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ON THE EXISTENCE OF AT LEAST A SOLUTION FOR FUNCTIONAL INTEGRAL EQUATIONS VIA MEASURE OF NONCOMPACTNESS

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Abstract. In this article, we use fixed-point methods and measure of noncompactness theory to focus on the problem of establishing the existence of at least a solution for the following functional integral equation

$$u(t) = g\big(t, u(t)\big) + \int_0^t G\big(t, s, u(s)\big)\, ds, \quad t \in [0, +\infty[,$$

in the space of all bounded and continuous real functions on $\mathbb{R}_+$, under suitable assumptions on g and G. Also, we establish an extension of Darbo's fixed-point theorem and discuss some consequences.

1. Introduction

In this article, we study the functional integral equation

$$u(t) = g\big(t, u(t)\big) + \int_0^t G\big(t, s, u(s)\big)\, ds, \quad t \in [0, +\infty[,$$

in the space of all bounded and continuous real functions on $[0, +\infty[$, under suitable assumptions on g and G. We obtain the existence of at least a solution by using fixed-point methods and measure of noncompactness theory. In fact, from the original paper of Kuratowski [16], the concept *measure of noncompactness*

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attracted the interest of mathematicians working in various fields such as differential equations, fixed-point theory, and optimization, among others. We give the formal definition of this concept later in this article. For now, a measure of noncompactness is a function suitable for measuring the degree of noncompactness of a given set; of course, to every compact set is associated a measure equal to zero. This theory is largely investigated in the setting of Banach spaces with interesting characterizations of various operators and consequences (see, e.g., [6]). In particular, we are interested in the applications of the measures of noncompactness to fixed-point theory, following a recent trend in the literature (see [3], [10]). The motivation is that the fixed-point theory furnishes efficient methods based on simple mathematical reasoning to solve (exactly or approximately) different problems arising in the applied sciences. The cornerstone at the junction between the theories of measure of noncompactness and fixed points is recognized in a theoretical result established by Darbo [12, p. 90]. In fact, Darbo gave a nice extension of a celebrated fixed-point theorem of Schauder. We recall its statement.

Theorem 1 (Schauder fixed-point theorem; [1, Theorem 4.14])**.** *Let M be a nonempty, bounded, closed, and convex subset of a Banach space X. Then each continuous and compact operator $T : M \to M$ has at least one fixed point in the set M.*

The main idea in the analogous theorem of Darbo is to relax the hypothesis of compactness of the operator T by using a Lipschitz condition involving a measure of noncompactness (see Theorem 3 in Section 2). We point out here that Theorem 1 above is itself an extension of the Brouwer fixed-point theorem from real-valued functions to topological vector spaces. Also, a consequence of Theorem 1, say, Schaefer's fixed-point theorem, is a powerful tool for proving the existence of at least a solution to nonlinear partial differential equations (see [6], [15], and the references therein). Our preceding few lines here underscore the relevance of these kind of results for the development of nonlinear analysis and topology in general.

Thus, we prove an existence result of a solution for the functional integral equation given at the beginning of this section. In so doing, we also establish an extension of Darbo's fixed-point theorem, by adapting the original proof to the relaxed assumption. The rest of this article is organized as follows. First, we provide the tools necessary to obtain the main result, then we present its proof, and finally, we discuss some auxiliary facts.

2. Preliminaries and statements

We start by recalling two ways to compute a measure of noncompactness (for more details, we refer the interested reader to [6], [16]). Later, the consolidated notation is used and hence, first, we assume that M is a bounded subset of a metric space X. Since a bounded set can be encased in a ball of fixed radius, then a basic measure of noncompactness is given by

$$\begin{aligned}\chi(M) = \inf\{&\varepsilon > 0 \text{ such that there exist finitely many balls of radius}\\ &\text{at most } \varepsilon \text{ which cover } M\}.\end{aligned}$$

On the other hand, Kuratowski [16] proposed the following definition:

$$\alpha(M) = \inf\{\delta > 0 \text{ such that there exist finitely many sets of diameter at most } \delta \text{ which cover } M\}.$$

From now on, we denote by

$$\operatorname{diam} M = \sup\{d(x,y) : x, y \in M\}$$

the diameter of a set $M \subset X$. Note that

$$\chi(M) \leq \alpha(M) \leq 2\chi(M).$$

However, our research setting is more generally represented by Banach spaces and hence in the following text we will denote by X a Banach space endowed with norm $\|\cdot\|$. This assumption is made based on [5], where the following definition is given (see also [4], [13], [18]). Again, M is a nonempty subset of X; in addition, we denote by $\overline{M}$ and $\overline{\operatorname{conv}(M)}$ the closure and the convex hull closure of M, respectively.

Definition 2. Let $B(X)$ and $K_r(X)$ be the family of all nonempty bounded subsets of X and the subfamily consisting of all relatively compact subsets of X, respectively. Then, we consider that $\phi : B(X) \to [0, +\infty[$ is a measure of noncompactness in X if the following conditions hold true:

(m1) $\ker\phi = \{A \in B(X) : \phi(A) = 0\}$ is nonempty and $\ker\phi \subset K_r(X)$;
(m2) $Z \subset Y$ implies $\phi(Z) \leq \phi(Y)$;
(m3) $\phi(\overline{A}) = \phi(A)$;
(m4) $\phi(\overline{\operatorname{conv}(A)}) = \phi(A)$;
(m5) $\phi(\lambda Z + (1-\lambda)Y) \leq \lambda\phi(Z) + (1-\lambda)\phi(Y)$ for all $\lambda \in [0,1]$;
(m6) if $\{M_n\}$ is a sequence of closed sets from $B(X)$ such that $M_{n+1} \subset M_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to+\infty}\phi(M_n) = 0$, then the intersection set $M_\infty = \bigcap_{n=1}^{+\infty} M_n$ is nonempty.

Note that the family $\ker\phi$ considered above is universally known as the kernel of the measure of noncompactness ϕ. Since $\phi(M_\infty) \leq \phi(M_n)$ for all $n \in \mathbb{N}$, we deduce that $\phi(M_\infty) = 0$, that is, $M_\infty \in \ker\phi$. Also, as M_∞ is a closed set, then it is compact.

Finally, we recall the statement of Darbo's fixed-point theorem.

Theorem 3 (Darbo's fixed-point theorem)**.** *Let M be a nonempty, bounded, closed, and convex subset of a Banach space X, and let $T : M \to M$ be a continuous operator. Assume that there exists a constant $k \in [0,1[$ such that*

$$\phi(TZ) \leq k\phi(Z) \quad \textit{for any nonempty } Z \subset M,$$

where ϕ is a measure of noncompactness defined in X. Then T has a fixed point in the set M.

Note that the inequality in Theorem 3 is the Lipschitz condition mentioned in the Introduction; more precisely, it is known as a *k-contraction condition.* The main statement of this section is an existence result of fixed point, which is inspired by Theorem 3. Precisely, we extend Theorem 3 by involving a V-ϕ-

contraction condition, which is more general than the k-contraction condition above. Thus, before establishing our result, we introduce a family $\mathcal{V}$ of V-functions. Indeed, a function $\rho : [0,+\infty[\times [0,+\infty[\to \mathbb{R}$ is a V-function if the following condition holds:

(v_1) if $\{a_n\} \subset]0,+\infty[$ is a sequence such that $\rho(a_{n+1}, a_n) > 0$ for all $n \in \mathbb{N}$, then $\lim_{n\to+\infty} a_n = 0$.

We use V-functions to define a V-ϕ-contraction condition as follows.

Definition 4. Let M be a nonempty, bounded, closed, and convex subset of a Banach space X, let ϕ be a measure of noncompactness in X, and let $T : M \to M$ be a continuous operator. Then, T is a V-ϕ-contraction if there exists a V-function $\rho : [0,+\infty[\times [0,+\infty[\to \mathbb{R}$ such that

$$\rho\big(\phi(TZ), \phi(Z)\big) > 0 \quad \text{for all } Z \subset M, \phi(Z) > 0. \tag{1}$$

As a result, we state the existence of a fixed point for a continuous (but not necessarily compact) operator satisfying a V-ϕ-contraction condition.

Theorem 5. *Let M be a nonempty, bounded, closed, and convex subset of a Banach space X, and let $T : M \to M$ be a continuous V-ϕ-contraction. Then T has a fixed point in the set M.*

Theorem 5 has the merit of being consistent with current ideas about how to generalize the fixed-point theory, by involving in a certain sense an implicit definition of the contraction condition. It furnishes a more flexible result than the existing ones, which can better represent specific situations, by particularizing the V-function.

3. Functional integral equations

Under the previous context, the main motivation of this section is to establish the existence of at least a solution for a functional integral equation in the space of all bounded and continuous real functions on $[0,+\infty[$, or $C_B([0,+\infty[)$ for short. We refer to consolidated arguments in the literature (see [8], [11]) and, according to the notation therein, $C_B([0,+\infty[)$ is endowed with the supremum norm

$$\|u\| = \sup\big\{\big|u(t)\big| : t \in [0,+\infty[\big\}.$$

Moreover, we consider a nonempty, bounded subset Z of $C_B([0,+\infty[)$ and a positive constant β. Thus, for $u \in Z$ and positive constant ε, we denote by $\omega^\beta(u,\varepsilon)$ the modulus of continuity of the function u on $[0,\beta]$, that is,

$$\omega^\beta(u,\varepsilon) = \sup\big\{\big|u(t) - u(s)\big| : t, s \in [0,\beta], |t-s| \le \varepsilon\big\}.$$

For further convenience, we denote

$$\omega^\beta(Z,\varepsilon) = \sup\big\{\omega^\beta(u,\varepsilon) : u \in Z\big\}, \qquad \omega_0^\beta(Z) = \lim_{\varepsilon\to0} \omega^\beta(Z,\varepsilon),$$

and

$$\omega_0(Z) = \lim_{\beta\to+\infty} \omega_0^\beta(Z).$$

Finally, for a fixed number $t \in [0, +\infty[$, we write $Z(t) = \{u(t) : u \in Z\}$ so that one can define the function ϕ on the family of all nonempty bounded subsets of $C_B([0, +\infty[)$, or $B(C_B([0, +\infty[))$ for short, as

$$\phi(Z) = \omega_0(Z) + \limsup_{t \to +\infty} \operatorname{diam} Z(t), \tag{2}$$

where, in light of the notation at the beginning of Section 2, $\operatorname{diam} Z(t)$ is given by

$$\operatorname{diam} Z(t) = \sup\big\{\big|u(t) - v(t)\big| : u, v \in Z\big\}.$$

Interest in the above functional is motivated by [2], [5], and [7], where it is shown that ϕ is a measure of noncompactness in $C_B([0, +\infty[)$. With all these elements in place, we are able to start the study of the functional integral equation

$$u(t) = g\big(t, u(t)\big) + \int_0^t G\big(t, s, u(s)\big)\, ds, \quad t \in [0, +\infty[, \tag{3}$$

where we assume that the following conditions hold:

(i) $g : [0, +\infty[\times \mathbb{R} \to \mathbb{R}$ is a continuous function; moreover, the function $t \to g(t, 0)$ is an element of $C_B([0, +\infty[)$;

(ii) there exists a nondecreasing right-continuous function $\varphi : [0, +\infty[\to [0, 1[$ such that, for each $t \in [0, +\infty[$ and for all $u, v \in \mathbb{R}$, we have

$$\big|g(t, u) - g(t, v)\big| \le \varphi\big(|u - v|\big)|u - v|;$$

(iii) $G : [0, +\infty[\times [0, +\infty[\times \mathbb{R} \to \mathbb{R}$ is a continuous function, and there exist two continuous functions $f, l : [0, +\infty[\to [0, +\infty[$ such that

$$\lim_{t \to +\infty} f(t) \int_0^t l(s)\, ds = 0 \quad \text{and} \quad \big|G(t, s, u)\big| \le f(t) l(s)$$

for all $t, s \in [0, +\infty[$ with $s \le t$, and for all $u \in \mathbb{R}$;

(iv) there exists a positive solution r_0 of the inequality $r\varphi(r) + \mu \le r$, with constant μ given by

$$\mu = \sup\Big\{\big|g(t, 0)\big| + f(t) \int_0^t l(s)\, ds : t \in [0, +\infty[\Big\}.$$

Before establishing the main theorem of this section, we prove an auxiliary proposition.

Proposition 6. *Suppose that the conditions (i)–(iv) hold true, and let r_0 be a positive solution of the inequality $r\varphi(r) + \mu \le r$. Then, the operator T defined by*

$$(Tu)(t) = g\big(t, u(t)\big) + \int_0^t G\big(t, s, u(s)\big)\, ds, \qquad \textit{for all } t \in [0, +\infty[\textit{ and } u \in C_B\big([0, +\infty[\big) \tag{4}$$

is continuous from $\mathcal{B}(r_0) = \{u \in C_B([0, +\infty[) : \|u\| \le r_0\}$ into itself.

Proof. The proof of Proposition 6 shows that the operator T is well defined and continuous on $\mathcal{B}(r_0)$. In fact, from (4), and by the above conditions on g and G, we deduce that Tu is a continuous function for each $u \in C_B([0,+\infty[)$. Also, we have

$$\begin{aligned}\big|Tu(t)\big| &= \Big|g\big(t,u(t)\big) - g(t,0) + g(t,0) + \int_0^t G\big(t,s,u(s)\big)\,ds\Big| \\ &\le \big|g\big(t,u(t)\big) - g(t,0)\big| + \big|g(t,0)\big| + \Big|\int_0^t G\big(t,s,u(s)\big)\,ds\Big| \\ &\le \varphi\big(\big|u(t)\big|\big)\big|u(t)\big| + \big|g(t,0)\big| + f(t)\int_0^t l(s)\,ds \\ &\le \varphi\big(\big|u(t)\big|\big)\big|u(t)\big| + \mu,\end{aligned}$$

where μ is given in condition (iv). Then, by condition (ii) the function φ is nondecreasing, and hence we get that

$$\|Tu\| \le \varphi\big(\|u\|\big)\|u\| + \mu \le \varphi(r_0)r_0 + \mu \le r_0.$$

Therefore, T maps the space $\mathcal{B}(r_0)$ into $\mathcal{B}(r_0)$. Next, we show that T is continuous on $\mathcal{B}(r_0)$. We proceed by fixing a positive number ε so that, for $u,v \in \mathcal{B}(r_0)$ with $\|u-v\| \le \varepsilon$, we have

$$\begin{aligned}&\big|Tu(t) - Tv(t)\big| \\ &\quad\le \varphi\big(\big|u(t)-v(t)\big|\big)\big|u(t)-v(t)\big| + \int_0^t \big|G\big(t,s,u(s)\big) - G\big(t,s,v(s)\big)\big|\,ds \\ &\quad\le \varphi\big(\big|u(t)-v(t)\big|\big)\big|u(t)-v(t)\big| + \int_0^t \big(\big|G\big(t,s,u(s)\big)\big| + \big|G\big(t,s,v(s)\big)\big|\big)\,ds \\ &\quad\le \varphi\big(\big|u(t)-v(t)\big|\big)\big|u(t)-v(t)\big| + 2f(t)\int_0^t l(s)\,ds \qquad (5)\end{aligned}$$

for all $t \in [0,+\infty[$. On the other hand, by condition (iii), it follows the existence of a positive constant β such that

$$2f(t)\int_0^t l(s)\,ds \le \varepsilon \quad \text{for all } t \ge \beta. \tag{6}$$

Consequently, by combining the inequalities (5) and (6), and by also keeping in mind that the function φ is nondecreasing, we deduce that

$$\big|Tu(t) - Tv(t)\big| \le 2\varepsilon \quad \text{for all } t \ge \beta. \tag{7}$$

Then, in light of the modulus of continuity at the beginning of this section, we have

$$\omega^\beta(G,\varepsilon) = \sup\big\{\big|G(t,s,u) - G(t,s,v)\big| : t,s \in [0,\beta], u,v \in [-r_0,r_0], |u-v| \le \varepsilon\big\}.$$

Note that $G(t,s,u)$ is uniformly continuous on $[0,\beta]\times[0,\beta]\times[-r_0,r_0]$, and that hence we conclude that

$$\lim_{\varepsilon\to 0}\omega^\beta(G,\varepsilon) = 0.$$

By returning to the inequality in (5), for an arbitrarily fixed $t \in [0, \beta]$, one quickly arrives at the inequality

$$\big|Tu(t) - Tv(t)\big| \leq \varepsilon + \int_0^t \omega^\beta(G, \varepsilon)\, ds = \varepsilon + \beta\omega^\beta(G, \varepsilon),$$

which in light of (7) and the above fact concerning $\omega^\beta(G, \varepsilon)$, leads one to conclude that T is a continuous operator on $\mathcal{B}(r_0)$. □

The existence of at least a solution of the functional integral equation (3) can be established in the form of the following theorem. We use arguments of fixed-point theory in Banach spaces.

Theorem 7. *Suppose that conditions (i)–(iv) hold true. Then, the functional integral equation (3) has at least one solution in the space* $C_B([0, +\infty[)$.

Proof. The proof of Theorem 7 shows that the operator T defined by (4) has a fixed point in $\mathcal{B}(r_0)$. In fact, let Z be an arbitrary nonempty subset of $\mathcal{B}(r_0)$. Thus, fixing two positive numbers ε and β, one can choose arbitrarily $t, s \in [0, \beta]$ such that $|t - s| \leq \varepsilon$. Here, it is not restrictive to suppose that $s < t$ and hence, for $u \in Z$, we write

$$\begin{aligned}
&\big|Tu(t) - Tu(s)\big| \\
&\quad\leq \big|g\big(t, u(t)\big) - g\big(s, u(s)\big)\big| + \bigg|\int_0^t G\big(t, \tau, u(\tau)\big)\, d\tau - \int_0^s G\big(s, \tau, u(\tau)\big)\, d\tau\bigg| \\
&\quad\leq \big|g\big(t, u(t)\big) - g\big(s, u(t)\big)\big| + \big|g\big(s, u(t)\big) - g\big(s, u(s)\big)\big| \\
&\qquad + \int_0^t \big|G\big(t, \tau, u(\tau)\big) - G\big(s, \tau, u(\tau)\big)\big|\, d\tau + \int_s^t \big|G\big(s, \tau, u(\tau)\big)\big|\, d\tau \\
&\quad\leq \omega_1^\beta(g, \varepsilon) + \varphi\big(\omega^\beta(u, \varepsilon)\big)\omega^\beta(u, \varepsilon) + \int_0^t \omega_1^\beta(G, \varepsilon)\, d\tau + f(s)\int_s^t l(\tau)\, d\tau \\
&\quad\leq \omega_1^\beta(g, \varepsilon) + \varphi\big(\omega^\beta(u, \varepsilon)\big)\omega^\beta(u, \varepsilon) + \beta\omega_1^\beta(G, \varepsilon) \\
&\qquad + \varepsilon \sup\big\{f(s)l(t) : t, s \in [0, \beta]\big\}, \qquad (8)
\end{aligned}$$

where

$$\omega_1^\beta(g, \varepsilon) = \sup\big\{\big|g(t, u) - g(s, u)\big| : t, s \in [0, \beta], u \in [-r_0, r_0], |t - s| \leq \varepsilon\big\}$$

and

$$\omega_1^\beta(G, \varepsilon) = \sup\big\{\big|G(t, \tau, u) - G(s, \tau, u)\big| : t, s, \tau \in [0, \beta], u \in [-r_0, r_0], |t - s| \leq \varepsilon\big\}.$$

Moreover, in light of the uniform continuity of g on $[0, \beta] \times [-r_0, r_0]$ and G on $[0, \beta] \times [0, \beta] \times [-r_0, r_0]$, we obtain

$$\lim_{\varepsilon \to 0} \omega_1^\beta(g, \varepsilon) = 0 \quad \text{and} \quad \lim_{\varepsilon \to 0} \omega_1^\beta(G, \varepsilon) = 0.$$

By the hypotheses on functions f and l, we get that $\sup\{f(s)l(t) : t, s \in [0, \beta]\}$ is finite. All these remarks and the inequalities in (8) imply that

$$\omega_0^\beta(TZ) \leq \lim_{\varepsilon \to 0} \varphi\big(\omega^\beta(Z, \varepsilon)\big)\omega^\beta(Z, \varepsilon).$$

Since the function φ is right-continuous, we can write

$$\omega_0^\beta(TZ) \le \varphi\big(\omega_0^\beta(Z)\big)\omega_0^\beta(Z)$$

and hence

$$\omega_0(TZ) \le \varphi\big(\omega_0(Z)\big)\omega_0(Z). \tag{9}$$

The next step is realized by choosing two arbitrary functions $u, v \in Z$ so that, for $t \in [0, +\infty[$, we have

$$\begin{aligned}
&\big|Tu(t) - Tv(t)\big| \\
&\quad \le \big|g\big(t, u(t)\big) - g\big(t, v(t)\big)\big| + \int_0^t \big|G\big(t, s, u(s)\big)\big|\,ds + \int_0^t \big|G\big(t, s, v(s)\big)\big|\,ds \\
&\quad \le \varphi\big(\big|u(t) - v(t)\big|\big)\big|u(t) - v(t)\big| + 2f(t)\int_0^t l(s)\,ds.
\end{aligned}$$

Starting from the above inequality, using the notion of diameter of a set, we deduce that

$$\operatorname{diam}(TZ)(t) \le \varphi\big(\operatorname{diam} Z(t)\big)\operatorname{diam} Z(t) + 2f(t)\int_0^t l(s)\,ds.$$

Since the function φ is right-continuous and nondecreasing, we obtain

$$\limsup_{t\to+\infty} \operatorname{diam}(TZ)(t) \le \varphi\big(\limsup_{t\to+\infty} \operatorname{diam} Z(t)\big) \limsup_{t\to+\infty} \operatorname{diam} Z(t). \tag{10}$$

Combining (2), (9), and (10), and again taking into account that the function φ is nondecreasing, we deduce that

$$\phi(TZ) \le \varphi\big(\phi(Z)\big)\phi(Z).$$

Thus, it remains to show that the function $\rho : [0, +\infty[\times [0, +\infty[\to \mathbb{R}$ defined by

$$\rho(t, s) = s\varphi(s) - t \quad \text{for all } t, s \in [0, +\infty[$$

is a V-function. In fact, if $\{a_n\} \subset]0, +\infty[$ is such that

$$0 < \rho(a_{n+1}, a_n) = a_n\varphi(a_n) - a_{n+1}$$

for all $n \in \mathbb{N}$, then $\{a_n\}$ is a decreasing sequence, and hence there exists a nonnegative real number r such that $\lim_{n\to+\infty} a_n = r$. Since $\varphi : [0, +\infty[\to [0, 1[$ is a right-continuous function, if $r > 0$, we get

$$0 \le \limsup_{n\to+\infty}\big[a_n\varphi(a_n) - a_{n+1}\big] \le r\varphi(r) - r < 0,$$

which is a contradiction, and hence $r = 0$. Therefore, by an application of Theorem 5 we conclude that the operator T has a fixed point in $\mathcal{B}(r_0)$ and hence in $C_B([0, +\infty[)$. Thus, by the definition of the operator T, the existence of at least a solution of (3) in $C_B([0, +\infty[)$ is proved. □

It is an easy matter to show that Theorem 7 continues to be true by replacing the above conditions (ii) and (iv) with the following:

(ii)′ there exists a nondecreasing right-continuous function $\varphi : [0,+\infty[\to [0,+\infty[$ such that $\varphi(t) < t/2$ for all $t \in [0,+\infty[$ and

$$\big|g(t,u) - g(t,v)\big| \le \varphi\big(|u-v|\big) \quad \text{for all } t \in [0,+\infty[\text{ and } u,v \in \mathbb{R};$$

(iv)′ there exists a positive solution r_0 of the inequality $\varphi(r) + \mu \le r$, with constant μ given by

$$\mu = \sup\Big\{\big|g(t,0)\big| + f(t)\int_0^t l(s)\,ds : t \in [0,+\infty[\Big\}.$$

We leave it to the reader to check this claim by adapting the proof of Theorem 7. Here, we give a numerical example.

Example 8. Consider the following functional integral equation

$$u(t) = \frac{1+t^2}{2+t^2}\frac{\ln(1+|u(t)|)}{2+\ln(1+|u(t)|)} + 2e^{-t} + \int_0^t \frac{\cos u(t)}{1+t^2}e^{-t}e^{\frac{s}{2}}\,ds, \quad t \in [0,+\infty[,$$

in the space $C_B([0,+\infty[)$. Clearly, $g : [0,+\infty[\times \mathbb{R} \to \mathbb{R}$ given by

$$g(t,u) = \frac{1+t^2}{2+t^2}\frac{\ln(1+|u(t)|)}{2+\ln(1+|u(t)|)} + 2e^{-t}$$

is continuous and is such that the function $t \to g(t,0)$ is an element of $C_B([0,+\infty[)$. Moreover, with respect to $\varphi : [0,+\infty[\to [0,+\infty[$ given by

$$\varphi(t) = \frac{\ln(1+t)}{2+\ln(1+t)} \quad \text{for all } t \in [0,+\infty[,$$

we have

$$\begin{aligned}
0 \le \big|g(t,u) - g(t,v)\big| &= \frac{1+t^2}{2+t^2}\Big|\frac{\ln(1+|u(t)|)}{2+\ln(1+|u(t)|)} - \frac{\ln(1+|v(t)|)}{2+\ln(1+|v(t)|)}\Big| \\
&\le 2\frac{|\ln(1+|u(t)|) - \ln(1+|v(t)|)|}{[2+\ln(1+|u(t)|)][2+\ln(1+|v(t)|)]} \\
&\le \frac{|\ln(1+\frac{1+|u(t)|-1-|v(t)|}{1+|v(t)|})|}{2+\ln(1+|u(t)|)+\ln(1+|v(t)|)} \\
&\le \frac{\ln(1+|u(t)-v(t)|)}{2+\ln(1+|u(t)|+|v(t)|)} \\
&\le \frac{\ln(1+|u(t)-v(t)|)}{2+\ln(1+|u(t)-v(t)|)} \\
&= \varphi\big(|u(t)-v(t)|\big).
\end{aligned}$$

Let $f, l : [0,+\infty[\to [0,+\infty[$ be defined by

$$f(t) = e^{-t} \text{ and } l(s) = e^{\frac{s}{2}} \quad \text{for all } t,s \in [0,+\infty[.$$

Then we have

$$\big|G(t,s,u)\big| = \frac{|\cos u(t)|}{1+t^2}e^{-t}e^{\frac{s}{2}} \le e^{-t}e^{\frac{s}{2}}$$

for all $t, s \in [0, +\infty[$; clearly,

$$\lim_{t\to+\infty} e^{-t} \int_0^t e^{\frac{s}{2}}\, ds = \lim_{t\to+\infty} 2e^{-t}(e^{\frac{t}{2}} - 1) = 0.$$

Also,

$$\begin{aligned}\mu &= \sup\Big\{ \big|g(t,0)\big| + f(t) \int_0^t l(s)\, ds : t \in [0, +\infty[\Big\} \\ &= \sup\big\{2e^{-t} + 2e^{-t}(e^{\frac{t}{2}} - 1) : t \in [0, +\infty[\big\} = 2,\end{aligned}$$

and hence $r_0 = 3$ is a solution of the inequality $\varphi(r) + \mu \le r$. Thus, conditions (i) and (iii) of Theorem 7 hold true; moreover conditions (ii)$'$ and (iv)$'$ are satisfied, and so the functional integral equation has at least a solution in $C_B([0, +\infty[)$, in light of the preceding facts.

4. Examples of V-functions and proof of Theorem 5

We can easily give examples where the condition (v_1) is fulfilled to show the fairness of this assumption in the practical context.

Definition 9 ([9, Definition 6]). Let $\varphi, \psi : [0, +\infty[\to \mathbb{R}$ be two functions. The pair (φ, ψ) is said to be a pair of *shifting distance functions* if the following conditions hold:

(i) for $u, v \in [0, +\infty[$, if $\varphi(u) \le \psi(v)$, then $u \le v$;
(ii) for $\{u_n\}, \{v_n\} \subset [0, +\infty[$ with $\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} v_n = r$, if $\varphi(u_n) \le \psi(v_n)$ for all $n \in \mathbb{N}$, then $r = 0$.

Consider a pair of shifting distance functions (φ, ψ) and a function $h \in \mathcal{H}$, where

$$\mathcal{H} = \big\{h : [0, +\infty[\to [0, +\infty[: \lim_{n\to+\infty} h(a_n) = 0 \text{ implies } \lim_{n\to+\infty} a_n = 0\big\}.$$

Then, define the function $\rho : [0, +\infty[\times [0, +\infty[\to \mathbb{R}$ by

$$\rho(t, s) = \psi\big(h(s)\big) - \varphi\big(h(t)\big) \quad \text{for all } t, s \in [0, +\infty[.$$

Clearly, ρ is a V-function. In fact, if $\{a_n\} \subset]0, +\infty[$ and

$$0 < \rho(a_{n+1}, a_n) = \psi\big(h(a_n)\big) - \varphi\big(h(a_{n+1})\big),$$

then

$$h(a_{n+1}) \le h(a_n) \quad \text{for all } n \in \mathbb{N}.$$

This implies that there exists a real number $r \ge 0$ such that $\lim_{n\to+\infty} h(a_n) = r$. Moreover, by Definition 9(ii) with $u_n = h(a_{n+1})$ and $v_n = h(a_n)$, we deduce that $r = 0$ and hence, since $h \in \mathcal{H}$, we get that $\lim_{n\to+\infty} a_n = 0$.

Here, we give two numerical examples.

Example 10. The function $\rho : [0, +\infty[\times [0, +\infty[\to \mathbb{R}$ defined by

$$\rho(t, s) = \ln \frac{1+s}{3} - \ln \frac{1+2t}{3} \quad \text{for all } t, s \in [0, +\infty[$$

is a V-function.

Example 11. The function $\rho : [0, +\infty[\times [0, +\infty[\to \mathbb{R}$ defined by

$$\rho(t, s) = \ln \frac{3 + \int_0^s \gamma(\tau)\, d\tau}{3} - \ln \frac{3 + 2\int_0^t \gamma(\tau)\, d\tau}{3} \quad \text{for all } t, s \in [0, +\infty[$$

is a V-function, where $\gamma : [0, +\infty[\to [0, +\infty[$ is a Lebesgue-integrable function such that $\int_0^\varepsilon \gamma(t)\, dt > \varepsilon$ for all $\varepsilon > 0$.

Again, we merge some classes of functions to get an implicit statement of V-function.

Consider four functions $h, k, \varphi, \psi : [0, +\infty[\to [0, +\infty[$ such that the following conditions hold:

(i) φ is nondecreasing and continuous,
(ii) ψ is a lower-semicontinuous function such that $\psi^{-1}(\{0\}) = \{0\}$,
(iii) $h, k \in \mathcal{H}$ and $k^{-1}(\{0\}) = \{0\}$.

Then, define the function $\rho : [0, +\infty[\times [0, +\infty[\to \mathbb{R}$ by

$$\rho(t, s) = \varphi\big(h(s)\big) - \psi\big(k(s)\big) - \varphi\big(h(t)\big) \quad \text{for all } t, s \in [0, +\infty[.$$

Clearly, ρ is a V-function. In fact, if $\{a_n\} \subset]0, +\infty[$ and

$$0 < \rho(a_{n+1}, a_n) = \varphi\big(h(a_n)\big) - \psi\big(k(a_n)\big) - \varphi\big(h(a_{n+1})\big), \tag{11}$$

then

$$\varphi\big(h(a_{n+1})\big) < \varphi\big(h(a_n)\big)$$

since $\psi(k(a_n)) > 0$. By (i), this implies that

$$h(a_{n+1}) < h(a_n) \quad \text{for all } n \in \mathbb{N}$$

and hence there exists $r \in [0, +\infty[$ such that $\lim_{n\to+\infty} h(a_n) = r$. Moreover, passing to the limit as $n \to +\infty$ in (11), by using the continuity of the function φ, one has that there exists $\lim_{n\to+\infty} \psi(k(a_n)) = 0$. By the hypotheses on function ψ in (ii), we deduce that there exists $\lim_{n\to+\infty} k(a_n) = 0$ and hence $\lim_{n\to+\infty} a_n = 0$ too, in light of (iii).

By arguments similar to those in the proof of Theorem 3, we obtain the existence of a fixed point for the continuous V-ϕ-contraction operator $T : M \to M$. Thus, we give the following proof of Theorem 5.

Proof of Theorem 5. Our proof shows that there exists a sequence $\{M_n\}$ of nonempty, closed, and convex subsets of M such that

$$TM_n \subset M_n \subset M_{n-1} \quad \text{for all } n \in \mathbb{N}.$$

In fact, let $M_0 = M$, and define the sequence $\{M_n\}$ by adopting the iterative rule

$$M_n = \overline{\mathrm{conv}(TM_{n-1})} \quad \text{for all } n \in \mathbb{N}. \tag{12}$$

Since T is a self-operator on M, then $TM_0 \subset M_0$ trivially. Thus, suppose that the condition $TM_n \subset M_n \subset M_{n-1}$ holds for a finite set of natural numbers, say, up to n. Therefore, in light of the iterative rule in (12), we get that $TM_n \subset M_n$ implies that $M_{n+1} = \overline{\mathrm{conv}(TM_n)} \subset M_n$ and hence

$$TM_{n+1} \subset TM_n \subset M_{n+1} \quad \text{for all } n \in \mathbb{N} \cup \{0\}.$$

Of course, if there exists a natural number m such that $\phi(M_m) = 0$, then M_m is a compact set. Moreover, since T is a self-operator on M_m, the existence of a fixed point of T in M_m follows by an application of Theorem 1; also, from the above range restrictions, it is clear that the fixed point belongs to M. Now, we suppose that $\phi(M_n) > 0$ for all $n \in \mathbb{N}$ and we prove that $\lim_{n\to+\infty} \phi(M_n) = 0$. In fact, by using the V-ϕ-contraction condition (1) with $Z = M_n$, we get

$$0 < \rho\big(\phi(TM_n), \phi(M_n)\big) = \rho\big(\phi\big(\overline{\mathrm{conv}(TM_n)}\big), \phi(M_n)\big) = \rho\big(\phi(M_{n+1}), \phi(M_n)\big),$$

and so the property (v_1) of the function ρ ensures that $\lim_{n\to+\infty} \phi(M_n) = 0$. Furthermore, in light of property (m6) of a measure of noncompactness (see Definition 2), we have $M_\infty = \bigcap_{n=1}^{+\infty} M_n \neq \emptyset$. Since $TM_n \subset M_n$, then $TM_\infty \subset M_\infty$. Moreover, $M_\infty \in \ker \phi$. Indeed, from property (m2) of a measure of noncompactness, one has that $\phi(M_\infty) \leq \phi(M_n)$ for all $n \in \mathbb{N}$, which leads to $\phi(M_\infty) = 0$ and so $M_\infty \in \ker \phi$. Finally, since M_∞ is a closed set, then it is a compact set. Thus, by an application of Theorem 1, we conclude that T has a fixed point in M_∞ and hence in M. $\square$

This short proof does not upset the simplicity and effectiveness of the original proof in [12], but with a few focused changes we think that it is more effective and general. We note that Darbo's fixed-point theorem follows immediately from Theorem 5 by choosing as V-function $\rho(t, s) = ks - t$ for all $t, s \in [0, +\infty[$ with $k \in [0, 1[$.

5. Consequences in fixed-point theory

Furthermore, there are some interesting consequences of Theorem 5 which deserve to be stated, such as the following corollary.

Corollary 12. *Let M be a nonempty, bounded, closed, and convex subset of a Banach space X, and let $T : M \to M$ be a continuous operator. Assume that there exist a pair of shifting distance functions (φ, ψ) and a function $h \in \mathcal{H}$ such that*

$$\varphi\big(h\big(\phi(TZ)\big)\big) \leq \psi\big(h\big(\phi(Z)\big)\big) \quad \text{for all } Z \subset M, \phi(Z) > 0.$$

Then T has a fixed point in the set M.

By appropriately choosing the function h in Corollary 12, one can obtain known results, for instance, Theorem 2.1 of [10] and Theorem 6 of [17]. On the other hand, as a particular case of Corollary 12, we give the following result, which is a generalization of Corollary 2.1 of [11].

Corollary 13. *Let M be a nonempty, bounded, closed, and convex subset of a Banach space X, let ϕ be a measure of noncompactness on X, and let $T : M \to M$ be a continuous operator. Assume that there exist two functions $\psi, \varphi : [0, +\infty[\to [0, +\infty[$ such that*

$$\varphi\big(\phi(TZ)\big) \leq \psi\big(\phi(Z)\big) \quad \text{for all } Z \subset M, \phi(Z) > 0.$$

If $\psi(t) < t \leq \varphi(t)$ for all $t > 0$, $\varphi^{-1}(\{0\}) = \{0\}$, $\psi^{-1}(\{0\}) = \{0\}$, ψ is upper-semicontinuous, and φ is lower-semicontinuous, then T has a fixed point in the set M.

The proof of Corollary 13 shows that (φ, ψ) is a pair of shifting distance functions. In fact, condition (i) of Definition 9 is a consequence of the inequalities $\psi(t) < t \leq \varphi(t)$, for all $t > 0$. Moreover, if $\{u_n\}$ and $\{v_n\}$ are as in condition (ii) of Definition 9, then we conclude that

$$\varphi(r) \leq \liminf_{n\to+\infty} \varphi(u_n) \leq \limsup_{n\to+\infty} \psi(v_n) \leq \psi(r).$$

But this condition holds only if $r = 0$.

By keeping in mind the implicit statement of V-function given in Section 4, we propose the following corollary.

Corollary 14. *Let M be a nonempty, bounded, closed, and convex subset of a Banach space X, let ϕ be a measure of noncompactness on X, and let $T : M \to M$ be a continuous operator. Assume that there exist four functions $h, k, \varphi, \psi : [0, +\infty[\to [0, +\infty[$ such that*

$$\varphi\big(h\big(\phi(TZ)\big)\big) \leq \varphi\big(h\big(\phi(Z)\big)\big) - \psi\big(k\big(\phi(Z)\big)\big) \quad \textit{for all } Z \subset M, \phi(Z) > 0,$$

where

(i) *φ is nondecreasing and continuous,*
(ii) *ψ is a lower-semicontinuous function such that $\psi^{-1}(\{0\}) = \{0\}$,*
(iii) *$h, k \in \mathcal{H}$ and $k^{-1}(\{0\}) = \{0\}$.*

Then T has a fixed point in the set M.

Corollary 14 is a generalization of Theorem 2.1 of [14], Theorem 2.7 of [10], and Corollary 2.2 of [11]. In the recent literature on fixed points, there are various classes of functions fulfilling the condition (v_1). Here, we use some functions in these classes for establishing other corollaries; but we leave to the reader the simple task of recognizing which options are adopted. The following result is a generalization of Corollary 2.5 of [11].

Corollary 15. *Let M be a nonempty, bounded, closed, and convex subset of a Banach space X, let ϕ be a measure of noncompactness on X, and let $T : M \to M$ be a continuous operator. Assume that there exists a function $\varphi : [0, +\infty[\to [0, +\infty[$ such that*

$$\varphi\big(\phi(TZ)\big) \leq \phi(Z) \quad \textit{for all } Z \subset M, \phi(Z) > 0.$$

If φ is a lower-semicontinuous function such that $\varphi(t) > t$ for all $t \in [0, +\infty[$, then T has a fixed point in the set M.

The proof of Corollary 15 shows that the function $\rho : [0, +\infty[\times [0, +\infty[\to \mathbb{R}$ defined by

$$\rho(t, s) = s - \varphi(t) \quad \text{for all } t, s \in [0, +\infty[$$

is a V-function. In fact, if $\{a_n\} \subset]0, +\infty[$ is such that

$$0 < \rho(a_{n+1}, a_n) = a_n - \varphi(a_{n+1}) < a_n - a_{n+1}$$

for all $n \in \mathbb{N}$, then $\{a_n\}$ is a decreasing sequence and hence there exists $r \in [0, +\infty[$ such that $\lim_{n\to+\infty} a_n = r$. The lower-semicontinuity of φ ensures that

$$\varphi(r) \leq \liminf_{n\to+\infty} \varphi(a_n) \leq \lim_{n\to+\infty} a_n = r,$$

which is a contradiction if $r > 0$ and hence $r = 0$.

Corollary 16. *Let M be a nonempty, bounded, closed, and convex subset of a Banach space X, let ϕ be a measure of noncompactness on X, and let $T : M \to M$ be a continuous operator. Assume that there exists a function $\varphi : [0,+\infty[\to [0,1[$ such that*

$$\phi(TZ) \leq \varphi\big(\phi(Z)\big)\phi(Z) \quad \textit{for all } Z \subset M, \phi(Z) > 0.$$

If φ is a function such that $\limsup_{t\to r^+} \varphi(t) < 1$ for all $r > 0$, then T has a fixed point in the set M.

The proof of Corollary 16 shows that the function $\rho : [0,+\infty[\times [0,+\infty[\to \mathbb{R}$ defined by

$$\rho(t,s) = s\varphi(s) - t \quad \text{for all } t,s \in [0,+\infty[$$

is a V-function. In fact, if $\{a_n\} \subset]0,+\infty[$ is such that

$$0 < \rho(a_{n+1}, a_n) = a_n\varphi(a_n) - a_{n+1}$$

for all $n \in \mathbb{N}$, then $\{a_n\}$ is a decreasing sequence and hence there exists $r \in [0,+\infty[$ such that $\lim_{n\to+\infty} a_n = r$. Now, if $r > 0$, we get

$$0 \leq \limsup_{n\to+\infty} \big[a_n\varphi(a_n) - a_{n+1}\big] \leq r \limsup_{n\to+\infty} \varphi(a_n) - r < 0,$$

which is a contradiction and hence $r = 0$.

Remark 17. The previous corollary holds also if $\varphi : [0,+\infty[\to [0,1[$ is a monotone function, as it is shown at the end of the proof of Theorem 7.

Corollary 18. *Let M be a nonempty, bounded, closed, and convex subset of a Banach space X, let ϕ be a measure of noncompactness on X, and let $T : M \to M$ be a continuous operator. Assume that there exists a function $\varphi : [0,+\infty[\to [0,+\infty[$ such that*

$$\phi(TZ) \leq \varphi\big(\phi(Z)\big) \quad \textit{for all } Z \subset M, \phi(Z) > 0.$$

If φ is an upper-semicontinuous function such that $\varphi(t) < t$ for all $t > 0$, then T has a fixed point in the set M.

The proof of Corollary 18 shows that the function $\rho : [0,+\infty[\times [0,+\infty[\to \mathbb{R}$ defined by

$$\rho(t,s) = \varphi(s) - t \quad \text{for all } t,s \in [0,+\infty[$$

is a V-function. In fact, if $\{a_n\} \subset]0,+\infty[$ is such that

$$0 < \rho(a_{n+1}, a_n) = \varphi(a_n) - a_{n+1}$$

for all $n \in \mathbb{N}$, then $\{a_n\}$ is a decreasing sequence and hence there exists $r \in [0,+\infty[$ such that $\lim_{n\to+\infty} a_n = r$. Now, if $r > 0$, we get that

$$r = \limsup_{n\to+\infty} a_{n+1} \leq \limsup_{n\to+\infty} \varphi(a_n) \leq \varphi(r),$$

which is a contradiction and hence $r = 0$.

6. Conclusions

The measures of noncompactness in Banach spaces represent an interesting way for improving the study of functional integral equations by focusing on the problem of the existence of at least one solution. Here, we extend Darbo's fixed-point theorem and work with methods of fixed-point theory to establish the main result of the article. The proposed approach is useful for generalizing and interrelating various results in the existing literature.

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