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A Completed System for Robin Smith's Incomplete Ecthetic Syllogistic

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Abstract In this paper we first show that Robin Smith's ecthetic system *SE* for Aristotle's assertoric syllogistic is not complete, despite what is claimed by Smith. *SE* is then not adequate to establish that ecthesis allows one to dispense with indirect or per impossibile deductions in Aristotle's assertoric logic. As an alternative to *SE*, we then present a stronger system *EC* which is adequate for this purpose. *EC* is a nonexplosive ecthetic system which is shown to be sound and complete with respect to all valid syllogistic arguments with a consistent set of premises.

1 Background

In the article Joray [5], we have argued that Aristotle's assertoric syllogistic is independent of the principle of contradiction (PC). Contrary to Łukasiewicz [6], who defended the same thesis, but with arguments we have shown to be too weak for this purpose, we have adopted the view that syllogistic is not an axiomatic system, but can be treated as a natural deduction system for the categorical propositions, usually labeled A, E, I, and O. Following Corcoran [3], [4], Smiley [11], and Smith [13], we consider the means presented in Aristotle [1] for the construction of syllogistic as inference rules. In this perspective, the main system which can be extracted from [1] has seven inference rules corresponding to the four perfect moods (traditionally labeled *Barbara*, *Celarent*, *Darii*, and *Ferio*), the three conversion laws (for propositions A, E, and I), and two kinds of deductions: direct deductions and indirect deductions (or deductions by reductio ad absurdum). In this paper, we will adopt such a system as a point of departure, namely, Corcoran's system D, which can be stated with the following elements.

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1.1 Corcoran's formal language and semantics The formal language L relies on the vocabulary $V = V_1 \cup V_2$ with $V_1 = \{A, E, I, O\}$ (the set of constants) and $V_2 = \{a, b, c, \ldots\}$ (the set of terms). The set of formulas of L is defined as the smallest set E such that $A\alpha\beta$, $E\alpha\beta$, $I\alpha\beta$, $O\alpha\beta \in E$ for all $\alpha, \beta \in V_2$, $\alpha \neq \beta$. Intuitively, $A\alpha\beta$, $E\alpha\beta$, $I\alpha\beta$, and $O\alpha\beta$ can be read as the traditional categorical propositions: respectively, *β belongs to all α*, *β belongs to no α*, *β belongs to some α*, and *β does not belong to some α*. Formally, when a nonempty universe of discourse U is stated, an interpretation i of the language L is a function from V_2 to $P(U) \setminus \{\emptyset\}$ (the set of nonempty parts of U).[1] The truth value of any formula of L under an interpretation i is given by the set of truth values $\{\top, \bot\}$ and the following five clauses:

(i) $\mathrm{Val}_i(A\alpha\beta) = \top$ if and only if $i(\alpha) \cup i(\beta) = i(\beta)$,
(ii) $\mathrm{Val}_i(E\alpha\beta) = \top$ if and only if $i(\alpha) \cap i(\beta) = \emptyset$,
(iii) $\mathrm{Val}_i(I\alpha\beta) = \top$ if and only if $i(\alpha) \cap i(\beta) \neq \emptyset$,
(iv) $\mathrm{Val}_i(O\alpha\beta) = \top$ if and only if $i(\alpha) \cup i(\beta) \neq i(\beta)$,
(v) for every formula e, $\mathrm{Val}_i(e) = \bot$ if and only if $\mathrm{Val}_i(e) \neq \top$.

For convenience, when $\mathrm{Val}_i(e) = \top$ for all formulas $e \in \Gamma$ (a certain set of formulas) we will simply write $\mathrm{Val}_i(\Gamma) = \top$. A formula c is said to be a *logical consequence* of a set Γ if and only if every interpretation i is such that $\mathrm{Val}_i(c) = \top$ when $\mathrm{Val}_i(\Gamma) = \top$. In this case, we will write $\Gamma \models c$ or say that the argument (Γ, c) is *valid*. It is also convenient to state the following notation for contradictory formulas: $\overline{A\alpha\beta} = O\alpha\beta$, $\overline{O\alpha\beta} = A\alpha\beta$, $\overline{I\alpha\beta} = E\alpha\beta$, and $\overline{E\alpha\beta} = I\alpha\beta$. Obviously, for any formula e, $\overline{\overline{e}} = e$, and $\mathrm{Val}_i(e) = \top$ if and only if $\mathrm{Val}_i(\overline{e}) = \bot$ for all i's. The following definitions will also be useful.

Definition 1.1 A set Γ of formulas is said to be *consistent* if and only if there is at least one interpretation i such that $\mathrm{Val}_i(\Gamma) = \top$. (Otherwise Γ is *inconsistent*.)

Definition 1.2 A deductive system S with the formal language L is said to be *c-complete* if and only if, for every valid argument (Γ, c), there is a deduction $\Gamma \vdash_S c$ (i.e., a deduction of c from Γ in S).

Definition 1.3 A deductive system S with the formal language L is said to be *s-complete* if and only if, for every valid argument (Γ, c) with Γ consistent, there is a deduction $\Gamma \vdash_S c$.

Definition 1.4 A deductive system S with the formal language L is said to be *sound* if and only if, when $\Gamma \vdash_S c$, the argument (Γ, c) is valid.

1.2 Corcoran's system *D* Corcoran [3] presents a deductive system called D. This system has the formal language L and the following seven inference rules:

Barbara: $A\alpha\beta, A\gamma\alpha \vdash A\gamma\beta$,
Celarent: $E\alpha\beta, A\gamma\alpha \vdash E\gamma\beta$,
Darii: $A\alpha\beta, I\gamma\alpha \vdash I\gamma\beta$,
Ferio: $E\alpha\beta, I\gamma\alpha \vdash O\gamma\beta$,
E-conv: $E\alpha\beta \vdash E\beta\alpha$,
A-conv: $A\alpha\beta \vdash I\beta\alpha$,
I-conv: $I\alpha\beta \vdash I\beta\alpha$.

Note that D has no reductio ad absurdum rule, but two distinct notions of deduction: a *direct deduction* $\Gamma \vdash_D^{\mathrm{Dir}} c$ (of a formula c from a set of formulas Γ) is a sequence of formulas ending with c, beginning with elements of Γ, and such that each subsequent formula is either a repetition or the result of the application of an inference rule on previous formulas; an *indirect deduction* $\Gamma \vdash_D^{\text{Ind}} c$ is a sequence ending in two contradictory formulas f, $\overline{f}$, beginning with elements of Γ immediately followed by $\overline{c}$ (the reductio assumption) and such that each subsequent formula is either a repetition or the result of the application of an inference rule on previous formulas. A *deduction* $\Gamma \vdash_D c$ is either a direct deduction $\Gamma \vdash_D^{\text{Dir}} c$ or an indirect deduction $\Gamma \vdash_D^{\text{Ind}} c$.

Corcoran [3] proves that D is sound and c-complete.

Theorem 1.1 *For all formulas c and set of formulas Γ, $\Gamma \models c$ if and only if $\Gamma \vdash_D c$.*

It is worth noting that Aristotle himself has shown in [1] that such a basis for assertoric syllogistic can be simplified. First, it is clear that the particular perfect moods (*Darii* and *Ferio*) are derivable from the remaining rules (see [1, I.7]). Second, Aristotle also considered the three conversions to be reducible to other means. The proofs he gives for the conversions (see [1, I.2]) crucially depend on the use of reductio ad absurdum and also on a procedure he calls elsewhere *ecthesis*. As we show in [5], no interpretation of syllogistic excluding ecthesis or treating it out of the system itself (as in Łukasiewicz [7], Corcoran [3], [4], and Smiley [11]) can account for all of Aristotle's proofs of conversions. Nevertheless, modern scholars have largely considered ecthesis as a redundant tool in [1]. This explains why most of the modern accounts of syllogistic rest on a basis which only contains perfect syllogisms, reductions, and the possibility of indirect proofs.

1.3 Corcoran's system *RD* In this perspective, Corcoran shows that D can be simplified in a system he calls *RD* by deleting *Darii*, *Ferio*, and I-conv. The set of inference rules of *RD* is then:

Barbara, *Celarent*,
E-conv, A-conv.

Corcoran [3] proves that *RD* is equivalent to D and, hence, preserves soundness and c-completeness.

Theorem 1.2 *For any formula c and set of formulas Γ, $\Gamma \vdash_{RD} c$ if and only if $\Gamma \vdash_D c$.*

Concerning *RD*, it is worth noting that all the inference rules take only universal operands (A- and E-formulas). Hence, once a particular formula (I- or O-formula) gets into a deduction, thereafter it can only be repeated.

To show, from a systematic point of view, that syllogistic is independent of PC, we had the idea of using Smith's ecthetic system *SE*. In [13], Smith intended to show that the addition of sound inference rules for ecthesis into Corcoran's system D allows one to dispense with indirect deductions, losing only the possibility to deduce an arbitrary conclusion from an inconsistent set of premises. Our argument in [5] for the independence of syllogistic vis-à-vis PC relies centrally on the possibility to account for syllogistic without using reductio ad absurdum. Hence, Smith's system *SE* seemed to offer the expected solution. Moreover, *SE* is interesting in another respect: without reductio ad absurdum, *SE* is *not explosive*. (It is not possible to deduce an arbitrary formula from an inconsistent set of premises.) Given this feature, it was clear that *SE* was not c-complete and that Smith's intention was to prove *SE* to be sound and s-complete. For that purpose, Smith intended to prove the following conjecture.

Smith's Conjecture For every formula c and every consistent set of formulas Γ, $\Gamma \vdash_{SE} c$ if and only if $\Gamma \vdash_{RD} c$.

Nevertheless, studying *SE*, we discovered it has two defects. The first one is just an esthetic imperfection: the system has too many primitive inference rules. The second defect is deeper: contrary to what is claimed in Smith [13], *SE* is not s-complete. As we shall see further, it is easy to find a counterexample to Smith's conjecture.

In this paper, we will present a system called *EC* as an alternative to Smith's ecthetic syllogistic. The main part of the article will be devoted to the soundness and s-completeness proofs of *EC*. Before going into the proofs, we will shortly recall Smith's system and point out its imperfections. Despite these imperfections, we would like to insist here on the fact that the main ideas in the construction of our system *EC* and in our proofs are only improvements of Smith's original ideas presented in Smith [12], [13].

2 Ecthesis and Ecthetic Syllogistic

Aristotle uses or alludes to ecthesis in five passages in [1]: (1) in the proof of E-conv (see [1, 25^a15–17]); (2) in the reduction of the mood *Darapti* (see [1, 28^a24–26]); (3) in the reduction of moods *Disamis* and *Datisi* (see [1, 28^b14–15]); (4) in the reduction of *Bocardo* (see [1, 28^b20–21]); (5) in the reduction of *Baroco* and *Bocardo* with necessary premises (see [1, 30^a9–14]). The main characteristic of ecthesis is the introduction, or setting out, of a letter which does not occur before in the deduction. In the rest of the deduction, this letter plays the role of an unassigned middle term to be eliminated further by the application of a syllogistic mood. If one leaves aside the untypical case of *Darapti* (the only place where ecthesis rests on a universal premise) and concentrates on the cases where the procedure is an alternative to a per impossibile reduction (mainly in the reductions of *Baroco* and *Bocardo*), it is reasonable to follow Smith, considering it concerns particular propositions and works according to the two following patterns (where γ is the term set out):

- IA If α belongs to some β, then there is a γ to which α and β both belong.
- IB If α does not belong to some β, then there is a γ to which α does not belong and to which β belongs.

The controversial question is that of the interpretation of the new letter. On the one hand, it can be seen as a regular syllogistic term.[2] But, on the other hand, it can also be considered as an individual term.[3] Here, once again, if one concentrates on the cases where ecthesis is an alternative to reductio ad absurdum, the interpretation of γ as a general syllogistic term fits better. As an example, the reconstruction of the reduction by ecthesis of *Bocardo* (see [1, 28^b21]) gives the following deduction schema:

Bocardo

(1)	$O\alpha\beta$	Premise
(2)	$A\alpha\gamma$	Premise
(3)	$A\lambda\alpha$	1, O-ecth (λ is the term set out)
(4)	$E\lambda\beta$	1, O-ecth
(5)	$A\lambda\gamma$	2, 3, *Barbara*
(6)	$I\gamma\lambda$	5, A-conv
(7)	$O\gamma\beta$	4, 6, *Ferio* (conclusion)

Our reading of Smith's patterns IA and IB then gives the following ecthetic rules:

O-ecth: $O\alpha\beta \vdash A\gamma\alpha, E\gamma\beta$
with no previous occurrence of γ in the deduction;
I-ecth: $I\alpha\beta \vdash A\gamma\alpha, A\gamma\beta$
with no previous occurrence of γ in the deduction.

The first rule issues directly from the above example. The new letter γ is introduced to designate one of the parts of α of which $O\alpha\beta$ assures it is not β. From this we infer $A\gamma\alpha$ (γ being a part of α) and $E\gamma\beta$ (since this part is selected out of β).[4] Starting from a particular affirmative proposition, we analogously infer with the second rule two A-propositions, since the part γ to be selected is one which is common to both first terms. In addition to these two rules, Smith also considers their converses as ecthetic rules:

O-intro: $A\gamma\alpha, E\gamma\beta \vdash O\alpha\beta$,
I-intro: $A\gamma\alpha, A\gamma\beta \vdash I\alpha\beta$.

2.1 Smith's system *SE* Smith's ecthetic deductive system *SE* has formal language L with its usual semantic interpretation. Its rules are the seven rules of D, with the addition of the four above-mentioned ecthetic rules: O-ecth, I-ecth, O-intro, and I-intro. A *deduction* in *SE* is defined like a *direct deduction* in D with the additional restriction that there must be no occurrence in the conclusion of a term introduced by ecthesis. Now, let us prove that *SE* is sound.

Theorem 2.1 *For any* Γ *and* c, *if* $\Gamma \vdash_{SE} c$, *then* (Γ, c) *is valid.*

Proof Since D is sound and *SE* has only direct deductions, it is sufficient to show that the four added rules are sound. O-intro and I-intro are nothing but the traditional valid moods *Felapton* and *Darapti*, which are clearly sound in Corcoran's semantics. For the soundness of O-ecth, consider that we have a valid argument $(\Gamma \cup \{A\gamma\alpha, E\gamma\beta\}, c)$ with no occurrence of γ in Γ or in c and with $\alpha, \beta \neq \gamma$. We have to show that $(\Gamma \cup \{O\alpha\beta\}, c)$ is then also valid. Suppose this last argument is *not* valid. Hence, there is an interpretation i such that $\text{Val}_i(\Gamma \cup \{O\alpha\beta\}) = \top$ and $\text{Val}_i(c) = \bot$. Now, consider j which differs from i only by the following assignation to γ: $j(\gamma) = i(\alpha) \setminus i(\beta)$ (not empty, for $\text{Val}_i(O\alpha\beta) = \top$). For $\alpha, \beta \neq \gamma$, we have $(i(\alpha) \setminus i(\beta)) \cup j(\alpha) = j(\alpha)$ and $(i(\alpha) \setminus i(\beta)) \cap j(\beta) = \emptyset$. Then $\text{Val}_j(A\gamma\alpha) = \text{Val}_j(E\gamma\beta) = \top$. Since Γ and c are without γ, $\text{Val}_j(\Gamma) = \top$ and $\text{Val}_j(c) = \bot$ (like with i). The existence of j is in contradiction with the assumed validity of $(\Gamma \cup \{A\gamma\alpha, E\gamma\beta\}, c)$. Then $(\Gamma \cup \{O\alpha\beta\}, c)$ is valid. Similar reasoning works for I-ecth. Consider a valid argument $(\Gamma \cup \{A\gamma\alpha, A\gamma\beta\}, c)$ with the same requirements on γ. Suppose that $(\Gamma \cup \{I\alpha\beta\}, c)$ is not valid. Hence, there is i such that $\text{Val}_i(\Gamma \cup \{I\alpha\beta\}) = \top$ and $\text{Val}_i(c) = \bot$. Now, consider j which differs from i only by $j(\gamma) = i(\alpha) \cap i(\beta)$ (not empty, for $\text{Val}_i(I\alpha\beta) = \top$). For $\alpha, \beta \neq \gamma$, we have $(i(\alpha) \cap i(\beta)) \cup j(\alpha) = j(\alpha)$ and $(i(\alpha) \cap i(\beta)) \cup j(\beta) = j(\beta)$. This means that $\mathrm{Val}_j(A\gamma\alpha) = \mathrm{Val}_j(A\gamma\beta) = \top$. Since Γ and c are without γ, $\mathrm{Val}_j(\Gamma) = \top$ and $\mathrm{Val}_j(c) = \bot$ (like with i). The existence of j is in contradiction with the assumed validity of $(\Gamma \cup \{A\gamma\alpha, A\gamma\beta\}, c)$. Then $(\Gamma \cup \{I\alpha\beta\}, c)$ is valid. All the inference rules are sound, so *SE* is sound. □

Now let us show that Smith's conjecture is false by establishing that *SE* is not s-complete.

Theorem 2.2 *There is a valid argument* (Γ, c) *with* Γ *consistent and* $\Gamma \nvdash_{SE} c$.

Proof Consider an argument $(\{E\alpha\beta\}, O\alpha\beta)$. The set $\{E\alpha\beta\}$ is obviously consistent. The following sequence shows that $E\alpha\beta \vdash^{\mathrm{Ind}}_{RD} O\alpha\beta$:

(1)	$E\alpha\beta$	Premise
(2)	$A\alpha\beta$	Reductio hypothesis
(3)	$I\beta\alpha$	2, A-conv
(4)	$E\beta\alpha$	1, E-conv (contradiction with 3)

RD is sound; hence, $(\{E\alpha\beta\}, O\alpha\beta)$ is valid. Now, in *SE* the only rule applicable on this sole premise $E\alpha\beta$ is E-conv, resulting in $E\beta\alpha$. Since there is no rule applicable to two E-formulas, the only rule which can be applied further is again E-conv, with an E-formula as the result. Hence, there will be no O-formula in the deduction. Then $E\alpha\beta \nvdash_{SE} O\alpha\beta$. □

3 An s-Complete Ecthetic Syllogistic

As an alternative to *SE*, we will present in this section the ecthetic system *EC*. It will be proved to be sound and s-complete. It is worth noting first that *SE* has three redundant rules: O-intro, I-intro, and I-conv.[5] As *EC* must be stronger than *SE*, the rule to be added is the following (which corresponds to the traditional *subalternation* of E-propositions):[6]

E-sub: $E\alpha\beta \vdash O\alpha\beta$.

3.1 System *EC* The ecthetic system *EC* has the formal language L with the usual semantic interpretation. Its inference rules are:

Barbara, *Celarent*, *Darii*, *Ferio*,
E-conv, A-conv,
O-ecth, I-ecth,
E-sub.

A *deduction* $\Gamma \vdash_{EC} c$ is a sequence of formulas, ending with c, beginning with elements of Γ, each subsequent formula being either a repetition or the result of the application of an inference rule to previous formulas, and with no term introduced by O-ecth or I-ecth occurring in the last formula c.

It is easy to show now that *EC* is sound.

Theorem 3.1 *For any* Γ *and* c*, if* $\Gamma \vdash_{EC} c$*, then* (Γ, c) *is valid.*

Proof We have shown (in the proof of Theorem 2.2) that $(\{E\alpha\beta\}, O\alpha\beta)$ is valid. Hence, E-sub is sound. Since all the other inference rules of *EC* are rules in the sound system *SE*, which has the same notion of deduction, *EC* is sound. □

Let us also show that *EC* (like *SE*) is not explosive.

Theorem 3.2 *There are* Γ *inconsistent and* c *such that* $\Gamma \nvdash_{EC} c$.

Proof Consider Γ to be an inconsistent set of formulas, and consider c to be a formula with terms which do not occur in Γ. In *EC*-deductions, ecthesis is the only way to introduce new terms. But, there must be no occurrence of these terms in the conclusion. Hence, obviously, $\Gamma \nvdash_{EC} c$. □

For the proof of *EC* s-completeness, it will be convenient to have the moods *Bocardo* and *Baroco* as derived inference rules. The schema for *Bocardo* already presented above is available in *EC*, so it remains to show that *Baroco* is derivable in *EC*.

Baroco $A\alpha\beta, O\gamma\beta \vdash_{EC} O\gamma\alpha$

(1)	$A\alpha\beta$	Premise
(2)	$O\gamma\beta$	Premise
(3)	$A\lambda\gamma$	2, O-ecth
(4)	$E\lambda\beta$	2, O-ecth
(5)	$E\beta\lambda$	4, E-conv
(6)	$E\alpha\lambda$	5,1, *Celarent*
(7)	$E\lambda\alpha$	6, E-conv
(8)	$I\gamma\lambda$	3, A-conv
(9)	$O\gamma\alpha$	7, 8, *Ferio*

EC will be shown to be s-complete relative to the c-completeness of Corcoran's system *RD*. For a simplified proof, it will be useful to introduce here a definition of a *chain*[7] of A-formulas and to prove two lemmas concerning chains in the deductions of *RD*.

Definition 3.1 A set C of A-formulas is said to be a *chain* from α to β (in symbols $\widehat{\alpha\beta}$) if and only if either the members of C can be arranged in a sequence $\langle A\alpha\lambda_1, A\lambda_1\lambda_2, \ldots, A\lambda_n\beta\rangle$ with $n \geq 0$ and $\alpha \neq \beta$ (C is then said to be a *nonempty* chain $\widehat{\alpha\beta}$) or $\alpha = \beta$ (C is then said to be an *empty* chain $\widehat{\alpha\beta}$).

Despite the fact that $\widehat{\alpha\beta}$ is clearly not a singular name, it will be convenient, when dealing with a certain determined chain C, to write $\widehat{\alpha\beta} \subseteq E$, $f \in \widehat{\alpha\beta}$, $\widehat{\alpha\beta} = \{A\alpha\beta\}$, and $\widehat{\alpha\beta} = \emptyset$ in order to signify, respectively, $C \subseteq E$, $f \in C$, $C = \{A\alpha\beta\}$, and C is an empty chain.[8] From these definition and conventions, the following results are obvious.

If $\widehat{\alpha\beta} \subseteq E$ and $\widehat{\beta\gamma} \subseteq E$, then $\widehat{\alpha\gamma} \subseteq E$.
If $\widehat{\alpha\beta} \subseteq E$ and $A\gamma\delta \in \widehat{\alpha\beta}$, then $\widehat{\alpha\gamma}, \widehat{\delta\beta} \subseteq E \setminus \{A\gamma\delta\}$.
If $\widehat{\alpha\gamma}, \widehat{\delta\beta} \subseteq E$, then $\widehat{\alpha\beta} \subseteq E \cup \{A\gamma\delta\}$.

Now come the two lemmas.

Lemma 1 *We have that* $\Gamma \vdash^{\text{Dir}}_{RD} A\alpha\beta$ *if and only if there is a nonempty chain* $\widehat{\alpha\beta} \subseteq \Gamma$.

Proof The if clause is quite obvious since if there is a nonempty $\widehat{\alpha\beta}$ chain with n members in Γ, then clearly $\Gamma \vdash^{\text{Dir}}_{RD} A\alpha\beta$, by $n - 1$ applications of *Barbara*. The only if clause will be proved by induction on the number of lines of the given deduction. Suppose first that $\Gamma \vdash^{\text{Dir}}_{RD} A\alpha\beta$ is in one line. In this case, $A\alpha\beta$ must be in the premises. Hence, we have a nonempty $\widehat{\alpha\beta}$ chain in Γ. Now, assume that the lemma is correct for all direct deductions with at most n lines (induction hypothesis). Then, consider a deduction $\Gamma \vdash^{\text{Dir}}_{RD} A\alpha\beta$ with $n + 1$ lines. Since the only rule in

RD which results in an A-proposition is *Barbara*, there are only three possibilities: (a) $A\alpha\beta \in \Gamma$ (which is like in the 1-line case), (b) $A\alpha\beta$ results from the repetition of a line $i \leq n$ (this case falls directly under the induction hypothesis), or (c) $A\alpha\beta$ results by *Barbara* from two formulas $A\alpha\gamma$, $A\gamma\beta$ at lines $i, j \leq n$. Then, by the induction hypothesis, we have the nonempty chains $\widehat{\alpha\gamma}, \widehat{\gamma\beta} \subseteq \Gamma$ and, hence, a nonempty chain $\widehat{\alpha\beta} \subseteq \Gamma$. □

Lemma 2 *We have that* $\Gamma \vdash^{\mathrm{Dir}}_{RD} E\alpha\beta$ *if and only if there is some* E*-formula* $E\gamma\delta \in \Gamma$ *and* Γ *contains one pair of (possibly empty) chains: either* $\widehat{\alpha\gamma}, \widehat{\beta\delta} \subseteq \Gamma$ *or* $\widehat{\alpha\delta}, \widehat{\beta\gamma} \subseteq \Gamma$.

Proof For the if clause, consider that we have $E\gamma\delta \in \Gamma$ and $\widehat{\alpha\gamma}, \widehat{\beta\delta} \subseteq \Gamma$. Suppose first that $\alpha \neq \gamma$ and $\beta \neq \delta$. We then have the following direct deductions in *RD*:

(1)	$\Gamma \vdash^{\mathrm{Dir}}_{RD} E\gamma\delta$	for $E\gamma\delta \in \Gamma$
(2)	$\Gamma \vdash^{\mathrm{Dir}}_{RD} A\alpha\gamma$	for $\widehat{\alpha\gamma} \subseteq \Gamma, \widehat{\alpha\gamma} \neq \emptyset$, Lemma 1
(3)	$\Gamma \vdash^{\mathrm{Dir}}_{RD} A\beta\delta$	for $\widehat{\beta\delta} \subseteq \Gamma, \widehat{\beta\delta} \neq \emptyset$, Lemma 1
(4)	$\Gamma \vdash^{\mathrm{Dir}}_{RD} E\alpha\delta$	1, 2, *Celarent*
(5)	$\Gamma \vdash^{\mathrm{Dir}}_{RD} E\delta\alpha$	4, E-conv
(6)	$\Gamma \vdash^{\mathrm{Dir}}_{RD} E\beta\alpha$	3, 5, *Celarent*
(7)	$\Gamma \vdash^{\mathrm{Dir}}_{RD} E\alpha\beta$	6, E-conv

Now if $\alpha = \gamma$ (resp., $\beta = \delta$), we do not have line 2 (resp., line 3), but line 1 is $E\alpha\delta$ (resp., $E\gamma\beta$) and $E\alpha\beta$ results without the first *Celarent* (resp., with only the first *Celarent*). The case with $\alpha = \gamma$ and $\beta = \delta$ is trivial for $E\alpha\beta = E\gamma\delta$. A similar proof works for the other pair of chains $\widehat{\alpha\delta}, \widehat{\beta\gamma}$. The only if clause will be proved by induction on the number of lines of the given deduction. Suppose first that $\Gamma \vdash^{\mathrm{Dir}}_{RD} E\alpha\beta$ is in one line. In this case, obviously $E\alpha\beta \in \Gamma$ and the lemma is correct, since $E\gamma\delta = E\alpha\beta$ and $\widehat{\alpha\gamma}, \widehat{\beta\delta}$ are simply empty. Now, assume that the lemma is correct for all direct deductions with at most n lines (induction hypothesis). Then, consider a deduction $\Gamma \vdash^{\mathrm{Dir}}_{RD} E\alpha\beta$ with $n+1$ lines. Since the only rules in *RD* which result in an E-formula are E-conv and *Celarent*, there are four possibilities: (a) $E\alpha\beta \in \Gamma$, (b) $E\alpha\beta$ is a repetition, (c) $E\alpha\beta$ results by E-conv, or (d) $E\alpha\beta$ results by *Celarent*. Case (a) is exactly like in the 1-line case. Cases (b) and (c) fall directly under the induction hypothesis, since we must have $E\alpha\beta$ or $E\beta\alpha$ at a line $i \leq n$. In case (d), $E\alpha\beta$ results by *Celarent* from two formulas $E\lambda\beta$ and $A\alpha\lambda$ at lines $i, j \leq n$. From $E\lambda\beta$, we have by the induction hypothesis: $E\gamma\delta \in \Gamma$ and either $\widehat{\lambda\gamma}, \widehat{\beta\delta} \subseteq \Gamma$ or $\widehat{\beta\gamma}, \widehat{\lambda\delta} \subseteq \Gamma$. But, from $A\alpha\lambda$, we have by Lemma 1: $\widehat{\alpha\lambda} \subseteq \Gamma$. Hence, we have $E\gamma\delta \in \Gamma$ and either $\widehat{\alpha\lambda}, \widehat{\lambda\gamma}, \widehat{\beta\delta} \subseteq \Gamma$ or $\widehat{\beta\gamma}, \widehat{\alpha\lambda}, \widehat{\lambda\delta} \subseteq \Gamma$. Considering the union of the chains including the term λ in both cases, we finally have $E\gamma\delta \in \Gamma$ and either $\widehat{\alpha\gamma}, \widehat{\beta\delta} \subseteq \Gamma$ or $\widehat{\alpha\delta}, \widehat{\beta\gamma} \subseteq \Gamma$. □

3.2 Completeness proof We can now present the s-completeness proof of *EC*.

Theorem 3.3 *If* $\Gamma \models c$ *with* Γ *consistent, then* $\Gamma \vdash_{EC} c$.

Proof Since[9] Corcoran's *RD* is c-complete, we have to show that, for every deduction $\Gamma \vdash_{RD} c$ with Γ consistent, there is a deduction $\Gamma \vdash_{EC} c$. First, note that all the inference rules in *RD* are also rules in *EC*. Obviously, every direct deduction in *RD* is a deduction in *EC*. Then, it will be sufficient here to show that there is a deduction $\Gamma \vdash_{EC} c$, for every indirect deduction $\Gamma \vdash_{RD}^{\text{Ind}} c$, with Γ consistent. An indirect deduction in *RD* is such that there is a direct deduction $\Gamma \cup \{\overline{c}\} \vdash_{RD}^{\text{Dir}} p, \overline{p}$, that is, a direct deduction of some couple of contradictory formulas $(p, \overline{p})$, from the set $\Gamma \cup \{\overline{c}\}$. There are four cases to be examined according to the possible forms of the conclusion c of the given indirect deduction. Those forms are (1) $A\alpha\beta$, (2) $E\alpha\beta$, (3) $I\alpha\beta$, and (4) $O\alpha\beta$.

Case 1: $c = A\alpha\beta$. The reductio assumption is then $\overline{c} = O\alpha\beta$. Since Γ is consistent, $O\alpha\beta$ occurs essentially in the derivation of the contradiction $(p, \overline{p})$. But in *RD*, an O-formula can only be repeated and the contradiction must be $(O\alpha\beta, A\alpha\beta)$. Now, $A\alpha\beta$ cannot be obtained with the help of the reductio assumption $O\alpha\beta$. Hence, there must be a direct deduction $\Gamma \vdash_{RD}^{\text{Dir}} A\alpha\beta$. Thus, we also have $\Gamma \vdash_{EC} c$.

Case 2: $c = E\alpha\beta$. The reductio assumption here is $\overline{c} = I\alpha\beta$. Since Γ is consistent, $I\alpha\beta$ occurs essentially in the derivation of the contradiction. As such a formula can only be repeated in a deduction in *RD*, the contradiction is $(I\alpha\beta, E\alpha\beta)$. Like in Case 1, there must be a direct deduction $\Gamma \vdash_{RD}^{\text{Dir}} E\alpha\beta$. Hence, we also have $\Gamma \vdash_{EC} c$.

Case 3: $c = I\alpha\beta$. The reductio assumption is $\overline{c} = E\alpha\beta$. Since Γ is consistent, $E\alpha\beta$ occurs essentially in the derivation of $(p, \overline{p})$. Since all the inference rules in *RD* with an E-premise (*Celarent*, E-conv) give an E-result, necessarily the lines depending on $\overline{c}$ are E-formulas. Hence, one of the members of the contradiction is an E-formula $E\gamma\delta$, the other member being an I-formula $I\gamma\delta$, directly deduced from Γ. In other words, we have $\Gamma \cup \{E\alpha\beta\} \vdash_{RD}^{\text{Dir}} E\gamma\delta$, $\Gamma \nvdash_{RD}^{\text{Dir}} E\gamma\delta$, and $\Gamma \vdash_{RD}^{\text{Dir}} I\gamma\delta$. Now, by Lemma 2, there is an E-formula $E\gamma'\delta' \in \Gamma \cup \{E\alpha\beta\}$ and two (possibly empty) chains $\widehat{\gamma\gamma'}$, $\widehat{\delta\delta'}$ or $\widehat{\delta\gamma'}$, $\widehat{\gamma\delta'}$ in Γ. Since $E\alpha\beta$ is essential in the deduction of $E\gamma\delta$, obviously $E\gamma'\delta' = E\alpha\beta$ and the chains in Γ are $\widehat{\gamma\alpha}$, $\widehat{\delta\beta}$ or $\widehat{\delta\alpha}$, $\widehat{\gamma\beta}$. Consider that it is the first pair of chains $\widehat{\gamma\alpha}$, $\widehat{\delta\beta}$, and suppose they are not empty (i.e., $\gamma \neq \alpha$ and $\delta \neq \beta$). We can then construct the following deductions in *EC*.

(1)	$\Gamma \vdash_{EC} A\gamma\alpha$	for $\widehat{\gamma\alpha} \subseteq \Gamma$, $\widehat{\gamma\alpha} \neq \emptyset$, Lemma 1
(2)	$\Gamma \vdash_{EC} A\delta\beta$	for $\widehat{\delta\beta} \subseteq \Gamma$, $\widehat{\delta\beta} \neq \emptyset$, Lemma 1
(3)	$\Gamma \vdash_{EC} I\gamma\delta$	for $\Gamma \vdash_{RD}^{\text{Dir}} I\gamma\delta$
(4)	$\Gamma \vdash_{EC} A\lambda\gamma$	3, I-ecth
(5)	$\Gamma \vdash_{EC} A\lambda\delta$	3, I-ecth
(6)	$\Gamma \vdash_{EC} A\lambda\alpha$	1, 4, *Barbara*
(7)	$\Gamma \vdash_{EC} A\lambda\beta$	2, 5, *Barbara*
(8)	$\Gamma \vdash_{EC} I\alpha\lambda$	6, A-conv
(9)	$\Gamma \vdash_{EC} I\alpha\beta$	7, 8, *Darii*

Now, if exactly one of the chains is empty $\widehat{\gamma\alpha} = \emptyset$ (resp., $\widehat{\delta\beta} = \emptyset$), the derivation is simplified. In that case, $\gamma = \alpha$ (resp., $\delta = \beta$), and we do not have line 1 (resp., line 2). Line 6 (resp., line 7) is then directly inferred by the ecthesis of $I\gamma\delta = I\alpha\delta$ (resp., $= I\gamma\beta$), without *Barbara*. Lastly, if both chains are empty, the derivation is trivial for $\gamma = \alpha$, $\delta = \beta$, and we have directly $\Gamma \vdash_{EC} I\alpha\beta$ from $\Gamma \vdash_{RD}^{\text{Dir}} I\gamma\delta$. With the other pair of chains $\widehat{\delta\alpha}$, $\widehat{\gamma\beta}$, a similar proof works, requiring only the inversion of the use of the results of ecthesis in the applications of *Barbara* (lines 6 and 7).

Case 4: $c = O\alpha\beta$. The reductio assumption is $\overline{c} = A\alpha\beta$. Here, there are two subcases: the contradiction which is derived from $\Gamma \cup \{A\alpha\beta\}$ is either (4.1)[10] $(A\gamma\delta, O\gamma\delta)$ or (4.2) $(I\gamma\delta, E\gamma\delta)$.

(4.1) The contradiction for the reductio is $(A\gamma\delta, O\gamma\delta)$. Since there is no inference rule in *RD* which results in a O-formula, necessarily $O\gamma\delta \in \Gamma$. Since Γ is consistent, the assumption $A\alpha\beta$ is essential in the derivation of the other member of the contradiction: $A\gamma\delta$. We have then $\Gamma \vdash^{\text{Dir}}_{RD} O\gamma\delta$, $\Gamma \cup \{A\alpha\beta\} \vdash^{\text{Dir}}_{RD} A\gamma\delta$, and $\Gamma \nvdash^{\text{Dir}}_{RD} A\gamma\delta$. By Lemma 1, there is a (nonempty) chain $\widehat{\gamma\delta} \subseteq \Gamma \cup \{A\alpha\beta\}$ and $A\alpha\beta \in \widehat{\gamma\delta}$. Suppose first that $\gamma \neq \alpha$ and $\delta \neq \beta$. In that case, the head and the bottom parts of the chain are in the premises: $\widehat{\gamma\alpha}, \widehat{\beta\delta} \subseteq \Gamma$. We can then construct the following deductions in *EC*:

(1)	$\Gamma \vdash_{EC} A\gamma\alpha$	for $\widehat{\gamma\alpha} \subseteq \Gamma$, $\widehat{\gamma\alpha} \neq \emptyset$, Lemma 1
(2)	$\Gamma \vdash_{EC} A\beta\delta$	for $\widehat{\beta\delta} \subseteq \Gamma$, $\widehat{\beta\delta} \neq \emptyset$, Lemma 1
(3)	$\Gamma \vdash_{EC} O\gamma\delta$	for $O\gamma\delta \in \Gamma$
(4)	$\Gamma \vdash_{EC} O\alpha\delta$	1, 3, *Bocardo*
(5)	$\Gamma \vdash_{EC} O\alpha\beta$	2, 4, *Baroco*

Now, if $\gamma = \alpha$ (resp., $\delta = \beta$), there is only one nonempty chain in the premises: $\widehat{\beta\delta} \subseteq \Gamma$ (resp., $\widehat{\gamma\alpha} \subseteq \Gamma$) and the derivation is simplified. We do not have line 1 (resp., line 2). At line 3, $O\gamma\delta = O\alpha\delta$ (resp., $= O\gamma\beta$), and we obtain the conclusion $O\alpha\beta$ by applying only *Baroco* (resp., only *Bocardo*). The case with both $\gamma = \alpha$ and $\delta = \beta$ is trivial, since we directly have $\Gamma \vdash_{EC} O\alpha\beta$ from $O\gamma\delta \in \Gamma$.

(4.2) The contradiction for the reductio is now $(I\gamma\delta, E\gamma\delta)$. Hence, we have $\Gamma \cup \{A\alpha\beta\} \vdash^{\text{Dir}}_{RD} I\gamma\delta, E\gamma\delta$. Since Γ is consistent, the reductio assumption $A\alpha\beta$ occurs essentially in the derivation of at least one of the members of the contradiction and possibly both. There are then three situations to consider:

(a) $A\alpha\beta$ is essential for $I\gamma\delta$, but not for $E\gamma\delta$:
$\Gamma \nvdash^{\text{Dir}}_{RD} I\gamma\delta$ and $\Gamma \vdash^{\text{Dir}}_{RD} E\gamma\delta$;

(b) $A\alpha\beta$ is essential for $E\gamma\delta$, but not for $I\gamma\delta$:
$\Gamma \nvdash^{\text{Dir}}_{RD} E\gamma\delta$ and $\Gamma \vdash^{\text{Dir}}_{RD} I\gamma\delta$;

(c) $A\alpha\beta$ is essential for both $E\gamma\delta$ and $I\gamma\delta$:
$\Gamma \nvdash^{\text{Dir}}_{RD} E\gamma\delta$ and $\Gamma \nvdash^{\text{Dir}}_{RD} I\gamma\delta$.

In situation (a), we have $\Gamma \vdash^{\text{Dir}}_{RD} E\gamma\delta$ and $\Gamma \cup \{A\alpha\beta\} \vdash^{\text{Dir}}_{RD} I\gamma\delta$, but $\Gamma \nvdash^{\text{Dir}}_{RD} I\gamma\delta$. Obviously, $I\gamma\delta \notin \Gamma \cup \{A\alpha\beta\}$. Hence, the only way in *RD* to derive $I\gamma\delta$ is by A-conv from $A\delta\gamma$, previously deduced from $\Gamma \cup \{A\alpha\beta\}$ with the essential use of $A\alpha\beta$. By Lemma 1, there is a nonempty chain $\widehat{\delta\gamma} \subseteq \Gamma \cup \{A\alpha\beta\}$ and $A\alpha\beta \in \widehat{\delta\gamma}$. Suppose first that $\delta \neq \alpha$ and $\gamma \neq \beta$. Hence, the head and the bottom parts of the chain are in the premises: $\widehat{\delta\alpha}, \widehat{\beta\gamma} \subseteq \Gamma$. We have then the following deductions in *EC*:

(1)	$\Gamma \vdash_{EC} E\gamma\delta$	for $\Gamma \vdash^{\text{Dir}}_{RD} E\gamma\delta$
(2)	$\Gamma \vdash_{EC} A\delta\alpha$	for $\widehat{\delta\alpha} \subseteq \Gamma$, $\widehat{\delta\alpha} \neq \emptyset$, Lemma 1
(3)	$\Gamma \vdash_{EC} A\beta\gamma$	for $\widehat{\beta\gamma} \subseteq \Gamma$, $\widehat{\beta\gamma} \neq \emptyset$, Lemma 1
(4)	$\Gamma \vdash_{EC} E\delta\gamma$	1, E-conv
(5)	$\Gamma \vdash_{EC} O\delta\gamma$	4, E-sub
(6)	$\Gamma \vdash_{EC} O\alpha\gamma$	2, 5, *Bocardo*
(7)	$\Gamma \vdash_{EC} O\alpha\beta$	3, 6, *Baroco*

Now, if $\delta = \alpha$ (resp., $\gamma = \beta$), there is only one nonempty chain in the premises: $\widehat{\beta\gamma} \subseteq \Gamma$ (resp., $\widehat{\delta\alpha} \subseteq \Gamma$) and the derivation is simplified. We do not have line 2 (resp., line 3). Line 5 is $O\alpha\gamma$ (resp., $O\delta\beta$), and the conclusion results with only *Baroco* (resp., only *Bocardo*). Consider the case with both $\delta = \alpha$ and $\gamma = \beta$.[11] We have then the following deductions in *EC*:

(1)	$\Gamma \vdash_{EC} E\beta\alpha$	for $\Gamma \vdash^{\text{Dir}}_{RD} E\gamma\delta$ and $\delta = \alpha, \gamma = \beta$
(2)	$\Gamma \vdash_{EC} E\alpha\beta$	1, E-conv
(3)	$\Gamma \vdash_{EC} O\alpha\beta$	2, E-sub

In situation (b), we have $\Gamma \vdash^{\text{Dir}}_{RD} I\gamma\delta$ and $\Gamma \nvdash^{\text{Dir}}_{RD} E\gamma\delta$. But, since we also have $\Gamma \cup \{A\alpha\beta\} \vdash^{\text{Dir}}_{RD} E\gamma\delta$, by Lemma 2, there are some $E\gamma'\delta' \in \Gamma$ and a pair of (possibly empty) chains in $\Gamma \cup \{A\alpha\beta\}$: either $\widehat{\gamma\gamma'}$, $\widehat{\delta\delta'}$ or $\widehat{\delta\gamma'}$, $\widehat{\gamma\delta'}$. Moreover, since $\Gamma \nvdash^{\text{Dir}}_{RD} E\gamma\delta$, the reductio assumption $A\alpha\beta$ must be a link either in the first or in the second chain of the pair. For example, if it is $\widehat{\gamma\gamma'}, \widehat{\delta\delta'} \subseteq \Gamma \cup \{A\alpha\beta\}$ and $A\alpha\beta \in \widehat{\gamma\gamma'}$, then Γ contains $E\gamma'\delta'$, $\widehat{\gamma\alpha}$, $\widehat{\beta\gamma'}$, and $\widehat{\delta\delta'}$ (the three chains being possibly empty). Considering all the possibilities, we have then four cases in which the chains are all possibly empty:

(i) $\widehat{\gamma\alpha}, \widehat{\beta\gamma'}, \widehat{\delta\delta'} \subseteq \Gamma$ and, since $E\gamma'\delta' \in \Gamma$, by Lemma 2, $\Gamma \vdash^{\text{Dir}}_{RD} E\delta\beta$;
(ii) $\widehat{\gamma\gamma'}, \widehat{\delta\alpha}, \widehat{\beta\delta'} \subseteq \Gamma$ and, since $E\gamma'\delta' \in \Gamma$, by Lemma 2, $\Gamma \vdash^{\text{Dir}}_{RD} E\gamma\beta$;
(iii) $\widehat{\delta\alpha}, \widehat{\beta\gamma'}, \widehat{\gamma\delta'} \subseteq \Gamma$ and, since $E\gamma'\delta' \in \Gamma$, by Lemma 2, $\Gamma \vdash^{\text{Dir}}_{RD} E\gamma\beta$;
(iv) $\widehat{\delta\gamma'}, \widehat{\gamma\alpha}, \widehat{\beta\delta'} \subseteq \Gamma$ and, since $E\gamma'\delta' \in \Gamma$, by Lemma 2, $\Gamma \vdash^{\text{Dir}}_{RD} E\delta\beta$.

It is worth noting that, in each case, Lemma 2 uses only two chains, which are not required to be nonempty. The remaining chain is always either $\widehat{\gamma\alpha}$ or $\widehat{\delta\alpha}$. Hence, we can construct the following deductions in *EC*:

(1)	$\Gamma \vdash_{EC} E\delta\beta$ (or $\Gamma \vdash_{EC} E\gamma\beta$)	for $\Gamma \vdash^{\text{Dir}}_{RD} E\delta\beta$ (or $\Gamma \vdash^{\text{Dir}}_{RD} E\gamma\beta$)
(2)	$\Gamma \vdash_{EC} A\gamma\alpha$ (or $\Gamma \vdash_{EC} A\delta\alpha$)	for $\widehat{\gamma\alpha} \subseteq \Gamma$ (or $\widehat{\delta\alpha} \subseteq \Gamma$), Lemma 1
(3)	$\Gamma \vdash_{EC} I\gamma\delta$	for $\Gamma \vdash^{\text{Dir}}_{RD} I\gamma\delta$
(4)	$\Gamma \vdash_{EC} A\lambda\gamma$	3, I-ecth
(5)	$\Gamma \vdash_{EC} A\lambda\delta$	3, I-ecth
(6)	$\Gamma \vdash_{EC} E\lambda\beta$	1, 5, *Celarent* (or 1, 4, *Celarent*)
(7)	$\Gamma \vdash_{EC} O\lambda\beta$	6, E-sub
(8)	$\Gamma \vdash_{EC} A\lambda\alpha$	2, 4, *Barbara* (or 2, 5, *Barbara*)
(9)	$\Gamma \vdash_{EC} O\alpha\beta$	7, 8, *Bocardo*

Line 2 is problematic, because Lemma 1 requires $\widehat{\gamma\alpha} \neq \emptyset$ (resp., $\widehat{\delta\alpha} \neq \emptyset$). But, if $\widehat{\gamma\alpha} = \emptyset$ (resp., $\widehat{\delta\alpha} = \emptyset$), then we have $\gamma = \alpha$ (resp., $\delta = \alpha$) and the derivation goes as follows:

(1)	$\Gamma \vdash_{EC} E\delta\beta$ (or $\Gamma \vdash_{EC} E\gamma\beta$)	for $\Gamma \vdash^{\text{Dir}}_{RD} E\delta\beta$ (or $\Gamma \vdash^{\text{Dir}}_{RD} E\gamma\beta$)
(2)	$\Gamma \vdash_{EC} I\alpha\delta$ (or $\Gamma \vdash_{EC} I\gamma\alpha$)	for $\Gamma \vdash^{\text{Dir}}_{RD} I\gamma\delta$ and $\gamma = \alpha$ (or $\delta = \alpha$)
(3)	$\Gamma \vdash_{EC} A\lambda\alpha$ (or $\Gamma \vdash_{EC} A\lambda\gamma$)	2, I-ecth
(4)	$\Gamma \vdash_{EC} A\lambda\delta$ (or $\Gamma \vdash_{EC} A\lambda\alpha$)	2, I-ecth
(5)	$\Gamma \vdash_{EC} E\lambda\beta$	1, 4, *Celarent* (or 1, 3, *Celarent*)
(6)	$\Gamma \vdash_{EC} O\lambda\beta$	5, E-sub
(7)	$\Gamma \vdash_{EC} O\alpha\beta$	3, 6, *Bocardo* (or 4, 6, *Bocardo*)

Situation (c)[12] is a blend of the two preceding situations, since we have $\Gamma \nvdash^{\text{Dir}}_{RD} E\gamma\delta$ and $\Gamma \nvdash^{\text{Dir}}_{RD} I\gamma\delta$. Like in (b), we have the four cases (i)–(iv). Like in (a), we do not have $I\gamma\delta$ (hence, no possibility to apply I-ecth). Instead, we have the two (possibly empty) chains $\widehat{\delta\alpha}$, $\widehat{\beta\gamma}$ in Γ.

Consider first the cases (i) and (iv) from situation (b). As we have seen, in these cases $\Gamma \vdash^{\text{Dir}}_{RD} E\delta\beta$. If $\delta = \alpha$, we immediately have $\Gamma \vdash_{EC} E\alpha\beta$ and then $\Gamma \vdash_{EC} O\alpha\beta$ results by E-sub. Now, if $\delta \neq \alpha$, since $\widehat{\delta\alpha} \subseteq \Gamma$, we can construct the following deductions in *EC*:

(1)	$\Gamma \vdash_{EC} E\delta\beta$	for $\Gamma \vdash_{RD}^{\mathrm{Dir}} E\delta\beta$
(2)	$\Gamma \vdash_{EC} A\delta\alpha$	for $\widehat{\delta\alpha} \subseteq \Gamma$, $\widehat{\delta\alpha} \neq \emptyset$, Lemma 1
(3)	$\Gamma \vdash_{EC} O\delta\beta$	1, E-sub
(4)	$\Gamma \vdash_{EC} O\alpha\beta$	2, 3, *Bocardo*

The two remaining cases, (ii) and (iii), are impossible with Γ consistent. As we have seen, in these cases we have $\Gamma \vdash_{RD}^{\mathrm{Dir}} E\gamma\beta$. But, since $\widehat{\beta\gamma} \subseteq \Gamma$, if $\beta \neq \gamma$, we have $\Gamma \vdash_{RD}^{\mathrm{Dir}} A\beta\gamma$ by Lemma 1 and $\Gamma \vdash_{RD}^{\mathrm{Dir}} I\gamma\beta$ by A-conv. Hence, $\Gamma \vdash_{RD}^{\mathrm{Dir}} E\gamma\beta, I\gamma\beta$; Γ is inconsistent. Now, if $\beta = \gamma$, since we have $\widehat{\gamma\gamma'}$, $\widehat{\beta\delta'} \subseteq \Gamma$ in case (ii) and $\widehat{\beta\gamma'}$, $\widehat{\gamma\delta'} \subseteq \Gamma$ in case (iii), in both cases we have $\widehat{\beta\gamma'}$, $\widehat{\beta\delta'} \subseteq \Gamma$. Hence, by Lemma 1, $\Gamma \vdash_{RD}^{\mathrm{Dir}} A\beta\gamma', A\beta\delta'$ (unless $\beta = \gamma'$ and $\beta = \delta'$), and by A-conv and *Darii*, $\Gamma \vdash_{RD}^{\mathrm{Dir}} I\gamma'\delta'$. As $E\gamma'\delta' \in \Gamma$, Γ is inconsistent. If $\beta = \gamma'$ (resp., $\beta = \delta'$), we would only have $\Gamma \vdash_{RD}^{\mathrm{Dir}} A\beta\delta'$ (resp., $\Gamma \vdash_{RD}^{\mathrm{Dir}} A\beta\gamma'$) by Lemma 1 and, hence, $\Gamma \vdash_{RD}^{\mathrm{Dir}} I\delta'\beta$ (resp., $\Gamma \vdash_{RD}^{\mathrm{Dir}} I\gamma'\beta$) by A-conv. This case is also inconsistent because $E\gamma'\delta'$ would be $E\beta\delta'$ (resp., $E\gamma'\beta$), and so we would have $\Gamma \vdash_{RD}^{\mathrm{Dir}} E\delta'\beta$ by E-conv (resp., $\Gamma \vdash_{RD}^{\mathrm{Dir}} E\gamma'\beta$). The case with $\beta = \gamma'$ and $\beta = \delta'$ is absurd since $E\gamma'\delta'$ has distinct terms. □

The system *EC* is then sound, s-complete, and not explosive. This system is of course a speculative construction. Even if there are several indications in [1] positioning ecthesis as an alternative to per impossibile deductions, there is no explicit evidence for the idea that Aristotle would have thought of ecthesis as such an alternative. Nevertheless, by resting only on syllogistic tools, *EC* shows that reductio ad absurdum is not essential in Aristotle's assertoric logic. Moreover, the fact that *EC* is not explosive makes this system close to Aristotle's views on the role of contradictions in logic. As has been pointed out by Priest [10], these views are not compatible with the *ex contradictione sequitur quodlibet* principle.

Notes

1. The exclusion of the empty set from the possible values of terms is an important feature of Corcoran's semantics. This is one of the solutions to the so-called problem of the existential import. For a recent discussion of the different solutions to this widespread problem, see Malink [8, pp. 41–44].

2. See, for example, [7] and [13]. For an extended discussion of this first kind of interpretation, see [5], [12], and [8].

3. For this second interpretation, see for example [11], Thom [14], [15], and Żarnecka-Biały [16]. More recently, Parsons [9, Chapter 1] has given a stimulating individual-term interpretation with which ecthesis, expository syllogism (the converse of ecthesis), and reductio ad absurdum are considered as the only primitive tools of Aristotle's assertoric syllogistic.

4. As we will show in the next section, O-ecth is sound with respect to Corcoran's semantics. This depends on the rejection of empty terms from this semantics. According to Parsons [9, pp. 36–37], O-ecth is not valid if the subject α of the premise $O\alpha\beta$ is possibly empty (in the sense that it has no individuals falling under it), since $O\alpha\beta$ is true and $A\gamma\alpha$ is false when α is empty. This is only correct if you adopt a semantics in which a term is interpreted as the set of individuals falling under it. But, as has been pointed

out by Malink [8], this is not the only possibility, since the entities associated with a term can also be species, or even terms. Malink makes very interesting suggestions in this direction. His preorder semantics has several important virtues. In our opinion, it remains unclear whether preorder and Corcoran's semantics are really extensionally different. A detailed examination of this model-theoretic question would be beyond the scope of this paper.

5. O-intro reduces to *Ferio* by conversion of the premise $A\gamma\alpha$, I-intro reduces to *Darii* also by conversion of $A\gamma\alpha$, and I-conv reduces to *Darii* by the deduction: (1) $I\alpha\beta$ (Premise), (2) $A\lambda\alpha$ (1, I-ecth), (3) $A\lambda\beta$ (1, I-ecth), (4) $I\beta\lambda$ (3, A-conv), (5) $I\beta\alpha$ (2, 4, *Darii*).

6. *Subalternation* is not present in [1], but it is explicitly mentioned by Aristotle [2, 109^a1–6 and 119^a34–37].

7. This notion (in fact, a very similar one) is due to Smiley [11].

8. It is worth noting that this last expression is just a convenient notation, since an empty chain is not necessarily the empty set. In fact, "$\widehat{\alpha\beta} = \emptyset$" simply signifies $\alpha = \beta$.

9. This proof follows roughly the main lines already given in Smith [13]. The exact locations where Smith's proof had to be corrected or improved are indicated further in the paper.

10. This case (4.1) is incorrectly considered as impossible in [13, p. 229].

11. This case is not examined in [13]. This constitutes the main flaw in Smith's proof, since there is no possibility here to get the expected conclusion in *SE* without a supplementary rule like E-sub.

12. This last situation is incorrectly considered as impossible in [13, p. 230].

References

[1] Aristotle, *Prior Analytics*, *Book I*, English translation and commentary by G. Striker, Oxford University Press, New York, 2009. 329, 331, 332, 340, 341

[2] Aristotle, *Topics*, pp. 167–277 in *The Complete Works of Aristotle*, vol. 1, edited by J. Barnes, Princeton University Press, Princeton, 1984. 341

[3] Corcoran, J., "Completeness of an ancient logic," *Journal of Symbolic Logic*, vol. 37 (1972), pp. 696–702. Zbl 0261.02004. MR 0317886. DOI 10.2307/2272415. 329, 330, 331

[4] Corcoran, J., "Aristotle's natural deduction system," pp. 85–131 in *Ancient Logic and its Modern Interpretations*, vol. 9 of *Synthese Historical Library*, edited by J. Corcoran, Reidel, Dordrecht, 1974. Zbl 0286.02002. MR 0497848. 329, 331

[5] Joray, P., "The principle of contradiction and ecthesis in Aristotle's syllogistic," *History and Philosophy of Logic*, vol. 35 (2014), pp. 219–36. Zbl 1312.03002. MR 3227810. DOI 10.1080/01445340.2014.907976. 329, 331, 340

[6] Łukasiewicz, J., *O zasadzie sprzeczności u Arystotelesa* (in Polish), Polska Akademia Umiejętnosci, Kraków, 1910. 329

[7] Łukasiewicz, J., *Aristotle's Syllogistic from the Standpoint of Modern Formal Logic*, Oxford University Press, Oxford, 1951. Zbl 0043.24601. MR 0053039. 331, 340

[8] Malink, M., *Aristotle's Modal Syllogistic*, Harvard University Press, Cambridge, Mass., 2013. Zbl 06353892. MR 3185722. 340, 341

[9] Parsons, T., *Articulating Medieval Logic*, Oxford University Press, Oxford, 2014. Zbl 1302.03002. MR 3445208. DOI 10.1093/acprof:oso/9780199688845.001.0001. 340

[10] Priest, G., "What's so bad about contradictions?," pp. 23–38 in *The Law of Non-Contradiction*, edited by G. Priest, J. C. Beall, and B. Armour-Garb, Oxford University Press, New York, 2004. Zbl 1066.03018. MR 2203481. DOI 10.1093/acprof:oso/9780199265176.003.0002. 340

[11] Smiley, T., "What is a syllogism?," *Journal of Philosophical Logic*, vol. 2 (1973), pp. 136–54. Zbl 0259.02005. DOI 10.1007/BF02115614. 329, 331, 340, 341

[12] Smith, R., "What is Aristotelian ecthesis?," *History and Philosophy of Logic*, vol. 3 (1982), pp. 113–27. MR 0691223. 332, 340

[13] Smith, R., "Completeness of an ecthetic syllogistic," *Notre Dame Journal of Formal Logic*, vol. 24 (1983), pp. 224–32. Zbl 0476.03004. MR 0687345. DOI 10.1305/ndjfl/1093870312. 329, 331, 332, 340, 341

[14] Thom, P., "Ecthesis," *Logique et Analyse, Nouvelle Série*, vol. 19 (1976), pp. 299–310. Zbl 0369.02008. MR 0450013. 340

[15] Thom, P., "Apodeictic ecthesis," *Notre Dame Journal of Formal Logic*, vol. 34 (1993), pp. 193–208. Zbl 0789.03007. MR 1231284. DOI 10.1305/ndjfl/1093634652. 340

[16] Żarnecka-Biały, E., "Aristotle's proof by ecthesis," *Bulletin of the Section of Logic (Łódź)*, vol. 21 (1993), pp. 40–44. 340

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