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NEW RESULTS ON KOTTMAN'S CONSTANT

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Abstract. We present new results on Kottman's constant of a Banach space, showing (i) that every Banach space is isometric to a hyperplane of a Banach space having Kottman's constant 2 and (ii) that Kottman's constant of a Banach space and of its bidual can be different. We say that a Banach space is a Diestel space if the infimum of Kottman's constants of its subspaces is greater that 1. We show that every Banach space contains a Diestel subspace and that minimal Banach spaces are Diestel spaces.

1. Introduction

In this article, we continue our study [7] of Kottman's constant of a Banach space

$$K(X)=\sup\{\sigma>0:\exists (x_n)_{n\in\mathbb{N}}\in B_X:\forall n\neq m, \|x_n-x_m\|\geq\sigma\}.$$

Following Kottman [18, Section 3], we also define the *isomorphic* Kottman's constant

$$\tilde{K}(X)=\inf\{K(\tilde{X}):\tilde{X}\simeq X\},$$

where the infimum is taken over all renormings (or isomorphic copies) of X. In what follows, a *subspace* means an infinite-dimensional closed subspace and $\tilde{X}\simeq X$ means that X and $\tilde{X}$ are isomorphic. For our purposes, it is better to view an isomorphism like the one above as a renorming of X. A λ-*renorming* of $(X,\|\cdot\|)$ means a new norm $r(\cdot)$ on X so that $\lambda^{-1}r(\cdot)\leq\|\cdot\|\leq\lambda r(\cdot)$. Of course, every equivalent renorming is a λ-renorming for some $\lambda>0$. We will consider

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the following three groups of problems about K and $\tilde{K}$: Diestel's problem (see [10]), 3-space-like problems for $\tilde{K}$ and stability properties of K on hyperplanes, and the bidual problem (see [7], [8]).

Kottman's constant was introduced and studied in [18], [19]. It is clear that $K(X) = 0$ if and only if X is finite-dimensional; the exact value of Kottman's constant for different classical Banach spaces has been computed in several papers. A well-known but highly nontrivial result of Elton and Odell [12] (see also [10]) establishes that $K(X) > 1$ for every infinite-dimensional Banach space. A previous combinatorial result of Kottman [18] showed that there always exists in the unit ball of an infinite-dimensional Banach space a sequence of elements (x_n) such that $\|x_n - x_m\| > 1$.

2. Diestel's problem

We do not known whether it is possible to have $\tilde{K}(X) = 1$. If we consider the constant $\tilde{K}_\lambda(X) = \inf\{K(X_r)\}$, where the infimum is taken over all λ-renormings of X, a formal application of the Elton–Odell theorem yields this proposition.

Proposition 2.1. *For every space X and every λ, we have $\tilde{K}_\lambda(X) > 1$.*

Proof. Let r_n be a λ-renorming of X for which $K(X_{r_n}) \leq 1 + 1/n$. Let $\mathscr{U}$ be a free ultrafilter on $\mathbb{N}$, and renorm X with $\|x\| = \lim_{\mathscr{U}} r_n(x)$. By the Elton–Odell theorem, $K(X, \|\cdot\|) = 1 + \alpha$ for some $\alpha > 0$, and thus let (x_k) be a sequence of norm 1 elements of $(X, \|\cdot\|)$ for which $\|x_i - x_j\| \geq 1 + \alpha/2$ when $i \neq j$. Recall that $\lim_{\mathscr{U}} r_n(x) = p$ means that for all $\varepsilon > 0$, $\{n : |r_n(x) - p| \leq \varepsilon\} \in \mathscr{U}$. Thus, $r_n(x_k) \leq 1 + \alpha/4$ for all k in some $U \in \mathscr{U}$. Passing to subsequences, diagonalizing and then relabeling, we can assume that $r_n(x_k) \leq 1 + \alpha/4$ for all $k \geq n$. Therefore $r_n(x_i - x_j) \leq (1 + 1/n)(1 + \alpha/4)$, and thus $\lim_{\mathscr{U}} r_n(x_i - x_j) \leq 1 + \alpha/4$, which is in contradiction with the choice of (x_n). ☐

Following [23, p. 229], we set $s(X) = \inf\{K(Y) : Y \subset X\}$, and we set its isomorphic analogue

$$\tilde{s}(X) = \inf\bigl\{\tilde{K}(Y) : Y \subset X\bigr\} = \inf\bigl\{K(\tilde{Y}) : \tilde{Y} \simeq Y \subset X\bigr\}.$$

It is also clear that $s(X) \geq \tilde{s}(X)$. In [10, Problems, p. 254], Diestel posed the problems of characterizing those Banach spaces X for which $s(X) > 1$ or $\tilde{s}(X) > 1$. These problems were considered by Prus in [23], where he introduced the following parameter $\lambda(X)$ for non-Schur spaces with the purpose of obtaining the estimate $\lambda(X) \leq \tilde{s}(X)$: if $\mathcal{N}$ denotes the set of sequences (x_n) of norm 1 elements of X such that

$$L(\alpha_1, \dots, \alpha_m) = \lim_{\substack{n_1 < \cdots < n_m \\ n_1 \to \infty}} \Bigl\| \sum_1^m \alpha_i x_{n_i} \Bigr\|$$

exists for all scalars $\alpha_1, \dots, \alpha_m$, and $\mathcal{N}_1$ the subset of weakly null sequences in $\mathcal{N}$, then

$$\lambda(X) = \inf_{\mathcal{N}_1} \limsup_{m \to \infty} \lim_{\substack{n_1 < \cdots < n_{2^m} \\ n_1 \to \infty}} \Bigl\| \sum_1^{2^m} x_{n_i} \Bigr\|^{1/m}.$$

We can reformulate this in the language of *spreading models*. Recall that, given a sequence $(x_n) \in \mathcal{N}$, the spreading model μ_x generated by the sequence $x = (x_n)$ is the completion of the space of finitely supported sequences endowed with the norm

$$\Big\|\sum_{i=1}^{i=m} \alpha_i e_i\Big\| = L(\alpha_1, \dots, \alpha_m),$$

where e_i denotes the scalar sequence having only a 1 in position i and 0 elsewhere. In this language

$$\lambda(X) = \inf_{\mathcal{N}_1} \limsup_m \Big\|\sum_1^{2^m} e_n\Big\|^{1/m},$$

where the infimum is taken over all spreading models generated by sequences of $\mathcal{N}_1$. Let us introduce a variation of $\lambda(\cdot)$ able to provide a lower bound for $\tilde{K}(X)$. We set

$$\kappa(X) = \sup_{\mathcal{N}_1} \limsup_m \Big\|\sum_1^{2^m} e_n\Big\|^{1/m}.$$

Let us also consider the parameter $\mathfrak{b}(X) = \inf_{\mathcal{N}_1} \|e_1 - e_2\|$. The parameters have certainly been introduced to work on non-Schur spaces (i.e., spaces admitting weakly null normalized sequences). One the other hand, a Schur space X is hereditarily ℓ_1, and therefore $\tilde{s}(X) = 2$.

Proposition 2.2. *We have $\mathfrak{b}(X) \le \lambda(X) \le \tilde{s}(X)$ and $\lambda(X) \le \kappa(X) \le \tilde{K}(X)$.*

Proof. We show first that $\kappa(X) \le K(X)$. It is easy to observe that $K(X) = \sup_{\mathcal{N}} \|e_1 - e_2\| \ge \sup_{\mathcal{N}_1} \|e_1 - e_2\|$. Since

$$\|e_1 - e_2 + e_3 - e_4\| = \|e_1 - e_2\| \Big\| \frac{e_1 - e_4}{\|e_1 - e_4\|} - \frac{e_2 - e_3}{\|e_2 - e_3\|} \Big\|$$

it follows that

$$\sup_{\mathcal{N}_1} \|e_1 - e_2 + e_3 - e_4\| \le K(X)^2.$$

When the starting sequence is weakly null, the spreading model sequence (e_n) is unconditional with unconditional constant at most 3 (see [3, Lemma 2]), and thus

$$\sup_{\mathcal{N}_1} \|e_1 + e_2 + e_3 + e_4\|^{1/2} \le 3^{1/2} K(X).$$

Analogously,

$$\sup_{\mathcal{N}_1} \Big\|\sum_1^{2^m} e_n\Big\|^{1/m} \le 3^{1/m} K(X),$$

from which the estimate $\kappa(X) \le K(X)$ follows. Since κ is invariant under renorming, we also get $\kappa(X) \le \tilde{K}(X)$.

We show now that $\mathfrak{b}(X) \le \lambda(X)$. For a given $\varepsilon > 0$, let $(z_n) \in \mathcal{N}_1$ be a sequence producing an almost optimal spreading (b_n) in the sense that $\|b_1 - b_2\| \le \|e_1 - e_2\| + \varepsilon$ for any other spreading (e_n) produced by a sequence $(x_n) \in \mathcal{N}_1$.

We can assume without loss of generality that $\|b_1 - b_2\| \leq \|e_1 - e_2\|$ to avoid a cumbersome ε. Since $\|e_i - e_j\| = \|e_1 - e_2\|$, we have

$$\begin{aligned}\|e_1 - e_2 + e_3 - e_4\| &= \|e_1 - e_2\| \Big\| \frac{e_1 - e_4}{\|e_1 - e_4\|} - \frac{e_2 - e_3}{\|e_2 - e_3\|} \Big\| \\ &\geq \|b_1 - b_2\| \Big\| \frac{e_1 - e_4}{\|e_1 - e_4\|} - \frac{e_2 - e_3}{\|e_2 - e_3\|} \Big\|.\end{aligned}$$

Now let $y_j = \frac{x_j - x_{j+1}}{\|x_j - x_{j+1}\|}$. The sequence $(y_{2n+1}) \in \mathcal{N}_1$ (or some subsequence) produces a spreading model $(\imath_n)$, and thus

$$\Big\| \frac{e_1 - e_4}{\|e_1 - e_4\|} - \frac{e_2 - e_3}{\|e_2 - e_3\|} \Big\| = \|\imath_1 - \imath_2\| \geq \|b_1 - b_2\|,$$

which yields $\|e_1 - e_2 + e_3 - e_4\| \geq \|b_1 - b_2\|^2$. By iteration, we get

$$\Big\| \sum_{n=1}^{2^m} (-1)^{n+1} e_n \Big\| \geq \|b_1 - b_2\|^m,$$

and hence

$$\limsup_{m\to\infty} \lim_{\substack{n_1 < \cdots < n_{2^m} \\ n_1 \to \infty}} \Big\| \sum_{i=1}^{2^m} (-1)^{i+1} x_{n_i} \Big\|^{1/m} \geq \|b_1 - b_2\|$$

which immediately yields $\lambda(X) \geq \mathfrak{b}(X)$. The fact that $\lambda(X) \leq \kappa(X)$ is obvious. Now, if $Y \subset X$, then $\lambda(X) \leq \lambda(Y)$ while $K(Y) \leq K(X)$. Therefore, we have $\lambda(X) \leq s(X)$. Since $\lambda(\cdot)$ is invariant under renorming (see [23, p. 229]), we get $\lambda(X) \leq \tilde{s}(X)$, which is precisely Prus's result (see [23, Theorem 3]). $\square$

Definition 2.3. A Banach space X will be called a *Diestel space* if $s(X) > 1$.

Among other results, Prus shows in [23] that super-reflexive spaces verify $\lambda(X) > 1$, and therefore they are Diestel spaces. From the results of Prus, it follows that $K(\ell_p) = \tilde{K}(\ell_p) = \tilde{s}(\ell_p) = 2^{1/p}$ for $1 \leq p < +\infty$, while clearly $K(c_0) = \tilde{K}(c_0) = \tilde{s}(c_0) = 2 = K(\ell_1) = \tilde{K}(\ell_1) = \tilde{s}(\ell_1)$. So all of them are Diestel spaces. Let us show that Diestel spaces are ubiquitous, as follows.

Lemma 2.4. *Every infinite-dimensional Banach space X contains an infinite-dimensional subspace X_0 so that $\tilde{s}(X_0) = \tilde{K}(X_0)$ and also contains an infinite-dimensional subspace X_1 so that $s(X_1) = K(X_1)$.*

Proof. We set the real intervals as

$$\begin{aligned}\sigma(X) &= \big[s(X), K(X)\big] \subset [1,2], \\ \tilde{\sigma}(X) &= \big[\tilde{s}(X), \tilde{K}(X)\big] \subset [1,2].\end{aligned}$$

When $A \subset B$, then $K(A) \leq K(B)$ and $s(A) \geq s(B)$ and thus $\sigma(A) \subset \sigma(B)$. Let us now show that also $\tilde{K}(A) \leq \tilde{K}(B)$ and $\tilde{s}(A) \geq \tilde{s}(B)$ and thus that $\tilde{\sigma}(A) \subset \tilde{\sigma}(B)$. Set the following order on the set $S_\infty(X)$ of all infinite-dimensional subspaces of X: $A \leq B$ if there is a finite-dimensional space F such that $(A \cap B) + F = A$. Since $K(F) = 0$ for F finite-dimensional, it also holds that $A \leq B$ implies that $\sigma(A) \subset \sigma(B)$. Let us also show that $\tilde{\sigma}(A) \subset \tilde{\sigma}(B)$. If $A \subset B$, then

$K(A) \le K(B)$, and if $f : B \to B^f$ is an isomorphism, then $f(A) = A^f \subset B^f$ is an isomorphic copy of A, and thus $s(A^f) \ge s(B^f)$; hence $\inf_f\{s(A^f)\} \ge \inf_f s(B^f) = \tilde{s}(B)$. Thus we are done once we show that $\tilde{s}(A) = \inf_f\{s(A^f)\}$; for that, we need to show that every isomorphism $g : A \to A^g$ can be extended to an isomorphism $f : B \to B^f$. This follows from the existence of the pushout construction (for detailed information, see [2]). What we need here is to recall that, given g, i, the pushout space PO provides a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\ i\ } & B \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle g'} \\ A^g & \xrightarrow[\ i'\ ]{} & PO \end{array}$$

The space PO is defined as the quotient $(A^g \oplus_1 B)/C$, where $C = \{(ga, -ia) : a \in A\}$, and the operators g', i' are naturally defined as $g'(x) = (x, 0) + C$ and $i'(y) = (0, y) + C$. Since i is the canonical isometric embedding, i' is also an isometric embedding; since g is an isomorphism, the operator g' is also an isomorphism.

Thus, σ and $\tilde{\sigma}$ are order-continuous maps from the ordered set $(S_\infty(X), \le)$ to the ordered set of all compact subsets of $[0, 2]$ in its natural order given by $\subset$. The set $(S(X), \le)$ is "σ-grounded" (using the term from [4]) in the sense that every countable chain $A_1 \ge A_2 \ge A_3 \cdots$ admits a lower bound: take $a_n \in A_n$ and set $A = [a_n]$, which clearly verifies $A \le A_n$ for all n. Thus, by Behrends [4], there must be a point X_0 (resp., X_1) where $\tilde{\sigma}$ (resp., σ) stabilizes; that is, $\tilde{\sigma}(X_0) = \tilde{\sigma}(Z)$ for all $Z \le X_0$ (resp., $\sigma(X_1) = \sigma(Z)$ for all $Z \le X_1$).

Get this X_0. Thus, for every subspace $Y_0 \subset X_0$, we have $\tilde{s}(X_0) = \tilde{s}(Y_0)$ and $\tilde{K}(X_0) = \tilde{K}(Y_0)$. If $\tilde{s}(X_0) < \tilde{K}(X_0)$, then pick $\varepsilon > 0$ so that $\tilde{s}(X_0) + \varepsilon < \tilde{K}(X_0) - \varepsilon$, and then pick a subspace $Y_0 \subset X_0$ and an isomorphism f of Y_0 so that $K(Y_0^f) \le \tilde{s}(X_0) + \varepsilon < \tilde{K}(X_0) - \varepsilon$. In these conditions,

$$\tilde{\sigma}(Y_0) = \left[\tilde{s}(Y_0), \tilde{K}(Y_0)\right] \subset \left[\tilde{s}(Y_0), \tilde{K}(X_0) - \varepsilon\right] \subsetneq \left[\tilde{s}(X_0), \tilde{K}(X_0)\right] = \tilde{\sigma}(X_0)$$

contrarily to our assumption. Thus, $\tilde{s}(X_0) = \tilde{K}(X_0)$. Reasoning as above with $\sigma(\cdot)$ and X_1, we obtain the proof of the second assertion. □

Since $K(X_1) > 1$, we get the following. (See Problem (3) at the end of this article.)

Proposition 2.5. *Every Banach space contains an infinite-dimensional Diestel subspace.*

Fonf and Zanco showed in [15, Theorem 2.1, Corollary 2.2] that any Kottman sequence in a Banach space X (i.e., a sequence (x_n) so that $\liminf \|x_n - x_m\| = K(X)$) must contain a subsequence whose span is infinite-codimensional. In our case, this means that every Banach space contains infinite-codimensional subspaces having the same Kottman constant as the whole space.

Recall that a Banach space X is considered minimal if every subspace contains an isomorphic copy of X. It is considered C-minimal if every closed subspace contains a C-isomorphic copy of X. Ferenczi and Rosendal show in [14] that minimal implies C-minimal for some $C > 1$. We have the following.

Proposition 2.6. *A minimal space is a Diestel space.*

Proof. Let X be minimal. Hence, it must contain a (minimal) subspace X_0 for which

$$\tilde{\sigma}(X) = \tilde{\sigma}(X_0) = \{\tilde{K}(X_0)\} = \{\tilde{K}(X)\}.$$

Since minimal means C-minimal for some C, only C-renormings have to be considered; hence, by Proposition 2.1, it holds that $\tilde{s}(X) = \tilde{K}(X) > 1$. □

3. Kottman's constant of twisted Hilbert spaces

A twisted sum of two Banach spaces Y and Z is a Banach space X such that it admits a subspace Y so that the corresponding quotient X/Y is isomorphic to Z. In other words, the middle space X in an exact sequence $0 \to Y \xrightarrow{j} X \xrightarrow{q} Z \to 0$ (i.e., a diagram formed by spaces and operators with the additional property that the kernel of each arrow coincides with the image of the preceding one). Since Kottman's constant is an isometric notion, trying to estimate the Kottman constant of a space X only knowing those of a subspace Y of X and of the corresponding quotient X/Y is an ill-posed problem. Moreover, recall that every Banach space, in particular the twisted sum space X, can always be renormed to have Kottman's constant 2 (see [18]). This suggests a couple of complementary approaches, as follows.

Definition 3.1. An exact sequence $0 \to Y \xrightarrow{j} X \xrightarrow{q} Z \to 0$ will be called an *isometric exact sequence* if j is an injective isometry and the image by q of the unit open ball of X is the unit open ball of Z. A renorming r of a twisted sum space X will be called an *exact renorming* if $0 \to Y \xrightarrow{j} (X, r) \xrightarrow{q} Z \to 0$ is an isometric exact sequence.

Recall that a Banach space is called a *twisted Hilbert space* if there is an isometric exact sequence $0 \to \ell_2 \to X \to \ell_2 \to 0$. A twisted Hilbert space is regarded as trivial if it is isomorphic to a Hilbert space. The most important twisted Hilbert space is known as the *Kalton–Peck Z_2 space*. Let us briefly recall its construction. With the aid of the so-called *z-linear map* $\Omega : \ell_2 \to \ell_2$ defined on the finitely supported sequences as

$$\Omega(x) = x \log \frac{\|x\|_2}{|x|}$$

(with the meaning that $\Omega(x)(n) = x(n) \log \frac{\|x\|_2}{|x(n)|}$ and the understanding that $\log 0 = 0$), we get the quasinorm $\|(y,z)\|_\Omega = \|y - \Omega(z)\| + \|z\|$ on $\ell_2 \times \ell_2$. This quasinorm is actually equivalent to the norm having as unit ball the closed convex hull of the points $(y, 0)$ with $\|y\| = 1$ and $(\Omega z, z)$ with $\|z\| = 1$. This is the Kalton–Peck twisted Hilbert space Z_2 [16]. We have the following proposition.

Proposition 3.2.

(1) *There is an exact renorming r of Z_2 for which $K([Z_2, r]) > \sqrt{2}$.*

(2) *There is a nontrivial twisted Hilbert space E_2 for which $K(E_2) = \sqrt{2}$.*
(3) *There is an isometric sequence $0 \to \ell_2 \to \ell_2^R \to \ell_2 \to 0$ in which $K(\ell_2^R) = 2$.*

Proof. The associated exact sequence $0 \to \ell_2 \to Z_2 \to \ell_2 \to 0$ has the embedding $y \to (y, 0)$ and the quotient map $(y, z) \to z$. The quasinorm $\|\cdot\|_\Omega$ is exact although the norm having unit ball B may be not be exact. To make $y \to (y, 0)$ an isometric embedding, we need to replace Ω by $Z_0^{-1}\Omega$, where $Z_0 = \sup\{\|\sum \Omega(z_i)\|\}$ and where the supremum is taken over all finite sets $z_1, \ldots, z_n$ so that $\sum z_i = 0$ and $\sum \|z_i\| \le 1$. The norm $\|\cdot\|_c$, whose unit ball is the closed convex hull B_c of the points $(y, 0)$ with $\|y\| = 1$ and $(Z_0^{-1}\Omega z, z)$ with $\|z\| = 1$, yields an exact renorming of Z_2: indeed, that $B_Y \subset B_c$ is in the definition; now let $\|(y, 0)\|_c \le 1$, which means that

$$y = \sum_i \theta_i y_i + \sum_j \theta_j Z_0^{-1}\Omega z_j,$$
$$0 = \sum_i \theta_j z_j.$$

Thus, $\|\sum_i \theta_i y_i + \sum_j \theta_j Z_0^{-1}\Omega z_j\| \le \sum_i \theta_i + \sum_j \theta_j = 1$.

Let us now show that Kottman's constant of Z_2 renormed with $\|\cdot\|_c$ is strictly greater than $\sqrt{2}$. We will do this by showing that the sequence points $(0, e_n) \in B_c$ verify $\inf_{n,m} \|(0, e_n) - (0, e_m)\| > \sqrt{2}$. That $(0, e_n) \in B_c$ is clear, since $\Omega(e_n) = 0$. Fix n, m, and assume that $\|(0, e_n - e_m))\| = \|(0, e_n) - (0, e_m)\| = \sqrt{2}$. This means that there is a convex combination $\sum_i \theta_i + \sum_j \theta_j = 1$ and points $\|y_i\| = 1$ in ℓ_2 and $\|z_i\| = 1$ in ℓ_2 such that

$$0 = \sqrt{2}\Big(\sum_i \theta_i y_i + \sum_j \theta_j Z_0^{-1}\Omega z_j\Big),$$
$$e_n - e_m = \sqrt{2}\sum_j \theta_j z_j.$$

Since $\|e_n - e_m\| = \sqrt{2}$ and all points in the unit sphere of ℓ_2 are extreme points, necessarily $\sum_j \theta_j = 1$ and all z_j are just $\sqrt{2}^{-1}(e_n - e_m)$. Therefore, $\sum_i \theta_i = 0$ and consequently

$$0 = \sqrt{2}\Big(\sum_j \theta_j Z_0^{-1}\Omega z_j\Big) = Z_0^{-1}\Omega(e_n - e_m),$$

which is false. That $\inf_{n,m} \|(0, e_n) - (0, e_m)\| > \sqrt{2}$ is a consequence of the fact that Ω is symmetric, in the sense that, given any isometry $\widehat{\sigma}$ of ℓ_2 induced by a permutation σ of $\mathbb{N}$, we have $\Omega(e_{\sigma(n)} - e_{\sigma(m)}) = \widehat{\sigma}\Omega(e_n - e_m)$; this means that $\|(0, e_n - e_m)\| = \|e_1 - e_2\|$ for all n, m.

The second assertion is a consequence of the Enflo–Lindenstrauss–Pisier construction of a twisted sum E of Hilbert spaces (see [13]) having the form $E = \ell_2(W_n)$, where W_n are finite-dimensional spaces, plus the estimate [7]

$$K\big(\lambda(X_n)\big) = \max\big\{K(\lambda), K(X_n)\big\}$$

for a λ-vector sum of spaces X_n when λ has an unconditional basis.

The third assertion is a consequence of our Theorem 4.2 below, and more precisely, of the Naidu–Sastry construction in [22] of a renorming N of $\ell_2 \times \mathbb{R}$ having Kottman's constant 2 and such that $N_{|\ell_2} = \|\cdot\|_2$. This yields an isometric exact sequence $0 \to \ell_2 \to (\ell_2 \oplus_N \mathbb{R}) \to \mathbb{R} \to 0$ and therefore $0 \to \ell_2(\ell_2) \to \ell_2(\ell_2 \oplus_N \mathbb{R}) \to \ell_2(\mathbb{R}) \to 0$ or else $0 \to \ell_2 \to (\ell_2 \oplus_N \mathbb{R}) \oplus_2 \ell_2 \to \mathbb{R} \oplus_2 \ell_2 \to 0$ to yield the desired exact renorming. $\square$

Kalton showed in [17, Proposition 5.8] (see also [7, Proposition 4.1]) that $K(X)$ is the infimum of the λ such that every norm 1 operator (equivalently, every Lipschitz map with Lipschitz constant 1) defined on a subspace of X with values on c_0 can be extended to the whole X with norm λ (resp., with Lipschitz constant λ). Therefore, there are norm 1 c_0-valued operators on some subspace of Z_2 that cannot be extended to the whole space with norm $\sqrt{2}$. Of course, by Sobczyk's theorem all such operators can be extended with norm 2. Moreover, Kalton shows in [17, Theorem 7.12] that all $C[0,1]$-valued operators defined on subspaces of Z_2 can be extended to the whole Z_2.

4. Kottman's constant on hyperplanes

The behavior of Kottman's constant on hyperplanes is an intriguing topic. In [7, Lemma 1.2] (see also [9, Claim 4.4]), the following appears.

Lemma 4.1. *If E is a c-complemented finite-codimensional subspace of X, then $K(E) \geq K(X)/c$.*

Thus, Kottman's constant of a Banach space and those of its 1-complemented hyperplanes coincide. Brown in [6, Theorem 1.3] claims a (wrong) proof for the fact that the Kottman's constant of a Banach space and all its hyperplanes must coincide. This is, however, wrong, since Naidu and Sastry show in [22] that there is a norm N on $\ell_2 \oplus \mathbb{R}$ that induces the original norm on ℓ_2 for which $K(\ell_2 \oplus_{\mathbb{N}} \mathbb{R}) = 2$. A refinement of their proof shows the following.

Theorem 4.2. *Every Banach space is isometric to a hyperplane of a Banach space with Kottman's constant* 2.

Proof. Recall the well-known fact [11, Lemma 3.4] that it is possible to choose in the unit ball of X an infinite set $\{x_1, \dots, x_n, \dots\}$ such that we then have $K(X) - \varepsilon \leq \|x_i - x_j\| \leq K(X) + \varepsilon$ for every pair $i, j \in N$ $(i \neq j)$. Set in $X \oplus \mathbb{R}$ the norm r whose unit ball is the absolutely convex hull $\mathcal{B}$ of

$$\big\{(x,0) : \|x\| \leq 1\big\} \cup \big\{(\mu x_n, 1)\big\} \cup (0,1),$$

where $\mu = 2\max\|x_n - x_m\|^{-1}$. The key point is to show that if B_X denotes the unit ball of X (seen as the canonical copy $X \times 0$), then $\mathcal{B} \cap X = B_X$. After that, it is clear that $K((X \oplus \mathbb{R}, r)) = 2$, since for $i \neq j$ we have

$$\begin{aligned} r\big((\mu x_i, 1) - (\mu x_j, 1)\big) &= r\big((\mu(x_i - x_j), 0)\big) = \big\|\mu(x_i - x_j)\big\| \\ &\geq \frac{2}{\max\|x_n - x_m\|}\big(K(X) - \varepsilon\big) \\ &\geq 2\frac{K(X) - \varepsilon}{K(X) + \varepsilon}. \end{aligned}$$

Thus let $(x,0) \in \mathcal{B} \cap X$ be a convex combination

$$(x,0) = \sum_{a\in A} \theta_a(p_a,0) + \sum_{b\in B} \theta_b(\mu x_b,1) + \delta(0,1),$$

where $A \cup B$ is finite and $\sum_{i\in A\cup B} |\theta_i| + |\delta| = 1$. Then, we get the two conditions

$$x = \sum_{a\in A} \theta_a p_a + \mu \sum_{b\in B} \theta_b x_b,$$
$$0 = \sum_{b\in B} \theta_b + \delta,$$

and we have

$$\|x\| = \Big\|\sum_{a\in A} \theta_a p_a + \mu \sum_{b\in B} \theta_b x_b\Big\|$$
$$\leq \sum_{a\in A} |\theta_a| + \mu \Big\|\sum_{b\in B} \theta_b x_b\Big\|.$$

Since $|\delta| = |\sum_{b\in B}\theta_b|$, we have $\sum_{b\in B} |\theta_b| + |\sum_{b\in B}\theta_b| = 1 - \sum_{a\in A}|\theta_a|$. Let us call $c = 1 - \sum_{a\in A}|\theta_a|$. We need to estimate

$$\sup_{\sum_n |\theta_n| + |\sum_n \theta_n| = c} \Big\|\mu \sum_n \theta_n x_n\Big\| = c \sup_{\sum_n |\theta_n| + |\sum_n \theta_n| = 1} \Big\|\mu \sum_n \theta_n x_n\Big\|. \tag{4.1}$$

One can assume without loss of generalization that $\sum \theta_n = 0$. Otherwise, let $s = \sum \theta_n$ and $m = s/M$, where M is now the number of elements θ_n, and replace θ_n by $\theta_n - m$. Clearly, $\sum \theta_n - m = 0$ and $\sum |\theta_n - m| + |\sum(\theta_n - m)| = \sum |\theta_n - m| \leq \sum |\theta_n| + |s| \leq 1$. Moreover,

$$\Big\|\sum(\theta_n - m)x_n\Big\| = \Big\|\sum \theta_n x_n + m \sum x_n\Big\|$$
$$= \Big\|\sum \theta_n x_n\Big\|.$$

Let us show an auxiliary result, as follows.

Claim. *Let $(\theta_n)_{1\leq n\leq N}$ be a finite set of scalars so that $\sum_n |\theta_n| = 1$ and $\sum_n \theta_n = 0$, and let $(x_n)_{1\leq n\leq N}$ be elements of a Banach space. Then*

$$\Big\|\sum \theta_n x_n\Big\| \leq \frac{1}{2} \max \|x_n - x_m\|.$$

Proof. It is obviously enough to make the proof when all the coefficients θ_n are rationals. So let $\theta_n = a_n/b_n$. Reduced to common denominators, we have $\theta_n = k_n/d$; since $1 = \sum^N |\theta_n| = \sum^N |k_n|/d$, we have $\sum |k_n| = d$. Let $A(1), \ldots, A(N)$ be the associated partition of $\{1, \ldots, d\}$ with cardinal $A(j) = |k_j|$ such that $\theta_n = \sum_{i\in A(n)} (\varepsilon_i/d)$ with $\varepsilon_i = \pm 1$. Replace the N original $x_n's$ by new d elements $y_i = x_n$ when $i \in A(n)$. Since $\sum_{n=1}^N \theta_n x_n = \sum_{i=1}^d (\varepsilon_i/d) y_i$, we get $\sum_{i=1} n^d(\varepsilon_i/d) = 0$, which also means that $\sum_{i=1}^{d}\varepsilon_i = 0$ or, which is the same, that there are $d/2$ positive ε, which we will call ε_i, and $d/2$ negative ε, which that we will call ε_i':

$$\begin{aligned}\Big\|\sum \theta_n x_n\Big\| &= \Big\|\sum_i^d \frac{\varepsilon_i}{d} y_i\Big\| \\ &= \Big\|\sum_i^{d/2} \frac{\varepsilon_i}{d} y_i - \sum_i^{d/2}\frac{|\varepsilon_i'|}{d} y_i\Big\| \\ &\le \frac{1}{2}\max_{n,m}\|y_n - y_m\|.\end{aligned}$$

Therefore, for every ε, we have

$$\begin{aligned}\sum_A |\theta_a| + \mu\Big\|\sum_B \theta_b x_b\Big\| &= \sum_A |\theta_a| + \mu c \sup_{\sum_n |\theta_n| + |\sum_n \theta_n| = 1}\Big\|\sum_n \theta_n x_n\Big\| \\ &\le \sum_A |\theta_a| + \mu c \frac{1}{2}\max\|x_n - x_m\| \\ &= \sum_A |\theta_a| + c \\ &= 1.\end{aligned}$$

Thus, $\mathcal{B} \cap X = B_X$. $\square$

Corollary 4.3. *A Banach space X is 1-complemented in $X \oplus_r \mathbb{R}$ if and only if $K(X) = 2$.*

Proof. The only "if" part is a consequence of Lemma 4.1: a Banach space and its 1-complemented hyperplanes have the same Kottman's constant. As for the if part, we will show that the operator $(x, t) \to x$ has norm 1: just pick (x, t) in the unit ball of r, which means that

$$(x,t) = \sum_A \theta_a(p_a, 0) + \sum_B \theta_b(\mu x_b, 1) + \delta(0, 1),$$

where $A \cup B$ is finite and $\sum_{i\in A\cup B}|\theta_i| + |\delta| = 1$. Therefore, we have

$$x = \sum_A \theta_a p_a + \mu \sum_B \theta_b x_b.$$

Since $K(X) = 2$, we have $\mu = 2/(KX) = 1$, and therefore

$$\|x\| = \Big\|\sum_A \theta_a p_a + \sum_B \theta_b x_b\Big\| \le \sum_A |\theta_a| + \sum_B |\theta_b| \le 1.$$

$\square$

5. Kottman's constant on biduals

Now, we solve one of the main problems left open in [7] and [8]: Does $K(X) = K(X^{**})$ always hold? The corresponding question for Whitley constant $T(\cdot)$ (see [24]) was negatively solved in [1] showing that $T(C[0,1]) \neq T(C[0,1]^{**})$. We show now that $K(J_p(\ell_1^n)) < K(J_p(\ell_1^n)^{**})$ for certain James-like spaces introduced by Bellenot in [5].

Let us denote by ℓ_1^n the subspace of ℓ_1 generated by $\{e_1,\ldots,e_n\}$, and we denote by $Q_n:\ell_1\to\ell_1^n$ the natural projection. For $1<p<\infty$, we define

$$J_p(\ell_1^n) := \big\{(x_n) : x_n\in\ell_1^n, \|x_n\|_1\to 0 \text{ and } \big\|(x_n)\big\|_{J_p}<\infty\big\},$$

where

$$\big\|(x_n)\big\|_{J_p} := \sup\Big\{\Big(\sum_{i=1}^{k}\|x_{n_{i+1}}-x_{n_i}\|_1^p\Big)^{1/p} : k\in\mathbb{N}, n_1<n_2<\cdots<n_{k+1}\Big\}.$$

For each $k\in\mathbb{N}$, we will identify ℓ_1^k with the subspace of all $(x_n)\in J_p(\ell_1^n)$ with $x_n=0$ for $n\neq k$.

Proposition 5.1. *Let $1<p<\infty$. Then $(J_p(\ell_1^n), \|(x_n)\|_{J_p})$, is a Banach space, and the quotient $J_p(\ell_1^n)^{**}/J_p(\ell_1^n)$ is isomorphic to ℓ_1.*

Proof. The proof of the first part is a standard argument. Moreover, as in the case of James' space [21, Example 1.d.2], it is easy to check that the sequence of subspaces $(\ell_1^k)_{k\in\mathbb{N}}$ is a shrinking decomposition of $J_p(\ell_1^n)$. Thus the arguments in the proof of [21, Proposition 1.b.2] show that

$$J_p(\ell_1^n)^{**} = \big\{(x_n) : x_n\in\ell_1^n \text{ and } \big\|(x_n)\big\|_{J_p}<\infty\big\}.$$

Let $(x_n)\in J_p(\ell_1^n)^{**}$. From $\|(x_n)\|_{J_p}<\infty$ it follows that (x_n) is a convergent sequence in ℓ_1. Therefore, there is a linear bounded operator $U: J_p(\ell_1^n)^{**}\to\ell_1$ given by $U((x_n))=\lim_n x_n$ for which $\ker(U)=J_p(\ell_1^n)$. Moreover, U is surjective because for each $x\in\ell_1$ we have $(Q_n(x))\in J_p(\ell_1^n)^{**}$ and $\lim_n Q_n(x)=x$. $\square$

Corollary 5.2. *Let $1<p<\infty$. Then $K(J_p(\ell_1^n)^{**})=2$.*

Proof. The previous result implies that $J_p(\ell_1^n)^{**}$ contains a subspace isomorphic to ℓ_1. So it is enough to observe that $K(\ell_1)=2$ and that every space isomorphic to ℓ_1 contains almost isometric copies of ℓ_1. $\square$

The proof of the next result was inspired by the proof of [20, Theorem 2] for the James' spaces J_p, but it is technically more complicated. Therefore, we give a detailed proof.

We denote by $e_{n,i}$ $(i=1,\ldots,n)$ the ith unit vector of ℓ_1^n as a subspace of $J_p(\ell_1^n)$. Moreover, P_m is the norm 1 projection on $J_p(\ell_1^n)$ defined as $P_m((x_n)) = (x_1,\ldots,x_m,0,0,\ldots)$, and we will say that $R_m=I-P_m$, where I is the identity operator. Note that $\|R_m(x)\|_{J_p}\to 0$ as $m\to\infty$ for every $x\in J_p(\ell_1^n)$.

Proposition 5.3. *Let $1<p<\infty$. Then $K(J_p(\ell_1^n))=(1+2^{p-1})^{1/p}$.*

Proof. From $\|e_{m,1}\|_{J_p}=2^{1/p}$ and $\|e_{m,1}-e_{n,1}\|_{J_p}=(1+2^p+1)^{1/p}$ for $1<m<n$, we get $K(J_p(\ell_1^n))\geq(1+2^{p-1})^{1/p}$.

To prove the converse inequality, let $(x^k)_{k\in\mathbb{N}}$ be a sequence in the unit ball of $J_p(\ell_1^n)$ with $x^k=(x_n^k)_{n\in\mathbb{N}}$. Since the quotient $J_p(\ell_1^n)^{**}/J_p(\ell_1^n)$ is separable, $J_p(\ell_1^n)$ contains no copies of ℓ_1. So passing to a subsequence, we can assume that $(x^k)_{k\in\mathbb{N}}$ is weakly Cauchy; hence, $(x_n^k)_{k\in\mathbb{N}}$ is convergent in ℓ_1^n for each n. We set $x_n=\lim_{k\to\infty}x_n^k\in\ell_1^n$ and $x=(x_n)$. Note that $x\in J_p(\ell_1^n)^{**}$ because it is the weak*-limit of $(x^k)_{k\in\mathbb{N}}$. Thus, as we observed in the proof of Proposition 5.1, the sequence (x_n) is convergent to some $\omega \in \ell_1$. We set $\alpha = (Q_n(\omega))$. Clearly, $x - \alpha = (x_n - Q_n(\omega)) \in J_p(\ell_1^n)$.

We fix a number $\varepsilon > 0$ with $0 < \varepsilon < 1$, and we take $m_0 \in \mathbb{N}$ such that

$$\big\|R_{m_0}(x-\alpha)\big\|_{J_p} < \varepsilon/7 \qquad \text{and} \qquad \big\|Q_{m_0}(\omega) - \omega\big\|_1 < \varepsilon/7.$$

In the proof of [20, Theorem 2] for J_p, the sequence (α_n) is constant. This is not true in our case, but we have chosen m_0 so that

$$\big\|Q_{m_0}(\omega) - \omega\big\|_1 = \sum_{n=m_0}^{\infty} |\omega_{n+1}| = \sum_{n=m_0}^{\infty} \|\alpha_{n+1} - \alpha_n\|_1 < \varepsilon/7.$$

So, if we consider the vector $\beta = (\beta_n)$ given by $\beta_n = \alpha_n$ for $n \le m_0$ and $\beta_n = \alpha_{m_0}$ for $n > m_0$, then $\beta \in J_p(\ell_1^n)^{**}$ and

$$\|\alpha - \beta\|_{J_p} \le \sum_{n=m_0}^{\infty} \|\alpha_{n+1} - \alpha_n\|_1 < \varepsilon/7.$$

Since (x^k) is weak*-convergent to x and the projections P_m are finite rank operators, we can find k_1 such that $\|P_{m_0}^{**}(x^{k_1}-x)\|_{J_p} < \varepsilon/7$ and $\|P_{m_0}(x^k-x^l)\|_{J_p} < \varepsilon/7$ for $k, l \ge k_1$. Now we pick $m_1 > m_0$ such that $\|R_{m_1}(x^{k_1})\| < \varepsilon/7$, and we choose $k_2 > k_1$ such that $\|P_{m_1}^{**}(x^{k_2} - x)\|_{J_p} < \varepsilon/7$. Next, we pick $m_2 > m_1$ such that $\|R_{m_2}(x^{k_2})\| < \varepsilon/7$, and we choose $k_3 > k_1$ such that $\|P_{m_2}^{**}(x^{k_3} - x)\|_{J_p} < \varepsilon/7$.

In this way we obtain (m_i) and (x^{k_i}) such that each of the terms $P_{m_0}(x^{k_i} - x^{k_j})$, $P_{m_{i-1}}^{**}(x^{k_i} - x)$, and $R_{m_i}(x^{k_i})$ has norm smaller than $\varepsilon/7$. Let us set

$$u^i = P_{m_0}x^{k_1} + P_{m_{i-1}}^{**}R_{m_0}^{**}\beta + P_{m_i}R_{m_{i-1}}x^{k_i}$$

for $i > 1$ in $\mathbb{N}$. Since $x^{k_i} = P_{m_0}x^{k_i} + P_{m_{i-1}}R_{m_0}x^{k_i} + P_{m_i}R_{m_{i-1}}x^{k_i} + R_{m_i}x^{k_i}$, we get

$$\begin{aligned}\|x^{k_i} - u^i\|_{J_p} &\le \big\|P_{m_0}(x^{k_i} - x^{k_1})\big\|_{J_p} + \big\|P_{m_{i-1}}^{**}R_{m_0}^{**}(x^{k_i} - \beta)\big\|_{J_p} + \|R_{m_i}x^{k_i}\|_{J_p} \\ &< 2(\varepsilon/7) + \big\|P_{m_{i-1}}^{**}R_{m_0}^{**}(x^{k_i} - \beta)\big\|_{J_p}.\end{aligned}$$

Since $P_{m_{i-1}}^{**}R_{m_0}^{**} = R_{m_0}P_{m_{i-1}}^{**}$ and $\|R_{m_0}\| \le 2$, we have

$$\begin{aligned}&\big\|P_{m_{i-1}}^{**}R_{m_0}^{**}(x^{k_i} - \beta)\big\|_{J_p} \\ &\quad \le \|R_{m_0}\|\big\|P_{m_{i-1}}^{**}(x^{k_i} - x)\big\|_{J_p} \\ &\qquad + \|P_{m_{i-1}}\|\big\|R_{m_0}(x-\alpha)\big\|_{J_p} + \|P_{m_{i-1}}\|\|R_{m_0}\|\big\|(\alpha-\beta)\big\|_{J_p} \\ &\quad \le 5(\varepsilon/7),\end{aligned}$$

and thus we get $\|x^{k_i} - u^i\|_{J_p} < \varepsilon$.

Now we set $w = u^3 - u^2$ and we write $u^2 = (u_n^2)$ and $u^3 = (u_n^3)$. Note that $w = (w_n)$ with $w_n = 0$ for $n \le m_1$ and $m_3 < n$, $w_n = \alpha_{m_0} - x_n^{k_2}$ for $m_1 < n \le m_2$, and $w_n = x_n^{k_3}$ for $m_2 < n \le m_3$. We choose $n_1 < \cdots < n_{k+1}$ such that

$$(1-\varepsilon)^p\|w\|_{J_p}^p \le \sum_{i=1}^{k} \|w_{n_{i+1}} - w_{n_i}\|_1^p.$$

Clearly, we can assume that $n_1 = m_1$ and that $n_{l-1} \leq m_2 < n_l$ for some l with $1 < l \leq k+1$. Assuming that, and since $w_j = \alpha_{m_0} - u_j^2$ for $m_1 < j \leq m_2$, we have

$$\sum_{i=1}^{l-1} \|w_{n_{i+1}} - w_{n_i}\|_1^p + \|u_{n_{l-1}}^2\|_1^p \leq \|u^2\|_{J_p}^p \leq (1+\varepsilon)^p.$$

Similarly, since $w_j = u_j^3$ for $m_2 < j$, we have

$$\sum_{i=l+1}^{k} \|w_{n_{i+1}} - w_{n_i}\|_1^p + \|\alpha_{m_0} - u_{n_l}^3\|_1^p \leq \|u^3\|_{J_p}^p \leq (1+\varepsilon)^p.$$

Therefore, $(1-\varepsilon)^p \|w\|_{J_p}^p$ is smaller than

$$2(1+\varepsilon)^p - \|u_{n_{l-1}}^2\|_1^p - \|\alpha_{m_0} - u_{n_l}^3\|_1^p + \left\|u_{n_l}^3 - (\alpha_{m_0} - u_{n_{l-1}}^2)\right\|_1^p.$$

Now taking into account the classical inequality $(x+y)^p \leq 2^{p-1}(x^p + y^p)$ for $x, y \geq 0$, we get

$$\begin{aligned} 2^{1-p}\left\|u_{n_l}^3 - (\alpha_{m_0} - u_{n_{l-1}}^2)\right\|_1^p &\leq 2^{1-p}\left(\|u_{n_{l-1}}^2\|_1 + \|\alpha_{m_0} - u_{n_l}^3\|_1\right)^p \\ &\leq \|u_{n_{l-1}}^2\|_1^p + \|\alpha_{m_0} - u_{n_l}^3\|_1^p. \end{aligned}$$

Hence

$$(1-\varepsilon)^p \|w\|_{J_p}^p \leq 2(1+\varepsilon)^p + (1 - 2^{1-p})\left\|u_{n_l}^3 - (\alpha_{m_0} - u_{n_{l-1}}^2)\right\|_1^p.$$

As in the proof of [20, Theorem 2], with arguments similar to the previous ones, we get

$$\left\|u_{n_l}^3 - (\alpha_{m_0} - u_{n_{l-1}}^2)\right\|_1^p \leq 2^{p-1}(1+\varepsilon)^p,$$

and hence

$$(1-\varepsilon)^p \|w\|_{J_p}^p \leq (1+\varepsilon)^p (1 + 2^{p-1}).$$

Since $\|x^{k_3} - x^{k_2}\|_{J_p} \leq \|w\|_{J_p} + 2\varepsilon$, we get $K(J_p(\ell_1^n)) \leq (1+2^{p-1})^{1/p}$, and the proof is complete. □

6. Open problems

A few problems have appeared in the course of our work that remain open.
Regarding Diestel spaces:

(1) We do not know if there is an analogue for the Elton–Odell theorem in the context of the isomorphic Kottman's constant (i.e., whether $\tilde{K}(X) > 1$ for every infinite-dimensional Banach space). The estimate $\kappa(X) \leq \tilde{K}(X)$ easily provides examples of spaces such that $\tilde{K}(X) > 1$; for example, spaces containing a subspace Y admitting a noncompact operator $Y \to \ell_p$ for some p, $\ell_2(\ell_1^n)$, and so forth.

(2) Analogously to Diestel spaces, a Banach space X can be defined as a $\tilde{\text{D}}$iestel space if $\tilde{s}(X) > 1$. It is clear that a Banach space isomorphic to a $\tilde{\text{D}}$iestel space is itself a $\tilde{\text{D}}$iestel space. We do not know, however, if Diestel and $\tilde{\text{D}}$iestel spaces coincide, or even if $\tilde{s}(X) = s(X)$. From the results of Prus [23], it follows that super-reflexive spaces are $\tilde{\text{D}}$iestel spaces.

(3) If $\tilde{K}(X) > 1$ held for every infinite-dimensional Banach space, then Proposition 2.5 could be proved for the isomorphic Kottman's constant; that is, *Every infinite-dimensional Banach space contains an infinite-dimensional* $\tilde{D}$*iestel subspace*.

Regarding twisted sum spaces:

(1) It will be interesting to know whether the following 3-space result for the isomorphic Kottman's constant holds. Show that for any given exact sequence $0 \to Y \to X \to Z \to 0$, we have
$$\tilde{K}(X) = \max\{\tilde{K}(Y), \tilde{K}(Z)\}.$$

(2) The case of twisted Hilbert spaces is especially interesting. Is it true that $\tilde{K}(X) = \sqrt{2}$ for every twisted Hilbert space X?

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