [12], and [13].

Directed graphs. A directed graph is a quadruple $E = (E^0, E^1, d, r)$, where E^0 is the set of vertices, E^1 is the set of edges, and d, r are maps from E^1 to E^0 describing the domain and the range of edges. A directed graph E is called *finite* if E^0 and E^1 are finite sets. A vertex is called a *sink* if it does not emit any edge and is called a *source* if it does not receive any edge.

Suppose that E is a directed graph. A *finite path* of length $n \geq 1$ in E is any finite sequence $a = a_1 \cdots a_n$, where $a_i \in E^1$ and $r(a_i) = d(a_{i+1})$ for every i. The *domain* of a is defined by $d(a) = d(a_1)$, and the *range* of a is defined by $r(a) = r(a_n)$. A vertex $v \in E^0$ will be considered as a path of length zero with $d(v) = r(v) = v$. For every integer $n \geq 0$, we denote by E^n the set of paths of length n in E and denote by E^* the set of finite paths in E; that is,

$$E^* = \bigcup_{n=0}^{\infty} E^n.$$

If a and b are paths in E such that $r(a) = d(b)$, then ab denotes the path obtained by concatenating a and b.

We will also consider the space $E^{-\omega}$ of left-infinite paths of the form $\cdots a_{-2}a_{-1}$ over E, where $a_i \in E^1$ and $r(a_i) = d(a_{i+1})$. The product topology of the discrete set E^1 is given on $E^{-\omega}$. A cylinder set $Z(a)$ for each $a \in E^*$ is defined as follows:

$$Z(a) = \{\alpha \in E^{-\omega} : \alpha = \cdots a_{-n-1}a_{-n} \cdots a_{-1} \text{ such that } a_{-n} \cdots a_{-1} = a\}.$$

Then the collection of all such cylinder sets forms a basis for the product topology on $E^{-\omega}$.

Group actions on graphs. Suppose that G is a group, E is a directed graph, and $\operatorname{Aut} E$ is the group of graph automorphisms of E. We denote the (left) actions of G on E^0 and E^1 by

$$(g, v) \mapsto gv \qquad \text{and} \qquad (g, e) \mapsto ge \quad \text{for } g \in G, v \in E^0 \text{ and } e \in E^1.$$

We say that G acts on E if there is a group homomorphism $G \to \operatorname{Aut} E$ that satisfies

$$d(ge) = gd(e) \qquad \text{and} \qquad r(ge) = gr(e) \quad \text{for } e \in E^1.$$

Self-similar graph actions. We follow [7], [15], and [16] to modify the definition of self-similar groups acting on finite sets to directed graphs.

Definition 2.1 (see [7], [15], [16]). Suppose that E is a finite directed graph with no sink or source and that G is a countable group acting faithfully on E^*. We call the pair (G,E) a *self-similar graph action* if, for all $g \in G$ and $e \in E^1$, there exists a unique $h \in G$ such that

$$g(ea) = g(e)h(a)$$

for every $a \in E^*$ with $r(e) = d(a)$.

By induction, we may obtain that, for all $g \in G$ and $b \in E^*$, there is a unique element $h \in G$ such that $g(ba) = g(b)h(a)$, where $a \in E^*$ with $r(b) = d(a)$. The unique element h is called the *restriction* of g at b and is denoted by $g|_b$. For $c = g(b)$ and $h = g|_b$, we formally write the above equality as

$$g \cdot b = c \cdot h.$$

We summarize basic properties of restrictions as follows (see [15], [16]): for $g, h \in G$ and $a, b \in E^*$,

$$g|_{ab} = (g|_a)|_b, \qquad (gh)|_a = g|_{h(a)}h|_a, \qquad (g|_a)^{-1} = g^{-1}|_{g(a)}.$$

Definition 2.2 (see [15], [16]). A self-similar graph action (G,E) is said to be *contracting* if there is a finite subset N of G satisfying the following: for every $g \in G$, there is $n \geq 0$ such that $g|_a \in N$ for every $a \in E^*$ of length $|a| \geq n$. If the action is contracting, then the smallest finite subset of G satisfying this condition is called the *nucleus* of the group and is denoted by $\mathcal{N}$.

Standing assumption. In this article, we assume that every group is a finitely generated countable group, every graph is a connected finite directed graph with no sink or source, and our self-similar graph action is contracting.

For general structures and properties of self-similar graph actions and self-similar groups, we refer the reader to [7], [15], and [16], and we mention only what we will use in this article.

Lemma 2.3. *Suppose that (G,E) is a self-similar graph action.*

(1) *For all $g \in G$ and $a \in E^n$ for some $n \geq 0$, $g(a) \in E^n$.*
(2) *For every $g \in G$, $\lambda_g : E^n \to E^n$ defined by $a \mapsto g(a)$ is a bijective map.*
(3) *For all $g \in G$ and $a \in E^n$, $d(a) = r(a)$ if and only if $d(g(a)) = r(g(a))$.*
(4) *For all $g \in G$ and $a \in E^n$, $d(a) \neq r(a)$ if and only if $d(g(a)) \neq r(g(a))$.*

Proof. (1) By the definition of group actions on graphs, $g(x) \in E^0$ for every $x \in E^0$ and $g(e) \in E^1$ for every $e \in E^1$. If $a = e_1 \cdots e_n \in E^n$, then

$$g(a) = g(e_1 \cdots e_n) = g(e_1)g|_{e_1}(e_2 \cdots e_n) = g(e_1)g|_{e_1}(e_2) \cdots g|_{e_1 \cdots e_{n-1}}(e_n)$$

implies $g(a) \in E^n$.

(2) If $a, b \in E^n$ and $g(a) = g(b)$, then we have $g^{-1}(g(a)) = g^{-1}(g(b))$ and λ_g is one-to-one. For every $a \in E^n$, $g^{-1}(a) \in E^n$ by (1) and $\lambda_g(g^{-1}(a)) = a$. Hence λ_g is a bijection.

(3) and (4) These statements are trivial because $d(g(a)) = g(d(a))$ and $r(g(a)) = g(r(a))$ hold and because λ_g is a one-to-one map. $\square$

The following lemma is trivial because the nucleus $\mathcal{N}$ is a finite set.

Lemma 2.4 ([15, Lemma 2.11.2])**.** *If (G,E) is a contracting self-similar action with the nucleus $\mathcal{N}$, then there is a $k \in \mathbb{N}$ such that $g|_{E^n} \subset \mathcal{N}$ for every $g \in \mathcal{N}$ and $n \geq k$.*

We now consider the space $E^{-\omega}$ of left-infinite paths $\cdots a_{-2}a_{-1}$ over E.

Definition 2.5 (see [15], [16]). Two paths $\cdots a_{-2}a_{-1}$ and $\cdots b_{-2}b_{-1}$ in $E^{-\omega}$ are said to be *asymptotically equivalent* if there is a finite set $I \subset G$ and a sequence $g_n \in I$ such that

$$g_n(a_{-n} \cdots a_{-1}) = b_{-n} \cdots b_{-1}$$

for every $n \in \mathbb{N}$.

Instead of an arbitrary finite subset of G, the nucleus $\mathcal{N}$ of G may be used to determine asymptotic equivalence.

Proposition 2.6 ([16, Section 2.3])**.** *Two paths $\cdots a_{-2}a_{-1}$ and $\cdots b_{-2}b_{-1}$ in $E^{-\omega}$ are asymptotically equivalent if and only if there is a sequence g_n of elements of the nucleus $\mathcal{N}$ such that*

$$g_n \cdot a_{-n} = b_{-n} \cdot g_{n-1}$$

for every $n \in \mathbb{N}$.

Remark 2.7. If two paths $\cdots a_{-2}a_{-1}$ and $\cdots b_{-2}b_{-1}$ in $E^{-\omega}$ are asymptotically equivalent to each other and $g_n \in \mathcal{N}$ is the group element satisfying the equality in Proposition 2.6, then we have

$$g_n \cdot (a_{-n} \cdots a_{-m}) = (b_{-n} \cdots b_{-m}) \cdot g_{m-1}.$$

Remark 2.8. By Proposition 3.2.6 of [15] every asymptotic equivalence class on $E^{-\omega}$ has no more than $|\mathcal{N}|$ elements, where $\mathcal{N}$ is the nucleus of the group G.

The quotient of the space $E^{-\omega}$ by the asymptotic equivalence relation is called the *limit space* of (G,E) and denoted by $J_{(G,E)}$. Since the asymptotic equivalence relation is invariant under the shift map $\sigma : \cdots a_{-2}a_{-1} \mapsto \cdots a_{-3}a_{-2}$ (see Lemma 2.9 below), the shift map induces a continuous map $J_{(G,E)} \to J_{(G,E)}$. By abuse of notation, we use σ to denote the induced shift map on $J_{(G,E)}$. The dynamical system $(J_{(G,E)}, \sigma)$ is called the *limit dynamical system* of (G,E) (see [15], [16] for details).

The following property is trivial and used several times in [15] and [16], but it was not mentioned explicitly.

Lemma 2.9. *For a self-similar graph action (G,E), let $q : E^{-\omega} \to J_{(G,E)}$ be the quotient map, and let $\sigma : E^{-\omega} \to E^{-\omega}$ and $\sigma : J_{(G,E)} \to J_{(G,E)}$ be the shift maps. Then we have $\sigma \circ q = q \circ \sigma$.*

Proof. If $\xi = \cdots a_{-2}a_{-1}$ and $\eta = \cdots b_{-2}b_{-1}$ in $E^{-\omega}$ are asymptotically equivalent, then there is a finite set $I \subset G$ such that, for each n, we can find a $g_n \in I$ satisfying $g_n(a_{-n} \cdots a_{-1}) = b_{-n} \cdots b_{-1}$. Then the self-similar condition implies that there are unique elements $h_1 \in G$, determined by g_n and a_{-n}, and $h_i \in G$, determined inductively by h_{i-1} and a_{-n+i-1}, such that

$$\begin{aligned} g_n(a_{-n} \cdots a_{-1}) &= g_n(a_{-n})h_1(a_{-n+1} \cdots a_{-1}) \\ &= \cdots \\ &= g_n(a_{-n})h_1(a_{-n+1})h_2(a_{-n+2}) \cdots h_{n-1}(a_{-1}) \\ &= b_{-n}b_{-n+1}b_{-n+2} \cdots b_{-1}. \end{aligned}$$

Hence we have

$$g_n(a_{-n} \cdots a_{-2}) = g_n(a_{-n})h_1(a_{-n+1}) \cdots h_{n-2}(a_{-2}) = b_{-n}b_{-n+1} \cdots b_{-2}.$$

Therefore $\sigma(\xi) = \cdots a_{-3}a_{-2}$ is also asymptotically equivalent to $\sigma(\eta) = \cdots b_{-3}b_{-2}$, and $\sigma \circ q = q \circ \sigma$ holds. □

3. Topological spaces from graph actions

First we study topological structures of $J_{(G,E)}$. We show that $J_{(G,E)}$ is compact metrizable but neither connected nor locally connected in a general situation.

The proof of the following proposition is identical to the proofs of Proposition 3.2.8 and Theorem 3.6.3 in [15], and so we omit it.

Proposition 3.1 (see [15])**.** *If (G, E) is a self-similar graph action, then $J_{(G,E)}$ is a compact metrizable space.*

We observe a relatively simple method to obtain open sets in $J_{(G,E)}$.

Lemma 3.2. *Let (G, E) be a self-similar graph action, and let $q : E^{-\omega} \to J_{(G,E)}$ be the quotient map. Suppose that, for some $n \geq 0$, E^n has two nonempty subsets A and B such that*

(1) *$A \cup B = E^n$;*
(2) *$A \cap B = \emptyset$; and*
(3) *for any $a \in A$ and $b \in B$, there is no $g \in G$ such that $g(a) = b$.*

Let

$$C = \bigcup_{a \in A} Z(a) \qquad \text{and} \qquad D = \bigcup_{b \in B} Z(b).$$

Then $q(C)$ and $q(D)$ are nonempty open sets in $J_{(G,E)}$ such that

$$q(C) \cup q(D) = J_{(G,E)} \qquad \text{and} \qquad q(C) \cap q(D) = \emptyset.$$

Proof. As C and D are unions of cylinder sets, they are open in $E^{-\omega}$. And because q is a surjective map and $C \cup D = E^{-\omega}$, it is trivial that $q(C)$ and $q(D)$ are nonempty subsets of $J_{(G,E)}$ with

$$q(C) \cup q(D) = q(C \cup D) = q(E^{-\omega}) = J_{(G,E)}.$$

We check that $q(C)$ and $q(D)$ are disjoint to each other: If $q(C) \cap q(D) \neq \emptyset$, then there are two infinite paths $\xi = \cdots e_{-2}e_{-1} \in C$ and $\eta = \cdots f_{-2}f_{-1} \in D$ such that $q(\xi) = q(\eta) \in q(C) \cap q(D)$. So ξ and η are asymptotically equivalent to each other, and there is a finite subset I of G such that, for every $k \in \mathbb{N}$,

$$g_k(e_{-k} \cdots e_{-1}) = f_{-k} \cdots f_{-1} \quad \text{for some } g_k \in I.$$

But $\xi = \cdots e_{-2}e_{-1} \in C$ and $\eta = \cdots f_{-2}f_{-1} \in D$ mean $e_{-n}\cdots e_{-1} = a \in A$ and $f_{-n}\cdots f_{-1} = b \in B$. Thus there cannot exist any $g \in G$ with $g(a) = b$ by condition (3), which is a contradiction. Hence we have

$$q(C) \cap q(D) = \emptyset.$$

Now we show that $q(C)$ and $q(D)$ are open in $J_{(G,E)}$. First we recall that

$$C \subset q^{-1} \circ q(C) \qquad \text{and} \qquad D \subset q^{-1} \circ q(D).$$

If $C \subsetneq q^{-1} \circ q(C)$, then we have $q^{-1} \circ q(C) \cap D \neq \emptyset$ as $C \cup D = E^{-\omega}$, and

$$\big(q^{-1} \circ q(C)\big) \cap \big(q^{-1} \circ q(D)\big) \neq \emptyset.$$

But $q(C) \cap q(D) = \emptyset$ implies that

$$q^{-1}\big(q(C) \cap q(D)\big) = \big(q^{-1} \circ q(C)\big) \cap \big(q^{-1} \circ q(D)\big) = \emptyset,$$

which is a contradiction. Thus we obtain $C = q^{-1} \circ q(C)$, and $q(C)$ is open in $J_{(G,E)}$. By the same argument, $q(D)$ is also open in $J_{(G,E)}$. $\square$

Remark 3.3. If (G, E) is a contracting self-similar graph action with its nucleus $\mathcal{N}$, then it is not difficult to observe that we can use elements of $\mathcal{N}$ instead of arbitrary group elements in condition (3) of Lemma 3.2.

Nekrashevych showed that for a graph E of one vertex (i.e., E is an n-bouquet), $J_{(G,E)}$ is connected if the G-action is *level-transitive* and is locally connected if the G-action is *recurrent* (see [15, Theorem 3.6.3]). Contrary to the n-bouquet case, it turns out that $J_{(G,E)}$ may not be connected or locally connected in general.

Proposition 3.4. *If a graph E has two edges e_1, e_2 such that $d(e_1) = r(e_1)$ and $d(e_2) \neq r(e_2)$, then $J_{(G,E)}$ is neither connected nor locally connected for any group G.*

Proof. We use Lemma 3.2. Divide E^1 into two classes:

$$A = \big\{e \in E^1 \text{ with } d(e) = r(e)\big\};$$
$$B = \big\{f \in E^1 \text{ with } d(f) \neq r(f)\big\}.$$

Then A and B are nonempty subsets of E^1 as $e_1 \in A$ and $e_2 \in B$, and it is trivial that $A \cup B = E^1$ and $A \cap B = \emptyset$. For condition (3) of Lemma 3.2, Lemma 2.3 says that $g(e) \in A$ and $g(f) \in B$ for all $e \in A$, $f \in B$, and $g \in G$. Therefore $J_{(G,E)}$ is a disjoint union of nonempty open sets by Lemma 3.2, and it is disconnected.

For non–local connectedness, we first remark that there is at least one pair of edges $e \in A$ and $f \in B$ such that $d(f) \neq r(f) = d(e) = r(e)$, as we assumed that our graph is a finite connected one with no sink or source. Let

$$v = r(f) = d(e) = r(e).$$

Then again by our standing assumption we can find an infinite path $\xi = \cdots a_{-2}a_{-1}$ such that $v = d(a_{-i})$ for infinitely many i (e.g., $\xi = \cdots ee$). We show that any open neighborhood of $q(\xi)$ in $J_{(G,E)}$ is disconnected.

Let U be an open neighborhood of $q(\xi)$. Then $q^{-1}(U)$ is open in $E^{-\omega}$, and there is an $n \in \mathbb{N}$ such that $Z(a_{-n} \cdots a_{-1}) \subset q^{-1}(U)$. We choose a $k \in \mathbb{N}$ such that $k \geq n$ and $v = d(a_{-k})$. It is clear that, for any $e_i \in E^1$,

$$Z(e_{-k-1}a_{-k} \cdots a_{-n} \cdots a_{-1}) \subset Z(a_{-n} \cdots a_{-1}).$$

For A and B defined above, we divide E^{k+1} into two classes,

$$C = \{e_{-k-1} \cdots e_{-1} \in E^{k+1} : e_{-k-1} \in A\},$$
$$D = \{e_{-k-1} \cdots e_{-1} \in E^{k+1} : e_{-k-1} \in B\},$$

and let

$$E = \bigcup_{e_{-k-1} \cdots e_{-1} \in C} Z(e_{-k-1} \cdots e_{-1}) \qquad \text{and} \qquad F = \bigcup_{e_{-k-1} \cdots e_{-1} \in D} Z(e_{-k-1} \cdots e_{-1}).$$

Then it is not difficult to check that C and D satisfy conditions of Lemma 3.2 so that $q(E)$ and $q(F)$ are disjoint open sets in $J_{(G,E)}$ with $q(E) \cup q(F) = J_{(G,E)}$. To show that U is disconnected, we need only to check that $q(E) \cap U \neq \emptyset$ and $q(F) \cap U \neq \emptyset$.

Consider $ea_{-k} \cdots a_{-1}$ and $fa_{-k} \cdots a_{-1}$. Since e, f, and a_{-k} are chosen to satisfy $v = r(f) = d(e) = r(e) = d(a_{-k})$, we have $ea_{-k} \cdots a_{-1} \in C$ and $fa_{-k} \cdots a_{-1} \in D$ with

$$Z(ea_{-k} \cdots a_{-1}) \subset E \cap Z(a_{-n} \cdots a_{-1}) \subset E \cap q^{-1}(U) \qquad \text{and}$$
$$Z(fa_{-k} \cdots a_{-1}) \subset F \cap Z(a_{-n} \cdots a_{-1}) \subset F \cap q^{-1}(U).$$

Therefore we obtain $q(E) \cap U \neq \emptyset$ and $q(F) \cap U \neq \emptyset$, and $J_{(G,E)}$ is not locally connected. $\square$

Remark 3.5. Level-transitive and recurrent conditions are special cases of the transitive G-action on E, where E is an n-bouquet. But they cannot be defined on general graphs, as we can see from Proposition 3.4. Instead of them, we will use the G-transitive condition (see Definition 4.1 below) of Exel and Pardo (see [7] for details).

4. Limit dynamical systems from graph actions

We follow Exel and Pardo [7] to study the limit dynamical system $(J_{(G,E)}, \sigma)$ of a self-similar graph action (G, E).

Definition 4.1 (see [7]). Suppose that (G, E) is a self-similar graph action. We say that E is *G-transitive* if, for any two vertices u and v of E, there is a finite sequence of vertices $u = u_0, u_1, \ldots, u_n = v$ such that, for each $i \in \{1, \ldots, n\}$, either there is a $g_i \in G$ such that

$$g_i(u_{i-1}) = u_i$$

or there is a path $a_i \in E^*$ such that

$$d(a_i) = u_{i-1} \qquad \text{and} \qquad r(a_i) = u_i.$$

Lemma 4.2 ([7, Proposition 13.2])**.** *If (G,E) is a self-similar graph action such that E is G-transitive, then for any vertices u and v of E there are $g \in G$ and $a \in E^*$ such that $g(u) = d(a)$ and $r(a) = v$.*

We recall that a dynamical system (Y, f) is called *topologically transitive* if, for every pair of nonempty open sets A, B in Y, there is an integer n such that $A \cap f^n(B) \neq \emptyset$, and it is called *topologically mixing* if there is an $m \in \mathbb{N}$ such that $A \cap f^k(B) \neq \emptyset$ for every $k \geq m$.

Proposition 4.3. *Suppose that (G,E) is a self-similar graph action. If E is G-transitive, then $(J_{(G,E)}, \sigma)$ is topologically transitive.*

Proof. Suppose that U and V are open sets in $J_{(G,E)}$. Then, for the quotient map $q : E^{-\omega} \to J_{(G,E)}$, $q^{-1}(U)$ and $q^{-1}(V)$ are open in $E^{-\omega}$ and there are cylinder sets $Z(a)$ and $Z(b)$ for some $a, b \in E^*$ such that

$$Z(a) \subset q^{-1}(U) \qquad \text{and} \qquad Z(b) \subset q^{-1}(V).$$

We will construct two infinite paths $\xi \in Z(a)$ and $\eta \in Z(b)$ such that ξ is asymptotically equivalent to $\sigma^n(\eta)$. Then $q(\xi) = q \circ \sigma^n(\eta) = \sigma^n \circ q(\eta) \in U \cap \sigma^n(V)$ as the quotient map and the shift map are commuting with each other.

Let $a \in E^m$. Since we assumed that E has no sink or source, we can find an infinite sequence of paths in E^m,

$$\ldots, \alpha_{-i}, \alpha_{-i+1}, \ldots, \alpha_{-1}, \alpha_0 = a,$$

such that $r(\alpha_{-i}) = d(\alpha_{-i+1})$ for every $i \geq 1$. As our graph E is a finite graph, E^m is also a finite set and there must be a repetition among α_{-i} so that, for some $0 \leq i < j$,

$$\gamma = \alpha_{-j}\alpha_{-j+1} \cdots \alpha_{-i}$$

is a loop; that is, $d(\gamma) = r(\gamma)$. For $\alpha = \alpha_{-i+1} \cdots \alpha_{-1}\alpha_0$, where $\alpha_0 = a$, and juxtaposing γ infinitely many times, we have an infinite path:

$$\xi = \cdots \gamma\gamma\alpha \in Z(a).$$

Let $\gamma_k = \gamma \cdots \gamma\alpha$ be a finite path, where γ is concatenated with itself k times. Then $r(\gamma_k) = r(\alpha) = r(a)$ for every $k \geq 0$ and, by Lemma 4.2, there are $g \in G$ and $c \in E^*$ such that $g(r(a)) = r(ga) = d(c)$ and $r(c) = d(b)$.

Since our group G satisfies the contracting condition, there is an $n_1 \in \mathbb{N}$ such that $g|_w \in \mathcal{N}$ for every $w \in E^*$ of length $|w| \geq n_1$. Here $\mathcal{N}$ is the nucleus of G. We recall that there is an $n_2 \in \mathbb{N}$ such that $h|_w \in \mathcal{N}$ for every $h \in \mathcal{N}$ and $w \in E^*$ of length $|w| \geq n_2$ by Lemma 2.4. Because $\gamma \cdots \gamma$ is also a loop whose length is increasing as the number of times γ is concatenated with itself is increasing, we may assume without loss of generality that $|\gamma| \geq \max\{n_1, n_2\}$.

We apply g to $\gamma_k = \gamma \cdots \gamma\alpha = \gamma\gamma_{k-1} = \gamma\gamma\gamma_{k-2} = \cdots$ to obtain

$$\begin{aligned} g(\gamma_k) &= g(\gamma\gamma_{k-1}) = g(\gamma)g_1(\gamma_{k-1}) \\ &= g(\gamma)g_1(\gamma\gamma_{k-2}) = g(\gamma)g_1(\gamma)g_2(\gamma_{k-2}) \\ &= \cdots \\ &= g(\gamma)g_1(\gamma) \cdots g_{k-1}(\gamma)g_k(\alpha), \end{aligned}$$

where $g_1 = g|_\gamma$ and $g_{i+1} = g_i|_\gamma$ for $i \geq 1$. Then $|\gamma| \geq \max\{n_1, n_2\}$ implies that every g_i is an element of the nucleus $\mathcal{N}$ of G. Hence, for k larger than the cardinality of $\mathcal{N}$, there must be repetition among the g_i, say,

$$g_s = g_t \quad \text{for } 1 \leq s < t \leq k.$$

To obtain a periodic pattern when g_s acts on γ_k, we need to treat two possible cases of $g_s = g_t$ separately, say, $t = s+1$ or $t > s+1$.

When $t = s+1$ (i.e., $g_{s+1} = g_s|_\gamma = g_s$), we have

$$\begin{aligned} g_s(\gamma_k) &= g_s(\gamma)g_{s+1}(\gamma_{k-1}) = g_s(\gamma)\big\{g_s(\gamma_{k-1})\big\} = g_s(\gamma)\big\{g_s(\gamma)g_{s+1}(\gamma_{k-2})\big\} \\ &= \cdots \\ &= g_s(\gamma)\cdots g_s(\gamma)g_s(\alpha). \end{aligned}$$

And when $t > s+1$, we write $k = i'(t-s-1) + j'$ with unique nonnegative integers i', j'. Then we have

$$g_s(\gamma_k) = \big\{g_s(\gamma)\cdots g_{t-1}(\gamma)\big\}\cdots\big\{g_s(\gamma)\cdots g_{t-1}(\gamma)\big\}g_s(\gamma)\cdots g_{s+j'-1}(\gamma)g_{s+j'}(\alpha),$$

where $\{g_s(\gamma)\cdots g_{t-1}(\gamma)\}$ is concatenated with itself i' times.

Before we go further, we recall that, by Lemma 2.3, $g(\gamma)$ and $g_i(\gamma)$ are also loops based at $g(d(\gamma))$ and that $g_j(\alpha)$ is a path with $d(g_j(\alpha)) = g(r(\gamma)) = g(d(\gamma))$. And $c \in E^*$ is such that $r(g_j(\alpha)) = d(c)$ and $r(c) = d(b)$.

For the $t = s+1$ case, let

$$\eta = \cdots g_s(\gamma)g_s(\gamma)g_s(\alpha)cb,$$

and for the $t > s+1$ case, let

$$\eta = \cdots\big\{g_s(\gamma)\cdots g_{t-1}(\gamma)\big\}\cdot\big\{g_s(\gamma)\cdots g_{t-1}(\gamma)\big\}g_s(\alpha)cb.$$

Then it is trivial that $\eta \in Z(b)$ in both cases, and for $n = |cb|$ we have

$$\sigma^n(\eta) = \cdots g_s(\gamma)g_s(\gamma)g_s(\alpha) \in \sigma^n\big(Z(b)\big) \quad \text{when } t = s+1$$

and

$$\begin{aligned} \sigma^n(\eta) &= \cdots\big\{g_s(\gamma)\cdots g_{t-1}(\gamma)\big\}\cdot\big\{g_s(\gamma)\cdots g_{t-1}(\gamma)\big\}g_s(\alpha) \\ &\in \sigma^n\big(Z(b)\big) \quad \text{when } t > s+1. \end{aligned}$$

Now we check that $\xi = \cdots\gamma\gamma\alpha$ and $\sigma^n(\eta)$ are asymptotically equivalent to each other. For the $t = s+1$ case,

$$g_s(\gamma\cdots\gamma\alpha) = g_s(\gamma)\cdots g_s(\gamma)g_s(\alpha)$$

implies that $\xi = \cdots\gamma\gamma\alpha$ is asymptotically equivalent to

$$\sigma^n(\eta) = \cdots g_s(\gamma)g_s(\gamma)g_s(\alpha).$$

For the $t > s+1$ case, we consider $\gamma' = \gamma\cdots\gamma$, concatenating γ with itself $t-s-1$ times. Then $\xi = \cdots\gamma\gamma\alpha = \cdots\gamma'\gamma'\alpha$ because γ is a loop, and we have

$$g_s(\gamma') = g_s(\gamma\cdots\gamma) = g_s(\gamma)g_{s+1}(\gamma\cdots\gamma) = \cdots = g_s(\gamma)\cdots g_{t-1}(\gamma).$$

Thus the periodic property implies that

$$\begin{aligned} g_s(\gamma' \cdots \gamma'\alpha) &= g_s(\gamma') \cdots g_s(\gamma')g_s(\alpha) \\ &= \{g_s(\gamma) \cdots g_{t-1}(\gamma)\} \cdots \{g_s(\gamma) \cdots g_{t-1}(\gamma)\} g_s(\alpha) \end{aligned}$$

and that $\xi = \cdots \gamma'\gamma'\alpha$ is asymptotically equivalent to $\sigma^n(\eta)$. Therefore $(J_{(G,E)}, \sigma)$ is a topologically transitive system. □

Proposition 4.4. *If the G-action on E^0 is transitive, then $(J_{(G,E)}, \sigma)$ is topologically mixing.*

Proof. As in the case of Proposition 4.3, we consider $Z(a) \subset q^{-1}(U)$ and $Z(b) \subset q^{-1}(V)$ for some $a, b \in E^*$. By the transitive condition, there is a $g \in G$ such that $g(r(a)) = d(b)$. So the only difference from the G-transitive case is that we do not need a path c connecting $r(ga)$ and $d(b)$. Then, for n being the length of b, it is not difficult to obtain that $\xi = \cdots \gamma\gamma\alpha$ and $\sigma^n(\eta)$, which were constructed in the proof of Proposition 4.3, are asymptotically equivalent to each other.

For $k > n$, our standing assumption that E is finite and connected with no sink or source implies that there is a path β of length $k - n$ such that $r(\beta) = d(b)$. Then there is a $g \in G$ such that $g(r(a)) = d(\beta)$, and our $\eta \in Z(b)$ will be given as $\eta = \cdots g_s(\alpha)\beta b$. Hence $\sigma^k(\eta) = \cdots g_s(\alpha)$ is asymptotically equivalent to ξ for every $k \geq n$, and $(J_{(G,E)}, \sigma)$ is topologically mixing. □

Definition 4.5 (see [5], [22]). Let X be a compact Hausdorff space, and let $f : X \to X$ be an endomorphism. The dynamical system (X, f) is called *topologically free* if

$$\{x \in X : f^n x \neq x \text{ for every } n > 0\}$$

is dense in X, and it is called *essentially free* if

$$\{x \in X : f^k x = f^l x \text{ for some } k, l \geq 0 \text{ implies } k = l\}$$

is dense in X.

Definition 4.6 (see [7], [9]). A self-similar graph action (G, E) is said to be *pseudofree* if $\mathrm{Fix}_g = \{a \in E^* : g(a) = a\}$ is a finite set for every $g \in G$.

Lemma 4.7. *For an infinite path $\xi \in E^{-\omega}$, if $q(\xi)$ is a periodic point of period n in $(J_{(G,E)}, \sigma)$, then ξ is an eventually periodic point in $(E^{-\omega}, \sigma)$. Moreover, if (G, E) is a pseudofree self-similar graph action, then ξ is a periodic point.*

Proof. If $q(\xi)$ is a periodic point of period n in $(J_{(G,E)}, \sigma)$, then $q(\xi) = \sigma^{kn}(q(\xi))$ and Lemma 2.9 imply that $q(\xi) = \sigma^{kn}(q(\xi)) = q(\sigma^{kn}(\xi))$. So ξ is asymptotically equivalent to $\sigma^{kn}(\xi)$ for every $k \geq 0$. Every asymptotic equivalence class on $E^{-\omega}$ has no more than $|\mathcal{N}|$ elements by Remark 2.8. Hence there are $k > l \geq 0$ such that $\sigma^{kn}(\xi) = \sigma^{ln}(\xi)$, and ξ is eventually periodic.

Now suppose that (G, E) is pseudofree. Let us denote $\xi = \cdots e_{-2}e_{-1}$ by $\cdots \gamma_{-2}\gamma_{-1}$, where $\gamma_{-k} = e_{-kn} \cdots e_{-(k-1)n-1} \in E^n$. Because

$$\sigma^{kn}(\xi) = \cdots \gamma_{-k-2}\gamma_{-k-1} = \cdots \gamma_{-l-2}\gamma_{-l-1} = \sigma^{ln}(\xi),$$

we have

$$\gamma_{-k}\cdots\gamma_{-l-1}=\gamma_{-2k+l}\cdots\gamma_{-k-1}=\cdots=\gamma_{-(i+1)k+l}\cdots\gamma_{-ik-1}=\cdots$$

for every $i\geq 1$. Thus

$$r(\gamma_{-l-1})=r(\gamma_{-k-1})=d(\gamma_{-k})$$

implies that $L=\gamma_{-k}\cdots\gamma_{-l-1}$ is a loop in E^* so that we have

$$\sigma^{ln}(\xi)=\cdots LL.$$

Then ξ is asymptotically equivalent to $\sigma^{ln}(\xi)$, which implies that, by Proposition 2.6 and Remark 2.7, there are $g_i\in\mathcal{N}$ such that

$$\begin{aligned} g_1(L)&=\gamma_{-(k-l)}\cdots\gamma_{-1} \qquad \text{and}\\ g_i\cdot L&=\gamma_{-i(k-l)}\cdots\gamma_{-(i-1)(k-l)-1}\cdot g_{i-1} \quad \text{for every } i\geq 2. \end{aligned}$$

Then $g_i(L)=\gamma_{-i(k-l)}\cdots\gamma_{-(i-1)(k-l)-1}=L_{-i}$ is a loop by Lemma 2.3, which implies

$$\xi=\cdots L_{-2}L_{-1}.$$

To show that ξ is periodic, we will check that $L_{-i-1}=L_{-i}$ for every $i\geq 1$ (i.e., $g_{i+1}=g_i$). For that purpose, as the pseudofree condition is related to fixed points of the group action, we use the number of fixed points to obtain $g_{i+1}^{-1}g_i=1$.

For each pair of $g,h\in\mathcal{N}$ with $gh^{-1}\neq 1$, let $m(gh^{-1})=|\mathrm{Fix}_{gh^{-1}}|$, the cardinality of fixed points in E^* by gh^{-1}. Since we assumed that our (G,E) is pseudofree, $m(gh^{-1})$ is a finite number as long as $gh^{-1}\neq 1$. Because $\mathcal{N}$ is a finite set, we have that

$$m=\max_{g,h\in\mathcal{N},gh^{-1}\neq 1} m(gh^{-1})$$

is a finite number.

Let us denote by L^i the concatenation of L with itself i times. Then there is an $s\in\mathbb{N}$ such that $|L^i|\gneq m$ for every $i\geq s$. We also note that, as ξ is eventually periodic, there is a $t\in\mathbb{N}$ such that $g_i(L)=g_j(L)=L'$ for all $i,j\geq t$.

Now we consider $i=s+t$ and $s+t+1$. Then $g_{i+1}|_L=g_i$ implies

$$\begin{aligned} g_{s+t}(L^s)&=g_{s+t}(L)g_{s+t-1}(L^{s-1})=\cdots=(L')^s \qquad \text{and}\\ g_{s+t+1}(L^s)&=g_{s+t+1}(L)g_{s+t}(L^{s-1})=\cdots=(L')^s. \end{aligned}$$

So we have

$$g_{s+t}^{-1}g_{s+t+1}(L^s)=L^s.$$

If we write $L^s=e_1\cdots e_n$, where $e_k\in E^1$ for every $k=1,\dots,n$, then we obtain

$$\begin{aligned} g_{s+t}^{-1}g_{s+t+1}(L^s)&=g_{s+t}^{-1}g_{s+t+1}(e_1\cdots e_ke_{k+1}\cdots e_n)\\ &=g_{s+t}^{-1}g_{s+t+1}(e_1\cdots e_k)g_{s+t}^{-1}g_{s+t+1}|_{e_1\cdots e_k}(e_{k+1}\cdots e_n)\\ &=e_1\cdots e_ke_{k+1}\cdots e_n. \end{aligned}$$

Thus we have $g_{s+t}^{-1}g_{s+t+1}(e_1\cdots e_k)=e_1\cdots e_k$, and $e_1\cdots e_k$ is a fixed point of $g_{s+t}^{-1}g_{s+t+1}$ for every $1\leq k\leq n=|L^s|$. Hence the number of fixed points by $g_{s+t}^{-1}g_{s+t+1}$ is at least $|L^s|\gneq m$. But this inequality cannot happen if $g_{s+t}^{-1}g_{s+t+1}\neq 1$.

Therefore we must have $g_{s+t}^{-1}g_{s+t+1} = 1$; in other words, $g_{s+t} = g_{s+t+1}$. Moreover, by induction we have

$$g_{s+t} = g_{s+t+i} \quad \text{for every } i \geq 1.$$

For $j < s+t$, we recursively show that $g_{j+1} = g_j$. Before we go further, we recall three basic properties of restrictions that we will use:

$$g_{j+1}|_L = g_j, \qquad (gh)|_a = g|_{h(a)}h|_a, \qquad g^{-1}|_{g(a)} = (g|_a)^{-1}.$$

Assume that we have $g_{j+1} = g_j$ for some $j \leq s+t$. Then $g_{j+1}^{-1}g_j = 1$ implies

$$\begin{aligned}(g_{j+1}^{-1}g_j)|_L = 1|_L &= 1 \\ &= g_{j+1}^{-1}|_{g_j(L)}g_j|_L \\ &= g_{j+1}^{-1}|_{g_{j+1}(L)}g_j|_L \\ &= (g_{j+1}|_L)^{-1}g_j|_L \\ &= g_j^{-1}g_{j-1}.\end{aligned}$$

Thus we have $g_j = g_{j-1}$, and $g_j = g_{j-1}$ is valid for every $2 \leq j \leq s+t$. Therefore we obtain $g_i = g_{i+1}$ for every $i \in \mathbb{N}$, and ξ is also periodic. □

Proposition 4.8. *If (G, E) is pseudofree and $(E^{-\omega}, \sigma)$ is topologically free, then $(J_{(G,E)}, \sigma)$ is topologically free.*

Proof. Let

$$\begin{aligned}A &= \{\xi \in E^{-\omega} : \sigma^n\xi \neq \xi \text{ for some } n \geq 1\};\\ B &= \{z \in J_{(G,E)} : \sigma^n z \neq z \text{ for some } n \geq 1\}.\end{aligned}$$

Then A^c and B^c are the sets of periodic points in $(E^{-\omega}, \sigma)$ and $(J_{(G,E)}, \sigma)$, respectively. Because (G, E) is pseudofree, we have $A^c = q^{-1}(B^c)$ by Lemma 4.7, and this equality implies $A = q^{-1}(B)$. As $(E^{-\omega}, \sigma)$ is topologically free, we have $\overline{A} = E^{-\omega}$, and $q(A) = B$ induces

$$\overline{B} = \overline{q(A)} \supset q(\overline{A}) = q(E^{-\omega}) = J_{(G,E)}.$$

Therefore $(J_{(G,E)}, \sigma)$ is topologically free. □

It is natural to expect that not only the action of a group G but also the structure of a graph E determines dynamical properties of the limit dynamical system $(J_{(G,E)}, \sigma)$.

Definition 4.9 (see [11]). A graph E is said to be *irreducible* if, for all $u, v \in E^0$, there is a finite path a such that $d(a) = u$ and $r(a) = v$. A graph E is said to be *primitive* if there is a $k \in \mathbb{N}$ such that, for every $\geq k$ and all $u, v \in E^0$, there is a path $a \in E^m$ with $d(a) = u$ and $r(a) = v$.

In symbolic dynamics, it is a well-known fact that an irreducible graph induces a topologically transitive shift space and a primitive graph induces a topological mixing shift space. We refer the reader to [14] for more details on symbolic dynamics. Then Lemma 2.9 implies the following.

Proposition 4.10. *Suppose that (G, E) is a self-similar graph action. If E is an irreducible graph, then $(J_{(G,E)}, \sigma)$ is topologically transitive. If E is a primitive graph, then $(J_{(G,E)}, \sigma)$ is topologically mixing.*

Proof. For two open sets U and V in $J_{(G,E)}$, we consider $\mu = \cdots a \in Z(a) \subset q^{-1}(U)$ and $Z(b) \subset q^{-1}(V)$, where $b \in E^n$ for some $n \geq 1$.

If G is an irreducible graph, then there is a finite path $c \in E^m$ for some $m \geq 1$ such that $r(a) = d(c)$ and $r(c) = d(b)$. So we have $\nu = \cdots acb$ such that

$$\sigma^{n+m}(\nu) = \mu \in Z(a) \cap \sigma^{n+m}\big(Z(b)\big)$$

and

$$\begin{aligned} q(\mu) &\in q\big(Z(a) \cap \sigma^{n+m}\big(Z(b)\big)\big) \subset q\big(Z(a)\big) \cap q\big(\sigma^{n+m}\big(Z(b)\big)\big) \\ &= q\big(Z(a)\big) \cap \sigma^{n+m}\big(q\big(Z(b)\big)\big) \subset U \cap \sigma^{n+m}(V) \end{aligned}$$

by Lemma 2.9. Therefore $(J_{(G,E)}, \sigma)$ is topologically transitive.

If E is a primitive graph, then for every $m \geq k$ there is a $c \in E^m$ with $r(a) = d(c)$ and $r(c) = d(b)$ so that $U \cap \sigma^{n+m}(V) \neq \emptyset$. Thus $(J_{(G,E)}, \sigma)$ is topologically mixing. □

Because every n-bouquet is a primitive graph, the next corollary is trivial.

Corollary 4.11. *If E is an n-bouquet, then $(J_{(G,E)}, \sigma)$ is topologically mixing.*

Proposition 4.12. *If $(E^{-\omega}, \sigma)$ is essentially free, then $(J_{(G,E)}, \sigma)$ is topologically free.*

Proof. Let

$$\begin{aligned} A &= \{\xi \in E^{-\omega} : \sigma^k \xi = \sigma^l \xi \text{ for some } k, l \geq 0 \text{ implies } k = l\}; \\ B &= \{z \in J_{(G,E)} : \sigma^n z \neq z \text{ for some } n \geq 1\}. \end{aligned}$$

Then A^c is the set of eventually periodic points in $(E^{-\omega}, \sigma)$, and B^c is the set of periodic points in $(J_{(G,E)}, \sigma)$. By Lemma 4.7, we have $q^{-1}(B^c) \subset A^c$, which implies $q^{-1}(B) \supset A$ and $B \supset q(A)$. If $(E^{-\omega}, \sigma)$ is essentially free, then we have $\overline{A} = E^{-\omega}$ so that

$$\overline{B} \supset \overline{q(A)} \supset q(\overline{A}) = q(E^{-\omega}) = J_{(G,E)}.$$

Hence B is dense in $J_{(G,E)}$, and $(J_{(G,E)}, \sigma)$ is topologically free. □

Definition 4.13 (see [12]). A directed graph E is said to satisfy Condition (L) if, for every loop $\gamma = \gamma_1 \cdots \gamma_n$ in E^*, there is an i such that $r^{-1}(d(\gamma_i))$ is not a singleton.

Lemma 4.14 ([12, Lemma 3.4])**.** *If E satisfies Condition (L), then $(E^{-\omega}, \sigma)$ is essentially free.*

So the following is trivial by Proposition 4.12.

Proposition 4.15. *If E satisfies Condition (L), then $(J_{(G,E)}, \sigma)$ is topologically free.*

5. Groupoid algebras

Suppose that (G, E) is a self-similar graph action. Then we have the limit dynamical system $(J_{(G,E)}, \sigma)$ such that $J_{(G,E)}$ is a compact metrizable Hausdorff space and σ is a continuous surjective map. To build C^*-algebras from limit dynamical systems, we follow Anantharaman-Delaroche [1] and Deaconu [5]. Let

$$\Gamma_{(G,E)} = \big\{(x, n, y) \in J_{(G,E)} \times \mathbb{Z} \times J_{(G,E)} : \exists k, l \geq 0, n = l - k, \sigma^k x = \sigma^l y\big\}.$$

A pair $\{(x, n, y), (w, m, z)\} \in \Gamma^2_{(G,E)}$ is composable if $y = w$, and multiplication and inverse are given by

$$(x, n, y)(y, m, z) = (x, n + m, z) \qquad \text{and} \qquad (x, n, y)^{-1} = (y, -n, x).$$

For $(x, n, y) \in \Gamma_{(G,E)}$, the range and domain are given by

$$r(x, n, y) = (x, 0, x) \qquad \text{and} \qquad d(x, n, y) = (y, 0, y).$$

The unit space of $\Gamma_{(G,E)}$ denoted by $\Gamma^{(0)}_{(G,E)}$ is identified with $J_{(G,E)}$ via the diagonal map, and the isotropy group bundle is given by $I = \{(x, n, x) \in \Gamma_{(G,E)}\}$. For open sets U, V of $J_{(G,E)}$ and $k, l \geq 0$, let

$$Z(U, k, l, V) = \big\{(x, l - k, y) : x \in U, y \in V, \sigma^k x = \sigma^l y\big\}.$$

Then the collection of these sets is a basis for a second countable, locally compact Hausdorff topology on $\Gamma_{(G,E)}$, and the counting measure is a Haar system of $\Gamma_{(G,E)}$.

Étale and amenable groupoids. To construct a groupoid algebra from $\Gamma_{(G,E)}$, we need $\Gamma_{(G,E)}$ to be an étale groupoid (i.e., r-discrete with the range map r being a local homeomorphism). But in the general graph case, as Thomsen observed in [4] and [21], it is not easy to obtain an étale property of $\Gamma_{(G,E)}$. We also need amenability to avoid the subtle argument of full and reduced groupoid algebras of $\Gamma_{(G,E)}$. When the graph E is an n-bouquet, Nekrashevych showed in [16, Proposition 6.1] that the *regular* condition on G implies that $\sigma : J_{(G,E)} \to J_{(G,E)}$ is a covering map so that $\Gamma_{(G,E)}$ is an étale groupoid. We follow [16] to obtain that $\Gamma_{(G,E)}$ is an étale and amenable groupoid.

The following lemma is from Lemmas 3.1 and 3.5 of [20] (see also [8, Section 3]).

Lemma 5.1. *If $\sigma : J_{(G,E)} \to J_{(G,E)}$ is a local homeomorphism, then $\Gamma_{(G,E)}$ is an étale and amenable groupoid.*

Definition 5.2 (see [16]). A self-similar graph action (G, E) is said to be *regular* if, for every $g \in G$ and every right-infinite path $w \in E^\omega$, either $g(w) \neq w$ or there is a neighborhood of w such that every point in the neighborhood is fixed by g.

Lemma 5.3 ([16, Lemma 6.3])**.** *If (G, E) is a contracting and regular self-similar graph action, then there is a $k \in \mathbb{N}$ such that, for every $v \in E^k$ and any two elements g, h of the nucleus, either $g(v) \neq h(v)$ or $g(v) = h(v)$ and $g|_v = h|_v$.*

Lemma 5.4. *If (G, E) is a contracting and regular self-similar graph action, then $\sigma : J_{(G,E)} \to J_{(G,E)}$ is a surjective local homeomorphism.*

Proof. It is trivial that σ is a continuous surjective map. Let $\mathcal{N}$ be the nucleus of G, and let k be the natural number given in Lemma 5.3. For a $\xi = \cdots \xi_{-k-1} \cdots \xi_{-1} \in E^{-\omega}$ and $q(\xi) \in J_{(G,E)}$, we define

$$A = \{a_{-k-1} \cdots a_{-1} \in E^{k+1} : g(a_{-k-1} \cdots a_{-1}) = \xi_{-k-1} \cdots \xi_{-1} \text{ for some } g \in \mathcal{N}\},$$
$$B = \{b_{-k-1} \cdots b_{-1} \in E^{k+1} : g(b_{-k-1} \cdots b_{-1}) \neq \xi_{-k-1} \cdots \xi_{-1} \text{ for any } g \in \mathcal{N}\}.$$

Then A and B satisfy conditions of Lemma 3.2 so that, for $C = \bigcup_{a \in A} Z(a)$, $q(C)$ is an open neighborhood of $q(\xi)$.

Now we show that $\sigma|_{q(C)}$ is an injective map. Suppose that there are two elements $\eta = \cdots \eta_{-2}\eta_{-1}$ and $\zeta = \cdots \zeta_{-2}\zeta_{-1}$ in $E^{-\omega}$ such that $q(\eta), q(\zeta) \in q(C)$ and $\sigma(q(\eta)) = \sigma(q(\zeta))$. To obtain that $\sigma|_{q(C)}$ is injective, we need to show that $q(\eta) = q(\zeta)$ (i.e., η is asymptotically equivalent to ζ). Here we observe that $\eta_{-k-1} \cdots \eta_{-1}$ and $\zeta_{-k-1} \cdots \zeta_{-1}$ are contained in A and $q \circ \sigma(\eta) = q \circ \sigma(\zeta)$ by Lemma 2.9. Then $\eta_{-k-1} \cdots \eta_{-1}, \zeta_{-k-1} \cdots \zeta_{-1} \in A$ implies that there is a $g \in \mathcal{N}$ such that

$$g(\eta_{-k-1} \cdots \eta_{-1}) = g(\eta_{-k-1} \cdots \eta_{-2}) g|_{\eta_{-k-1} \cdots \eta_{-2}}(\eta_{-1}) = \zeta_{-k-1} \cdots \zeta_{-2}\zeta_{-1}.$$

On the other hand, $q \circ \sigma(\eta) = q \circ \sigma(\zeta)$ means that $\sigma(\eta) = \cdots \eta_{-k-1} \cdots \eta_{-2}$ is asymptotically equivalent to $\sigma(\zeta) = \cdots \zeta_{-k-1} \cdots \zeta_{-2}$ so that, for $n \geq 2$, there are $h_n \in \mathcal{N}$ such that $h_{n+1} \cdot \eta_{-n-1} = \zeta_{-n-1} \cdot h_n$. Hence we have

$$h_{k+1}(\eta_{-k-1} \cdots \eta_{-2}) = \zeta_{-k-1} \cdots \zeta_{-2} = g(\eta_{-k-1} \cdots \eta_{-2}).$$

Then, by Lemma 5.3, regularity implies $h_{k+1}|_{\eta_{-k-1} \cdots \eta_{-2}} = g|_{\eta_{-k-1} \cdots \eta_{-2}}$, which induces

$$\begin{aligned} h_{k+1}(\eta_{-k-1} \cdots \eta_{-2}\eta_{-1}) &= h_{k+1}(\eta_{-k-1} \cdots \eta_{-2}) h_{k+1}|_{\eta_{-k-1} \cdots \eta_{-2}}(\eta_{-1}) \\ &= g(\eta_{-k-1} \cdots \eta_{-2}) g|_{\eta_{-k-1} \cdots \eta_{-2}}(\eta_{-1}) \\ &= \zeta_{-k-1} \cdots \zeta_{-2}\zeta_{-1}. \end{aligned}$$

Thus η is asymptotically equivalent to ζ, and $\sigma|_{q(C)}$ is an injective map.

We recall that $(\sigma|_{q(C)})^{-1}$ is continuous if and only if $(\sigma|_{q(C)})^{-1} \circ q$ is continuous. For $\eta \in C$ and $\sigma(\eta) \in \sigma(C)$, it is routine to have

$$\big((\sigma|_{q(C)})^{-1} \circ q\big)\big(\sigma(\eta)\big) = (\sigma|_{q(C)})^{-1} \circ \sigma \circ q(\eta) = q(\eta).$$

Since we showed in the preceding paragraph that, for $\eta, \zeta \in C$, $\sigma(\eta) \sim \sigma(\zeta)$ implies $\eta \sim \zeta$ where $\sim$ means the asymptotic equivalence, the above equality says that $(\sigma|_{q(C)})^{-1} \circ q$ is well defined and continuous. Therefore $(\sigma|_{q(C)})^{-1}$ is continuous, and $\sigma : J_{(G,E)} \to J_{(G,E)}$ is a local homeomorphism. □

The next property is obvious from Lemma 5.1 and Lemma 5.4.

Proposition 5.5. *If (G, E) is a contracting and regular self-similar graph action, then $\Gamma_{(G,E)}$ is an étale and amenable groupoid.*

Minimality and local contractivity. We follow Exel and Pardo [6], [7].

Definition 5.6 ([6, Definition 5.3]). Suppose that $\mathcal{G}$ is a groupoid with the domain and range maps $d, r : \mathcal{G} \to \mathcal{G}^{(0)}$. A subset U of $\mathcal{G}^{(0)}$ is called an *invariant* of $\mathcal{G}^{(0)}$ if, for every $\gamma \in \mathcal{G}$, we have $d(\gamma) \in U$ if and only if $r(\gamma) \in U$. We say that $\mathcal{G}$ is a *minimal* groupoid if the only invariant open subsets of $\mathcal{G}^{(0)}$ are the empty set and $\mathcal{G}^{(0)}$ itself.

To find a condition for $\Gamma_{(G,E)}$ to be a minimal groupoid, we need a groupoid defined in [7, Theorem 8.19]. Let

$$\begin{aligned}\mathcal{G}_{G,E} = \big\{ \big(\xi; \{g_n\}, l-k; \eta\big) :& \\ &\xi, \eta \in E^{-\omega}, \{g_n\} \subset G, l, k \in \mathbb{N}, g_{n+l} \cdot \xi_{-n-k} = \eta_{-n-l} \cdot g_{n+l-1} \big\}.\end{aligned}$$

We recall that our graph E has no sink or source by our standing assumption.

Lemma 5.7 ([7, Theorem 13.6]). *Suppose that (G, E) is a pseudofree self-similar graph action. Then $\mathcal{G}_{G,E}$ is minimal if and only if E is G-transitive.*

Proposition 5.8. *If (G, E) is a pseudofree self-similar graph action and E is G-transitive, then $\Gamma_{(G,E)}$ is a minimal groupoid.*

Proof. Suppose that $q : E^{-\omega} \to J_{(G,E)}$ is the quotient map and that U is an open invariant subset of $\Gamma^{(0)}_{(G,E)}$. We show that $q^{-1}(U)$ is an open invariant subset of $\mathcal{G}^{(0)}_{G,E}$.

We note that $(\xi; \{g_n\}, l-k; \eta) \in \mathcal{G}_{G,E}$ means that $(q(\xi), l-k, q(\eta)) \in \Gamma_{(G,E)}$. If $\xi = r((\xi; \{g_n\}, l-k; \eta))$ is an element of $q^{-1}(U)$, then we have $q(\xi) \in U$, which implies $q(\eta) \in U$ because U is an invariant subset of $\Gamma^{(0)}_{(G,E)}$. We then have $\eta \in q^{-1}(q(\eta)) \subset q^{-1}(U)$. The other implication is the same. Since $q^{-1}(U)$ is an open subset of $\mathcal{G}^{(0)}_{G,E}$, we conclude that $q^{-1}(U)$ is an open invariant subset of $\mathcal{G}^{(0)}_{G,E}$. As $\mathcal{G}_{G,E}$ is a minimal groupoid by Lemma 5.7, $q^{-1}(U)$ is either the empty set or $\mathcal{G}^{(0)}_{G,E}$. Thus U is also either the empty set or $\Gamma^{(0)}_{(G,E)}$, and $\Gamma_{(G,E)}$ is a minimal groupoid. $\square$

We recall Anantharaman-Delaroche's definition of locally contracting groupoids.

Definition 5.9 (see [1]). Suppose that $\mathcal{G}$ is an étale groupoid. We say that $\mathcal{G}$ is *locally contracting* if, for every open nonempty subset U of $\mathcal{G}^{(0)}$, there are an open subset V of U and an open bisection S in $\mathcal{G}$ such that $\overline{V} \subseteq r(S)$ and $\alpha_S(\overline{V})$ is a proper subset of V. Here $\alpha_S : r(S) \to d(S)$ is the homeomorphism given by $\alpha_S(r(\gamma)) = d(\gamma)$ for every $\gamma \in S$.

To show that $\Gamma_{(G,E)}$ is locally contracting, we need to refine the open sets of $J_{(G,E)}$ given in Lemmas 3.2 and 5.4. For each $x \in J_{(G,E)}$ and an $n \in \mathbb{N}$, we choose a $\xi = \cdots \xi_{-n} \cdots \xi_{-1} \in q^{-1}(x) \subset E^{-\omega}$. We let

$$A_n(x) = \big\{ a_{-n} \cdots a_{-1} \in E^n : g(a_{-n} \cdots a_{-1}) = \xi_{-n} \cdots \xi_{-1} \text{ for some } g \in \mathcal{N} \big\}$$

and

$$U_n(x) = q\Big(\bigcup_{a\in A_n(x)} Z(a)\Big).$$

Then $q^{-1}(x)$ is a subset of $A_n(x)$ for every natural number n, and $U_n(x)$ is a neighborhood of x by Lemmas 3.2 and 5.4. Moreover, Nekrashevych attained the following property for $U_n(x)$.

Lemma 5.10 ([15, Proposition 3.4.1])**.** *The collection $\{U_n(x)\}_{n\in\mathbb{N}}$ is a fundamental system of neighborhoods of x.*

Proposition 5.11. *Suppose that (G,E) is a contracting and regular self-similar graph action. If the graph E satisfies Condition (L), then $\Gamma_{(G,E)}$ is a locally contracting groupoid.*

Proof. For an open set U of $J_{(G,E)}$, choose an $x \in U$ and one of its inverse images $\xi = \cdots\xi_{-1} \in E^{-\omega}$. Then by Lemma 5.10, there is an n such that $U_n(x) \subset U$. Since we assumed that our graph E is a finite graph, for an infinite path $\xi \in E^{-\omega}$ there are two indices $n \le i \le j$ such that $d(\xi_{-j}) = r(\xi_{-i})$. In other words, $\gamma = \xi_{-j}\cdots\xi_{-i}$ is a loop included in ξ. Without loss of generality, we may say that $i = n+1$ so that $\xi = \cdots\xi_{-j-1}\gamma\xi_{-n}\cdots\xi_{-1}$. For simplicity, we denote $\xi_{-n}\cdots\xi_{-1}$ as $a \in E^n$ and the length of γ as k. Since we may use $\gamma\cdots\gamma$ instead of γ, we can choose k large enough so that Lemma 5.3 holds. Now we consider another path $\eta \in E^{-\omega}$ given by $\eta = \cdots\eta_{-n-2k-1}\gamma\gamma a$ and its image $y = q(\eta) \in J_{(G,E)}$. It is easy to see $y \in U_n(x)$ as $\eta \in Z(a) \subset q^{-1}(U_n(x))$.

We let

$$V = U_{n+k}(x), \qquad W = U_{n+2k}(y), \qquad \text{and} \qquad S = Z(V, n, n+k, W).$$

Then V and W are clopen subsets of $J_{(G,E)}$ by Lemma 3.2 so that S is an open set in $\Gamma_{(G,E)}$. It is trivial that $V \subset U_n(x) \subset U$ and $V = \overline{V} = r(S)$. So we need to show that S is a bijection in $\Gamma_{(G,E)}$ and that $\alpha_S(V)$ is a proper subset of V.

We first check

$$S = Z(V, n, n+k, W) = \big\{\big(q\big(\alpha g(\gamma a)\big), n, n+k, q\big(\alpha g(\gamma\gamma a)\big)\big) : \alpha g(\gamma a) \in q^{-1}(V)\big\}.$$

We note that

$$q^{-1}(V) = \big\{\alpha g(\gamma a) : \alpha \in E^{-\omega}, r(\alpha) = d\big(g(\gamma a)\big), g \in \mathcal{N}\big\};$$
$$q^{-1}(W) = \big\{\beta h(\gamma\gamma a) : \beta \in E^{-\omega}, r(\beta) = d\big(h(\gamma\gamma a)\big), h \in \mathcal{N}\big\}.$$

Then Lemmas 2.9 and 2.3 imply

$$\sigma^n\big(q\big(\alpha g(\gamma a)\big)\big) = q\big(\sigma^n\big(\alpha g(\gamma a)\big)\big) = q\big(\alpha g(\gamma)\big) \qquad \text{and}$$
$$\sigma^{n+k}\big(q\big(\beta h(\gamma\gamma a)\big)\big) = q\big(\sigma^{n+k}\big(\beta h(\gamma\gamma a)\big)\big) = q\big(\beta h(\gamma)\big)$$

because $\alpha g(\gamma a) = \alpha g(\gamma) g|_\gamma(a)$ and $\beta h(\gamma\gamma a) = \beta h(\gamma) h|_\gamma(\gamma a)$. For an element $(q(\alpha g(\gamma a)), n, n+k, q(\beta h(\gamma\gamma a))) \in S$, let us show that $q(\beta h(\gamma\gamma a)) = q(\alpha g(\gamma\gamma a))$. The definition of $Z(V, n, n+k, W)$ and Lemma 2.9 imply

$$\sigma^n\big(q\big(\alpha g(\gamma a)\big)\big) = q\big(\alpha g(\gamma)\big) = \sigma^{n+k}\big(q\big(\alpha g(\gamma\gamma a)\big)\big) = \sigma^{n+k}\big(q\big(\beta h(\gamma\gamma a)\big)\big).$$

We recall that $q(C)$ in the proof of Lemma 5.4 is equal to $U_{k+1}(q(\xi))$, and we obtained there that, for $q(\mu), q(\nu) \in q(C) = U_{k+1}(q(\xi))$, $q(\sigma(\mu)) = q(\sigma(\nu))$ induces $q(\mu) = q(\nu)$. So Lemma 2.9 and induction imply that, for $\alpha g(\gamma\gamma a)$ and $\beta h(\gamma\gamma a)$ in $U_{n+2k}(y) = U_{k+n+k}(y)$, $\sigma^{n+k}(q(\alpha g(\gamma\gamma a))) = \sigma^{n+k}(q(\beta h(\gamma\gamma a)))$ induces $q(\alpha g(\gamma\gamma a)) = q(\beta h(\gamma\gamma a))$. Thus we have

$$S = \big\{ \big(q\big(\alpha g(\gamma a)\big), n, n+k, q\big(\alpha g(\gamma\gamma a)\big)\big) : \alpha g(\gamma a) \in q^{-1}(V) \big\}.$$

Now it is clear from the above equality that $r|_S$ and $d|_S$ are bijective, and S is an open bijection in $\Gamma_{(G,E)}$.

To show $\alpha_S(V) \subsetneq V$, we note $\alpha_S(V) = W = U_{n+2k}(y)$. Because $\{U_m(y)\}_{m\in\mathbb{N}}$ is a fundamental system of neighborhoods of y by Lemma 5.10, Condition (L) ensures that there is an $m > n+k$ such that $U_m(y) \subsetneq U_{n+k}(y)$. As we can choose k sufficiently large, without loss of generality we say that $n+k < m < n+2k$. Then we have

$$\alpha_S(V) = U_{n+2k}(y) \subseteq U_m(y) \subsetneq U_{n+k}(y).$$

We recall that, referring to Lemma 5.4, $U_{n+k}(y)$ is determined by the first $n+k$ coordinates of η and the actions of elements of the nucleus. As the first $n+k$ coordinates of ξ and η are the same, we conclude that $U_{n+k}(y) = U_{n+k}(x) = V$. Hence we have $\alpha_S(V) \subsetneq V$, and this completes the proof. $\square$

Now we define our C^*-algebras.

Definition 5.12 (see [19]). Suppose that (G, E) is a contracting and regular self-similar graph action with a limit dynamical system $(J_{(G,E)}, \sigma)$ and corresponding groupoid $\Gamma_{(G,E)}$. Then we let $C^*(\Gamma_{(G,E)})$ be the groupoid C^*-algebra of $\Gamma_{(G,E)}$.

The following lemma is from Theorem 5.1, Corollary 5.11, and Proposition 7.5 of [3].

Lemma 5.13 (see [3])**.** *Suppose that (G, E) is a contracting and regular self-similar graph action.*

(1) *The limit dynamical system $(J_{(G,E)}, \sigma)$ is topologically free if and only if $\Gamma_{(G,E)}$ is topologically principal.*
(2) *The groupoid algebra $C^*(\Gamma_{(G,E)})$ is simple if and only if $\Gamma_{(G,E)}$ is topologically principal and minimal.*
(3) *If $\Gamma_{(G,E)}$ is locally contracting and $C^*(\Gamma_{(G,E)})$ is simple, then $C^*(\Gamma_{(G,E)})$ is purely infinite.*

Now we have our main result which generalizes Corollary 6.12 and Corollary 6.15 of [16].

Theorem 5.14 (see [16])**.** *Suppose that (G, E) is a contracting, regular and pseudofree self-similar graph action. If the graph E satisfies Condition (L) and E is G-transitive, then $C^*(\Gamma_{(G,E)})$ is a simple, separable, nuclear, and purely infinite C^*-algebra satisfying the UCT.*

Proof. Contracting and regular conditions imply that $\Gamma_{(G,E)}$ is étale and amenable by Proposition 5.5. Then Condition (L) implies that $\Gamma_{(G,E)}$ is locally contracting by Proposition 5.11 and topologically principal by Proposition 4.15 and

Lemma 5.13. On the other hand, pseudofree and G-transitive conditions imply that $\Gamma_{(G,E)}$ is minimal by Proposition 5.8. Hence $C^*(\Gamma_{(G,E)})$ is simple and purely infinite by Lemma 5.13. As $\Gamma_{(G,E)}$ is second countable, [19, p. 59, Remark (iii)] induces that $C^*(\Gamma_{(G,E)})$ is separable. Because $\Gamma_{(G,E)}$ is amenable by Proposition 5.5, $C^*(\Gamma_{(G,E)})$ is nuclear by [18, Corollary 2.17], and $C^*(\Gamma_{(G,E)})$ satisfies the UCT by [23, Lemma 3.5, Proposition 10.7]. □

Remark 5.15. When E is an n-bouquet, (G, E) is a contracting, regular, and recurrent self-similar group, and the G-action on E^* is faithful, Nekrashevych constructed a Smale space, called the *limit solenoid* of (G, E), from the limit dynamical system $(J_{(G,E)}, \sigma)$ (see [16, Section 6] for details). Then he showed in [16, Proposition 6.13] that $C^*(\Gamma_{(G,E)})$ is strongly Morita-equivalent to the Ruelle algebra of the stable equivalence on the limit solenoid, which is a simple, purely infinite, and nuclear C^*-algebra satisfying the UCT (see [17]). In comparison with the conditions of Theorem 5.14, it is easy to see that an n-bouquet satisfies Condition (L), the recurrent condition implies G-transitivity, and a faithful G-action induces pseudofreeness.

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