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ADDITIVE MAPS PRESERVING DRAZIN INVERTIBLE OPERATORS OF INDEX n

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Abstract. Given an integer $n \geq 2$, in this article we provide a complete description of all additive surjective maps on the algebra of all bounded linear operators acting on an infinite-dimensional complex Banach space, preserving in both directions the set of Drazin invertible operators of index n.

1. Introduction

Throughout this paper, X denotes an infinite-dimensional complex Banach space, and $\mathcal{B}(X)$ denotes the algebra of all bounded linear operators acting on X. An operator $T \in \mathcal{B}(X)$ is said to be *Drazin invertible* if there exist an operator $S \in \mathcal{B}(X)$ and a nonnegative integer k such that

$$TS = ST, \qquad S^2T = S, \qquad \text{and} \qquad T^{k+1}S = T^k. \tag{1.1}$$

Such an operator S is unique, and it is called the *Drazin inverse* of T, and it is denoted by $S = T^{\mathrm{D}}$. The *Drazin index* of T, designated by $\mathrm{i}(T)$, is the smallest nonnegative integer k satisfying (1.1). Clearly, every invertible operator is Drazin invertible with index zero.

The concept of Drazin inverse was introduced in [6], and it has numerous applications in matrix theory, iterative methods, singular differential equations, and Markov chains (see, for instance, [2], [4], [16], [22], and the references therein).

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Let Λ be a subset of $\mathcal{B}(X)$. A map $\Phi : \mathcal{B}(X) \to \mathcal{B}(X)$ is said to *preserve* Λ *in both directions* if, for every $T \in \mathcal{B}(X)$,

$$T \in \Lambda \qquad \text{if and only if} \qquad \Phi(T) \in \Lambda.$$

The problem of studying linear maps on Banach algebras that leave certain subsets invariant has attracted the attention of many mathematicians in the last decades. For excellent expositions on linear preserver problems, we refer the reader to [8], [10], [13], [15], [18], [20], [21], and the references therein.

In [13], the authors have characterized surjective additive continuous maps $\Phi : \mathcal{B}(H) \to \mathcal{B}(H)$, where H is a separable infinite-dimensional complex Hilbert space, preserving the set of all Drazin invertible operators in both directions. It should be mentioned that the characterization of such maps in the setting of Banach spaces is still an open question.

For each positive integer n, let $\mathcal{D}_n(X)$ denote the set of all Drazin invertible operators in $\mathcal{B}(X)$ of index n.

The authors of this note obtain in [15] a complete description of additive surjective maps $\Phi : \mathcal{B}(X) \to \mathcal{B}(X)$ preserving $\mathcal{D}_1(X)$ in both directions. In this paper, we establish a similar result for $\mathcal{D}_n(X)$ where $n \geq 2$, and hence we compliment and extend the main conclusion in [15]. Our arguments are influenced by ideas from [15] and the approaches given therein, but the proof of our main result requires new ingredients as illustrated in Example 2.3 and the remark preceding it. Contrary to what could be expected, the techniques used here do not allow us to include the special case of $\mathcal{D}_1(X)$ as a consequence. For these reasons reference [15] and this note are complementary papers which deserve their own independent study.

Let X^* denote the topological dual space of X, and let T^* denote the Banach space adjoint of T. The main result of this article can be stated as follows.

Theorem 1.1. *Let $\Phi : \mathcal{B}(X) \to \mathcal{B}(X)$ be an additive surjective map, and let $n \geq 2$ be an integer. Then Φ preserves $\mathcal{D}_n(X)$ in both directions if and only if one of the following assertions holds.*

(1) *There is a nonzero $\alpha \in \mathbb{C}$, and there is a bijective continuous mapping $A : X \to X$, either linear or conjugate linear, such that*

$$\Phi(T) = \alpha A T A^{-1} \quad \textit{for all } T \in \mathcal{B}(X).$$

(2) *There is a nonzero $\alpha \in \mathbb{C}$, and there is a bijective continuous mapping $B : X^* \to X$, either linear or conjugate linear, such that*

$$\Phi(T) = \alpha B T^* B^{-1} \quad \textit{for all } T \in \mathcal{B}(X).$$

For $T \in \mathcal{B}(X)$, we denote by $\mathcal{P}_n(T)$ the set of all the poles of order n of its resolvent. It follows by [23, Theorems 10.1 and 10.2] that $\lambda \in \mathcal{P}_n(T)$ if and only if $T - \lambda \in \mathcal{D}_n(X)$. As a consequence of the previous theorem, we derive the following corollary.

Corollary 1.2. *Let $\Phi : \mathcal{B}(X) \to \mathcal{B}(X)$ be an additive surjective map, and let $n \geq 2$ be an integer. Then $\mathcal{P}_n(\Phi(T)) = \mathcal{P}_n(T)$ for all $T \in \mathcal{B}(X)$ if and only if one of the following assertions holds.*

(1) *There is a bijective continuous mapping $A : X \to X$, either linear or conjugate linear, such that*

$$\Phi(T) = ATA^{-1} \quad \textit{for all } T \in \mathcal{B}(X).$$

(2) *There is a bijective continuous mapping $B : X^* \to X$, either linear or conjugate linear, such that*

$$\Phi(T) = BT^*B^{-1} \quad \textit{for all } T \in \mathcal{B}(X).$$

The present article is organized as follows. In the second section, we establish some useful results on the perturbation of Drazin invertible operators of index n. These results will be needed for proving the main theorem and its corollary in the last section.

2. $\mathcal{D}_n(X)$ under rank 1 perturbations

Throughout the rest of this paper, n is an integer greater than 1. For an operator $T \in \mathcal{B}(X)$, write $\ker(T)$ for its kernel, write $\operatorname{ran}(T)$ for its range, and write $\sigma(T)$ for its spectrum. The *ascent* $\mathrm{a}(T)$ and *descent* $\mathrm{d}(T)$ of T are defined respectively by

$$\mathrm{a}(T) = \inf\{k \geq 0 : \ker(T^k) = \ker(T^{k+1})\}$$

and

$$\mathrm{d}(T) = \inf\{k \geq 0 : \operatorname{ran}(T^k) = \operatorname{ran}(T^{k+1})\},$$

where the infimum over the empty set is taken to be infinite (see [19], [23]). From [7, Lemma 1.1], given a nonnegative integer k, we have

$$\mathrm{a}(T) \leq k \Leftrightarrow \ker(T^m) \cap \operatorname{ran}(T^k) = \{0\} \quad \text{for some (equivalently, all) } m \geq 1, \tag{2.1}$$

and

$$\mathrm{d}(T) \leq k \Leftrightarrow \ker(T^k) + \operatorname{ran}(T^m) = X \quad \text{for some (equivalently, all) } m \geq 1. \tag{2.2}$$

Remark 2.1. Let $T \in \mathcal{B}(X)$. Then T is Drazin invertible if and only if T has finite ascent and descent (see [11, Theorem 4]). Moreover, we have in this case the following well-known assertions (see [19, Corollary 20.5 and Theorem 22.10]):

(1) $\mathrm{a}(T) = \mathrm{d}(T)$, and this value coincides with the Drazin index $\mathrm{i}(T)$;
(2) $X = \ker(T^k) \oplus \operatorname{ran}(T^k)$, where $k = \mathrm{i}(T)$ and the direct sum is topological;
(3) 0 is a pole of T of order k when $k \geq 1$.

Let $z \in X$, and let $f \in X^*$. As usual, we denote by $z \otimes f$ the rank 1 operator given by $(z \otimes f)(x) = f(x)z$ for all $x \in X$. Note that every rank 1 operator in $\mathcal{B}(X)$ can be written in this form.

Proposition 2.2. *Let $T \in \mathcal{B}(X)$ be such that $\mathrm{a}(T) \leq m$ where $m \geq 1$ is an integer, and let $F \in \mathcal{B}(X)$ be a rank 1 operator. Assume that $\mathrm{a}(T + \alpha F) > m$ and $\mathrm{a}(T + \beta F) > m$ for two different nonzero scalars $\alpha, \beta \in \mathbb{C}$. Then $\mathrm{a}(T + cF) > m$ for every nonzero $c \in \mathbb{C}$.*

Proof. Let $F = z \otimes f$, where $z \in X$ and $f \in X^*$ are nonzero. Then it follows from [13, Lemma 2.2] that there exist two sequences $\{x_k\}_{k=0}^m$ and $\{y_k\}_{k=0}^m$ of linearly independent vectors and two integers $0 \le i, j \le m$ such that

$$\begin{cases} (T+\alpha F)x_0 = (T+\beta F)y_0 = 0, \\ (T+\alpha F)x_k = x_{k-1} \quad \text{and} \quad (T+\beta F)y_k = y_{k-1} \quad \text{for } 1 \le k \le m, \\ f(x_k) = \delta_{ki} \quad \text{and} \quad f(y_k) = \delta_{kj} \quad \text{for } 0 \le k \le m. \end{cases}$$

From this, one can easily see that

$$\begin{cases} Tx_i = x_{i-1} - \alpha z, \\ Ty_j = y_{j-1} - \beta z, \end{cases} \quad \text{and} \quad \begin{cases} Tx_k = x_{k-1} \quad \text{for } 0 \le k \ne i \le m, \\ Ty_k = y_{k-1} \quad \text{for } 0 \le k \ne j \le m, \end{cases} \tag{2.3}$$

where we set formally $x_s = y_s = 0$ for $s < 0$.

We claim that $i = j$. Suppose to the contrary that $i \ne j$. We may assume without loss of generality that $i < j$. Let $u_k = \alpha y_k - \beta x_{k+i-j}$ for $0 \le k \le m$. We have $u_0 = \alpha y_0$ and $u_j = \alpha y_j - \beta x_i$. Hence $Tu_0 = 0$ because $j \ne 0$, and, using (2.3), we also get

$$Tu_j = \alpha(y_{j-1} - \beta z) - \beta(x_{i-1} - \alpha z) = u_{j-1}$$

and

$$Tu_k = u_{k-1} \quad \text{for } 1 \le k \ne j \le m.$$

Consequently, $T^m u_m = u_0 \ne 0$, and hence $u_0 \in \ker(T) \cap \operatorname{ran}(T^m)$. Thus $\mathrm{a}(T) \ge m+1$ by (2.1), the desired contradiction.

Fix an arbitrary nonzero $c \in \mathbb{C}$. Let $v_k = -\alpha y_k + \beta x_k$, and let $w_k = v_k + c(y_k - x_k)$ for $0 \le k \le m$, and put $v_s = w_s = 0$ for $s < 0$. In particular, we have

$$f(w_i) = f(v_i) = -\alpha + \beta \quad \text{and} \quad f(w_k) = f(v_k) = 0 \quad \text{for } 0 \le k \ne i \le m.$$

Furthermore, using (2.3), we obtain that

$$\begin{cases} Tv_i = -\alpha(y_{j-1} - \beta z) + \beta(x_{i-1} - \alpha z) = v_{i-1}, \\ (T+cF)w_i = v_{i-1} + c(y_{j-1} - \beta z - x_{i-1} + \alpha z) + c(\beta - \alpha)z = w_{i-1}, \\ Tv_k = v_{k-1} \quad \text{and} \quad (T+cF)w_k = Tw_k = w_{k-1} \quad \text{for } 0 \le k \ne i \le m. \end{cases}$$

Hence we get that $T^m v_m = v_0$ and $T^{m+1} v_m = 0$. Since $\mathrm{a}(T) \le m$, it follows that $v_0 = -\alpha y_0 + \beta x_0 = 0$, and so $y_0 = \alpha^{-1}\beta x_0$ and $w_0 = v_0 + c(y_0 - x_0) = c(\alpha^{-1}\beta - 1)x_0 \ne 0$. Finally, since $(T+cF)^m w_m = w_0 \ne 0$ and $(T+cF)^{m+1} v_m = 0$, we obtain that $\mathrm{a}(T+cF) > m$. □

Let T be an operator in $\mathcal{B}(X)$. One can easily show that it follows from Remark 2.1 and (2.1) that

$$T \in \mathcal{D}_n(X) \quad \text{if and only if} \quad T^n \in \mathcal{D}_1(X) \quad \text{and} \quad T^{n-1} \notin \mathcal{D}_1(X). \tag{2.4}$$

We also note that $T \in \mathcal{D}_n(X)$ if and only if $T^* \in \mathcal{D}_n(X^*)$. Indeed, it follows easily from [19, Theorem A.1.14 and Corollary A.1.17] that $\mathrm{a}(T) = \mathrm{d}(T) = n$ if and only if $\mathrm{a}(T^*) = \mathrm{d}(T^*) = n$.

It is noteworthy that the study of additive maps Φ on $\mathcal{B}(X)$ preserving $\mathcal{D}_n(X)$ in both directions is based on the characterization of rank 1 operators in terms of elements in $\mathcal{D}_n(X)$. Namely, we establish in [15] that a nonzero operator F is of rank 1 if and only if for every $T \in \mathcal{D}_1(X)$ such that $T+F \in \mathcal{D}_1(X)$ at least one of the operators $T+2F$ or $T-2F$ belongs to $\mathcal{D}_1(X)$. The following example shows that this characterization does not hold for $\mathcal{D}_n(X)$ where $n \geq 2$, which constrains to search additional conditions for obtaining a similar characterization in Proposition 2.4 and Theorem 2.7.

Example 2.3. Consider the following matrices:

$$T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad \text{and} \qquad F = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 \end{bmatrix}.$$

Clearly, F is of rank 1. Furthermore, by computing the minimal polynomial of each matrix T, $T+F$, $T-2F$, and $T+2F$, it follows from [2, Theorem 1, p. 136] that $\mathrm{i}(T) = \mathrm{i}(T+F) = 2$, $\mathrm{i}(T-2F) = 4$ and $\mathrm{i}(T+2F) = 1$.

Proposition 2.4. *Let $T \in \mathcal{D}_n(X)$, and let $F \in \mathcal{B}(X)$ be a rank 1 operator such that $T+F \in \mathcal{D}_n(X)$ and $T-F \notin \mathcal{D}_n(X)$. Then $T+2F \in \mathcal{D}_n(X)$ or $T-2F \in \mathcal{D}_n(X)$.*

Before proving this proposition, we need to establish the following lemma which has been already proved for $\mathcal{D}_1(X)$ in [15, Lemma 2.5].

Lemma 2.5. *Let $T \in \mathcal{D}_n(X)$, and let $F \in \mathcal{B}(X)$ be a finite rank operator. Then*

$$T+F \in \mathcal{D}_n(X) \;\Leftrightarrow\; \mathrm{d}(T+F) = n \;\Leftrightarrow\; \mathrm{a}(T+F) = n.$$

Proof. Observe that $(T+F)^n = T^n + K$, where $K \in \mathcal{B}(X)$ is finite rank. Note also that it suffices to show that $T+F \in \mathcal{D}_n(X)$ whenever $\mathrm{a}(T+F) = n$ or $\mathrm{d}(T+F) = n$. Suppose that $\mathrm{a}(T+F) = n$. Then $\mathrm{a}((T+F)^n) = \mathrm{a}(T^n+K) = 1$ and $\mathrm{a}((T+F)^{n-1}) \neq 1$. Since $T^n \in \mathcal{D}_1(X)$ by (2.4), it follows from [15, Lemma 2.5] that $(T+F)^n = T^n + K \in \mathcal{D}_1(X)$. On the other hand, as $\mathrm{a}((T+F)^{n-1}) \neq 1$, we have $(T+F)^{n-1} \notin \mathcal{D}_1(X)$. Hence we obtain again by (2.4) that $T+F \in \mathcal{D}_n(X)$.

The case $\mathrm{d}(T+F) = n$ can be dealt with in a similar way. □

Proof of Proposition 2.4. Let F be a rank 1 operator such that $T+F \in \mathcal{D}_n(X)$ and $T-F \notin \mathcal{D}_n(X)$. According to the previous lemma, it suffices to show that $\mathrm{a}(T+2F) = n$ or $\mathrm{a}(T-2F) = n$. Since $\mathrm{a}(T) = \mathrm{a}(T+F) = n$, it follows from Proposition 2.2 that

$$\mathrm{a}(T+2F) \leq n \qquad \text{or} \qquad \mathrm{a}(T-2F) \leq n.$$

There is no loss of generality in assuming that $\mathrm{a}(T+2F) \leq n$. If $\mathrm{a}(T+2F) = n$, then the proposition is proved. Assume that $\mathrm{a}(T+2F) \leq n-1$. Since

$$\mathrm{a}(T+2F-2F) = \mathrm{a}(T+2F-F) = n,$$

Proposition 2.2 implies that

$$\mathrm{a}(T+2F+cF) > n-1 \quad \text{for every nonzero } c \in \mathbb{C}.$$

In particular, we have $\mathrm{a}(T-2F) \geq n$ and $\mathrm{a}(T-F) > n$ because $T-F \notin \mathcal{D}_n(X)$. Now, using again Proposition 2.2 for T, we obtain that

$$\mathrm{a}(T-2F) \leq n \qquad \text{or} \qquad \mathrm{a}(T-F) \leq n.$$

This shows that $\mathrm{a}(T-2F) = n$, which completes the proof. □

For an integer $k \geq 2$, we denote by J_k the $k \times k$ nilpotent matrix of order k with 1 in the diagonal directly below the main diagonal and 0 elsewhere. Notice that a nilpotent operator T of order k is Drazin invertible of index k and $T^{\mathrm{D}} = 0$.

The following example shows both that $\mathcal{D}_n(X)$ is not stable under rank 1 perturbations, and that the assumptions $T+F \in \mathcal{D}_n(X)$ and $T-F \notin \mathcal{D}_n(X)$ in Proposition 2.4 are necessary.

Example 2.6. Let $Y \subset X$ be a subspace of dimension n, and write $X = Y \oplus Z$, where Z is a closed subspace. With respect to an arbitrary basis of Y, consider the operators $T, F \in \mathcal{B}(X)$ given by

$$T = J_n \oplus I \qquad \text{and} \qquad F = E_{1,n} \oplus 0,$$

where $E_{1,n}$ is the $n \times n$ matrix whose only nonzero entry is 1 in position $(1, n)$. Clearly, F is rank 1, and $T \in \mathcal{D}_n(X)$. However, the matrix $J_n + \alpha E_{1,n}$ is invertible, and so $T + \alpha F$ is invertible for every nonzero $\alpha \in \mathbb{C}$.

The following theorem, which is interesting in itself, allows us to establish in the next section that every additive surjective map Φ preserving $\mathcal{D}_n(X)$ in both directions is bijective and preserves rank 1 operators in both directions.

Theorem 2.7. *Let $F \in \mathcal{B}(X)$ be nonzero. Then the following assertions hold:*

(1) *there exists $T \in \mathcal{D}_n(X)$ such that $T + 2F \notin \mathcal{D}_n(X)$;*
(2) *if $\dim \operatorname{ran}(F) \geq 2$, then there exists $T \in \mathcal{D}_n(X)$ such that $T + F \in \mathcal{D}_n(X)$ and $T - cF \notin \mathcal{D}_n(X)$ for every $c \in \{1, \pm 2\}$.*

Before proving this theorem, some auxiliary results should be established first.

Lemma 2.8. *Let Y and Z be two nontrivial closed subspaces such that $X = Y \oplus Z$, and let $T \in \mathcal{B}(X)$ have the operator matrix form*

$$T = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix},$$

where C is invertible. Then T is Drazin invertible if and only if A is Drazin invertible, and in this case $\mathrm{i}(T) = \mathrm{i}(A)$.

Proof. See [5, Corollary 5.2] and [24, Theorem 2.1]. □

We would like to mention that the previous lemma remains true for finite-dimensional spaces. Indeed, every operator acting on a finite-dimensional space has finite ascent and descent, and, consequently, it is Drazin invertible. The equality of the indices follows from [17, Corollary 2.1]. It should also be noted that, by passing to the adjoint, one can easily show that the same conclusion holds true for lower triangular operator matrix

$$T = \begin{bmatrix} A & 0 \\ D & C \end{bmatrix}.$$

The following lemma shows that the proof of Theorem 2.7 can be reduced to the case where the space is finite-dimensional.

Lemma 2.9. *Let $F \in \mathcal{B}(X)$, and let F_1 be its restriction to an F-invariant subspace Y of finite dimension. Then every operator $S \in \mathcal{B}(Y)$ can be extended to an operator $T \in \mathcal{B}(X)$ such that $T + \alpha F$ is Drazin invertible and*

$$\mathrm{i}(T + \alpha F) = \mathrm{i}(S + \alpha F_1) \quad \textit{for every } \alpha \in \{0, \pm 1, \pm 2\}.$$

Proof. Let Z be a closed subspace such that $X = Y \oplus Z$. With respect to this decomposition, the operator F can be expressed as follows:

$$F = \begin{bmatrix} F_1 & F_2 \\ 0 & F_3 \end{bmatrix}.$$

Consider also the operator $T \in \mathcal{B}(X)$ represented by the matrix

$$T = \begin{bmatrix} S & 0 \\ 0 & cI \end{bmatrix},$$

where c is a nonzero complex number such that $cI + \alpha F_3$ is invertible for every $\alpha \in \{0, \pm 1, \pm 2\}$. Now, using Lemma 2.8, we obtain that $T + \alpha F$ is Drazin invertible and $\mathrm{i}(T + \alpha F) = \mathrm{i}(S + \alpha F_1)$ for every $\alpha \in \{0, \pm 1, \pm 2\}$. ☐

Lemma 2.10. *Let Y be a complex Banach space such that $\dim(Y) \geq n + 3$. Let $F \in \mathcal{B}(Y)$ be a rank 1 operator, and let $c \in \{-2, 2\}$. Then there exists $T \in \mathcal{B}(Y)$ such that $T + \alpha F$ is Drazin invertible for every $\alpha \in \{0, \pm 1, c\}$ and*

$$\mathrm{i}(T) = \mathrm{i}(T + F) = n, \qquad \mathrm{i}(T - F) = n - 1, \qquad \textit{and} \qquad \mathrm{i}(T + cF) > n.$$

Proof. Let $F = z \otimes f$ where $z \in Y$ and $f \in Y^*$ are nonzero. Choose $x_{n+2} \in Y$ linearly independent of z and such that $f(x_{n+2}) = 1$. Since $Y = \mathrm{Span}\{x_{n+2}, z\} + \ker(f)$, there are linearly independent vectors $x_{n+1}, \ldots, x_3, x_2, x_0$ forming with $\{x_{n+2}, z\}$ a linearly independent set and such that $f(x_2) = 1$ and $f(x_i) = f(x_0) = 0$ for $3 \leq i \leq n + 1$. Let $x_1 = (1 + c)x_{n+1} - cz$. Then $\{x_{n+2}, \ldots, x_0\}$ is a linearly independent set, and

$$\begin{cases} f(x_{n+2}) = f(x_2) = 1, \qquad f(x_1) = -cf(z), \\ f(x_i) = f(x_0) = 0 \quad \text{for } 3 \leq i \leq n + 1, \\ z = (1 + c^{-1})x_{n+1} - c^{-1}x_1. \end{cases} \tag{2.5}$$

Put $Z = \mathrm{Span}\{x_{n+2}, \ldots, x_0\}$, and consider the operator $S \in \mathcal{B}(Z)$ represented by the matrix

$$S = \begin{bmatrix} J_n & 0 \\ 0 & U \end{bmatrix} \quad \text{where } U = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Let $F_\text{o} = F_{|Z}$, $\varepsilon \in \{0, \pm 1\}$, and let $u_{n+2} = \varepsilon c^{-1} x_2 + (1 - \varepsilon c^{-1}) x_{n+2}$. We get by (2.5) that

$$\begin{cases} (S + \varepsilon F_\text{o}) u_{n+2} = \varepsilon c^{-1} x_1 + (1 - \varepsilon c^{-1}) x_{n+1} + \varepsilon z = (1 + \varepsilon) x_{n+1}, \\ (S + \varepsilon F_\text{o}) x_i = S x_i = x_{i-1} \quad \text{for } 4 \leq i \leq n+1, \\ (S + \varepsilon F_\text{o}) x_3 = 0. \end{cases}$$

With respect to the basis $\{u_{n+2}, x_{n+1}, \ldots, x_0\}$, we have

$$S + \varepsilon F_\text{o} = \begin{bmatrix} N_\varepsilon & A \\ 0 & V_\varepsilon \end{bmatrix},$$

where N_ε and V_ε are given by

$$N_\varepsilon = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 \\ 1+\varepsilon & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{bmatrix} \qquad \text{and} \qquad V_\varepsilon = \begin{bmatrix} 0 & 0 & 1 \\ 1 - \varepsilon c^{-1} & \varepsilon f(z) & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Obviously, N_ε is nilpotent and V_ε is invertible. Since $\mathrm{i}(N_0) = \mathrm{i}(N_1) = n$ and $\mathrm{i}(N_{-1}) = n - 1$, we obtain that $\mathrm{i}(S) = \mathrm{i}(S + F_\text{o}) = n$ and $\mathrm{i}(S - F_\text{o}) = n - 1$. On the other hand, we have

$$\begin{cases} (S + cF_\text{o}) x_0 = x_2, \qquad (S + cF_\text{o}) x_2 = (1 + c) x_{n+1}, \\ (S + cF_\text{o}) x_k = x_{k-1} \quad \text{for } 4 \leq k \leq n+1, \\ (S + cF_\text{o}) x_3 = 0. \end{cases}$$

Then $(S + cF_\text{o})^n x_0 = (1 + c) x_3 \neq 0$ and $(S + cF_\text{o})^{n+1} x_0 = 0$. Thus $\mathrm{a}(S + cF_\text{o}) > n$, and so $\mathrm{i}(S + cF_\text{o}) > n$. Finally, using Lemma 2.9, we get the desired operator T. $\square$

We mention that this lemma does not hold for $n = 1$. Indeed, let $F \in \mathcal{B}(X)$ be a rank 1 operator, and let $T \in \mathcal{B}(X)$. If $T - F$ is invertible, then it follows from [21, Lemma 2.1] that $(T - F) + F = T$ is invertible or $(T - F) + 2F = T + F$ is invertible.

If T has a diagonal representation $T = T_1 \oplus T_2$, then one can easily show that T is Drazin invertible if and only if T_1 and T_2 are Drazin invertible, and in this case $\mathrm{i}(T) = \max\{\mathrm{i}(T_1), \mathrm{i}(T_2)\}$.

Lemma 2.11. *Let $F \in \mathcal{B}(X)$ be such that $\dim \ker(F) = \infty$ and $\dim \operatorname{ran}(F) \geq 2$. Assume that there exist $x_1, x_2 \in X$ such that*

(1) *the sum $\operatorname{Span}\{x_1, Fx_1\} + \operatorname{Span}\{x_2, Fx_2\}$ is direct,*
(2) *$F_{|\operatorname{Span}\{x_i, Fx_i\}}$ has rank 1 for $1 \leq i \leq 2$.*

Then there exists $T \in \mathcal{D}_n(X)$ satisfying $T + F \in \mathcal{D}_n(X)$ and $T - cF \notin \mathcal{D}_n(X)$ for every $c \in \{1, \pm 2\}$.

Proof. We can pick two subspaces $X_i \subseteq \ker(F)$, $1 \leq i \leq 2$, of dimension $n + 2$ such that the sum

$$\operatorname{Span}\{x_1, Fx_1\} + \operatorname{Span}\{x_2, Fx_2\} + X_1 + X_2$$

is direct. Let $Y_i = \operatorname{Span}\{x_i, Fx_i\} \oplus X_i$, and let $F_i = F_{|Y_i}$, $i = 1, 2$. Write $Y = Y_1 \oplus Y_2$, and write $F_o = F_1 \oplus F_2$. Since F_i has rank 1 and $\dim Y_i \geq n+3$ for $i = 1, 2$, it follows from Lemma 2.10 that there are two linear operators S_1 and S_2 acting on Y_1 and Y_2, respectively, such that

$$\begin{cases} \mathrm{i}(S_k) = \mathrm{i}(S_k + F_k) = n \quad \text{and} \quad \mathrm{i}(S_k - F_k) = n-1 \quad \text{for } 1 \leq k \leq 2, \\ \mathrm{i}(S_1 + 2F_1) > n \qquad \text{and} \qquad \mathrm{i}(S_2 - 2F_2) > n. \end{cases}$$

Letting $S = S_1 \oplus S_2$, we have

$$\mathrm{i}(S + cF_o) = \max\bigl\{\mathrm{i}(S_1 + cF_1), \mathrm{i}(S_2 + cF_2)\bigr\} \quad \text{for every } c \in \mathbb{C},$$

and so $\mathrm{i}(S) = \mathrm{i}(S+F_o) = n$, $\mathrm{i}(S-F_o) = n-1$, $\mathrm{i}(S+2F_o) > n$, and $\mathrm{i}(S-2F_o) > n$. Thus Lemma 2.9 gives the desired operator T. $\square$

For a positive integer k we denote by I_k the $k \times k$ identity complex matrix.

The following lemma is a special case of Theorem 2.7(2) with F being algebraic. Recall that an operator $F \in \mathcal{B}(X)$ is said to be *algebraic* if there exists a nonzero complex polynomial p such that $p(F) = 0$.

Lemma 2.12. *Let $F \in \mathcal{B}(X)$ be an algebraic operator such that* $\dim \operatorname{ran}(F) \geq 2$. *Then there exists $T \in \mathcal{D}_n(X)$ such that $T + F \in \mathcal{D}_n(X)$ and $T - cF \notin \mathcal{D}_n(X)$ for every $c \in \{1, \pm 2\}$.*

Proof. Assume that $\ker(F)$ has finite dimension. Then there exists a nonzero $\lambda \in \mathbb{C}$ such that $\ker(F - \lambda)$ is infinite-dimensional. Let $L \subset \ker(F - \lambda)$ be a subspace of dimension $2n$. Let $G = F_{|L} = \lambda I_n$, and define $R \in \mathcal{B}(L)$ by $R = J_n \oplus (J_n - \lambda I_n)$ with respect to an arbitrary basis of L. Clearly, R and $R + G$ are Drazin invertible of index n. However, $R - cG$ is invertible for every $c \in \{1, \pm 2\}$. The proof is completed by using Lemma 2.9.

Assume now that $\ker(F)$ has infinite dimension. We note that $\sigma(F)$ is contained in $\{0, \alpha_1, \dots, \alpha_r\}$, where 0 and α_i, $1 \leq i \leq r$, are the zeros of a nonzero complex polynomial annihilating T. Put

$$m = \dim\bigl[\bigl(\operatorname{ran}(F) \cap \ker(F)\bigr) \oplus \ker(F - \alpha_1) \oplus \cdots \oplus \ker(F - \alpha_r)\bigr].$$

Then $m \geq 1$. In fact, if F is nilpotent, then $X = \ker(F^p)$ for some $p \geq 2$ because $\dim \operatorname{ran}(F) \geq 2$, and so $\operatorname{ran}(F) \cap \ker(F) \neq \{0\}$. We shall discuss three cases.

Case 1. If $m \geq 2$, then there are $x_1, x_2 \in X$ such that Fx_1 and Fx_2 are linearly independent, and $F^2 x_i = 0$ or Fx_i is collinear with x_i for $1 \leq i \leq 2$. Perturbing x_1, x_2 by suitable elements of $\ker(F)$, we may assume that $\{x_1, Fx_1, x_2, Fx_2\}$ is linearly independent. Thus, using the previous lemma, we get the desired operator.

Case 2. If $m = \dim \operatorname{ran}(F) \cap \ker(F) = 1$, then $\sigma(T) = \{0\}$ and $\operatorname{ran}(F) \nsubseteq \ker(F)$. Thus $F^2 \neq 0$, and hence there are linearly independent vectors y_2, y_1, y_0 such that

$$Fy_2 = y_1, \qquad Fy_1 = y_0, \qquad \text{and} \qquad Fy_0 = 0.$$

Choose vectors $y_i \in \ker(F)$, $3 \leq i \leq 2n$, forming with $\{y_2, y_1, y_0\}$ a linearly independent set. Let $Y = \operatorname{Span}\{y_{2n}, \dots, y_0\}$, and let $S \in \mathcal{D}_n(Y)$ be the operator given by $S = J_n \oplus (-J_n) \oplus 0$. If we put $F_o = F_{|Y}$, then it follows that $S + F_o =$

$J_n \oplus (-J_{n-1}) \oplus J_2$, and hence $S + F_{\mathrm{o}} \in \mathcal{D}_n(Y)$. Letting $c \in \{1, \pm 2\}$, we have

$$\begin{cases} (S - cF_{\mathrm{o}})y_i = Sy_i = -y_{i-1} \quad \text{for } 3 \le i \le n, \\ (S - cF_{\mathrm{o}})y_2 = (-1-c)y_1, \qquad (S - cF_{\mathrm{o}})y_1 = -cy_0, \qquad \text{and} \qquad (S + cF_{\mathrm{o}})y_0 = 0. \end{cases}$$

Therefore, $\mathrm{a}(S - cF_{\mathrm{o}}) \ge n + 1$, and so $S - cF_{\mathrm{o}} \notin \mathcal{D}_n(Y)$. Using Lemma 2.9, we get the desired operator T.

Case 3. Assume that $m = \dim\ker(F - \alpha) = 1$ with $\alpha \neq 0$. From this and the fact that $\dim\ker(F) = \infty$, we infer that $\sigma(F) = \{0, \alpha\}$. Furthermore, we have $\mathrm{a}(F) = 1$ because $\mathrm{ran}(F) \cap \ker(F) = \{0\}$, and hence $X = \ker(F) \oplus \ker(F - \alpha)^k$ for some positive integer $k \ge 1$. Consequently, $\mathrm{ran}(F) = \ker(F - \alpha)^k$, and since $\dim\mathrm{ran}(F) \ge 2$ and $\dim\ker(F - \alpha) = 1$, we obtain that $\ker(F - \alpha) \subsetneq \ker(F - \alpha)^2$. Hence there are linearly independent vectors $\{x_0, x_1\}$ such that

$$(F - \alpha)x_1 = x_0 \qquad \text{and} \qquad (F - \alpha)x_0 = 0.$$

Choose vectors $x_i \in \ker(F)$, $2 \le i \le n$, that constitute with $\{x_0, x_1\}$ a linearly independent set. Let $Z = \mathrm{Span}\{x_n, \dots, x_0\}$, and let $F_1 = F_{|Z}$. Consider also the operator $K \in \mathcal{B}(Z)$ defined by

$$Kx_0 = 0, \qquad Kx_1 = -\alpha x_1 - x_0, \qquad \text{and} \qquad Kx_i = x_{i-1} \quad \text{for } 2 \le i \le n.$$

We have $K + F_1 = J_n \oplus \alpha I_1$, and hence $K + F_1 \in \mathcal{D}_n(Z)$. Let $c \in \{1, \pm 2\}$. Then we can express $K - cF_1$ as follows:

$$K - cF_1 = \begin{bmatrix} J_{n-1} & 0 \\ A & B \end{bmatrix} \quad \text{where } B = \begin{bmatrix} -(c+1)\alpha & 0 \\ -c-1 & -c\alpha \end{bmatrix}.$$

Since B is invertible, Lemma 2.8 yields that $\mathrm{i}(K - cF_1) = n - 1$. Let $u_i = \alpha x_i + x_{i-1}$ for $2 \le i \le n$, $u_1 = -x_0$, and $u_0 = \alpha x_1 + x_0$. Then we get easily that

$$Ku_i = u_{i-1} \quad \text{for } 2 \le i \le n, \qquad Ku_1 = 0, \qquad \text{and} \qquad Ku_0 = -\alpha u_0.$$

Thus $K = J_n \oplus (-\alpha I_1)$ relative to the basis $u_n, \dots, u_0$, and so $K \in \mathcal{D}_n(Z)$. Using again Lemma 2.9, we get the desired operator T. □

Proof of Theorem 2.7. (2) Suppose that F has at least rank 2. According to Lemma 2.11, we may assume that F is not algebraic. It follows by [1, Theorem 4.2.7] that there is $x \in X$ such that $\{F^i x : 0 \le i \le 2n + 4\}$ is a linearly independent set. Write $X = X_1 \oplus X_2 \oplus X_3$, where $X_1 = \mathrm{Span}\{x, \dots, F^{n+3}x\}$, $X_2 = \mathrm{Span}\{F^{n+4}x, \dots, F^{2n+3}x\}$, and X_3 is a closed subspace containing $F^{2n+4}x$. With respect to this decomposition, F can be expressed as follows:

$$F = \begin{bmatrix} J_{n+4} & 0 & A \\ B & J_n & C \\ 0 & D & E \end{bmatrix}.$$

Consider a nonzero $\alpha \in \mathbb{C}$ such that $\alpha I + E$ is invertible, and let $T \in \mathcal{B}(X)$ be the operator given by

$$T = \begin{bmatrix} S & 0 & -A \\ 0 & J_n & -C \\ 0 & 0 & \alpha I \end{bmatrix},$$

where the operator S is defined by

$$\begin{cases} SF^k x = 3F^{k+1}x \quad \text{for } 0 \le k \le n-1, \\ SF^n x = F^{n+1}x \qquad \text{and} \qquad SF^{n+1}x = -2F^{n+2}x, \\ SF^{n+2}x = 2F^{n+3}x \qquad \text{and} \qquad SF^{n+3}x = x. \end{cases}$$

One can easily verify that S and $S + J_{n+4}$ are invertible. Hence we obtain by Lemma 2.8 that T and $T + F$ are Drazin invertible, and $\mathrm{i}(T) = \mathrm{i}(T + F) = n$. Furthermore, for any $c \in \{1, \pm 2\}$, we have

$$\begin{cases} (T - cF)F^i x = (3 - c)F^{i+1}x \quad \text{for } 0 \le i \le n-1, \\ (T - cF)F^n x = (1 - c)F^{n+1}x, \\ (T - cF)F^{n+1}x = (-2 - c)F^{n+2}x, \\ (T - cF)F^{n+2}x = (2 - c)F^{n+3}x. \end{cases}$$

Therefore, $(T - F)^n x = 2^n F^n x \neq 0$ and $(T - F)^{n+1}x = 0$, and, consequently, $\mathrm{a}(T - F) > n$ and $T - F \notin \mathcal{D}_n(X)$. Similarly, we get that $T + 2F \notin \mathcal{D}_n(X)$ and $T - 2F \notin \mathcal{D}_n(X)$.

(1) If $\dim \mathrm{ran}(F) \ge 2$, then the second assertion implies the first one. If F has rank 1, then Lemma 2.10 ensures the existence of the desired operator. □

3. Proof of main result

With these results at hand, we are ready to prove our main results in this section.

Lemma 3.1. *Let $\Phi : \mathcal{B}(X) \to \mathcal{B}(X)$ be an additive surjective map preserving $\mathcal{D}_n(X)$ in both directions. Then*

(1) *Φ is injective,*

(2) *Φ preserves the set of rank 1 operators in both directions.*

Proof. (1) Suppose to the contrary that there exists $F \neq 0$ such that $\Phi(F) = 0$. Then, by Theorem 2.7, we can find $T \in \mathcal{D}_n(X)$ satisfying $T + 2F \notin \mathcal{D}_n(X)$. But $\Phi(T + 2F) = \Phi(T) \in \mathcal{D}_n(X)$, the desired contradiction.

(2) Let $F \in \mathcal{B}(X)$ with $\dim \mathrm{ran}(F) \ge 2$. Then it follows again by Theorem 2.7 that there exists $T \in \mathcal{D}_n(X)$ such that $T + F \in \mathcal{D}_n(X)$ and $T - cF \notin \mathcal{D}_n(X)$ for every $c \in \{1, \pm 2\}$. Thus $\Phi(T)$ and $\Phi(T) + \Phi(F)$ belong to $\mathcal{D}_n(X)$, but $\Phi(T) - c\Phi(F) \notin \mathcal{D}_n(X)$ for every $c \in \{1, \pm 2\}$. Therefore, we obtain by Proposition 2.4 that $\dim \mathrm{ran}(\Phi(F)) \ge 2$. Since Φ is bijective and Φ^{-1} satisfies the same properties as Φ, we obtain that Φ preserves the set of rank 1 operators in both directions. □

Recall that an operator $T \in \mathcal{B}(X)$ is said to be *semi-Fredholm* if $\mathrm{ran}(T)$ is closed and either $\dim \ker(T)$ or $\mathrm{codim}\, \mathrm{ran}(T)$ is finite. For such an operator T, the *index* is defined by

$$\mathrm{ind}(T) = \dim \ker(T) - \mathrm{codim}\, \mathrm{ran}(T),$$

and if the index is finite, T is said to be *Fredholm*.

Remark 3.2. Let $T \in \mathcal{B}(X)$ be a semi-Fredholm operator. The following assertions hold.

(1) If $K \in \mathcal{B}(X)$ is a compact operator, then $T+K$ is semi-Fredholm of the same index as T (see [19, Theorem 16.16]).
(2) If $\operatorname{ind}(T)=0$, then $\mathrm{a}(T)=\mathrm{d}(T)$ (see [14, Lemma 2.3]). In particular, in this case, $T \in \mathcal{D}_n(X)$ if and only if $\mathrm{a}(T)=n$.
(3) If $T \in \mathcal{D}_n(X)$, then $\operatorname{ind}(T)=0$ (see [19, Theorem 16.12]).

Proposition 3.3. *Let $T \in \mathcal{B}(X)$ be invertible, and let $x \in X$ and $f \in X^*$ be nonzero. Then $T + x \otimes f \in \mathcal{D}_n(X)$ if and only if*

$$f(T^{-i}x) = -\delta_{i1} \quad \text{for } 1 \le i \le n \qquad \text{and} \qquad f(T^{-(n+1)}x) \neq 0. \tag{3.1}$$

Proof. Let $F = x \otimes f$. Suppose that $T + F \in \mathcal{D}_n(X)$. It follows from [13, Lemma 2.2] that there exist linearly independent vectors $x_0, \ldots, x_{n-1}$ such that

$$(T+F)x_0 = 0, \qquad (T+F)x_i = x_{i-1} \quad \text{for } 1 \le i \le n-1,$$

and

$$f(x_i) = \delta_{i0} \quad \text{for } 0 \le i \le n-1.$$

Hence $Tx_0 = -x$ and $Tx_i = x_{i-1}$ for $1 \le i \le n-1$. Consequently, $x = -T^i x_{i-1}$, and so $f(T^{-i}x) = f(-x_{i-1}) = -\delta_{i1}$ for $1 \le i \le n$. If $f(T^{-(n+1)}x) = 0$, then

$$\begin{aligned}(T+F)^n T^{-(n+1)}x &= (T+F)^{n-1}T^{-n}x \\ &= -(T+F)^{n-1}x_{n-1} \\ &= -x_0 \in \ker(T+F),\end{aligned}$$

and hence $\mathrm{a}(T+F) \ge n+1$. This contradiction shows that $f(T^{-(n+1)}x) \neq 0$.

Conversely, assume that (3.1) holds. Let $u_i = -T^{-(i+1)}x$ for $0 \le i \le n-1$. Then it follows that $f(u_i) = \delta_{i0}$ for $0 \le i \le n-1$, $Tu_0 = -x$, and $Tu_i = u_{i-1}$ for $1 \le i \le n-1$. Hence

$$(T+F)u_0 = 0 \qquad \text{and} \qquad (T+F)u_i = u_{i-1} \quad \text{for } 1 \le i \le n-1.$$

In particular, this implies that $\mathrm{a}(T+F) \ge n$. To finish, let us show that $\mathrm{a}(T+F) = n$. Suppose to the contrary that $\mathrm{a}(T+F) > n$, and let $y_0, \ldots, y_n$ be linearly independent vectors such that

$$(T+F)y_0 = 0, \qquad (T+F)y_i = y_{i-1} \quad \text{for } 1 \le i \le n,$$

and

$$f(y_i) = \delta_{i0} \quad \text{for } 0 \le i \le n.$$

Then, just as above, we get that $f(T^{-(n+1)}x) = f(-y_n) = 0$. This contradiction completes the proof. □

Let $T \in \mathcal{B}(X)$. We associate for each $x \in X$ the following subset:

$$\mathrm{M}_x(T) = \big\{ f \in X^* : T + x \otimes f \in \mathcal{D}_n(X) \big\}.$$

Corollary 3.4. *Let T be a bounded invertible operator on X, and let $x \in X$ be nonzero. Then*

$$\mathrm{M}_x(T) \neq \emptyset \iff \{T^{-k}x : 1 \leq k \leq n\} \text{ is linearly independent.}$$

Moreover, in this case we have $\mathrm{M}_y(T) \neq \emptyset$ for all $y \in \{T^i x : i \in \mathbb{Z}\}$.

Proof. The direct implication follows immediately from the previous proposition. Conversely, assume that $\{T^{-k}x : 1 \leq k \leq n\}$ is linearly independent. If $T^{-(n+1)}x$ is not a linear combination of $T^{-k}x$, $1 \leq k \leq n$, then the existence of a linear form $f \in X^*$ satisfying (3.1) is obvious. Suppose that $T^{-(n+1)}x = \sum_{k=1}^{n} \alpha_k T^{-k}x$. Since T is bijective and $T^{-k}x$, $1 \leq k \leq n$, are linearly independent, we infer that $\alpha_1 \neq 0$. Choose an arbitrary linear form $f \in X^*$ satisfying $f(T^{-k}x) = -\delta_{k1}$ for $1 \leq k \leq n$. It follows that $f(T^{-(n+1)}x) = -\alpha_1 \neq 0$, and thus f fulfils (3.1). This shows that $f \in \mathrm{M}_x(T)$.

Now, let $y = T^i x$ where i is an arbitrary integer. Since also the set $T^{-k}y$, $1 \leq k \leq n$, is linearly independent, we get that $\mathrm{M}_y(T)$ is not empty. $\square$

Let $T, S \in \mathcal{B}(X)$. We will write $T \sim S$ if the following equivalence holds:

$$T + F \in \mathcal{D}_n(X) \iff S + F \in \mathcal{D}_n(X)$$

for every finite rank operator $F \in \mathcal{B}(X)$. Clearly, $(\sim)$ defines an equivalence relation on $\mathcal{B}(X)$. Furthermore, if $T \sim S$, then $\mathrm{M}_x(T) = \mathrm{M}_x(S)$ for all $x \in X$, and $T + F \sim S + F$ for all finite rank operators $F \in \mathcal{B}(X)$.

Remark 3.5. Let $T \in \mathcal{B}(X)$ be invertible, and let $x \in X$ be nonzero. The following assertions follow immediately from Proposition 3.3 and Corollary 3.4:

(1) $\mathrm{M}_x(T) = \{f \in X^* : f(T^{-i}x) = -\delta_{i1} \text{ for } 1 \leq i \leq n \text{ and } f(T^{-(n+1)}x) \neq 0\}$;
(2) $\mathrm{M}_x(T) \neq \emptyset$ if and only if $\mathrm{M}_{T^i x}(T) \neq \emptyset$ for every $i \in \mathbb{Z}$.

Proposition 3.6. *Let $T, S \in \mathcal{B}(X)$ be invertible operators such that $T \sim S$. Then $T = S$.*

Before presenting the proof of this proposition, we need the following lemma.

For a subset $G \subseteq X$, $G^{\perp} = \{f \in X^* : G \subseteq \ker(f)\}$ is the polar or annihilator of G.

Lemma 3.7. *Let $T, S \in \mathcal{B}(X)$ be invertible operators such that $T \sim S$. If there exists a vector $x \in X$ such that $\{x, Tx, \ldots, T^{2n}x\}$ is linearly independent, then $Ty = Sy$ for all $y \in \mathrm{Span}\{T^i x : i \in \mathbb{Z}\}$.*

Proof. Note that since $T^i x$ satisfies the same hypothesis as x for all $i \in \mathbb{Z}$, it suffices to show that $ST^{-n}x = T^{-(n-1)}x$. Let $y \in \{T^i x, S^i x : i \in \mathbb{Z}\}$. It follows from the previous corollary that $\mathrm{M}_y(T) = \mathrm{M}_y(S)$ is not empty. Let $f \in \mathrm{M}_y(T)$, and consider an arbitrary $g \in \{T^{-j}y : 2 \leq j \leq n\}^{\perp}$. Multiplying g by a suitable scalar, we may assume that

$$g(T^{-1}y) \neq -1 \qquad \text{and} \qquad g(T^{-(n+1)}y) \neq -\big(g(T^{-1}y) + 1\big) f(T^{-(n+1)}y).$$

Let $h = g + (g(T^{-1}y) + 1)f$. Then we have

$$h(T^{-1}y) = -1, \qquad h(T^{-i}y) = 0 \quad \text{for } 2 \leq i \leq n, \qquad \text{and} \qquad h(T^{-(n+1)}y) \neq 0,$$

and so $h \in \mathrm{M}_y(T) = \mathrm{M}_y(S)$. Therefore, $h(S^{-1}y) = -1$ and $h(S^{-i}y) = 0$ for $2 \le i \le n$, and, consequently, $g(S^{-1}y - T^{-1}y) = g(S^{-i}y) = 0$ for $2 \le i \le n$. This implies that

$$\{S^{-1}y - T^{-1}y, S^{-i}y : 2 \le i \le n\} \subseteq \operatorname{Span}\{T^{-j}y : 2 \le j \le n\}. \tag{3.2}$$

Let us show that

$$S^{-i}x - T^{-i}x \in \operatorname{Span}\{T^{-k}x : i+1 \le k \le n\} \quad \text{for } 1 \le i \le n. \tag{3.3}$$

Clearly, replacing y by x in (3.2), we obtain that (3.3) is satisfied for $i = 1$. Suppose that (3.3) holds for $i < n$. We have

$$S^{-(i+1)}x - T^{-(i+1)}x = S^{-1}(S^{-i}x - T^{-i}x) + S^{-1}T^{-i}x - T^{-1}T^{-i}x.$$

Utilizing (3.3) and (3.2) for $y = T^{-k}x$, we get that

$$\begin{aligned} S^{-1}(S^{-i}x - T^{-i}x) &\in \operatorname{Span}\{S^{-1}T^{-k}x : i+1 \le k \le n\} \\ &\in \operatorname{Span}\{T^{-(j+k)}x : 1 \le j \le n, i+1 \le k \le n\} \\ &\in \operatorname{Span}\{T^{-p}x : i+2 \le p \le 2n\}. \end{aligned}$$

Moreover, formula (3.2) for $y = T^{-i}x$ asserts that

$$\begin{aligned} S^{-1}T^{-i}x - T^{-1}T^{-i}x &\in \operatorname{Span}\{T^{-(j+i)}x : 2 \le j \le n\} \\ &\subseteq \operatorname{Span}\{T^{-p}x : i+2 \le p \le 2n\}. \end{aligned}$$

Thus $S^{-(i+1)}x - T^{-(i+1)}x \in \operatorname{Span}\{T^{-p}x : i+2 \le p \le 2n\}$. On the other hand, replacing y by x in (3.2), we recover that $S^{-(i+1)}x$ is a linear combination of $T^{-j}x$, $2 \le j \le n$, and hence so is $S^{-(i+1)}x - T^{-(i+1)}x$. Therefore,

$$\begin{aligned} S^{-(i+1)}x - T^{-(i+1)}x &\in \operatorname{Span}\{T^{-p}x : i+2 \le p \le 2n\} \cap \operatorname{Span}\{T^{-j}x : 2 \le j \le n\} \\ &\subseteq \operatorname{Span}\{T^{-k}x : i+2 \le k \le n\}, \end{aligned}$$

which establishes (3.3). Hence $S^{-n}x = T^{-n}x$ and $S^{-(n-1)}x = T^{-(n-1)}x + \beta T^{-n}x$ for some $\beta \in \mathbb{C}$. Moreover, it follows from (3.2) with $y = S^{-(n-1)}x$ that there exist complex numbers $\alpha_1, \dots, \alpha_n$ such that $\alpha_1 = 1$ and

$$S^{-1}S^{-(n-1)}x = \sum_{j=1}^{n} \alpha_j T^{-j} S^{-(n-1)}x.$$

Therefore,

$$\begin{aligned} S^{-n}x &= \sum_{j=1}^{n} \alpha_j T^{-j}(T^{-(n-1)}x + \beta T^{-n}x) \\ &= \sum_{j=1}^{n} \alpha_j (T^{-(n-1+j)}x + \beta T^{-(n+j)}x) \\ &= T^{-n}x + (\alpha_1\beta + \alpha_2)T^{-(n+1)}x + \cdots \\ &\quad + (\alpha_{n-1}\beta + \alpha_n)T^{-(2n-1)}x + \alpha_n \beta T^{-(2n)}x. \end{aligned}$$

Since $S^{-n}x = T^{-n}x$ and $\{T^{-(n+1)}x, \dots, T^{-2n}x\}$ is a linearly independent set, we infer that $\beta + \alpha_2 = \alpha_2\beta + \alpha_3 = \cdots = \alpha_{n-1}\beta + \alpha_n = \alpha_n\beta = 0$ so that $\beta = -\alpha_2$ and $\alpha_i = \alpha_2^{i-1}$ for $2 \le i \le n$. But, as $\alpha_n\beta = -\alpha_2^n = 0$, we obtain that $\beta = \alpha_i = 0$ for $2 \le i \le n$. Thus $S^{-(n-1)}x = T^{-(n-1)}x$. Finally, we have $ST^{-n}x = SS^{-n}x = S^{-(n-1)}x = T^{-(n-1)}x$. □

Proof of Proposition 3.6. Notice first that, for a finite-codimensional subspace Y of X, it is an elementary fact that

$$\dim[Y \cap T^{-1}Y \cap \cdots \cap T^{-2n}Y \cap T^{-1}SY] = \infty.$$

Let $x_0 \in X$ be nonzero, and let us show that $Tx_0 = Sx_0$. Let Y be a complement of $\operatorname{Span}\{x_0, Tx_0, \dots, T^{2n}x_0, S^{-1}Tx_0\}$. Then $Y \cap T^{-1}Y \cap \cdots \cap T^{-2n}Y \cap T^{-1}SY$ contains a nonzero vector x_1, and the sum

$$\operatorname{Span}\{x_0, Tx_0, \dots, T^{2n}x_0, S^{-1}Tx_0\} + \operatorname{Span}\{x_1, Tx_1, \dots, T^{2n}x_1, S^{-1}Tx_1\}$$

is direct. Repeating the same argument, we get the existence of nonzero vectors $x_2, \dots, x_{2n} \in X$ such that the sum of the subspaces

$$Z_i = \operatorname{Span}\{x_i, Tx_i, \dots, T^{2n}x_i, S^{-1}Tx_i\}, \quad 0 \le i \le 2n,$$

is direct. Let $f_0, \dots, f_{2n-1} \in X^*$ be such that $f_i \in Z_j^\perp$ for $i \ne j$, and let $f_0(x_0) = f_i(Tx_i) = 1$ for $1 \le i \le 2n-1$. Consider also the operators $H, R \in \mathcal{B}(X)$ defined by

$$H = T + \sum_{i=1}^{2n} Tx_i \otimes f_{i-1} \qquad \text{and} \qquad R = S + \sum_{i=1}^{2n} Tx_i \otimes f_{i-1}.$$

Clearly, we have $H \sim R$. Note also that

$$I + \sum_{i=1}^{2n} x_i \otimes f_{i-1} = \prod_{i=1}^{2n} (I + x_i \otimes f_{i-1})$$

and

$$I + \sum_{i=1}^{2n} S^{-1}Tx_i \otimes f_{i-1} = \prod_{i=1}^{2n} (I + S^{-1}Tx_i \otimes f_{i-1}).$$

Since $f_{i-1}(x_i) = f_{i-1}(S^{-1}Tx_i) = 0$ for $1 \le i \le 2n$, we obtain that these operators are invertible. Therefore, H and R are invertible. Furthermore, one can easily verify that $H^k x_0 = v_{k-1} + Tx_k$ for $1 \le k \le 2n$, where $v_{k-1} \in Z_0 \oplus \cdots \oplus Z_{k-1}$. Consequently, the vectors $x_0, \dots, H^{2n}x_0$ are linearly independent. Thus $Hx_0 = Rx_0$ by Lemma 3.7. But, we have also $Hx_0 = Tx_0 + Tx_1$ and $Rx_0 = Sx_0 + Tx_1$. Hence $Tx_0 = Sx_0$. This completes the proof. □

Proposition 3.8. *Let $\Phi : \mathcal{B}(X) \to \mathcal{B}(X)$ be an additive surjective map. If Φ preserves $\mathcal{D}_n(X)$ in both directions, then there exists a nonzero $\alpha \in \mathbb{C}$ such that $\Phi(I) = \alpha I$.*

For proving this proposition, we need to establish some auxiliary lemmas. Throughout the sequel, we shall denote by $\mathcal{F}_n(X)$ the set of all operators $F \in \mathcal{B}(X)$ with $\dim \operatorname{ran}(F) < n$.

Remark 3.9. Let $F \in \mathcal{F}_n(X)$, and let $U \in \mathcal{B}(X)$ be an invertible operator. If $UF = F$, then $U + F \notin \mathcal{D}_n(X)$. Indeed, we have $(U + F)^n = U^n + FV$ for some $V \in \mathcal{B}(X)$. Hence it follows that $\ker((U + F)^n) \subseteq U^{-n}\operatorname{ran}(F)$, and so $\dim\ker((U + F)^n) \leq n - 1$. Consequently, $U + F \notin \mathcal{D}_n(X)$.

Lemma 3.10. *Let $x \in X$, $f \in X^*$ and $x_1, \ldots, x_{n-1} \in \ker(f)$ be linearly independent vectors. Then $f(x) = -1$ if and only if there exist $f_1, \ldots, f_{n-1} \in X^*$ such that*

$$I + x \otimes f + x_1 \otimes f_1 + \cdots + x_{n-1} \otimes f_{n-1} \in \mathcal{D}_n(X).$$

Proof. Suppose that $f(x) = -1$. Let $x_0 = x$, and write $X = \operatorname{Span}\{x_0, \ldots, x_{n-1}\} \oplus Z$ where Z is a closed subspace of $\ker(f)$. Consider the linear forms $f_1, \ldots, f_{n-1} \in Z^\perp$ given by

$$f_i(x_{i-1}) = 1, \qquad f_i(x_i) = -1, \qquad \text{and} \qquad f_i(x_j) = 0 \quad \text{otherwise for } 1 \leq i \leq n-1.$$

If we let $F = x \otimes f + x_1 \otimes f_1 + \cdots + x_{n-1} \otimes f_{n-1}$, then we get easily that $Fx_i = x_{i+1} - x_i$ for $0 \leq i \leq n-2$, and $Fx_{n-1} = -x_{n-1}$. Consequently, $F = (J_n - I_n) \oplus 0$ with respect to the above decomposition. Thus $I + F \in \mathcal{D}_n(X)$ as desired.

Conversely, suppose that $f(x) \neq -1$. Let $f_1, \ldots, f_{n-1} \in X^*$ be arbitrary. Then $U = I + x \otimes f$ is invertible, and $K = x_1 \otimes f_1 + \cdots + x_{n-1} \otimes f_{n-1}$ belongs to $\mathcal{F}_n(X)$. Furthermore, we have $UK = K$, and so $U + K \notin \mathcal{D}_n(X)$ by the previous remark. This finishes the proof. □

Lemma 3.11. *Let $S \in \mathcal{B}(X)$ be such that, for every $T \in \mathcal{D}_n(X)$, there exists $\varepsilon_0 > 0$ such that $T + \varepsilon S \notin \mathcal{D}_n(X)$ for all rational number $0 < \varepsilon < \varepsilon_0$. Then* $\dim\ker(S^n) \leq n - 1$.

Proof. Suppose to the contrary that $\dim\ker(S^n) \geq n$. Then there exist linearly independent vectors x_i, $0 \leq i \leq n - 1$, such that $Sx_0 = 0$ and $Sx_i = \varepsilon_i x_{i-1}$ for $1 \leq i \leq n - 1$ where $\varepsilon_i \in \{0, 1\}$. Write $X = \operatorname{Span}\{x_{n-1}, \ldots, x_0\} \oplus Y$ where Y is a closed subspace, and write

$$S = \begin{bmatrix} S_1 & S_2 \\ 0 & S_3 \end{bmatrix}.$$

Consider also the operator $T \in \mathcal{B}(X)$ represented by the matrix

$$T = \begin{bmatrix} J_n & 0 \\ 0 & I \end{bmatrix}.$$

For $\eta > 0$ choose a rational number $0 < r < \min\{\eta, \|S_3\|^{-1}\}$. Then it follows that $I + rS_3$ is invertible. We have $(J_n + rS_1)x_0 = 0$ and $(J_n + rS_1)x_i = (1 + r\varepsilon_i)x_{i-1}$ for $1 \leq i \leq n-1$, and hence the matrix $J_n + rS_1$ is nilpotent with $\mathrm{i}(J_n + rS_1) = n$. Since

$$T + rS = \begin{bmatrix} J_n + rS_1 & rS_2 \\ 0 & I + rS_3 \end{bmatrix},$$

we get by Lemma 2.8 that $T + rS \in \mathcal{D}_n(X)$, the desired contradiction. □

We note that the set of algebraic operators is invariant under finite rank perturbation (see [3, Proposition 3.6]).

Lemma 3.12. *Let $S \in \mathcal{B}(X)$ be an invertible operator such that $S+F \notin \mathcal{D}_n(X)$ for all $F \in \mathcal{F}_n(X)$. Then $S = \alpha I$, where $\alpha \in \mathbb{C}$ is nonzero.*

Proof. It follows immediately that $\mathrm{M}_x(S) = \emptyset$, and hence $\{S^{-i}x : 1 \le i \le n\}$ is a linearly dependent set for every $x \in X$. Consequently, we obtain by [1, Theorem 4.2.7] that S is an algebraic operator. Suppose to the contrary that S is not a scalar multiple of the identity. Then there exists $x \in X$ such that $S^{-1}x$ and $S^{-2}x$ are linearly independent. It follows from Corollary 3.4 that there exists a rank 1 operator $F \in \mathcal{B}(X)$ such that $S+F$ is the Drazin invertible of index 2. This yields to a contradiction if $n = 2$, and hence we may assume that $n \ge 3$. Since the restriction of F to $\ker(S+F)$ is an injective rank 1 operator, we get easily that $\dim\ker(S+F) = 1$. But we also have $X = \ker(S+F)^2 \oplus \mathrm{ran}(S+F)^2$. Therefore, $\dim\ker(S+F)^2 = 2$. On the other hand, since $S+F$ is algebraic, then so is its restriction to $\mathrm{ran}(S+F)^2$, and hence there exist an $(S+F)$-invariant subspace Y of dimension $n-2$ and a closed subspace Z such that $\mathrm{ran}(S+F)^2 = Y \oplus Z$. With respect to this decomposition, we have

$$(S+F)_{|\,\mathrm{ran}(S+F)^2} = \begin{bmatrix} U & V \\ 0 & C \end{bmatrix},$$

where U and C are invertible operators (see [9, Corollary 8]). Let $\{x_2, x_1\}$ be a basis of $\ker((S+F)^2)$ such that $(S+F)x_2 = x_1$ and $(S+F)x_1 = 0$, and let $\{x_n, \dots, x_3\}$ be an arbitrary basis of Y. Relative to the decomposition $X = \mathrm{Span}\{x_n, \dots, x_1\} \oplus Z$, the operator $S+F$ can be expressed as follows:

$$S+F = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}.$$

Consider now the operator $R \in \mathcal{B}(X)$ represented by

$$R = \begin{bmatrix} J_n - A & 0 \\ 0 & 0 \end{bmatrix}.$$

Clearly, $\dim\mathrm{ran}(R) \le n-2$ because $Rx_2 = Rx_1 = 0$, and so $F + R \in \mathcal{F}_n(X)$. Since C is invertible, $S + F + R \in \mathcal{D}_n(X)$ by Lemma 2.8. This contradiction completes the proof. $\square$

The following lemma generalizes Lemma 3.12.

Lemma 3.13. *Let $S \in \mathcal{B}(X)$ be a Fredholm operator of index zero such that $S + F \notin \mathcal{D}_n(X)$ for all $F \in \mathcal{F}_n(X)$. If $\dim\ker(S^n) \le n-1$, then $S = \alpha I$, where $\alpha \in \mathbb{C}$ is nonzero.*

Proof. Note that since $\dim\ker(S^n) \le n-1$, the operator S has finite ascent $p \le n-1$, and so $\mathrm{a}(S) = \mathrm{d}(S) = p$. In particular, $X = \ker(S^p) \oplus \mathrm{ran}(S^p)$, $S_{\mathrm{o}} = S_{|\ker(S^p)}$ is nilpotent with $\mathrm{i}(S_{\mathrm{o}}) = p$, and $S_1 = S_{|\,\mathrm{ran}(S^p)}$ is invertible.

Let F_1 be a bounded operator on $\mathrm{ran}(S^p)$ with $\dim\mathrm{ran}(F_1) \le n-1$, and let $F = 0 \oplus F_1$ with respect to the above decomposition. Since $F \in \mathcal{F}_n(X)$ and $S+F = S_{\mathrm{o}} \oplus (S_1 + F_1) \notin \mathcal{D}_n(X)$, then $S_1 + F_1$ is not Drazin invertible of index n. It follows from the previous lemma that $S_1 = \alpha I$ for some nonzero $\alpha \in \mathbb{C}$.

To finish, let us show that $\ker(S^p) = \{0\}$. Suppose to the contrary that $\ker(S^p)$ is not trivial. Let $\{y_r, \ldots, y_0\}$ be a basis of $\ker(S^p)$ such that $Sy_0 = 0$, and let $y_{n-1}, \ldots, y_{r+1}$ be linearly independent vectors in $\operatorname{ran}(S^p)$. Write $X = \operatorname{Span}\{y_{n-1}, \ldots, y_0\} \oplus Y$, where Y is a closed subspace of $\operatorname{ran}(S^p)$ and $S = N \oplus \alpha I$. Consider the operator $R \in \mathcal{B}(X)$ given by $R = (J_n - N) \oplus 0$. Then $R \in \mathcal{F}_n(X)$ because $Ry_0 = 0$, and $S + R = J_n \oplus I \in \mathcal{D}_n(X)$, the desired contradiction. □

Proof of Proposition 3.8. Put $S = \Phi(I)$. First, we claim that $\dim\ker(S^n) \leq n-1$. Let $T \in \mathcal{D}_n(X)$, and let $R \in \mathcal{D}_n(X)$ be such that $T = \Phi(R)$. Since 0 is an isolated point of $\sigma(R)$, there exists $\varepsilon_0 > 0$ such that $R + \varepsilon I$ is invertible, and hence $\Phi(R + \varepsilon I) = T + \varepsilon S \notin \mathcal{D}_n(X)$ for all rational numbers $0 < \varepsilon < \varepsilon_0$. Thus, by Lemma 3.11, $\dim\ker(S^n) \leq n - 1$.

Next, let us show that $S + F \notin \mathcal{D}_n(X)$ for all $F \in \mathcal{F}_n(X)$. Since Φ is bijective and preserves rank 1 operators in both directions, it follows that Φ preserves $\mathcal{F}_n(X)$ in both directions. Let $F \in \mathcal{F}_n(X)$. Then there exists $R \in \mathcal{F}_n(X)$ such that $F = \Phi(R)$. Hence $I + R \notin \mathcal{D}_n(X)$ by Remark 3.9, and so $S + F = \Phi(I + R) \notin \mathcal{D}_n(X)$.

Now, according to the previous lemma, it suffices to establish that S is a Fredholm operator of index zero. By Lemma 3.10 there exists a finite rank operator $G \in \mathcal{B}(X)$ such that $I + G \in \mathcal{D}_n(X)$, and hence $\Phi(I + G) = S + K \in \mathcal{D}_n(X)$ where $K = \Phi(G)$ is finite rank. Since $\operatorname{ran}(S + K)^n$ is closed, then so is $\operatorname{ran}(S^n)$. On the other hand, as $\dim\ker(S^n) \leq n - 1$, the operator S, and so $S + K$, is semi-Fredholm. Finally, it follows from Remark 3.2 (3) that $S + K$, and thus S, is a Fredholm operator of index zero. □

Let τ be a field automorphism of $\mathbb{C}$. An additive map $A : X \to X$ will be called *τ-semilinear* if $A(\lambda x) = \tau(\lambda)Ax$ holds for all $x \in X$ and $\lambda \in \mathbb{C}$. Moreover, we simply say that A is *conjugate linear* when τ is the complex conjugation. Notice that if A is nonzero and bounded, then τ is continuous, and, consequently, τ is either the identity or the complex conjugation (see [12, Lemma 14.5.1]). Moreover, in this case, the adjoint operator $A^* : X^* \to X^*$ defined by

$$A^*(g) = \tau^{-1} \circ g \circ A \quad \text{for all } g \in X^*$$

is again τ-semilinear.

Lemma 3.14. *Let $\Phi : \mathcal{B}(X) \to \mathcal{B}(X)$ be an additive surjective map preserving $\mathcal{D}_n(X)$ in both directions. Then there exists a nonzero $\alpha \in \mathbb{C}$ such that $\Phi(I) = \alpha I$, and either*

(1) *there exists a bounded invertible linear, or conjugate linear, operator $A : X \to X$ such that $\Phi(F) = \alpha AFA^{-1}$ for all finite rank operators $F \in \mathcal{B}(X)$, or*
(2) *there exists a bounded invertible linear, or conjugate linear, operator $B : X^* \to X$ such that $\Phi(F) = \alpha BF^*B^{-1}$ for all finite rank operators $F \in \mathcal{B}(X)$. In this case, X is reflexive.*

Proof. The existence of a nonzero $\alpha \in \mathbb{C}$ such that $\Phi(I) = \alpha I$ is ensured by Proposition 3.8. Clearly, we can suppose without loss of generality that $\Phi(I) = I$. Since Φ is bijective and preserves the set of rank 1 operators in both directions (compare Lemma 3.1), then, by [20, Theorem 3.3], there exist a ring automorphism $\tau : \mathbb{C} \to \mathbb{C}$ and either two bijective τ-semilinear mappings $A : X \to X$ and $C : X^* \to X^*$ such that

$$\Phi(x \otimes f) = Ax \otimes Cf \quad \text{for all } x \in X \text{ and } f \in X^*, \tag{3.4}$$

or two bijective τ-semilinear mappings $B : X^* \to X$ and $D : X \to X^*$ such that

$$\Phi(x \otimes f) = Bf \otimes Dx \quad \text{for all } x \in X \text{ and } f \in X^*. \tag{3.5}$$

Suppose that Φ satisfies (3.4), and let us show that

$$C(f)(Ax) = \tau\big(f(x)\big) \quad \text{for all } x \in X \text{ and } f \in X^*. \tag{3.6}$$

Clearly, it suffices to establish that, for all $x \in X$ and $f \in X^*$,

$$f(x) = -1 \qquad \text{if and only if} \qquad C(f)(Ax) = -1.$$

Let $x \in X$, and let $f \in X^*$. Consider arbitrary linearly independent vectors z_i, $1 \le i \le n-1$, in $\ker(f) \cap \ker(C(f)A)$. Then it follows from Lemma 3.10 that

$$\begin{aligned} f(x) = -1 \ &\Leftrightarrow \ \exists \{g_i\}_{i=1}^{n-1} \subseteq X^* : I + x \otimes f + \sum_{i=1}^{n-1} z_i \otimes g_i \in \mathcal{D}_n(X) \\ &\Leftrightarrow \ \exists \{g_i\}_{i=1}^{n-1} \subseteq X^* : I + Ax \otimes Cf + \sum_{i=1}^{n-1} Az_i \otimes Cg_i \in \mathcal{D}_n(X) \\ &\Leftrightarrow \ C(f)(Ax) = -1. \end{aligned}$$

Thus equation (3.6) holds, and, arguing as in [20, p. 252], we get that τ, A, C are continuous, τ is the identity or the complex conjugation, and $C = (A^{-1})^*$. Therefore, $\tau^{-1} = \tau$ and, for every $u \in X$, we have

$$\Phi(x \otimes f)u = \tau(fA^{-1}u)Ax = A\big(f(A^{-1}u)x\big) = A(x \otimes f)A^{-1}u.$$

Thus $\Phi(x \otimes f) = A(x \otimes f)A^{-1}$ for all $x \in X$ and $f \in X^*$; that is, $\Phi(F) = AFA^{-1}$ for all finite rank operators $F \in \mathcal{B}(X)$.

Now, suppose that Φ satisfies (3.5), and let us show that

$$D(x)(Bf) = \tau\big(f(x)\big) \quad \text{for all } x \in X \text{ and } f \in X^*. \tag{3.7}$$

Let $x \in X$, and let $f \in X^*$. Let $h_1, \dots, h_{n-1} \in X^*$ be linearly independent linear forms such that $h_i(x) = (D(x)B)(h_i) = 0$ for $1 \le i \le n-1$. Then, using the fact that D is bijective, it follows from Lemma 3.10 that

$$\begin{aligned} D(x)(Bf) = -1 \ &\Leftrightarrow \ \exists \{u_i\}_{i=1}^{n-1} \subseteq X : I + Bf \otimes Dx + \sum_{i=1}^{n-1} Bh_i \otimes Du_i \in \mathcal{D}_n(X) \\ &\Leftrightarrow \ \exists \{u_i\}_{i=1}^{n-1} \subseteq X : I + x \otimes f + \sum_{i=1}^{n-1} u_i \otimes h_i \in \mathcal{D}_n(X) \\ &\Leftrightarrow \ \exists \{u_i\}_{i=1}^{n-1} \subseteq X : I + f \otimes \mathrm{J}x + \sum_{i=1}^{n-1} h_i \otimes \mathrm{J}u_i \in \mathcal{D}_n(X^*) \\ &\Leftrightarrow \ f(x) = -1, \end{aligned}$$

where $\mathrm{J} : X \to X^{**}$ is the natural embedding. Thus equation (3.7) holds, and, arguing as in [20, p. 252], we get that τ, B, D are continuous, τ is the identity or the complex conjugation, and $D = (B^{-1})^*\mathrm{J}$. But the operators D and $(B^{-1})^*$, and therefore also J, are bijections which imply the reflexivity of X. Furthermore, $\tau^{-1} = \tau$ and, for every $u \in X$, we have

$$\begin{aligned}\Phi(x \otimes f)u &= \big(Bf \otimes (B^{-1})^*\mathrm{J}(x)\big)u = (B^{-1})^*\mathrm{J}(x)(u) \cdot Bf \\ &= \tau\big(\mathrm{J}(x)(B^{-1}u)\big) \cdot Bf = B\big(\mathrm{J}(x)(B^{-1}u)f\big) \\ &= B\big(f \otimes \mathrm{J}(x)\big)B^{-1}u = B(x \otimes f)^*B^{-1}u.\end{aligned}$$

Thus $\Phi(x\otimes f) = B(x\otimes f)^*B^{-1}$ for all $x \in X$ and $f \in X^*$. Hence $\Phi(F) = BF^*B^{-1}$ for all finite rank operators $F \in \mathcal{B}(X)$. This completes the proof. $\square$

Proof of Theorem 1.1. The "if" part is obvious. We prove the "only if" part. Suppose that Φ preserves $\mathcal{D}_n(X)$ in both directions. It follows that Φ takes one of the two forms in Lemma 3.14.

Assume that $\Phi(F) = \alpha AFA^{-1}$ for all finite rank operators $F \in \mathcal{B}(X)$. Let

$$\Psi(T) = \alpha^{-1}A^{-1}\Phi(T)A \quad \text{for all } T \in \mathcal{B}(X).$$

Clearly, Ψ satisfies the same properties as Φ. Furthermore, $\Psi(I) = I$, and $\Psi(F) = F$ for all finite rank operators $F \in \mathcal{B}(X)$. Let $T \in \mathcal{B}(X)$, and choose a rational number λ such that $T - \lambda$ and $\Psi(T) - \lambda$ are invertible. We have

$$T - \lambda + F \in \mathcal{D}_n(X) \;\Leftrightarrow\; \Psi(T - \lambda) + F \in \mathcal{D}_n(X)$$

for all finite rank operators $F \in \mathcal{B}(X)$. Hence we get by Proposition 3.6 that $\Psi(T) = T$. This shows that $\Phi(T) = \alpha ATA^{-1}$ for all $T \in \mathcal{B}(X)$.

Now suppose that $\Phi(F) = \alpha BF^*B^{-1}$ for all finite rank operators $F \in \mathcal{B}(X)$. Then Lemma 3.14 ensures that X is reflexive. By considering

$$\Gamma(T) = \alpha^{-1}\mathrm{J}^{-1}\big(B^{-1}\Phi(T)B\big)^*\mathrm{J} \quad \text{for all } T \in \mathcal{B}(X),$$

we get in a similar way that $\Gamma(T) = T$ for all $T \in \mathcal{B}(X)$. Thus $\Phi(T) = \alpha BT^*B^{-1}$ for all $T \in \mathcal{B}(X)$, as desired. $\square$

Proof of Corollary 1.2. The "if" part is obvious. We prove the "only if" part. Suppose that $\mathcal{P}_n(\Phi(T)) = \mathcal{P}_n(T)$ for all $T \in \mathcal{B}(X)$. Clearly, Φ preserves $\mathcal{D}_n(X)$ in both directions, and so one of the two assertions in Theorem 1.1 holds. To show that the constant $\alpha = 1$, consider an arbitrary $F \in \mathcal{D}_n(X)$ of rank less than n, and let $T = F + I$. It follows from Remark 3.9 that

$$\{1\} = \mathcal{P}_n(T) = \mathcal{P}_n\big(\Phi(T)\big) = \{\alpha\}.$$

This completes the proof. $\square$

We close this article by the following remarks.

Remark 3.15. Let X and Y be infinite-dimensional complex Banach spaces. Theorem 1.1 can be without any change formulated for additive surjective mappings $\Phi : \mathcal{B}(X) \to \mathcal{B}(Y)$, preserving Drazin invertible operators of index n in both directions.

Remark 3.16. Combining Theorem 1.1 and [15, Theorem 1.1], we obtain a complete characterization of additive surjective mappings $\Phi : \mathcal{B}(X) \to \mathcal{B}(X)$, preserving Drazin invertible operators of index $m \geq 1$ in both directions.

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