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THE MULTIPLIER ALGEBRA OF THE NONCOMMUTATIVE SCHWARTZ SPACE

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ABSTRACT. We describe the multiplier algebra of the noncommutative Schwartz space. This multiplier algebra can be seen as the largest $*$ -algebra of unbounded operators on a separable Hilbert space with the classical Schwartz space of rapidly decreasing functions as the domain. We show in particular that it is neither a \mathcal{Q} -algebra nor m -convex. On the other hand, we prove that classical tools of functional analysis, for example, the closed graph theorem, the open mapping theorem, or the uniform boundedness principle, are still available.

1. INTRODUCTION

The aim of this article is to study algebraic and topological properties of some specific topological algebra with involution, called the *multiplier algebra of the noncommutative Schwartz space* and denoted by \mathcal{MS} . In particular, we will show that, in spite of the fact that \mathcal{MS} is neither Banach nor metrizable, the closed graph theorem, open mapping theorem, and uniform boundedness principle work on this space (see Theorem 4.5). Since it is a locally convex space the Hahn–Banach theorem obviously holds.

The algebra \mathcal{MS} can be described in many ways. For instance, it can be seen as the “intersection” $\mathcal{L}(\mathcal{S}(\mathbb{R})) \cap \mathcal{L}(\mathcal{S}'(\mathbb{R}))$, where $\mathcal{L}(\mathcal{S}(\mathbb{R}))$ (resp., $\mathcal{L}(\mathcal{S}'(\mathbb{R}))$) is the

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algebra of continuous linear operators on the well-known Schwartz space $\mathcal{S}(\mathbb{R})$ of smooth rapidly decreasing functions (resp., on the space $\mathcal{S}'(\mathbb{R})$ of tempered distributions). It appears that we can replace the above function and distribution spaces with the corresponding sequence spaces. This means that \mathcal{MS} is isomorphic (as a topological $*$ -algebra) to the “intersection” $\mathcal{L}(s) \cap \mathcal{L}(s')$, where s is the space of rapidly decreasing sequences, s' is the space of slowly increasing sequences (see Section 2 for definitions), and $\mathcal{L}(s)$, $\mathcal{L}(s')$ are the corresponding spaces of continuous linear operators. The “intersection”—whatever it is—allows us to introduce a natural involution, and this is why we consider $\mathcal{L}(s) \cap \mathcal{L}(s')$ instead of, for example, $\mathcal{L}(s)$. The algebra \mathcal{MS} also turns out to be the maximal \mathcal{O}^* -algebra with domain s , and thus it can be viewed as the largest algebra of unbounded operators on ℓ_2 with domain s (see the work of Schmüdgen [28] which deals with \mathcal{O}^* -algebras, especially [28, Part I.2]). Moreover, by Proposition 4.3, our algebra is isomorphic (as a topological $*$ -algebra) to the matrix algebra

$$\Lambda(\mathcal{A}) := \left\{ x = (x_{i,j})_{i,j \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}^2} : \forall N \in \mathbb{N}_0 \exists n \in \mathbb{N}_0 : \sup_{i,j \in \mathbb{N}} \left(|x_{i,j}| \max \left\{ \frac{i^N}{j^n}, \frac{j^N}{i^n} \right\} \right) < \infty \right\}.$$

As the name suggests, \mathcal{MS} can be also viewed as the algebra of “multipliers” for the so-called *noncommutative Schwartz space* \mathcal{S} , also known as the *algebra of smooth operators*. The noncommutative Schwartz space is a specific m -convex Fréchet $*$ -algebra isomorphic to several operator algebras naturally appearing in analysis, for example, to the algebra $\mathcal{L}(\mathcal{S}'(\mathbb{R}), \mathcal{S}(\mathbb{R}))$ of continuous linear operators from $\mathcal{S}'(\mathbb{R})$ to $\mathcal{S}(\mathbb{R})$, to the algebra $\mathcal{L}(s', s)$, or to the algebra

$$\mathcal{K}_\infty := \left\{ (x_{i,j})_{i,j \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}^2} : \forall N \in \mathbb{N}_0 \sup_{i,j \in \mathbb{N}} |x_{i,j}| i^N j^N < \infty \right\}$$

of *rapidly decreasing matrices* (see [11, Theorem 1.1] for more representations). It is also isomorphic—topologically, but not as an algebra—to the Fréchet space $\mathcal{S}(\mathbb{R})$ (which explains its name).

The noncommutative Schwartz space and the space $\mathcal{S}(\mathbb{R})$ itself play a role in a number of fields, for example: structure theory of Fréchet spaces and splitting of short exact sequences (see [22, Part IV]); K -theory (see [7]); C^* -dynamical systems (see [15]); cyclic cohomology for crossed products (see [29]); operator analogues of locally convex spaces (see [14]); and quantum mechanics, where it is called the *space of physical states* and its dual is the so-called *space of observables* (see [13]). Recently, some progress has been made in the investigation of the noncommutative Schwartz space. This area includes functional calculus, description of closed commutative $*$ -subalgebras, automatic continuity, amenability properties, and Grothendieck inequality (see [6], [26] and references therein).

However, the significance of the algebra \mathcal{MS} lies not only in the fact that it is a multiplier algebra of some well-known algebra of operators, but also in its resemblance to the C^* -algebra $\mathcal{B}(\ell_2)$ of bounded operators on ℓ_2 in the context of some class of Fréchet and topological $*$ -algebras. This can already be seen in the very definition of the maximal \mathcal{O}^* -algebra on the domain s (which is,

recall, isomorphic to \mathcal{MS}). It is also worth pointing out that \mathcal{S} is considered as a Fréchet analogue of the algebra $\mathcal{K}(\ell_2)$ of compact operators on ℓ_2 and $\mathcal{B}(\ell_2)$ is the multiplier algebra of $\mathcal{K}(\ell_2)$, whereas, as we have already noted, \mathcal{MS} is the multiplier algebra of \mathcal{S} —in fact, in order to prove this, we apply methods used in the case of $\mathcal{K}(\ell_2)$ and $\mathcal{B}(\ell_2)$ (compare Theorem 3.9 with [5, Proposition 2.5] and [23, Example 3.1.2]).

Finally, we find it quite interesting to compare some commutative sequence *-algebras with the corresponding noncommutative operator *-algebras. This is done by the following diagram with the horizontal continuous embeddings of algebras:

$$\begin{array}{ccccccccc}
 s & \hookrightarrow & l_1 & \hookrightarrow & l_2 & \hookrightarrow & c_0 & \hookrightarrow & l_\infty & \hookrightarrow & s' \\
 \updownarrow & & \updownarrow \\
 \mathcal{S} & \hookrightarrow & \mathcal{N}(\ell_2) & \hookrightarrow & \mathcal{HS}(\ell_2) & \hookrightarrow & \mathcal{K}(\ell_2) & \hookrightarrow & \mathcal{B}(\ell_2) & & \mathcal{MS}
 \end{array}$$

where $\mathcal{N}(\ell_2)$ and $\mathcal{HS}(\ell_2)$ are the algebras of nuclear and Hilbert–Schmidt operators, respectively. The “vertical correspondences” from s up to c_0 mean, for example, that every monotonic sequence of nonnegative numbers belonging to a commutative algebra from the first row is a sequence of singular numbers of some operator of its noncommutative analogues. Moreover, algebras from the first row are embedded in a canonical way, as the algebras of diagonal operators, into the corresponding algebras from the second row. It will be made clear below that $\mathcal{B}(\ell_2)$ is not embedded in \mathcal{MS} and vice versa.

This article is divided into four parts. Section 2 recalls basic notation and properties of the objects involved. In Section 3, we describe the multiplier algebra of the noncommutative Schwartz space. Section 4 deals with its topological properties and, finally, in Section 5 we consider properties of \mathcal{MS} as a topological algebra. (For more information on functional analysis, see [17], [22]; for more on general Banach algebra theory, see [8]; and for non-Banach operator algebras, see [28].)

2. NOTATION AND PRELIMINARIES

In what follows, we set

$$\begin{aligned}
 \mathbb{N} &:= \{1, 2, 3, \dots\}, \\
 \mathbb{N}_0 &:= \{0, 1, 2, \dots\}.
 \end{aligned}$$

For locally convex spaces E and F , we denote by $\mathcal{L}(E, F)$ the space of all continuous linear operators from E to F , and we set $\mathcal{L}(E) := \mathcal{L}(E, E)$. These spaces will be considered with the topology $\tau_{\mathcal{L}(E, F)}$ of uniform convergence on bounded sets.

By s we denote the *space of rapidly decreasing sequences*, that is, the Fréchet space

$$s := \left\{ \xi = (\xi_j)_{j \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : |\xi|_n := \left(\sum_{j=1}^{\infty} |\xi_j|^2 j^{2n} \right)^{1/2} < \infty \text{ for all } n \in \mathbb{N}_0 \right\}$$

with the topology given by the system $(|\cdot|_n)_{n \in \mathbb{N}_0}$ of norms. We will denote by s_n the Hilbert space corresponding to the norm $|\cdot|_n$.

By [22, Proposition 27.13], we may identify the strong dual of s , that is, the space of all continuous linear functionals on s with the topology of uniform convergence on bounded subsets of s , with the *space of slowly increasing sequences*

$$s' := \left\{ \xi = (\xi_j)_{j \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : |\xi|_{-n} := \left(\sum_{j=1}^{\infty} |\xi_j|^2 j^{-2n} \right)^{1/2} < \infty \text{ for some } n \in \mathbb{N}_0 \right\}$$

equipped with the inductive limit topology for the sequence $(s_{-n})_{n \in \mathbb{N}_0}$, where s_{-n} is the Hilbert space corresponding to the norm $|\cdot|_{-n}$. In other words, the locally convex topology on s' is given by the family $\{|\cdot|'_B\}_{B \in \mathcal{B}}$ of norms, $|\xi|'_B := \sup_{\eta \in B} |\langle \eta, \xi \rangle|$, where \mathcal{B} denotes the class of all bounded subsets of s .

The space $\mathcal{L}(s', s)$, denoted also by \mathcal{S} , is a Fréchet space, whose topology is described by the sequence $(\|\cdot\|_n)_{n \in \mathbb{N}_0}$ of norms,

$$\|x\|_n := \sup\{|x\xi|_n : \xi \in U_n^\circ\},$$

where $U_n^\circ = \{\xi \in s' : |\xi|_{-n} \leq 1\}$ are polars of the zero neighborhood basis $(U_n)_{n \in \mathbb{N}_0}$ in s . One can show that \mathcal{S} is in fact isomorphic (as a Fréchet space) to the space s (see [19, Proposition 41.7(5)] and [22, Lemma 31.1]). Since $\mathcal{L}(s', s) \hookrightarrow \mathcal{B}(\ell_2)$ (continuous embedding given by $\ell_2 \hookrightarrow s' \xrightarrow{x} s \hookrightarrow \ell_2$), we can endow our space with multiplication, involution, and order structure of the C^* -algebra $\mathcal{B}(\ell_2)$. With these operations $\mathcal{L}(s', s)$ becomes an m -convex Fréchet*-algebra and is called the *noncommutative Schwartz space* or the *algebra of smooth operators*.

We devote a large part of the present article to considering the spaces $\mathcal{L}(s)$, $\mathcal{L}(s')$ and their “intersection”

$$\mathcal{L}(s) \cap \mathcal{L}(s') := \{x \in \mathcal{L}(s) : x = \tilde{x}|_s \text{ for some (and hence unique) } \tilde{x} \in \mathcal{L}(s')\}$$

equipped with the topology $\tau_{\mathcal{L}(s) \cap \mathcal{L}(s')} := \tau_{\mathcal{L}(s)} \cap \tau_{\mathcal{L}(s')}$; $\tau_{\mathcal{L}(s) \cap \mathcal{L}(s')}$ is therefore determined by the family $\{q_{n,B}\}_{n \in \mathbb{N}_0, B \in \mathcal{B}}$ of seminorms, where

$$q_{n,B}(x) := \max\left\{ \sup_{\xi \in B} |x\xi|_n, \sup_{\xi \in U_n^\circ} |x\xi|'_B \right\} \tag{2.1}$$

and \mathcal{B} is the class of all bounded subsets of s . It is easy to show that $\mathcal{L}(s)$ and $\mathcal{L}(s')$ are topological algebras and, as we will show in Proposition 3.8, the same is true for $\mathcal{L}(s) \cap \mathcal{L}(s')$.

The spaces $\mathcal{L}(s)$ and $\mathcal{L}(s')$ can be seen as the completed tensor products $s' \widehat{\otimes} s$ and $s \widehat{\otimes} s'$, respectively (see [19, Corollary 43.3(7)]). Since s and s' are nuclear (see [22, Proposition 28.16]), the injective and the projective tensor product topologies coincide (see [17, Theorem 21.2.1]). Consequently (see [17, Theorem 15.4.2 and Corollary 15.5.4]), $\mathcal{L}(s)$ and $\mathcal{L}(s')$ admit natural PLS-topologies (see below) coinciding with $\tau_{\mathcal{L}(s)}$ and $\tau_{\mathcal{L}(s')}$, that is,

$$\begin{aligned} \mathcal{L}(s) &= \text{proj} \text{ ind}_{N \in \mathbb{N}_0} \mathcal{L}(s_n, s_N), \\ \mathcal{L}(s') &= \text{proj} \text{ ind}_{N \in \mathbb{N}_0} \mathcal{L}(s'_N, s'_n), \end{aligned} \tag{2.2}$$

where s_n (resp., s'_n) denotes the weighted Hilbert space

$$s_n := \left\{ \xi = (\xi_j)_{j \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : \sum_{j=1}^{\infty} |\xi_j|^2 j^{2n} < \infty \right\},$$

$$s'_n := \left\{ \xi = (\xi_j)_{j \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : \sum_{j=1}^{\infty} |\xi_j|^2 j^{-2n} < \infty \right\}.$$

Remark 2.1. To be precise, the above two representations of $\mathcal{L}(s)$ and $\mathcal{L}(s')$ are not of PLS-type. But this can be easily overcome following the first part of Remark 4.11.

As for the class of PLS-spaces (the best work for which is [9] and the references therein), we will need more information. Recall that by a PLB-space we mean a locally convex space X whose topology is given by

$$X = \text{proj}_{N \in \mathbb{N}_0} \text{ind}_{n \in \mathbb{N}_0} X_{N,n},$$

where all the $X_{N,n}$'s are Banach spaces and all the linking maps $\iota_{N,n}^{N,n+1} : X_{N,n} \hookrightarrow X_{N,n+1}$ are continuous linear inclusions. If all the $X_{N,n}$'s are Hilbert spaces, then we call X a PLH-space. If the linking maps $(\iota_{N,n}^{N,n+1})_{N,n \in \mathbb{N}_0}$ are compact (nuclear), then X is called a PLS-space (PLN-space).

Of particular importance for us will be the so-called *Köthe-type* PLB-spaces. Recall that a *Köthe* PLB-matrix is a matrix $\mathcal{C} := (c_{j,N,n})_{j \in \mathbb{N}, N,n \in \mathbb{N}_0}$ of nonnegative scalars satisfying

- (i) $\forall j \in \mathbb{N} \exists N \in \mathbb{N}_0 \forall n \in \mathbb{N}_0 : c_{j,N,n} > 0$,
- (ii) $\forall j \in \mathbb{N}, N, n \in \mathbb{N}_0 : c_{j,N,n+1} \leq c_{j,N,n} \leq c_{j,N+1,n}$.

We define

$$\Lambda^p(\mathcal{C}) := \left\{ x = (x_j)_j \mid \forall N \in \mathbb{N}_0 \exists n \in \mathbb{N}_0 : \|x\|_{N,n,p} < +\infty \right\} \quad (1 \leq p < \infty),$$

where

$$\|x\|_{N,n,p} := \left(\sum_{j=1}^{\infty} (|x_j| c_{j,N,n})^p \right)^{1/p}.$$

For $p = \infty$ we use the respective sup-norms. Then $\Lambda^p(\mathcal{C})$ can be identified with the space $\text{proj}_{N \in \mathbb{N}_0} \text{ind}_{n \in \mathbb{N}_0} \ell^p(c_{j,N,n})$, and it is called a *Köthe-type* PLB-space. If for all $N, n \in \mathbb{N}_0$ we have $\lim_j \frac{c_{j,N,n+1}}{c_{j,N,n}} = 0$ (compact linking maps), then $\Lambda^p(\mathcal{C})$ is a PLS-space, and if for all $N, n \in \mathbb{N}_0$ we have $\sum_j \frac{c_{j,N,n+1}}{c_{j,N,n}} < \infty$ (nuclear linking maps), then $\Lambda^p(\mathcal{C})$ is a PLN-space. If $\Lambda^p(\mathcal{C}) = \Lambda^q(\mathcal{C})$, as sets, for all $1 \leq p, q \leq \infty$, then we simply write $\Lambda(\mathcal{C})$.

For later use, we distinguish specific Köthe PLB-matrices on $\mathbb{N} \times \mathbb{N}$, defined as follows:

$$\begin{aligned} \mathcal{B} &= (b_{ij;N,n})_{i,j \in \mathbb{N}, N,n \in \mathbb{N}_0}, & b_{ij;N,n} &:= \frac{i^N}{j^n}, \\ \mathcal{B}' &= (b'_{ij;N,n})_{i,j \in \mathbb{N}, N,n \in \mathbb{N}_0}, & b'_{ij;N,n} &:= \frac{j^N}{i^n} \end{aligned} \tag{2.3}$$

and

$$\mathcal{A} = (a_{ij;N,n})_{i,j \in \mathbb{N}, N,n \in \mathbb{N}_0}, a_{ij;N,n} := \max\{b_{ij;N,n}, b'_{ij;N,n}\} = \max\left\{\frac{i^N}{j^n}, \frac{j^N}{i^n}\right\}. \quad (2.4)$$

Proposition 2.2. *Let \mathcal{M} be any of the Köthe-type PLB-matrices given by (2.3) or (2.4). Then $\Lambda^p(\mathcal{M}) = \Lambda^q(\mathcal{M})$ topologically, for all $1 \leq p, q \leq \infty$.*

Proof. Since $m_{ij;N+2,n} = (ij)^2 m_{ij;N,n+2}$ for all indices $i, j \in \mathbb{N}, N, n \in \mathbb{N}_0$, we have the inequalities

$$\|\cdot\|_{N,n,\infty} \leq \|\cdot\|_{N,n,1} \quad \text{and} \quad \|\cdot\|_{N,n+2,1} \leq C \|\cdot\|_{N+2,n,\infty},$$

which hold for all $N, n \in \mathbb{N}_0$ with $C = \sum_{i,j=1}^\infty (ij)^{-2} = \frac{\pi^4}{36}$. □

Using the notion of Köthe-type PLB-spaces, we get from (2.2) and Proposition 2.2 topological isomorphisms

$$\mathcal{L}(s) \cong \Lambda(\mathcal{B}) \quad \text{and} \quad \mathcal{L}(s') \cong \Lambda(\mathcal{B}'). \quad (2.5)$$

In both cases, the isomorphism is given by $x \mapsto (\langle xe_j, e_i \rangle)_{i,j \in \mathbb{N}}$.

3. REPRESENTATIONS OF THE MULTIPLIER ALGEBRA

In this section, we want to describe the so-called *multiplier algebra* of \mathcal{S} which is, in some sense, the largest algebra of operators acting on \mathcal{S} . The algebra $\mathcal{L}(s) \cap \mathcal{L}(s')$ seems to be a good candidate, because if $x \in \mathcal{S}$ and $y \in \mathcal{L}(s) \cap \mathcal{L}(s')$, then clearly $xy, yx \in \mathcal{S}$. Now, using heuristic arguments, we may show that the algebra $\mathcal{L}(s) \cap \mathcal{L}(s')$ is optimal. Assume that $y \in \mathcal{L}(E, F)$ for some locally convex spaces E, F . If $xy \in \mathcal{S}$ for every $x \in \mathcal{S}$, then, in particular, $(\langle \cdot, \xi \rangle \xi)y \in \mathcal{S}$ for all $\xi \in s$, and therefore $\langle y(\eta), \xi \rangle$ has to be well defined for every $\xi \in s$ and $\eta \in s'$, which shows that $y: s' \rightarrow s'$. Similarly, we show that if $yx \in \mathcal{S}$ for every $x \in \mathcal{S}$, then $y: s \rightarrow s$. Hence, $y \in \mathcal{L}(s) \cap \mathcal{L}(s')$.

Another, more abstract approach to multipliers goes through the so-called *double centralizers* (see Definition 3.1) and is due to Johnson [18]. Theorem 3.9 below shows that this approach leads again to $\mathcal{L}(s) \cap \mathcal{L}(s')$. In Corollary 4.4, we show that the multiplier algebra of \mathcal{S} has other representations—also important in further investigation.

The theory of double centralizers of C^* -algebras was developed by Busby [5] (see also [23, pp. 38–39, 81–83]). Our exposition for the noncommutative Schwartz space will follow that of C^* -algebras.

Definition 3.1. Let A be a $*$ -algebra over \mathbb{C} . A pair (L, R) of maps from A to A (neither linearity nor continuity is required) such that $xL(y) = R(x)y$ for $x, y \in A$ is called a *double centralizer* on A . We denote the set of all double centralizers on A by $\mathcal{DC}(A)$. Furthermore, for a map $T: A \rightarrow A$, we define $T^*: A \rightarrow A$ by $T^*(x) := (T(x^*))^*$.

Now, let $(L_1, R_1), (L_2, R_2) \in \mathcal{DC}(A)$, $\lambda \in \mathbb{C}$. We define

- (i) $(L_1, R_1) + (L_2, R_2) := (L_1 + L_2, R_1 + R_2)$,
- (ii) $\lambda(L_1, R_1) := (\lambda L_1, \lambda R_1)$,
- (iii) $(L_1, R_1) \cdot (L_2, R_2) := (L_1 L_2, R_2 R_1)$,

(iv) $(L_1, R_1)^* := (R_1^*, L_1^*)$.

A straightforward computation shows that $\mathcal{DC}(A)$ with the operations defined above is a $*$ -algebra. The elements of A correspond to the elements of $\mathcal{DC}(A)$ via the map, called the *double representation* of A (see [18, p. 301]),

$$\varrho: A \rightarrow \mathcal{DC}(A), \quad \varrho(x) := (L_x, R_x), \tag{3.1}$$

where $L_x(y) := xy$ and $R_x(y) := yx$ are the left and right multiplication maps, respectively. One can easily show that ϱ is a homomorphism of $*$ -algebras.

Definition 3.2. Let A be an algebra over \mathbb{C} , and let I be an ideal in A .

- (i) We say that A is *faithful* if for every $x \in A$ we have: $xA = \{0\}$ implies $x = 0$ and $Ax = \{0\}$ implies $x = 0$.
- (ii) An ideal I is called *essential* in A if for every $x \in A$ the following implications hold: if $xI = \{0\}$, then $x = 0$ and if $Ix = \{0\}$, then $x = 0$.

It is well known that every C^* -algebra is an essential ideal in its multiplier algebra (see [23, p. 82]), so it is faithful (see also [5, Corollary 2.4]). For example, the C^* -algebra of compact operators on ℓ_2 is an essential ideal in $\mathcal{L}(\ell_2)$. The analogue of this result holds for \mathcal{S} and $\mathcal{L}(s) \cap \mathcal{L}(s')$.

Proposition 3.3. *An algebra \mathcal{S} is an essential ideal in $\mathcal{L}(s) \cap \mathcal{L}(s')$. In particular, \mathcal{S} is faithful.*

Proof. Clearly, \mathcal{S} is an ideal in $\mathcal{L}(s) \cap \mathcal{L}(s')$. Assume that $xz = 0$ for all $z \in \mathcal{S}$. Then, in particular, for $\xi \in s$ and $z := \langle \cdot, \xi \rangle \xi$, we get $\langle \cdot, \xi \rangle x(\xi) = 0$. Thus $x(\xi) = 0$ for all $\xi \in s$; that is, $x = 0$. This gives $x\mathcal{S} = \{0\} \Rightarrow x = 0$. After applying the involution, we get the implication $\mathcal{S}x = \{0\} \Rightarrow x = 0$. □

The following result can be deduced from [18, Theorems 7, 14]. We present a more direct proof for the reader's convenience, following the proof of [5, Proposition 2.5] (the case of C^* -algebras).

Proposition 3.4. *Let A be a faithful m -convex Fréchet algebra, and let $(L, R) \in \mathcal{DC}(A)$. Then*

- (i) L and R are linear continuous maps on A ,
- (ii) $L(xy) = L(x)y$ for every $x, y \in A$,
- (iii) $R(xy) = xR(y)$ for every $x, y \in A$.

Proof. (i) Let $x, y, z \in A, \alpha, \beta \in \mathbb{C}$. Then

$$zL(\alpha x + \beta y) = R(z)(\alpha x + \beta y) = \alpha R(z)x + \beta R(z)y = z(\alpha L(x) + \beta L(y)),$$

and since A is faithful, $L(\alpha x + \beta y) = \alpha L(x) + \beta L(y)$.

Now, let $(x_j)_{j \in \mathbb{N}} \subset A$, and assume that $x_j \rightarrow 0$ and $L(x_j) \rightarrow y$ (convergence in the topology of A). Let $(\|\cdot\|_q)_{q \in \mathbb{N}_0}$ be a fundamental system of submultiplicative seminorms on A . Then

$$\begin{aligned} \|zy\|_q &\leq \|zy - zL(x_j)\|_q + \|zL(x_j)\|_q = \|z(y - L(x_j))\|_q + \|R(z)x_j\|_q \\ &\leq \|z\|_q \cdot \|y - L(x_j)\|_q + \|R(z)\|_q \cdot \|x_j\|_q \rightarrow 0, \end{aligned}$$

as $j \rightarrow \infty$, so $\|zy\|_q = 0$ for every $q \in \mathbb{N}_0$, and therefore $zy = 0$. Hence, by the assumption on A , $y = 0$. Now, by the closed graph theorem for Fréchet spaces (see, e.g., [22, Theorem 24.31]), L is continuous. Analogous arguments work for the map R .

(ii) Let $x, y, z \in A$. Then

$$zL(xy) = R(z)xy = (R(z)x)y = (zL(x))y = z(L(x)y),$$

and therefore, $L(xy) = L(x)y$.

(iii) This can be proved analogously as in (ii). \square

Now, we need to prove that elements of $\mathcal{L}(s) \cap \mathcal{L}(s')$ can be seen as some unbounded operators on ℓ_2 , namely, as the elements of the class

$$\mathcal{L}^*(s) := \{x: s \rightarrow s : x \text{ is linear, } s \subset \mathcal{D}(x^*) \text{ and } x^*(s) \subset s\}.$$

Here

$$\mathcal{D}(x^*) := \{\eta \in \ell_2 : \exists \zeta \in \ell_2 \forall \xi \in s \langle x\xi, \eta \rangle = \langle \xi, \zeta \rangle\}$$

and $x^*\eta := \zeta$ for $\eta \in \mathcal{D}(x^*)$ (one can show that such a vector ζ is unique). This $*$ -operation defines a natural involution on $\mathcal{L}^*(s)$. The algebra $\mathcal{L}^*(s)$ is considered the maximal \mathcal{O}^* -algebra with domain s , and it can be viewed as the largest $*$ -algebra of unbounded operators on ℓ_2 with domain s (see [28, Section I.2.1] for details). Let \mathcal{B} denote the set of all bounded subsets of s . We endow $\mathcal{L}^*(s)$ with the topology $\tau_{\mathcal{L}^*(s)}$ given by the family $\{p_{n,B}\}_{n \in \mathbb{N}_0, B \in \mathcal{B}}$ of seminorms,

$$p_{n,B}(x) := \max\left\{\sup_{\xi \in B} |x\xi|_n, \sup_{\xi \in B} |x^*\xi|_n\right\}. \quad (3.2)$$

It is well known that $\mathcal{L}^*(s)$ is a complete locally convex space and a topological $*$ -algebra (see [28, Proposition 3.3.15] and Remark 3.5 below).

Remark 3.5. The so-called *graph topology* on s of the \mathcal{O}^* -algebra $\mathcal{L}^*(s)$ is given by the system of seminorms $(\|\cdot\|_a)_{a \in \mathcal{L}^*(s)}$, $\|\xi\|_a := \|a\xi\|_{\ell_2}$ (see [28, Definition 2.1.1]). It is easy to see that the usual Fréchet space topology on s is equal to the graph topology (consider the diagonal map $s \ni (\xi_j)_{j \in \mathbb{N}} \mapsto (j^n \xi_j)_{j \in \mathbb{N}} \in s$, $n \in \mathbb{N}_0$), and therefore the topology $\tau_{\mathcal{L}^*(s)}$ on $\mathcal{L}^*(s)$ coincides with the topology τ^* (see [28, pp. 81–82]) defined by the seminorms $\{p^{a,B}\}_{a \in \mathcal{L}^*(s), B \in \mathcal{B}}$,

$$p^{a,B}(x) := \max\left\{\sup_{\xi \in B} \|ax\xi\|_{\ell_2}, \sup_{\xi \in B} \|ax^*\xi\|_{\ell_2}\right\}.$$

Lemma 3.6. *We have $\mathcal{L}^*(s) \subset \mathcal{L}(s)$, and every operator $x \in \mathcal{L}^*(s)$ can be extended to an operator $\tilde{x} \in \mathcal{L}(s')$. Moreover,*

$$\langle x^*\eta, \xi \rangle = \langle \eta, \tilde{x}\xi \rangle$$

for all $\xi \in s'$ and $\eta \in s$.

Proof. Take $x \in \mathcal{L}^*(s)$. Let $(\xi_j)_{j \in \mathbb{N}} \subset s$, and assume that $\xi_j \rightarrow 0$ and $x\xi_j \rightarrow \eta$ as $j \rightarrow \infty$. Then, for every $\zeta \in s$, we have

$$\langle x\xi_j, \zeta \rangle = \langle \xi_j, x^*\zeta \rangle \rightarrow 0$$

and, on the other hand,

$$\langle x\xi_j, \zeta \rangle \rightarrow \langle \eta, \zeta \rangle.$$

Hence, $\langle \eta, \zeta \rangle = 0$ for every $\zeta \in s$, and therefore $\eta = 0$. By the closed graph theorem for Fréchet spaces, $x: s \rightarrow s$ is continuous, and thus $\mathcal{L}^*(s) \subset \mathcal{L}(s)$.

Fix $x \in \mathcal{L}^*(s)$, $\xi \in s'$, and define a linear functional $\phi_\xi: s \rightarrow \mathbb{C}$, $\phi_\xi(\eta) := \langle x^*\eta, \xi \rangle$. From the continuity of $x^*: s \rightarrow s$, it follows that for every $q \in \mathbb{N}_0$ there is $r \in \mathbb{N}_0$ and $C > 0$ such that $|x^*\eta|_q \leq C|\eta|_r$ for all η in s . Hence, with the same quantifiers, we get

$$|\phi_\xi(\eta)| = |\langle x^*\eta, \xi \rangle| \leq |x^*\eta|_q \cdot |\xi|_{-q} \leq C|\eta|_r \cdot |\xi|_{-q}, \tag{3.3}$$

and thus ϕ_ξ is continuous. Consequently, for each $\xi \in s'$ we can find a unique $\zeta \in s'$ such that

$$\langle \eta, \zeta \rangle = \phi_\xi(\eta) = \langle x^*\eta, \xi \rangle$$

for all $\eta \in s$, and we may define $\tilde{x}: s' \rightarrow s'$ by $\tilde{x}\xi := \zeta$. Clearly, \tilde{x} is a linear extension of x , and moreover \tilde{x} is continuous. Indeed, by (3.3), for every $q \in \mathbb{N}_0$ there is $r \in \mathbb{N}_0$ and $C > 0$ such that

$$|\tilde{x}\xi|_{-r} = \sup_{|\eta|_r \leq 1} |\langle \eta, \tilde{x}\xi \rangle| = \sup_{|\eta|_r \leq 1} |\langle x^*\eta, \xi \rangle| \leq C|\xi|_{-q}$$

for all $\xi \in s'$. This shows the continuity of \tilde{x} , and the proof is complete. □

The following result follows also from [20, Proposition 2.2].

Proposition 3.7. *We have $\mathcal{L}^*(s) = \mathcal{L}(s) \cap \mathcal{L}(s')$ as sets.*

Proof. The inclusion $\mathcal{L}^*(s) \subset \mathcal{L}(s) \cap \mathcal{L}(s')$ follows directly from Lemma 3.6.

Let $x \in \mathcal{L}(s) \cap \mathcal{L}(s')$. For each $\eta \in s$, we define a linear functional $\psi_\eta: s' \rightarrow \mathbb{C}$, $\psi_\eta(\xi) := \langle \tilde{x}\xi, \eta \rangle$, where $\tilde{x}: s' \rightarrow s'$ is the continuous extension of x . By the continuity of the operator \tilde{x} and the Grothendieck factorization theorem (see [22, Theorem 24.33]), it follows that for every $r \in \mathbb{N}_0$ there is $q \in \mathbb{N}_0$ and $C > 0$ such that $|\tilde{x}\xi|_{-q} \leq C|\xi|_{-r}$ for $\xi \in s'$. Hence, for $\xi \in s'$, we have

$$|\psi_\eta(\xi)| = |\langle \tilde{x}\xi, \eta \rangle| \leq |\tilde{x}\xi|_{-q} \cdot |\eta|_q \leq C|\xi|_{-r} \cdot |\eta|_q.$$

This shows that ψ_η is continuous, and therefore there exists $\zeta \in s$ such that $\psi_\eta(\cdot) = \langle \cdot, \zeta \rangle$. Consequently, $\langle x\xi, \eta \rangle = \langle \xi, \zeta \rangle$ for $\xi \in s$, hence $s \subset \mathcal{D}(x^*)$ and $x^*(s) \subset s$, that is, $x \in \mathcal{L}^*(s)$. □

If we endow $\mathcal{L}(s) \cap \mathcal{L}(s')$ with the involution on $\mathcal{L}^*(s)$, we obtain the following.

Proposition 3.8. *We have $\mathcal{L}^*(s) = \mathcal{L}(s) \cap \mathcal{L}(s')$ as locally convex spaces and *-algebras. Consequently, $\mathcal{L}(s) \cap \mathcal{L}(s')$ is a complete locally convex space and a topological *-algebra.*

Proof. By definition and Proposition 3.7, $\mathcal{L}^*(s) = \mathcal{L}(s) \cap \mathcal{L}(s')$ as *-algebras. Let us compare fundamental systems $\{p_{n,B}\}_{n \in \mathbb{N}_0, B \in \mathcal{B}}$ (3.2) and $\{q_{n,B}\}_{n \in \mathbb{N}_0, B \in \mathcal{B}}$ (2.1) of seminorms on $\mathcal{L}^*(s)$ and $\mathcal{L}(s) \cap \mathcal{L}(s')$, respectively. Take $x \in \mathcal{L}^*(s)$ with its

unique extension $\tilde{x} \in \mathcal{L}(s')$. Let B be a bounded subset of s , and let $n \in \mathbb{N}_0$. Then, by Lemma 3.6,

$$\sup_{\eta \in B} |x^* \eta|_n = \sup_{\eta \in B} \sup_{\xi \in U_n^\circ} |\langle x^* \eta, \xi \rangle| = \sup_{\eta \in B} \sup_{\xi \in U_n^\circ} |\langle \eta, \tilde{x} \xi \rangle| = \sup_{\xi \in U_n^\circ} |\tilde{x} \xi|'_B. \tag{3.4}$$

This shows that $p_{n,B}(x) = q_{n,B}(x)$. Since $\mathcal{L}^*(s)$ is a complete locally convex space and a topological $*$ -algebra (see [28, Proposition 3.3.15] and Remark 3.5), the result follows. \square

By Propositions 3.3 and 3.4, $\mathcal{DC}(\mathcal{S}) \subset \mathcal{L}(\mathcal{S}) \times \mathcal{L}(\mathcal{S})$, and thus we may endow $\mathcal{DC}(\mathcal{S})$ with the corresponding subspace topology, denoted by $\tau_{\mathcal{DC}(\mathcal{S})}$. Note that a typical continuous seminorm of an element (L, R) of $\mathcal{L}(\mathcal{S}) \times \mathcal{L}(\mathcal{S})$ is given by the formula $\max\{\sup_{x \in M} \|L(x)\|_n, \sup_{x \in M} \|R(x)\|_n\}$, where M is a bounded subset of \mathcal{S} and $n \in \mathbb{N}_0$.

For $u \in \mathcal{L}(s) \cap \mathcal{L}(s')$, we define the left and right multiplication maps $L_u, R_u: \mathcal{S} \rightarrow \mathcal{S}$, $L_u(x) := ux$, $R_u(x) := x\tilde{u}$, where $\tilde{u}: s' \rightarrow s'$ is the extension of u according to the definition of $\mathcal{L}(s) \cap \mathcal{L}(s')$. Theorem 3.9 below states that the double representation (3.1) of \mathcal{S} can be extended to an isomorphism of topological $*$ -algebras $\mathcal{L}(s) \cap \mathcal{L}(s')$ and $\mathcal{DC}(\mathcal{S})$, and thus $\mathcal{L}(s) \cap \mathcal{L}(s')$ can be seen as the multiplier algebra of \mathcal{S} (see [23, Theorem 3.1.8, Example 3.1.2]).

Theorem 3.9. *The map $\tilde{\varrho}: \mathcal{L}(s) \cap \mathcal{L}(s') \rightarrow \mathcal{DC}(\mathcal{S})$, $u \mapsto (L_u, R_u)$ is an isomorphism of locally convex spaces and $*$ -algebras. Consequently, $\mathcal{DC}(\mathcal{S})$ is a complete locally convex space and a topological $*$ -algebra.*

Proof. Throughout the proof, for $\xi, \eta \in s$, $\xi \otimes \eta$ denotes the 1-dimensional operator $\langle \cdot, \eta \rangle \xi: s' \rightarrow s$.

Clearly, for $u \in \mathcal{L}(s) \cap \mathcal{L}(s')$, the left and right multiplication maps $L_u, R_u: \mathcal{S} \rightarrow \mathcal{S}$ are well defined. Furthermore, it is easy to see that $xL_u(y) = R_u(x)y$ for $x, y \in \mathcal{S}$ and $u \in \mathcal{L}(s) \cap \mathcal{L}(s')$. Hence, $(L_u, R_u) \in \mathcal{DC}(\mathcal{S})$ for every $u \in \mathcal{L}(s) \cap \mathcal{L}(s')$; that is, $\tilde{\varrho}$ is well defined.

The proof of the fact that $\tilde{\varrho}$ is a $*$ -algebra homomorphism is straightforward and $\tilde{\varrho}$ is injective, because \mathcal{S} is an essential ideal in $\mathcal{L}(s) \cap \mathcal{L}(s')$ (see Proposition 3.3). We will show that $\tilde{\varrho}$ is surjective.

Let $(L, R) \in \mathcal{DC}(\mathcal{S})$, and fix $e \in s$ with $\|e\|_{\ell_2} = 1$. We define a linear continuous map (use Propositions 3.3 and 3.4) $u: s \rightarrow s$ by

$$u\xi := L(\xi \otimes e)(e).$$

For $\xi, \eta \in s$, we have

$$\begin{aligned} \langle u\xi, \eta \rangle &= \langle L(\xi \otimes e)(e), \eta \rangle = \langle L(\xi \otimes e)(e), (\eta \otimes e)(e) \rangle \\ &= \langle (e \otimes \eta)[L(\xi \otimes e)(e)], e \rangle = \langle [(e \otimes \eta)L(\xi \otimes e)](e), e \rangle \\ &= \langle [R(e \otimes \eta)(\xi \otimes e)](e), e \rangle = \langle R(e \otimes \eta)[(\xi \otimes e)(e)], e \rangle \\ &= \langle R(e \otimes \eta)(\xi), e \rangle = \langle \xi, (R(e \otimes \eta))^*(e) \rangle. \end{aligned} \tag{3.5}$$

This means that $u^* \eta = (R(e \otimes \eta))^*(e) \in s$ for $\eta \in s$. Hence, $s \subset \mathcal{D}(u^*)$ and $u^*(s) \subset s$. Consequently, $u \in \mathcal{L}^*(s)$, and thus, by Proposition 3.7, u has the continuous extension $\tilde{u}: s' \rightarrow s'$.

By Propositions 3.3 and 3.4, for $\zeta \in s$, we obtain

$$\begin{aligned} L_u(\xi \otimes \eta)(\zeta) &= (u\xi \otimes \eta)(\zeta) = [L(\xi \otimes e)(e) \otimes \eta](\zeta) \\ &= \langle \zeta, \eta \rangle L(\xi \otimes e)(e) = L(\xi \otimes e)(\langle \zeta, \eta \rangle e) \\ &= L(\xi \otimes e)[(e \otimes \eta)(\zeta)] = [L(\xi \otimes e)(e \otimes \eta)](\zeta) \\ &= L((\xi \otimes e)(e \otimes \eta))(\zeta) = L(\xi \otimes \eta)(\zeta), \end{aligned}$$

and hence $L_u(\xi \otimes \eta) = L(\xi \otimes \eta)$. Since $\{\xi \otimes \eta : \xi, \eta \in s\}$ is linearly dense in \mathcal{S} , it follows that $L_u = L$. Likewise, (3.5) implies that, for $\zeta \in s$,

$$\begin{aligned} R_u(\xi \otimes \eta)(\zeta) &= [(\xi \otimes \eta)\tilde{u}](\zeta) = \langle u\zeta, \eta \rangle \xi = \langle R(e \otimes \eta)(\zeta), e \rangle \xi \\ &= (\xi \otimes e)(R(e \otimes \eta)(\zeta)) = [(\xi \otimes e)R(e \otimes \eta)](\zeta) \\ &= R((\xi \otimes e)(e \otimes \eta))(\zeta) = R(\xi \otimes \eta)(\zeta), \end{aligned}$$

and therefore that $R_u = R$. Hence, $\tilde{\varrho}(u) = (L_u, R_u) = (L, R)$, and thus $\tilde{\varrho}$ is surjective.

Next, we will prove that $\tilde{\varrho}$ is continuous. Let M be a bounded subset of \mathcal{S} , and let $n \in \mathbb{N}_0$. Since the involution on \mathcal{S} is continuous (see [6, p. 148]), the set M^* is bounded and there are $C > 0$, $k \geq n$ such that $\|y^*\|_n \leq C\|y\|_k$ for all $y \in \mathcal{S}$. Define

$$\begin{aligned} B_1 &:= \{x\xi : x \in M, \xi \in s', |\xi|_{-n} \leq 1\}, \\ B_2 &:= \{x^*\xi : x \in M, \xi \in s', |\xi|_{-k} \leq 1\}. \end{aligned}$$

Then, for all $m \geq k$, we have

$$\begin{aligned} \sup\{|\eta|_m : \eta \in B_1\} &\leq \sup_{x \in M} \|x\|_m < \infty, \\ \sup\{|\eta|_m : \eta \in B_2\} &\leq \sup_{x \in M^*} \|x\|_m < \infty, \end{aligned}$$

and therefore B_1 and B_2 are bounded subsets of s . Now,

$$\sup_{x \in M} \|L_u(x)\|_n = \sup_{x \in M} \|ux\|_n = \sup_{x \in M} \sup_{|\xi|_{-n} \leq 1} |u(x\xi)|_n = \sup_{\eta \in B_1} |u(\eta)|_n \leq q_{n, B_1}(u).$$

Moreover, by Lemma 3.6, $(x\tilde{u})^* = u^*x^*$ for $x \in \mathcal{S}$ and $u \in \mathcal{L}(s) \cap \mathcal{L}(s')$, and thus

$$\begin{aligned} \sup_{x \in M} \|R_u(x)\|_n &= \sup_{x \in M} \|x\tilde{u}\|_n \\ &\leq C \sup_{x \in M} \|(x\tilde{u})^*\|_k = C \sup_{x \in M} \|u^*x^*\|_k \\ &= \sup_{x \in M} \sup_{|\xi|_{-k} \leq 1} |u^*(x^*\xi)|_k = \sup_{\eta \in B_2} |u^*\eta|_k \\ &\leq p_{k, B_2}(u) = q_{k, B_2}(u), \end{aligned}$$

where the last identity follows from (3.4). Consequently,

$$\max\left\{\sup_{x \in M} \|L_u(x)\|_n, \sup_{x \in M} \|R_u(x)\|_n\right\} \leq \max\{q_{n, B_1}(u), q_{k, B_2}(u)\} \leq q_{k, B_1 \cup B_2}(u),$$

and thus $\tilde{\varrho}$ is continuous.

Finally, we show that the inverse of $\tilde{\varrho}$ is continuous. Let us take a bounded subset B of s and $n \in \mathbb{N}_0$. Define

$$M := \{x_\xi : \xi \in B \setminus \{0\}\} \cup \{\mathbf{0}\},$$

where $x_\xi := \|\xi\|_{\ell_2}^{-1} \xi \otimes \xi$ and $\mathbf{0}$ is the zero operator in \mathcal{S} . For all $m \in \mathbb{N}_0$, we have

$$\sup_{x \in M} \|x\|_m = \sup_{\xi \in B \setminus \{0\}} \|\xi\|_{\ell_2}^{-1} |\xi|_m^2 \leq \sup_{\xi \in B \setminus \{0\}} |\xi|_{2m} < \infty,$$

where the first inequality follows from the Cauchy–Schwarz inequality and the last one is a consequence of the boundedness of the set B . Hence, M is a bounded subset of \mathcal{S} .

Let

$$B' := \{x\xi : x \in M, \xi \in s', |\xi|_{-n} \leq 1\}.$$

Clearly, $0 \in B'$. If $\xi \in B \setminus \{0\}$, then $\xi = x_\xi(\|\xi\|_{\ell_2}^{-1} \xi)$ and $\|\|\xi\|_{\ell_2}^{-1} \xi\|_{-n} \leq 1$, hence $\xi \in B'$. Consequently, $B \subset B'$. Again by identity (3.4), we get, for all $u \in \mathcal{L}(s) \cap \mathcal{L}(s')$,

$$\begin{aligned} q_{n,B}(u) &= p_{n,B}(u) = \max\left\{\sup_{\eta \in B} |u\eta|_n, \sup_{\eta \in B} |u^*\eta|_n\right\} \\ &\leq \max\left\{\sup_{\eta \in B'} |u\eta|_n, \sup_{\eta \in B'} |u^*\eta|_n\right\} \\ &= \max\left\{\sup_{x \in M} \sup_{|\xi|_{-n} \leq 1} |u(x\xi)|_n, \sup_{x \in M} \sup_{|\xi|_{-n} \leq 1} |u^*(x\xi)|_n\right\} \\ &= \max\left\{\sup_{x \in M} \|ux\|_n, \sup_{x \in M} \|u^*x^*\|_n\right\} \\ &= \max\left\{\sup_{x \in M} \|ux\|_n, \sup_{x \in M} \|x\tilde{u}\|_n\right\} \\ &= \max\left\{\sup_{x \in M} \|L_u(x)\|_n, \sup_{x \in M} \|R_u(x)\|_n\right\}, \end{aligned}$$

and therefore $\tilde{\varrho}^{-1}$ is continuous. □

4. TOPOLOGICAL PROPERTIES OF THE MULTIPLIER ALGEBRA

We start by showing how the multiplier algebra of \mathcal{S} can be realized as a matrix algebra. Before that we make the following rather easy but very efficient observation.

Proposition 4.1. *The space $\mathcal{L}(s) \cap \mathcal{L}(s')$ is isomorphic as a locally convex space to a complemented subspace of $\mathcal{L}(s) \times \mathcal{L}(s')$.*

Proof. We use matrix representations (2.5). If $x = (x_{ij})_{i,j \in \mathbb{N}} \in \Lambda(\mathcal{B})$ and $y = (y_{ij})_{i,j \in \mathbb{N}} \in \Lambda(\mathcal{B}')$, then we denote $M(x, y) \in \mathbb{C}^{\mathbb{N} \times \mathbb{N}}$,

$$[M(x, y)]_{ij} := \begin{cases} x_{ij} & i \leq j, \\ y_{ij} & i > j, \end{cases}$$

and define a map $P: \Lambda(\mathcal{B}) \times \Lambda(\mathcal{B}') \rightarrow \Lambda(\mathcal{B}) \times \Lambda(\mathcal{B}')$ by

$$P(x, y) := (M(x, y), M(x, y)).$$

It is easily seen that P is a projection with

$$\text{im } P = \Delta(\Lambda(\mathcal{B}) \times \Lambda(\mathcal{B}')) := \{(x, x) \in \Lambda(\mathcal{B}) \times \Lambda(\mathcal{B}') : x \in \Lambda(\mathcal{B}) \cap \Lambda(\mathcal{B}')\}.$$

To get continuity, observe first that $\Lambda(\mathcal{B})$ and $\Lambda(\mathcal{B}')$ are webbed by [22, Lemma 24.28]. Moreover, $\mathcal{L}(s) \cong \mathcal{L}(s') \cong s \otimes s'$ by [19, Corollary 43.3(7)], and therefore $\mathcal{L}(s)$ and $\mathcal{L}(s')$ are ultrabornological by [16, Chapitre II, Proposition 15, Corollaire 2] (see also [21] for a homological proof of this fact). Since $\Lambda(\mathcal{B}) \cong \mathcal{L}(s)$ and $\Lambda(\mathcal{B}') \cong \mathcal{L}(s')$, $\Lambda(\mathcal{B})$ and $\Lambda(\mathcal{B}')$ are ultrabornological as well. This implies that $\Lambda(\mathcal{B}) \times \Lambda(\mathcal{B}')$ is ultrabornological by [27, Chapter II, Section 8.2, Corollary 1] and has a web by [19, Theorem 35.4(6)]. Continuity of P follows now by the closed graph theorem (see [22, Theorem 24.31]).

Now, let us consider the map

$$\Phi: \mathcal{L}(s) \times \mathcal{L}(s') \rightarrow \Lambda(\mathcal{B}) \times \Lambda(\mathcal{B}'), \quad \Phi(x, y) := (\Phi_1(x), \Phi_2(y)),$$

where $\Phi_1: \mathcal{L}(s) \rightarrow \Lambda(\mathcal{B})$, $\Phi_2: \mathcal{L}(s') \rightarrow \Lambda(\mathcal{B}')$ are isomorphisms given by $x \mapsto \langle xe_j, e_i \rangle_{i,j \in \mathbb{N}}$. Clearly, Φ is an isomorphism of locally convex spaces and

$$\Phi(\Delta(\mathcal{L}(s) \times \mathcal{L}(s'))) = \Delta(\Lambda(\mathcal{B}) \times \Lambda(\mathcal{B}')),$$

where

$$\Delta(\mathcal{L}(s) \times \mathcal{L}(s')) := \{(x, \tilde{x}) \in \mathcal{L}(s) \times \mathcal{L}(s') : x \in \mathcal{L}(s) \cap \mathcal{L}(s')\}.$$

This shows that the map $\Phi^{-1}P\Phi$ is a continuous projection onto $\Delta(\mathcal{L}(s) \times \mathcal{L}(s'))$. Finally, by comparing fundamental systems of seminorms on $\mathcal{L}(s) \cap \mathcal{L}(s')$ and $\Delta(\mathcal{L}(s) \times \mathcal{L}(s'))$, we see that the map

$$\Psi: \mathcal{L}(s) \cap \mathcal{L}(s') \rightarrow \Delta(\mathcal{L}(s) \times \mathcal{L}(s')), \quad \Psi(x) := (x, \tilde{x})$$

is an isomorphism, and thus $\mathcal{L}(s) \cap \mathcal{L}(s')$ is isomorphic to $\Delta(\mathcal{L}(s) \times \mathcal{L}(s'))$, a complemented subspace of $\mathcal{L}(s) \times \mathcal{L}(s')$. \square

Corollary 4.2. *The space $\mathcal{L}(s) \cap \mathcal{L}(s')$ is a nuclear, ultrabornological PLS-space.*

Proof. The spaces $\mathcal{L}(s)$ and $\mathcal{L}(s')$ are nuclear, ultrabornological PLS-spaces, and so is $\mathcal{L}(s) \times \mathcal{L}(s')$ as their product (see [22, Proposition 28.7], [24, Corollary 6.2.14]). The desired properties are inherited by complemented subspaces (see [22, Proposition 28.6], [27, Chapter II, 8.2, Corollary 1], [12, Proposition 1.2]), and thus, by Proposition 4.1, the proof is complete. \square

Let \mathcal{A} be as in (2.4), that is,

$$\mathcal{A} := (a_{ij;N,n})_{i,j \in \mathbb{N}, N, n \in \mathbb{N}_0}, \quad a_{ij;N,n} := \max \left\{ \frac{i^N}{j^n}, \frac{j^N}{i^n} \right\}.$$

Proposition 4.3. *We have $\mathcal{L}(s) \cap \mathcal{L}(s') \cong \Lambda(\mathcal{A})$ as topological $*$ -algebras.*

Proof. The map

$$T: \Lambda(\mathcal{A}) \rightarrow \mathcal{L}(s) \cap \mathcal{L}(s'), \quad \langle (Tx)e_j, e_i \rangle := x_{ij}$$

is a $*$ -algebra isomorphism. To see that it is continuous, observe that the embeddings $\Lambda(\mathcal{A}) \hookrightarrow \mathcal{L}(s)$ and $\Lambda(\mathcal{A}) \hookrightarrow \mathcal{L}(s')$ are continuous. Continuity of T^{-1} follows from the open mapping theorem (see [22, Theorem 24.30]), since $\Lambda(\mathcal{A})$ —as a PLS-space—has a web, and $\mathcal{L}(s) \cap \mathcal{L}(s')$ is ultrabornological by Corollary 4.2. \square

Combining Propositions 3.8 and 4.3 and Theorem 3.9, we obtain the following.

Corollary 4.4. *We have $\mathcal{DC}(\mathcal{S}) \cong \mathcal{L}^*(s) = \mathcal{L}(s) \cap \mathcal{L}(s') \cong \Lambda(\mathcal{A})$ as topological $*$ -algebras.*

From now on, by \mathcal{MS} we denote any topological $*$ -algebra isomorphic to $\mathcal{DC}(\mathcal{S})$ and we call it the *multiplier algebra of the noncommutative Schwartz space*.

We have just shown that \mathcal{MS} is webbed and ultrabornological. By [30, Theorem 4.2], this last property is equivalent to barrelledness. Therefore, by [22, Theorems 24.30, 24.31] and [24, Proposition 4.1.3], all the classical functional analytic tools are available for \mathcal{MS} . For convenience we state this result separately.

Theorem 4.5. *Let $X = \mathcal{MS}$ be the multiplier algebra of the noncommutative Schwartz space, and let E, F, G be locally convex spaces with E webbed and F ultrabornological. The following hold.*

- (1) *Uniform boundedness principle: every pointwise bounded set $B \subset \mathcal{L}(X, G)$ is equicontinuous.*
- (2) *Closed graph theorem: every linear map $T: X \rightarrow E$ and $S: F \rightarrow X$ with closed graph is continuous.*
- (3) *Open mapping theorem: every continuous linear surjection $T: E \rightarrow X$ and $S: X \rightarrow F$ is open.*

We will now need a characterization of those Fréchet spaces which are PLN-spaces. This characterization seems to be known for specialists; however, we were not able to find any reference to that result. We state it explicitly below for the reader's convenience.

Proposition 4.6. *A Fréchet space is a PLN-space if and only if it is strongly nuclear.*

Proof. Let X be a strongly nuclear Fréchet space. By [17, Theorem 21.8.8], X is a topological subspace of $(s')^I$ for a countable set I . Since s' is an LN-space, the product $(s')^I$ is a PLN-space. By [12, Proposition 1.2], X is a PLN-space.

Conversely, every PLN-space is a topological subspace of a countable product of LN-spaces and these are strongly nuclear by [17, Theorem 21.8.6]. Consequently, by [17, Propositions 21.1.3 and 21.1.5], X is strongly nuclear. \square

Corollary 4.7. *The multiplier algebra of the noncommutative Schwartz space is not a PLN-space.*

Proof. Suppose that the multiplier algebra is a PLN-space. Then so is s as its closed subspace. In fact, s is even complemented in $\Lambda(\mathcal{A})$ —consider the projection in $\Lambda(\mathcal{A})$ which cancels all but first row entries. By Proposition 4.6, s is strongly nuclear which leads to a contradiction by [17, 21.8, Example 3]. \square

The multiplier algebra of the noncommutative Schwartz space satisfies another useful property. Recall from [4, p. 433] that a PLS-space X is said to have the *dual interpolation estimate for big θ* if

$$\forall N \exists M \forall K \exists n \forall m \exists \theta_0 \in (0, 1) \forall \theta \geq \theta_0 \exists k, C > 0 \forall x' \in X'_N : \\ \|x' \circ \iota_N^M\|_{M,m}^* \leq C (\|x' \circ \iota_N^K\|_{K,k}^*)^{1-\theta} (\|x'\|_{N,n}^*)^\theta.$$

If we take $\theta \leq \theta_0$, then X has the *dual interpolation estimate for small θ* , and if we take $\theta \in (0, 1)$, then X has the *dual interpolation estimate for all θ* . For Köthe-type PLS-spaces $\Lambda^p(B)$ it is enough—see the proof of [3, Theorem 4.3]—to check the above condition for evaluation functionals $\phi_j(x) := x_j, j \in \mathbb{N}$. Examples of PLS-spaces with this property can be found in [2], [4], and [3]. The dual interpolation estimate plays an important role in partial differential equations, for example, in the surjectivity of operators, the existence of linear right inverses, and parameter dependence of solutions (see [10] for more details).

Proposition 4.8. *The multiplier algebra \mathcal{MS} has the dual interpolation estimate for big θ but not for small θ .*

Proof. For any $N \in \mathbb{N}_0$, take $M := N + 1$ and for any $K \in \mathbb{N}_0$, take $\theta_0 \in (0, 1)$ so that

$$(1 - \theta_0)K + N\theta_0 \leq M.$$

In a similar fashion, for $n = 1$, any $m \in \mathbb{N}_0$, and $\theta \geq \theta_0$, take $k \in \mathbb{N}_0$ so that

$$(1 - \theta)k + n\theta \geq m.$$

Then (with all the quantifiers in front)

$$\left(\frac{j^K}{i^k}\right)^{1-\theta} \left(\frac{j^N}{i^n}\right)^\theta \leq C \frac{j^M}{i^m}.$$

Exchanging indices i, j in the above inequality, we obtain by (2.4)

$$(a_{ij;K,k})^{1-\theta} (a_{ij;N,n})^\theta \leq C a_{ij;M,m}.$$

Since $\mathcal{MS} \cong \Lambda(\mathcal{A})$, the dual interpolation estimate for big θ follows. If it had the condition for small θ , then, by [2, Propositions 5.3(b) and 5.4(b)] and [22, Corollary 29.22], the space s of rapidly decreasing sequences would be a Banach space, which is a contradiction. \square

Remark 4.9. The above result together with [4, Proposition 1.1, Corollary 1.2(c)] gives another proof of the fact that the multiplier algebra of the noncommutative Schwartz space is ultrabornological.

We end this section with a technical lemma which characterizes when an arbitrary PLB-space is already a PLS-space; the proof uses interpolation theory and follows the idea of [25, Lemma 7]. As a consequence (see Remark 4.11(ii)), we obtain another proof of the fact that \mathcal{MS} is a PLS-space.

Lemma 4.10. *Let $X := \text{proj}_{M \in \mathbb{N}} \text{ind}_{m \in \mathbb{N}} X_{M,m}$ be a PLB-space. The following conditions are equivalent:*

- (i) X is a PLS-space,

(ii) $\forall M \exists L := L_M \forall l \exists m := m_l: X_{L,l} \hookrightarrow X_{M,m}$ is a compact inclusion.

Proof. We only need to show the implication (ii) \Rightarrow (i). There is no loss of generality in assuming that $L_M = M + 1$ and $m_l = l$. Consider the commutative diagram

$$\begin{array}{ccc} X_{2,1} & \xhookrightarrow{j_1} & X_{1,1} \\ \iota_1^2 \downarrow & & \downarrow \kappa_1^2 \\ X_{2,2} & \xhookrightarrow{j_2} & X_{1,2} \end{array}$$

where the inclusions ι_1^2, κ_1^2 are the respective linking maps and the inclusions j_1, j_2 are compact. Applying the real interpolation method with parameters $\theta_1, 1$ ($0 < \theta_1 < 1$) to the Banach couples $\overline{Y}_1 := (X_{2,1}, X_{1,1}), \overline{Y}_2 := (X_{2,2}, X_{1,2})$, we obtain, by [1, Theorem 3.11.8], a continuous map

$$\mathcal{J}_{\theta_1,1}(\overline{Y}_1) \rightarrow \mathcal{J}_{\theta_1,1}(\overline{Y}_2)$$

between the interpolation spaces $\mathcal{J}_{\theta_1,1}(\overline{Y}_1)$ and $\mathcal{J}_{\theta_1,1}(\overline{Y}_2)$. By [1, Corollary 3.8.2], we get for $0 < \theta_1 < \theta_2 < 1$ the compact inclusion

$$\mathcal{J}_{\theta_1,1}(\overline{Y}_2) \hookrightarrow \mathcal{J}_{\theta_2,1}(\overline{Y}_2),$$

and therefore the map

$$j_1^2: \mathcal{J}_{\theta_1,1}(\overline{Y}_1) \rightarrow \mathcal{J}_{\theta_2,1}(\overline{Y}_2)$$

is also compact. We apply the same procedure to the commutative diagram

$$\begin{array}{ccc} X_{2,2} & \xhookrightarrow{j_2} & X_{1,2} \\ \iota_2^3 \downarrow & & \downarrow \kappa_2^3 \\ X_{2,3} & \xhookrightarrow{j_3} & X_{1,3} \end{array}$$

and obtain a compact operator

$$j_2^3: \mathcal{J}_{\theta_2,1}(\overline{Y}_2) \rightarrow \mathcal{J}_{\theta_3,1}(\overline{Y}_3),$$

where $\overline{Y}_3 := (X_{2,3}, X_{1,3})$ and $\theta_2 < \theta_3 < 1$. Proceeding this way we obtain a countable inductive system $(j_n^{n+1}: \mathcal{J}_{\theta_n,1}(\overline{Y}_n) \rightarrow Y_1)$, where $Y_1 := \bigcup_n j_n^{n+1}(\mathcal{J}_{\theta_n,1}(\overline{Y}_n))$. Let us observe that the inductive topology of this system exists. Indeed, let $x \in Y_1$ be a nonzero element. Since $Y_1 \subset X_1 = \text{ind}_n X_{1,n}$, there exists, by [22, Lemma 24.6], a linear functional $\phi \in (\text{ind}_n X_{1,n})^*$ such that $\phi(x) \neq 0$ and $\phi \circ \kappa_n \in X'_{1,n}$ for all $n \in \mathbb{N}$ (by $(\kappa_n: X_{1,n} \rightarrow X_1)$ we denote the embedding spectrum of X_1). Recall that we distinguish here the space of linear functionals—denoted by $(\cdot)^*$ —and the space of continuous linear functionals—denoted by $(\cdot)'$. Therefore, $\phi \in Y_1^*$. Furthermore, for every $n \in \mathbb{N}$ we have the commutative diagram

$$\begin{array}{ccccc}
 \mathcal{J}_{\theta_n,1}(\overline{Y_n}) & \hookrightarrow & X_{1,n} & \xrightarrow{\phi \circ \kappa_n} & \mathbb{K} \\
 j_n^{n+1} \downarrow & & & & \downarrow \cong \\
 \mathcal{J}_{\theta_{n+1},1}(\overline{Y_{n+1}}) & \hookrightarrow & X_{1,n+1} & \xrightarrow{\phi \circ \kappa_{n+1}} & \mathbb{K}
 \end{array}$$

and therefore $\phi \circ j_n^{n+1} \in \mathcal{J}_{\theta_n,1}(\overline{Y_n})'$ for every $n \in \mathbb{N}$. Again, by [22, Lemma 24.6], this implies that the inductive topology of $(j_n^{n+1}: \mathcal{J}_{\theta_n,1}(\overline{Y_n}) \rightarrow Y_1)$ exists. Now, by [22, Lemma 24.34], we conclude that $(j_n^{n+1}: \mathcal{J}_{\theta_n,1}(\overline{Y_n}) \rightarrow Y_1)$ is an LB-space and that compactness of the linking maps $(j_n^{n+1})_n$ implies that it is even an LS-space. It follows that we have continuous linear maps

$$X_2 \rightarrow Y_1 \rightarrow X_1. \tag{4.1}$$

Indeed, since for every $n \in \mathbb{N}$, $\mathcal{J}_{\theta_n,1}(\overline{Y_n})$ is an interpolation space for the couple $\overline{Y_n} = (X_{2,n}, X_{1,n})$ with the compact inclusion $j_n: X_{2,n} \hookrightarrow X_{1,n}$, we get compact inclusions

$$X_{2,n} \hookrightarrow \mathcal{J}_{\theta_n,1}(\overline{Y_n}) \hookrightarrow X_{1,n}$$

(observe that we lose injectivity in (4.1) because the linking maps $(j_n^{n+1})_n$ are not, in general, injective). The above argument works for all LB-spaces X_M and X_{M+1} ; therefore

$$X = \text{proj}_{M \in \mathbb{N}} Y_M,$$

where all the Y_M 's are LS-spaces. Consequently, X is a PLS-space. □

Remark 4.11. Here we give two new proofs of the fact that \mathcal{MS} is a PLS-space. By Propositions 2.2 and 4.3, we may use the topological *-algebra isomorphism $\mathcal{MS} \cong \Lambda(\mathcal{A})$ (recall that the PLB-matrix \mathcal{A} is given by (2.4)).

(i) We do a slight perturbation of \mathcal{A} . Let $\mathcal{D} := (d_{ij;N,n})_{i,j \in \mathbb{N}, N, n \in \mathbb{N}_0}$ be a 4-indexed Köthe PLB-matrix given by

$$d_{ij;N,n} := \max_{i,j \in \mathbb{N}} \left\{ \frac{j^{N+\frac{1}{n}}}{i^n}, \frac{i^{N+\frac{1}{n}}}{j^n} \right\}.$$

One can easily show that for all $i, j \in \mathbb{N}, N, n \in \mathbb{N}_0$ we have

$$a_{ij;N,n} \leq d_{ij;N,n} \leq a_{ij;N+1,n}.$$

This implies the topological isomorphism $\mathcal{MS} \cong \Lambda(\mathcal{D})$. Since

$$\frac{d_{ij;N,n}}{d_{ij;N,n+1}} = \min\{i, j\} \max\{i, j\}^{\frac{1}{n(n+1)}},$$

we get that for all $N, n \in \mathbb{N}_0$

$$\lim_{i,j \rightarrow +\infty} \frac{d_{ij;N,n+1}}{d_{ij;N,n}} = 0.$$

Consequently, $\Lambda(\mathcal{D})$, and therefore also \mathcal{MS} , is a PLS-space.

(ii) Since $\frac{a_{ij,N,n+2}}{a_{ij,N+2,n}} = (ij)^{-2}$ for all i, j, N, n , the inclusion map

$$\ell_2((a_{ij,N+2,n})_{i,j \in \mathbb{N}}) \hookrightarrow \ell_2((a_{ij,N,n+2})_{i,j \in \mathbb{N}})$$

is compact. By Lemma 4.10, the result follows.

5. ALGEBRAIC PROPERTIES OF THE MULTIPLIER ALGEBRA

We say that a subalgebra B of an algebra A is *spectral invariant in A* if, for every $x \in B$, x is invertible in A if and only if it is invertible in B . We show that \mathcal{MS} contains a spectral invariant copy of the algebra s' , which implies that it is neither a \mathcal{Q} -algebra nor m -convex.

Proposition 5.1. *Let \mathcal{A} be a Köthe PLB-matrix given by (2.4), and let $\Delta(\mathcal{A})$ be the algebra of all diagonal matrices belonging to $\Lambda(\mathcal{A})$. Then*

- (i) $\Delta(\mathcal{A})$ is a complemented subspace of $\Lambda(\mathcal{A})$,
- (ii) $\Delta(\mathcal{A})$ is a closed commutative $*$ -subalgebra of $\Lambda(\mathcal{A})$,
- (iii) $\Delta(\mathcal{A}) \cong s'$ as topological $*$ -algebras,
- (iv) $\Delta(\mathcal{A})$ is spectral invariant in $\Lambda(\mathcal{A})$.

Proof. (i) Define $\pi: \Lambda(\mathcal{A}) \rightarrow \Lambda(\mathcal{A})$ by

$$\pi x := \sum_{j=1}^{\infty} e_{jj} x e_{jj},$$

where $(e_{ij})_{i,j \in \mathbb{N}}$ is a sequence of matrix units. Clearly, π is a projection. Note that continuity of a linear operator T on $\Lambda(\mathcal{A})$ follows from the condition

$$\forall N \exists M \forall m \exists n, C > 0 \forall x \in \Lambda(\mathcal{A}) \quad \|Tx\|_{N,n} \leq C \|x\|_{M,m}.$$

But

$$\|\pi x\|_{N,n} = \sum_{j=1}^{\infty} |x_{jj}| a_{jj,N,n} \leq \sum_{i,j=1}^{\infty} |x_{ij}| a_{ij,N,n} = \|x\|_{N,n},$$

so π is continuous, and thus $\Delta(\mathcal{A})$ is complemented in $\Lambda(\mathcal{A})$.

(ii) It is clear that $\Delta(\mathcal{A})$ is a commutative $*$ -subalgebra of $\Lambda(\mathcal{A})$, and by (i) it is closed in $\Lambda(\mathcal{A})$.

(iii) Since

$$\begin{aligned} \Delta(\mathcal{A}) &= \left\{ x \in \mathbb{C}^{\mathbb{N} \times \mathbb{N}} : x_{ij} = 0 \text{ for } i \neq j \text{ and } \forall N \exists n \sum_{j=1}^{\infty} |x_{jj}| j^{N-n} < \infty \right\} \\ &= \left\{ x \in \mathbb{C}^{\mathbb{N} \times \mathbb{N}} : x_{ij} = 0 \text{ for } i \neq j \text{ and } (x_{jj})_{j \in \mathbb{N}} \in s' \right\}, \end{aligned}$$

the operator

$$\phi: s' \rightarrow \Delta(\mathcal{A}), \quad \phi \xi := \sum_{j=1}^{\infty} \xi_j e_{jj}$$

is a $*$ -algebra isomorphism. Moreover, for all $N, n \in \mathbb{N}_0$ and all $\xi \in s'$, we have

$$\|\phi \xi\|_{N,n} = \sum_{j=1}^{\infty} |\xi_j| j^{N-n}.$$

Now, by Cauchy–Schwarz, we get

$$|\xi|_{-m} \leq \sum_{j=1}^{\infty} |\xi_j| j^{-m} \leq \frac{\pi}{\sqrt{6}} |\xi|_{-m+1}$$

for all $m \in \mathbb{N}_0$. Therefore ϕ is an isomorphism of locally convex spaces. Consequently, $\Delta(\mathcal{A}) \cong s'$ as topological $*$ -algebras.

(iv) Let us take $x \in \Delta(\mathcal{A})$ which is invertible in $\Lambda(\mathcal{A})$, and let y be its inverse. Then

$$\sum_{k=1}^{\infty} x_{ik} y_{kj} = x_{ii} y_{ij} = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, $x_{ii} \neq 0$, and thus $y_{ij} = 0$ for $i \neq j$. This shows that $y \in \Delta(\mathcal{A})$, and the proof is complete. \square

Proposition 5.2. *The following statements hold:*

- (i) \mathcal{MS} is not a \mathcal{Q} -algebra,
- (ii) \mathcal{MS} is not m -convex.

Proof. By Proposition 5.1, \mathcal{MS} contains a closed, spectral invariant $*$ -subalgebra M isomorphic to s' .

(i) Let F, G be the sets of invertible elements in M and $\mathcal{ML}(s', s)$, respectively. Then $F = G \cap M$. This shows that G is not open, because otherwise F would be open, which contradicts [4, Theorem 2.8]. Hence, $\mathcal{ML}(s', s)$ is not a \mathcal{Q} -algebra.

(ii) Suppose that there is a basis \mathcal{V} of zero neighborhoods in $\mathcal{ML}(s', s)$ such that $V^2 \subseteq V$ for all $V \in \mathcal{V}$. Then $\{V \cap M\}_{V \in \mathcal{V}}$ is a basis of zero neighborhoods in M and

$$(V \cap M)^2 \subseteq V^2 \cap M \subseteq V \cap M,$$

so M is m -convex, which is a contradiction (again apply [4, Theorem 2.8]). Hence, $\mathcal{ML}(s', s)$ is not m -convex. \square

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