

# Nonstandard Functional Interpretations and Categorical Models

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**Abstract** Recently, the second author, Briseid, and Safarik introduced *non-standard Dialectica*, a functional interpretation capable of eliminating instances of familiar principles of nonstandard arithmetic—including overspill, underspill, and generalizations to higher types—from proofs. We show that the properties of this interpretation are mirrored by first-order logic in a constructive sheaf model of nonstandard arithmetic due to Moerdijk, later developed by Palmgren, and draw some new connections between nonstandard principles and principles that are rejected by strict constructivism. Furthermore, we introduce a variant of the Diller–Nahm interpretation with two different kinds of quantifiers, similar to Hernest’s light Dialectica interpretation, and show that one can obtain nonstandard Dialectica by weakening the computational content of the existential quantifiers—a process called *herbrandization*. We also define a constructive sheaf model mirroring this new functional interpretation, and show that the process of herbrandization has a clear meaning in terms of these sheaf models.

## 1 Introduction

The focus of this article stands at the confluence of two quite different paths in mathematical logic.

On one end, there is nonstandard arithmetic and analysis: a subject that has been an upshot of classical model theory, and even after it was recognized as being amenable to a syntactic treatment, as in Nelson’s *internal set theory* (see [17]), it mostly remained within the boundaries of classical set theory. On the other end, there is the markedly proof-theoretic topic of functional interpretations, stemming from Gödel’s *Dialectica interpretation* (see [6]); and, in particular, its recent revival by Kohlenbach and Oliva [11] through the program of *proof mining*.

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Where these ends meet, is in a general inclination toward the *constructivization* of mathematics. The first explicit model of nonstandard analysis, due to Schmieden and Laugwitz [26], was actually fully constructive, but had a quite weak transfer property. On the other hand, Robinson’s model of nonstandard arithmetic in [25], and subsequent ones which were elementary extensions of the standard model, were built from nonconstructive objects, such as nonprincipal ultrafilters of sets. Even in the syntactic approach, it soon became apparent that many useful principles led to instances of the excluded middle. But did nonstandard analysis really have *nothing* to offer to constructive analysts?

Not everyone was convinced, including, notably, P. Martin-Löf, who pushed the question in the early 1990s. First, Palmgren [20] succeeded in building a model with a restricted, yet useful transfer principle. Then, in 1995, Moerdijk [16] described the first constructive model of nonstandard arithmetic with a *full* transfer principle—a topos of sheaves over a category of filters. Later, by working in this topos, Palmgren [21]–[24] provided simplified, nonstandard proofs of several theorems of constructive analysis, thereby demonstrating the usefulness of this model.

But if nonstandard proofs do provide some constructive information, we might as well try to extract it in an automated fashion. In 2012, the second author, Briseid, and Safarik [30] succeeded in defining a functional interpretation, *nonstandard Dialectica*, which could eliminate nonstandard principles from proofs of intuitionistic arithmetic in all finite types, enriched, à la Nelson, with a predicate  $\text{st}_\sigma(x)$ , “ $x$  is standard,” for all types  $\sigma$ , also yielding a proof of conservativity of these principles over the base system. Section 2 is a review.

Now, some of the principles validated by nonstandard *Dialectica* were known to hold in Moerdijk’s topos—including a form of Nelson’s idealization axiom, an under-*spill* principle, and the undecidability of the standardness predicate. Our first aim was to investigate how deep this connection would go.

And a deep connection it is: with the exception of the principle called  $\text{HAC}^{\text{st}}$  (herbrandized axiom of choice), which requires some form of the axiom of choice in the metatheory, *all* the characteristic principles of nonstandard *Dialectica* are true in the topos model, for free. Section 3 is devoted to showing this. During this investigation, we also chanced upon two new principles, *sequence overspill* and *sequence under-*spill**, which appear to be more natural equivalents of principles that have been taken into consideration, earlier, in the context of proof-theoretic nonstandard arithmetic. We map their relation to other familiar principles from nonstandard and constructive analysis in Section 2.

Several characteristic principles of nonstandard *Dialectica* have a peculiarity: they are *herbrandized*. This is explained in more detail in Section 4; in short, where “traditional” functional interpretations would produce a *single* witness of an existential statement, these principles produce a *finite sequence* of *potential* witnesses, of which at least one is an actual witness. This property destroys the computational meaning of intuitionistic disjunction, yet seems unavoidable in the interpretation of nonstandard arithmetic.

The categorical analysis of nonstandard *Dialectica* supplied a very convenient way of “de-herbrandizing,” through a simple change in the Grothendieck topology, down from *finite* covers to *singleton* covers. Full transfer is lost—in the new topos, disjunction is stronger than in the metatheory—as is the link to nonstandard arithmetic; but

the de-herbrandized principles induce a new functional interpretation, which we call *uniform Diller–Nahm*, and this is the main focus of Section 4.

Uniform Diller–Nahm can be seen as an extension of the Diller–Nahm variant of the Dialectica interpretation (see Diller and Nahm [5]), and has some striking similarities to light Dialectica, a variant of Dialectica with two different kinds of quantifiers—computational and noncomputational—introduced in 2005 by Hernest [7], for the purpose of more efficient program extraction from formal proofs. Yet, irrespective of its technical value, the characteristic proof system of uniform Diller–Nahm might have a dignity of its own.

In 1985, Lifschitz [13] proposed a simple extension of Heyting arithmetic, where a distinction could be made between calculable and noncalculable natural numbers—a synthesis of classical and intuitionistic arithmetic. Under the interpretation of the predicate  $\text{st}(x)$  as “ $x$  is calculable,” the proof system of uniform Diller–Nahm seems to be well suited for Lifschitz’s intended calculus. This is also discussed in Section 4.

Finally, in Section 5, we survey some open questions.

**Note** This work is based on research done by the first author, under the supervision of the second author, in partial fulfillment of the requirements for the degree of *Laurea Magistrale* in Mathematics at the University of Pavia.

## 2 The Nonstandard Dialectica Interpretation

We start by briefly recalling the definition of the system  $\text{E-HA}_{\text{st}}^{\omega*}$ , as introduced in [30]; we refer to the original article for a detailed presentation.

**2.1 The system  $\text{E-HA}_{\text{st}}^{\omega*}$**  We take  $\text{E-HA}^{\omega*}$  to be an extension of the system called “ $\text{E-HA}_0^{\omega}$ ” by Troelstra [27], with additional types and constants for handling *finite sequences*. More precisely, the collection of types  $\mathbf{T}^*$  is generated by the inductive clauses

- ▷ 0 is in  $\mathbf{T}^*$ ;
- ▷ if  $\sigma, \tau$  are in  $\mathbf{T}^*$ , then  $\sigma \rightarrow \tau$  and  $\sigma^*$  are in  $\mathbf{T}^*$ ;

and, for all types  $\sigma, \tau$  in  $\mathbf{T}^*$ , we have constants  $\langle \rangle_{\sigma} : \sigma$  (empty sequence),  $C : \sigma \rightarrow \sigma^* \rightarrow \sigma^*$  (prepending operator), and  $L_{\sigma, \tau} : \sigma \rightarrow (\sigma \rightarrow \tau \rightarrow \sigma) \rightarrow (\tau^* \rightarrow \sigma)$  (list recursor), with defining axioms

$$\text{SA} : \quad \forall s : \sigma^* (s = \langle \rangle_{\sigma} \vee \exists x : \sigma \exists s' : \sigma^* (s = C x s')),$$

$$\begin{cases} L_{\sigma, \tau} x y \langle \rangle_{\tau} =_{\sigma} x, \\ L_{\sigma, \tau} x y (C z s) =_{\sigma} y (L_{\sigma, \tau} x y s) \langle z \rangle, \end{cases} \quad x : \sigma, y : \sigma \rightarrow \tau \rightarrow \sigma, z : \tau, s : \tau^*,$$

where  $\langle z \rangle$  is the “singleton”  $C z \langle \rangle_{\tau}$ .

**Notation** We use  $s, t, u, v$  (and  $s', t', \dots$ ) as variables of sequence type.

This system has an extensionality axiom

$$\forall f, g : \sigma \rightarrow \tau (f =_{\sigma \rightarrow \tau} g \leftrightarrow \forall x : \sigma f x =_{\tau} g x)$$

for all types  $\sigma, \tau$ .

Using the projectors and combinators from the language of  $\text{E-HA}_0^{\omega}$ , it is possible, already in the latter system, to introduce a coding of finite sequences of elements of any type, as in Kohlenbach [10, p. 59]; therefore,  $\text{E-HA}^{\omega*}$  is a *definitional*, hence

conservative, extension of  $\text{E-HA}_0^\omega$ . However, finite sequences seem to be quite ubiquitous in arguments of nonstandard arithmetic, mostly due to the expanded notion of “finiteness” in a nonstandard model; so it seems preferable to have them built into our syntax.

Since every type is provably inhabited, we can conservatively add for every type  $\sigma$  a constant  $\emptyset_\sigma$ . Using the list recursor, one can define all the basic operations on finite sequences one needs in practice as follows:

- (i) a *length function*  $|\cdot| : \sigma^* \rightarrow 0$ , satisfying

$$|\langle \rangle_\sigma| = 0, \quad |Cas| = S|s|,$$

for  $s : \sigma^*$ ,  $a : \sigma$ ;

- (ii) a *projection function*  $(s, i) \mapsto s_i$  of type  $\sigma^* \rightarrow 0 \rightarrow \sigma$ , satisfying

$$\begin{aligned} (\langle \rangle_\sigma)_i &= \emptyset_\sigma \quad \text{for all } i, \\ (Cas)_0 &= a, \\ (Cas)_{si} &= s_i; \end{aligned}$$

- (iii) a *concatenation operation*  $\cdot : \sigma^* \rightarrow \sigma^* \rightarrow \sigma^*$ , such that

$$\langle \rangle_\sigma \cdot t = t, \quad Cas \cdot t = Ca(s \cdot t);$$

as expected, concatenation is provably associative, so we will iterate it without bothering with brackets.

The following easy properties are all established in [30].

**Lemma 2.1** ([3, Lemma 2.15]) *We have*

- (a)  $\text{E-HA}^{\omega*} \vdash \forall s : \sigma^* (|s| = 0 \leftrightarrow s = \langle \rangle_\sigma)$ ,  
 (b)  $\text{E-HA}^{\omega*} \vdash \forall n : 0 \forall s : \sigma^* (|s| = Sn \leftrightarrow \exists x : \sigma \exists t : \sigma^* (s = Cxt \wedge |t| = n))$ .

**Proof** Let  $s : \sigma^*$ . By the sequence axiom SA, either  $s = \langle \rangle_\sigma$  or  $s = Cxt$  for some  $x : \sigma$ ,  $t : \sigma^*$ . If  $|s| = 0$ , then the latter case leads to a contradiction, for  $|s| = S|t| > 0$ .

If  $|s| = Sn$ , then the former case leads to a contradiction, and we have proved the left-to-right directions. The converses are immediate.  $\square$

**Proposition 2.2** ([3, Proposition 2.6]) *The extension  $\text{E-HA}^{\omega*}$  proves the induction schema for sequences*

$$\text{IA}^* : (\varphi(\langle \rangle_\sigma) \wedge \forall x : \sigma \forall s : \sigma^* (\varphi(s) \rightarrow \varphi(Cxs))) \rightarrow \forall s : \sigma^* \varphi(s).$$

**Proof** Suppose that  $\varphi(\langle \rangle_\sigma)$  and  $\forall x : \sigma \forall s : \sigma^* (\varphi(s) \rightarrow \varphi(Cxs))$ . By the previous lemma,

$$\forall s : \sigma^* (|s| = 0 \rightarrow \varphi(s)).$$

Fix  $n : 0$ , and assume that  $\forall s : \sigma^* (|s| = n \rightarrow \varphi(s))$ . Let  $s$  be of length  $Sn$ . Again by the previous lemma,  $s = Cxt$  for some  $x : \sigma$ , and  $t : \sigma^*$  of length  $n$ , and  $\varphi(t)$  holds by hypothesis. Therefore,  $\varphi(Cxt) \equiv \varphi(s)$  holds as well, and we have proved

$$\forall s : \sigma^* (|s| = n \rightarrow \varphi(s)) \rightarrow \forall s : \sigma^* (|s| = Sn \rightarrow \varphi(s)).$$

By ordinary induction, it follows that  $\forall n : 0, s : \sigma^* (|s| = n \rightarrow \varphi(s))$ .  $\square$

**Definition 2.3** Let  $s, t : \sigma^*$ . We say that  $s$  and  $t$  are *extensionally equal* and write  $s =_e t$  if

$$|s| = |t| \wedge \forall i < |s| (s_i = t_i).$$

**Corollary 2.4** ([3, Proposition 2.18]) *We have  $\text{E-HA}^{\omega*} \vdash \forall s, t : \sigma^* (s =_e t \rightarrow s = t)$ .*

**Proof** This is by induction for sequences. Suppose that  $s =_e t$ . If  $s = \langle \rangle_\sigma$ , then  $|s| = |t| = 0$ , so, by Lemma 2.1,  $t = \langle \rangle_\sigma$ .

Otherwise,  $s = Cxs'$  for some  $x, s'$ . Then  $|s| = |t| = Sn$  for  $n = |s'|$ ; again, by Lemma 2.1,  $t = Cyt'$  for some  $y, t'$ . But  $x = s_0 = t_0 = y$ , and  $s' =_e t'$ ; by the inductive hypothesis,  $s' = t'$ . Therefore,  $s = Cxs' = Cyt' = t$ .  $\square$

Since finite sequences will be used as a replacement for finite sets, we will borrow some set-theoretic notation.

**Definition 2.5** Let  $a : \sigma, s, s' : \sigma^*$ . We define the abbreviations

- (i)  $a \in_\sigma s := \exists i < |s| (a =_\sigma s_i)$  ( $a$  is an *element* of  $s$ );
- (ii)  $s' \subseteq_\sigma s := \forall x : \sigma (x \in_\sigma s' \rightarrow x \in_\sigma s)$  ( $s'$  is *contained* in  $s$ ).

We will drop subscripts in most instances. We also extend the relation  $\subseteq_\sigma$  to sequence-valued functionals, pointwise: for  $s', s : \tau \rightarrow \sigma^*$ ,

- (iii)  $s' \subseteq s := \forall x : \tau (s'x \subseteq_\sigma sx)$ .

The relation  $\subseteq$  determines a preorder, provably in  $\text{E-HA}^{\omega*}$ .

In the definition of the nonstandard Dialectica translation, one needs a form of application for finite sequences—and an associated form of  $\lambda$ -abstraction—that is *monotone* in the first component with respect to the preorder we just defined.

**Definition 2.6 (Finite sequence application and abstraction)** Let  $s : (\sigma \rightarrow \tau^*)^*$ ,  $a : \sigma, t : \tau^*$ . Then

$$s[a] := (s_0a) \cdot \dots \cdot (s_{|s|-1}a) : \tau^*,$$

$$\Lambda x : \sigma.t := C(\lambda x : \sigma.t)\langle \rangle : (\sigma \rightarrow \tau^*)^*.$$

The new application and abstraction are interdefinable with the usual ones. In fact, we have the following easy compatibility result.

**Proposition 2.7** *The extension  $\text{E-HA}^{\omega*}$  proves that, for all  $s : \tau^*, a : \sigma$ ,*

$$(\Lambda x : \sigma.s)[a] = (\lambda x : \sigma.s)a = s[a/x].$$

**Lemma 2.8** *The extension  $\text{E-HA}^{\omega*}$  proves that, for all  $s, s' : (\sigma \rightarrow \tau^*)^*, a : \sigma$ ,*

$$s \subseteq s' \rightarrow s[a] \subseteq s'[a].$$

**Proof** See [30, Lemma 2.22].  $\square$

Since we do not have product types, we will often work with *tuples* of types and of terms, for which we follow the conventions of [10], as briefly summarized below.

**Notation** We write  $\underline{\sigma} := \sigma_1, \dots, \sigma_n$ ,  $\underline{x} : \underline{\sigma} := x_0 : \sigma_0, \dots, x_n : \sigma_n$  for tuples of types and terms. A pair of single square brackets  $[\ ]$  stands for the empty tuple. We write

$$f \underline{x} := (\dots (fx_0)x_1) \dots x_n,$$

with the appropriate types; while, if  $\underline{f} := f_0, \dots, f_m$ , then  $\underline{f} \underline{x}$  stands for  $f_0 \underline{x}, \dots, f_m \underline{x}$ . We will have, correspondingly,

$$\lambda \underline{x}. \underline{f} := \lambda \underline{x}. f_0, \dots, \lambda \underline{x}. f_m,$$

and the same for finite sequence application.

Relations distribute as expected: for instance, if  $\underline{y} := y_0, \dots, y_n$ , with the same length *and* types as  $\underline{x}$ , then

$$\underline{x} =_{\sigma} \underline{y} := \bigwedge_{i=0}^n x_i =_{\sigma_i} y_i;$$

and if  $\underline{s} := s_0 : \sigma_0^*, \dots, s_n : \sigma_n^*$  is a tuple of sequences, then

$$\underline{x} \in_{\sigma} \underline{s} := \bigwedge_{i=0}^n x_i \in_{\sigma_i} s_i.$$

Most of the results we have listed so far are easily extended to tuples of terms, in particular, those concerning finite sequence application and abstraction.

We now lay the syntactic groundwork for doing nonstandard arithmetic in our system.

**Definition 2.9** The system  $\text{E-HA}_{\text{st}}^{\omega^*}$  is an extension of  $\text{E-HA}^{\omega^*}$ , whose language includes a (unary) predicate  $\text{st}_{\sigma}(x)$ ,  $x : \sigma$ , for all types  $\sigma$  of  $\mathbf{T}^*$ , and the *external quantifiers*  $\forall^{\text{st}}x : \sigma$ ,  $\exists^{\text{st}}x : \sigma$ .

**Notation** Following Nelson, so-called *internal* formulae—those in the language of  $\text{E-HA}^{\omega^*}$ —are always denoted with small Greek letters, and generic, *external* formulae are denoted with capital Greek letters.

The following axioms are added to those of  $\text{E-HA}^{\omega^*}$ :

- (1) the defining axioms of the external quantifiers:

$$\begin{aligned} \forall^{\text{st}}x : \sigma \Phi(x) &\leftrightarrow \forall x : \sigma (\text{st}_{\sigma}(x) \rightarrow \Phi(x)), \\ \exists^{\text{st}}x : \sigma \Phi(x) &\leftrightarrow \exists x : \sigma (\text{st}_{\sigma}(x) \wedge \Phi(x)); \end{aligned}$$

- (2) axioms for the standardness predicate:

$$\begin{aligned} \text{st}_{\sigma}(x) \wedge x =_{\sigma} y &\rightarrow \text{st}_{\sigma}(y), \\ \text{st}_{\sigma}(a) & \text{ for all closed } a : \sigma, \\ \text{st}_{\sigma \rightarrow \tau}(f) \wedge \text{st}_{\sigma}(x) &\rightarrow \text{st}_{\tau}(fx); \end{aligned}$$

- (3) the *external induction* schema:

$$\text{IA}^{\text{st}} : (\Phi(0) \wedge \forall^{\text{st}}x : 0 (\Phi(x) \rightarrow \Phi(Sx))) \rightarrow \forall^{\text{st}}x : 0 \Phi(x).$$

Since it is part of  $\text{E-HA}^{\omega^*}$ , the system  $\text{E-HA}_{\text{st}}^{\omega^*}$  also contains, besides the external induction schema, an “internal” induction schema  $\text{IA}$ , which is assumed to hold for internal formulae only.

So far, there is nothing inherently nonstandard about the system we have defined. In fact, one could interpret  $\text{st}_{\sigma}(x)$  as  $x =_{\sigma} x$ , and all the new axioms would be provable in  $\text{E-HA}^{\omega^*}$ . This simple fact also implies that  $\text{E-HA}_{\text{st}}^{\omega^*}$  is a conservative extension of  $\text{E-HA}^{\omega^*}$ .

However, there are some simple results, of the kind we would expect from a “standardness property,” that can already be proved.

**Proposition 2.10** ([3, Lemma 2.3]) *For every formula  $\Phi(x)$ ,  $\text{E-HA}_{\text{st}}^{\omega^*}$  proves*

$$\Phi(x) \wedge x = y \rightarrow \Phi(y).$$

**Proof** The proof is by easy induction on the logical structure of  $\Phi$ , utilizing the fact that the standardness predicate is extensional.  $\square$

**Proposition 2.11** ([3, Lemma 2.4]) *We have  $\text{E-HA}_{\text{st}}^{\omega*} \vdash \forall n, m : 0 (st_0(n) \wedge m \leq n \rightarrow st_0(m))$ .*

**Proof** Apply external induction to the formula  $\Phi(n) := \forall m : 0 (m \leq n \rightarrow st_0(m))$ .  $\square$

Basically anything one can get from standard sequences is standard.

**Lemma 2.12** *We have*

- (a)  $\text{E-HA}_{\text{st}}^{\omega*} \vdash \forall s : \sigma^* (st(s) \rightarrow st(|s|))$ ,
- (b)  $\text{E-HA}_{\text{st}}^{\omega*} \vdash \forall s : \sigma^* (st(s) \rightarrow \forall i < |s| st(s_i))$ ,
- (c)  $\text{E-HA}_{\text{st}}^{\omega*} \vdash \forall s : \sigma^* \forall x : \sigma (st(s) \wedge x \in_{\sigma} s \rightarrow st(x))$ ,
- (d)  $\text{E-HA}_{\text{st}}^{\omega*} \vdash \forall s, t : \sigma^* (st(s) \wedge st(t) \rightarrow st(s \cdot t))$ ,
- (e)  $\text{E-HA}_{\text{st}}^{\omega*} \vdash \forall f : 0 \rightarrow \sigma^* \forall n : 0 (st(f) \wedge st(n) \rightarrow st(f0 \dots \cdot fn))$ .

**Proof** Everything follows from the standardness axioms, coupled with the fact that the list recursor is standard.  $\square$

A simple consequence of the lemma is that the operations of sequence application and abstraction, as defined in the previous section, preserve standardness.

**Corollary 2.13** ([3, Lemma 2.23]) *We have*

- (a)  $\text{E-HA}_{\text{st}}^{\omega*} \vdash \forall s : (\sigma \rightarrow \tau^*)^* \forall x : \sigma (st(s) \wedge st(x) \rightarrow st(s[x]))$ ,
- (b)  $\text{E-HA}_{\text{st}}^{\omega*} \vdash \forall s : \tau^* (st(s) \rightarrow st(\Lambda x.s))$ .

Finally, we prove that finite sequences of standard elements are standard; the converse is already a consequence of Lemma 2.12(a)–(b).

**Lemma 2.14** ([3, Corollary 2.19]) *The system  $\text{E-HA}_{\text{st}}^{\omega*}$  proves that*

$$\forall s : \sigma^* (st(|s|) \wedge \forall i < |s| st(s_i) \rightarrow st(s)).$$

**Proof** Suppose that  $s : \sigma^*$  is finite, and that, for all  $i < |s|$ ,  $s_i$  is standard. By an iteration of Lemma 2.12(d),  $s' := s_0 \dots \cdot s_{|s|-1}$  is also standard. Clearly,  $s$  and  $s'$  are extensionally equal; by Corollary 2.4,  $s = s'$ . Thus,  $s$  is standard.  $\square$

This, in turn, is used to prove an external induction schema for sequences.

**Proposition 2.15** ([3, Corollary 2.20]) *The system  $\text{E-HA}_{\text{st}}^{\omega*}$  proves the external induction schema for sequences*

$$\text{IA}^{*st} : (\Phi(\langle \rangle_{\sigma}) \wedge \forall^{st} x : \sigma \forall^{st} s : \sigma^* (\Phi(s) \rightarrow \Phi(Cxs))) \rightarrow \forall^{st} s : \sigma^* \Phi(s).$$

**Proof** From the previous lemma, one obtains that if  $s = Cxt$  and  $s$  is standard, then  $x$  and  $t$  are also standard. Then one argues precisely as in Proposition 2.2, applying external instead of ordinary induction.  $\square$

The linguistic blocks are in place for the definition of the nonstandard Dialectica interpretation.

## 2.2 The $D_{\text{st}}$ -translation

**Definition 2.16** To every formula  $\Phi(\underline{a})$  of the language of  $\text{E-HA}_{\text{st}}^{\omega*}$ , with free variables  $\underline{a}$ , we associate inductively its *nonstandard Dialectica* translation

$$\Phi(\underline{a})^{D_{\text{st}}} = \exists^{st} \underline{s} \forall^{st} \underline{y} \varphi_{D_{\text{st}}}(\underline{s}, \underline{y}, \underline{a}),$$

where  $\varphi_{D_{\text{st}}}$  is internal, and all the variables in  $\underline{s}$  are of sequence type:

- ▷  $\varphi(\underline{a})^{D_{st}} := \varphi_{D_{st}}(\underline{a}) := \varphi(\underline{a})$ , for  $\varphi$  internal atomic;
- ▷  $\text{st}_\sigma(x)^{D_{st}} := \exists^{\text{st}} \underline{s} : \sigma^*(x \in s)$ .

Let  $\Phi(\underline{a})^{D_{st}} = \exists^{\text{st}} \underline{s} \forall^{\text{st}} \underline{y} \varphi_{D_{st}}(\underline{s}, \underline{y}, \underline{a})$ ,  $\Psi(\underline{b})^{D_{st}} = \exists^{\text{st}} \underline{t} \forall^{\text{st}} \underline{v} \psi_{D_{st}}(\underline{t}, \underline{v}, \underline{b})$ :

- ▷  $(\Phi(\underline{a}) \wedge \Psi(\underline{b}))^{D_{st}} := \exists^{\text{st}} \underline{s}, \underline{t} \forall^{\text{st}} \underline{y}, \underline{v} (\varphi_{D_{st}}(\underline{s}, \underline{y}, \underline{a}) \wedge \psi_{D_{st}}(\underline{t}, \underline{v}, \underline{b}))$ ;
- ▷  $(\Phi(\underline{a}) \vee \Psi(\underline{b}))^{D_{st}} := \exists^{\text{st}} \underline{s}, \underline{t} \forall^{\text{st}} \underline{y}, \underline{v} (\varphi_{D_{st}}(\underline{s}, \underline{y}, \underline{a}) \vee \psi_{D_{st}}(\underline{t}, \underline{v}, \underline{b}))$ ;
- ▷  $(\Phi(\underline{a}) \rightarrow \Psi(\underline{b}))^{D_{st}} := \exists^{\text{st}} \underline{T}, \underline{Y} \forall^{\text{st}} \underline{s}, \underline{v} (\forall \underline{y} \in \underline{Y}[\underline{s}, \underline{v}] \varphi_{D_{st}}(\underline{s}, \underline{y}, \underline{a}) \rightarrow \psi_{D_{st}}(\underline{T}[\underline{s}], \underline{v}, \underline{b}))$ ;
- ▷  $(\exists z \Phi(z, \underline{a}))^{D_{st}} := \exists^{\text{st}} \underline{s} \forall^{\text{st}} \underline{t} \exists z \forall \underline{y} \in \underline{t} \varphi_{D_{st}}(\underline{x}, \underline{y}, z, \underline{a})$ ;
- ▷  $(\forall z \Phi(z, \underline{a}))^{D_{st}} := \exists^{\text{st}} \underline{s} \forall^{\text{st}} \underline{y} \forall z \varphi_{D_{st}}(\underline{s}, \underline{y}, z, \underline{a})$ ;
- ▷  $(\exists^{\text{st}} z \Phi(z, \underline{a}))^{D_{st}} := \exists^{\text{st}} \underline{u}, \underline{s} \forall^{\text{st}} \underline{t} \exists z \in \underline{u} \forall \underline{y} \in \underline{t} \varphi_{D_{st}}(\underline{s}, \underline{y}, z, \underline{a})$ ;
- ▷  $(\forall^{\text{st}} z \Phi(z, \underline{a}))^{D_{st}} := \exists^{\text{st}} \underline{S} \forall^{\text{st}} \underline{y}, z \varphi_{D_{st}}(\underline{S}[z], \underline{y}, z, \underline{a})$ .

The idea is that, in the  $D_{st}$ -interpretation, realizers should be finite sequences of *potential* realizers, of which at least one is an actual realizer. Hence, if  $s$  is a valid realizer, then any  $s'$  with  $s \subseteq s'$  should work as well. That this is the case is guaranteed by the following proposition.

**Definition 2.17** A formula  $\Phi(s)$  is *upward closed* in  $s : \sigma^*$  if

$$\Phi(s) \wedge s \subseteq s' \rightarrow \Phi(s').$$

**Proposition 2.18** Let  $\Phi(\underline{a})$  be a formula of  $\text{E-HA}_{st}^{\omega*}$ ,  $\Phi(\underline{a})^{D_{st}} = \exists^{\text{st}} \underline{s} \forall^{\text{st}} \underline{y} \varphi(\underline{s}, \underline{y}, \underline{a})$ . Then  $\text{E-HA}^{\omega*}$  proves that  $\varphi$  is upward closed in  $\underline{s}$ :

$$\text{E-HA}^{\omega*} \vdash \varphi(\underline{s}, \underline{y}, \underline{a}) \wedge \underline{s} \subseteq \underline{s}' \rightarrow \varphi(\underline{s}', \underline{y}, \underline{a}).$$

**Proof** The proof is by induction on the logical structure of  $\Phi(\underline{a})$ , using Lemma 2.8 in the clauses for  $\rightarrow$  and  $\forall^{\text{st}} z$ .  $\square$

In [30], the nonstandard Dialectica interpretation was given a characterization in terms of five principles. We provide here an alternative characterization, which keeps the following three principles from the former.

(1) The *herbrandized axiom of choice*:

$$\begin{aligned} \text{HAC}^{\text{st}} : \quad & \forall^{\text{st}} x : \sigma \exists^{\text{st}} y : \tau \Phi(x, y) \\ & \rightarrow \exists^{\text{st}} (f : \sigma \rightarrow \tau)^* \forall^{\text{st}} x : \sigma \exists y \in f[x] \Phi(x, y). \end{aligned}$$

(2) The *herbrandized independence of premise* principle:

$$\begin{aligned} \text{HIP}_{\forall}^{\text{st}} : \quad & (\forall^{\text{st}} x : \sigma \varphi(x) \rightarrow \exists^{\text{st}} y : \tau \Psi(y)) \\ & \rightarrow \exists^{\text{st}} t : \tau^* (\forall^{\text{st}} x : \sigma \varphi(x) \rightarrow \exists y \in t \Psi(y)). \end{aligned}$$

(3) The principle called *nonclassical realization* in [30]—which, as we will see in Section 4, could also be called *herbrandized nonstandard uniformity*:

$$\text{NCR} : \quad \forall y : \tau \exists^{\text{st}} x : \sigma \Phi(x, y) \rightarrow \exists^{\text{st}} s : \sigma^* \forall y : \tau \exists x \in s \Phi(x, y).$$

In addition to these, the former characterization had *idealization*

$$I : \quad \forall^{\text{st}} s : \sigma^* \exists y : \tau \forall x \in s \varphi(x, y) \rightarrow \exists y : \tau \forall^{\text{st}} x : \sigma \varphi(x, y),$$

whose dual

$$R : \quad \forall y : \tau \exists^{\text{st}} x : \sigma \varphi(x, y) \rightarrow \exists^{\text{st}} s : \sigma^* \forall y : \tau \exists x \in s \varphi(x, y)$$

is clearly a consequence of NCR, and the *herbrandized generalized Markov's principle*

$$\text{HGMP}^{\text{st}} : (\forall^{\text{st}} x : \sigma \varphi(x) \rightarrow \psi) \rightarrow \exists^{\text{st}} s : \sigma^* (\forall x \in s \varphi(x) \rightarrow \psi).$$

We will replace them as follows.

**Definition 2.19** Let  $s : \sigma^*$ . We say that  $s$  is a *hyperfinite enumeration* of the type  $\sigma$  if

$$\forall^{\text{st}} x : \sigma (x \in s).$$

We define, for all types  $\sigma$ , a predicate

$$\text{hyper}(s) := \forall^{\text{st}} x : \sigma (x \in s),$$

as well as quantifiers ranging over hyperfinite enumerations, with defining axioms

$$\forall^{\text{hyp}} s : \sigma^* \Phi(s) := \forall s : \sigma^* (\text{hyper}_\sigma(s) \rightarrow \Phi(s)),$$

$$\exists^{\text{hyp}} s : \sigma^* \Phi(s) := \exists s : \sigma^* (\text{hyper}_\sigma(s) \wedge \Phi(s)).$$

The most basic nonstandard principles are, arguably, overspill and underspill in the type of natural numbers:

$$\text{OS}_0 : \forall^{\text{st}} n : 0 \varphi(n) \rightarrow \exists n : 0 (\neg \text{st}(n) \wedge \varphi(n)),$$

$$\text{US}_0 : \forall n : 0 (\neg \text{st}(n) \rightarrow \varphi(n)) \rightarrow \exists^{\text{st}} n : 0 \varphi(n).$$

These principles are almost invariably used with formulae of the form  $\forall k < n \varphi(k)$ , stating that a certain property holds *up to* a number  $n$ . From the assumption  $\forall^{\text{st}} n \forall k < n \varphi(k)$ , which says that  $\varphi$  holds up to *any* standard natural number,  $\text{OS}_0$  allows one to derive that  $\varphi$  holds up to some nonstandard (infinite) number  $n$ .

From  $n$ , one can obtain a hyperfinite enumeration  $s := \langle 0, \dots, n \rangle$  of the natural numbers, so that  $\forall k < n \varphi(k) \leftrightarrow \forall k \in s \varphi(k)$ ; and, in a way, it is *this* fact—that  $n$  induces a hyperfinite enumeration—that is relevant to the argument, rather than  $n$  being nonstandard. This suggests the following generalization of overspill and underspill to all finite types.

We introduce the principle of *sequence overspill*

$$\text{OS}^* : \forall^{\text{st}} s : \sigma^* \varphi(s) \rightarrow \exists^{\text{hyp}} s : \sigma^* \varphi(s),$$

and its dual, *sequence underspill*

$$\text{US}^* : \forall^{\text{hyp}} s : \sigma^* \varphi(s) \rightarrow \exists^{\text{st}} s : \sigma^* \varphi(s).$$

**Proposition 2.20** We have  $\text{E-HA}_{\text{st}}^{\omega^*} \vdash \text{I} \leftrightarrow \text{OS}^*$ .

**Proof** Assume  $\text{I}$ , and suppose that  $\forall^{\text{st}} s : \sigma^* \varphi(s)$ . Let  $t : (\sigma^*)^*$  be a standard sequence of sequences; then  $s := t_0 \cdot \dots \cdot t_{|t|-1}$  is again standard, so  $\varphi(s)$  holds. Furthermore, by construction, for all  $i < |t|$ ,  $t_i \subseteq s$ ; in other words,

$$\forall^{\text{st}} t : (\sigma^*)^* \exists s : \sigma^* \forall t' \in t (t' \subseteq s \wedge \varphi(s)).$$

By idealization, we obtain

$$\exists s : \sigma^* \forall^{\text{st}} t : \sigma^* (t \subseteq s \wedge \varphi(s)).$$

It remains to prove that  $\forall^{\text{st}} t : \sigma^* (t \subseteq s) \leftrightarrow \text{hyper}(s)$ , an easy consequence of Lemma 2.12.

Conversely, assume  $OS^*$ , and suppose that  $\forall^{st} s : \sigma^* \exists y : \tau \forall x \in s \varphi(x, y)$ . By sequence overspill, it follows that

$$\exists y : \tau \exists^{hyp} s : \sigma^* \forall x \in s \varphi(x, y),$$

which implies that

$$\exists y : \tau \forall^{st} x : \sigma^* \varphi(x, y).$$

This concludes the proof.  $\square$

Several consequences of I are listed in [22] and in [30], which, by the previous proposition, are also consequences of  $OS^*$ . For us, it is particularly relevant that  $OS^*$  implies an external version of the *lesser limited principle of omniscience*, a nonconstructive principle well known in the area of reverse mathematics (see, e.g., [8]); namely,

$$LLPO^{st} : \forall^{st} x, y : \sigma (\varphi(x) \vee \psi(y)) \rightarrow (\forall^{st} x : \sigma \varphi(x) \vee \forall^{st} x : \sigma \psi(x)).$$

**Proposition 2.21** *We have  $E-HA_{st}^{\omega^*} + OS^* \vdash LLPO^{st}$ .*

**Proof** Suppose that  $\forall^{st} x, y : \sigma (\varphi(x) \vee \psi(y))$ . We prove by external sequence induction that

$$\forall^{st} s : \sigma^* (\forall x \in s \varphi(x) \vee \forall x \in s \psi(x)). \quad (1)$$

For  $s = \langle \rangle_{\sigma}$ ,  $\forall x \in s \varphi(x) \vee \forall x \in s \psi(x)$  is vacuously true. Suppose that it is true for some arbitrary, standard  $s$ , and pick any standard  $a : \sigma$ . We want to show that  $\forall x \in Cas \varphi(x) \vee \forall x \in Cas \psi(x)$ .

Suppose that  $\forall x \in s \varphi(x)$  (the case where  $\forall x \in s \psi(x)$  is true is similar). Since

$$\forall^{st} x, y : \sigma (\varphi(x) \vee \psi(y)),$$

we have

$$\forall b \in Cas (\varphi(a) \vee \psi(b));$$

since  $Cas$  is a finite sequence, we can run through all  $b \in Cas$  and see whether  $\varphi(a)$  holds. If so, then  $\forall x \in Cas \varphi(x)$  holds and we are done; otherwise, we will get that  $\psi(b)$  holds for all  $b \in Cas$  and we again achieve the desired disjunction. Now, applying sequence overspill to (1) gives

$$\exists^{hyp} s : \sigma^* (\forall x \in s \varphi(x) \vee \forall x \in s \psi(x)),$$

which implies  $LLPO^{st}$ .  $\square$

Note that  $OS_0$  alone would have sufficed to prove the restriction of  $LLPO^{st}$  to type 0.

Since I is equivalent to  $OS^*$ , it would make sense if R were equivalent to  $US^*$ ; yet things are not so simple. In fact, only one implication seems to hold.

**Proposition 2.22** *We have  $E-HA_{st}^{\omega^*} + US^* \vdash R$ .*

**Proof** Suppose that  $\forall y : \tau \exists^{st} x : \sigma \varphi(x, y)$ . Then

$$\forall^{hyp} s : \sigma^* \forall y : \tau \exists x \in s \varphi(x, y),$$

which, by sequence underspill, implies  $\exists^{st} s : \sigma^* \forall y : \tau \exists x \in s \varphi(x, y)$ .  $\square$

What is missing, to obtain an equivalence, is precisely the last characteristic principle.

**Proposition 2.23** *We have  $E-HA_{st}^{\omega^*} + US^* \vdash HGMP^{st}$ .*

**Proof** Suppose that  $\forall^{\text{st}}x : \sigma \varphi(x) \rightarrow \psi$ . Then

$$\exists^{\text{hyp}}s : \sigma^* \forall x \in s \varphi(x) \rightarrow \psi,$$

which is intuitionistically equivalent to

$$\forall^{\text{hyp}}s : \sigma^* (\forall x \in s \varphi(x) \rightarrow \psi).$$

An application of sequence underspill leads to the conclusion.  $\square$

We now complete the characterization of  $\text{US}^*$ .

**Proposition 2.24** *We have  $\text{E-HA}_{\text{st}}^{\omega^*} + \text{HGMP}^{\text{st}} + \text{R} \vdash \text{US}^*$ .*

**Proof** Suppose that  $\forall^{\text{hyp}}s : \sigma^* \varphi(s)$ ; that is,

$$\forall s : \sigma^* (\forall^{\text{st}}x : \sigma (x \in s) \rightarrow \varphi(s)).$$

By the herbrandized generalized Markov's principle, this is equivalent to

$$\forall s : \sigma^* \exists^{\text{st}}t : \sigma^* (t \subseteq s \rightarrow \varphi(s)),$$

which, by realization and intuitionistic logic, implies

$$\exists^{\text{st}}t : (\sigma^*)^* \forall s : \sigma^* (\forall t' \in t (t' \subseteq s) \rightarrow \varphi(s)).$$

Take a standard  $t : (\sigma^*)^*$  as in (2.2), and pick  $s := t_0 \dots t_{|t|-1}$ . By Lemma 2.12,  $s$  is standard, and for all  $t' \in t$ ,  $t' \subseteq s$ ; therefore, it holds that  $\varphi(s)$ . We thus prove

$$\exists^{\text{st}}s : \sigma^* \varphi(s)$$

and the sequence overspill principle.  $\square$

Replacing  $\psi$  with a contradiction, for example,  $0 =_0 1$ , and choosing a *negated*  $\varphi(x)$ , we see that  $\text{HGMP}^{\text{st}}$ —hence,  $\text{US}^*$  as well—implies an external version of *Markov's principle*, another noted principle that is rejected by strict constructivism:

$$\text{MP}^{\text{st}} : (\forall^{\text{st}}x : \sigma (\varphi(x) \vee \neg \varphi(x)) \wedge \neg \neg \exists^{\text{st}}x : \sigma \varphi(x)) \rightarrow \exists^{\text{st}}x : \sigma \varphi(x).$$

This is another instance of a principle whose nature appears markedly nonstandard, forcing a nonconstructive mode of reasoning.

**Theorem 2.25 (Soundness of the nonstandard Dialectica interpretation)** *Suppose that*

$$\text{E-HA}_{\text{st}}^{\omega^*} + \text{OS}^* + \text{US}^* + \text{NCR} + \text{HAC}^{\text{st}} + \text{HIP}_{\forall}^{\text{st}} + \Delta_{\text{int}} \vdash \Phi(\underline{a}),$$

where  $\Delta_{\text{int}}$  is a set of internal sentences. Let  $\Phi(\underline{a})^{\text{D}_{\text{st}}} = \exists^{\text{st}}\underline{s} \forall^{\text{st}}\underline{y} \varphi_{\text{D}_{\text{st}}}(\underline{s}, \underline{y}, \underline{a})$ . Then from the proof we can extract a tuple of closed terms  $\underline{t}$  such that

$$\text{E-HA}^{\omega^*} + \Delta_{\text{int}} \vdash \forall \underline{y} \varphi_{\text{D}_{\text{st}}}(\underline{t}, \underline{y}, \underline{a}).$$

**Proof** This is [30, Theorem 5.5], coupled with the fact that  $\text{OS}^* + \text{US}^*$  is equivalent to  $\text{I} + \text{R} + \text{HGMP}^{\text{st}}$  over  $\text{E-HA}_{\text{st}}^{\omega^*}$ . We provide explicit realizers for the new principles.

The interpretation of  $\text{OS}^*$  is

$$\exists^{\text{st}}S \forall^{\text{st}}s' (\forall s \in S[s'] \varphi(s) \rightarrow \exists s (s' \subseteq s \wedge \varphi(s))),$$

and we can take  $S := \Lambda s'. \langle s' \rangle$ .

The interpretation of  $\text{US}^*$  is

$$\exists^{\text{st}}T \forall^{\text{st}}s'' (\forall s \exists s' \in s'' (s' \subseteq s \rightarrow \varphi(s)) \rightarrow \exists t \in T[s''] \varphi(t));$$

since  $\forall s \exists s' \in s'' (s' \subseteq s \rightarrow \varphi(s))$  implies  $\varphi(s''_0 \cdot \dots \cdot s''_{|s''|-1})$ , unless  $s''$  is the empty sequence (in which case, the premise is false anyway), we can take

$$T := \Lambda s''.(s''_0 \cdot \dots \cdot s''_{|s''|-1}). \quad \square$$

**Corollary 2.26** *The system*

$$H := \text{E-HA}_{\text{st}}^{\omega*} + \text{OS}^* + \text{US}^* + \text{NCR} + \text{HAC}^{\text{st}} + \text{HIP}_{\forall}^{\text{st}}$$

*is a conservative extension of E-HA $^{\omega*}$ , hence of E-HA $^{\omega}$ .*

**Proof** This follows from the soundness theorem, noting that internal formulae are  $D_{\text{st}}$ -interpreted as themselves.  $\square$

**Theorem 2.27 (Characterization of nonstandard Dialectica)** *Let  $\Phi$  be a formula in the language of E-HA $_{\text{st}}^{\omega*}$ .*

- (a) *We have  $H \vdash \Phi \leftrightarrow \Phi^{D_{\text{st}}}$ .*  
 (b) *If for all formulae  $\Psi$  of the language of E-HA $_{\text{st}}^{\omega*}$ , with  $\Psi^{D_{\text{st}}} = \exists^{\text{st}} \underline{s} \forall^{\text{st}} \underline{y} \psi(\underline{s}, \underline{y})$ ,*

$$H + \Phi \vdash \Psi$$

*implies that there exist closed terms  $\underline{t}$  such that*

$$\text{E-HA}^{\omega*} \vdash \forall \underline{y} \psi(\underline{t}, \underline{y})$$

*holds, then  $H \vdash \Phi$ .*

**Proof** See [30, Theorem 5.8].  $\square$

Again, we refer to [30] for proofs of other consequences of the soundness and characterization theorems, including the closure of  $H$  under the *transfer rules*

$$\text{TR}_{\forall} : \frac{\forall^{\text{st}} x : \sigma \varphi(x)}{\forall x : \sigma \varphi(x)},$$

$$\text{TR}_{\exists} : \frac{\exists x : \sigma \varphi(x)}{\exists^{\text{st}} x : \sigma \varphi(x)}.$$

In summary, the  $D_{\text{st}}$ -interpretation is characterized by two reasonable nonstandard principles and three principles which share the attribute *herbrandized*—something we will later explain in detail. In the next section, we will show that, under the right interpretation of a first-order language, these principles are true in Moerdijk's topos of filters.

### 3 The Filter Topos $\mathcal{N}$

For this section, we assume some basic knowledge about Grothendieck topoi, what it means to interpret a first-order language in a Heyting category, and forcing semantics; van Oosten [31, Chapter 4] and Mac Lane and Moerdijk [14, Chapter 6] can be used as a reference.

**3.1 The filter construction** Blass [3] introduced a category of filters of sets and “continuous” maps between them; rediscovered by Moerdijk, it was used as the underlying category of a site, whose sheaves provided a model of nonstandard arithmetic.

This category arises from **Set** as a special case of a general construction—the *filter construction*—whose properties and functoriality were studied by Butz [4]. When applied on arbitrary categories with finite limits, it can be considered as a completion of the subobject posets under arbitrary meets. We will briefly discuss the general construction, following Butz, before specializing to the case of **Set**.

We start by recalling the definition of filter on a  $\wedge$ -semilattice, that is, on a poset with all finite meets.

**Definition 3.1** Let  $S$  be a  $\wedge$ -semilattice. A *filter* on  $S$  is an inhabited, upwards closed subset of  $S$  that is closed under binary meets.

We say that a filter is *proper* if it does not coincide with  $S$ ; otherwise, it is *non-proper*.

Following Palmgren, we would rather work with *filter bases*, indexed by a set  $I$ .

**Definition 3.2** A *filter base*  $\mathcal{F}_I$  on  $S$  is an inhabited set  $\{\mathcal{F}_i\}_{i \in I}$  of elements of  $S$  such that, for all  $i, j \in I$ , there exists  $k \in I$  such that  $\mathcal{F}_k \leq \mathcal{F}_i \wedge \mathcal{F}_j$ .

A filter base *generates* a filter as follows:  $A$  belongs to the filter if and only if there exists  $i \in I$  such that  $\mathcal{F}_i \leq A$ .

Note that a filter base generates a *nonproper* filter if and only if it contains the bottom element.

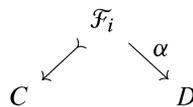
In every category  $\mathbf{C}$  with finite limits, the subobject posets are in fact  $\wedge$ -semilattices; it is therefore possible to speak of filters of subobjects. That is sufficient to perform the filter construction.

**Definition 3.3** Let  $\mathbf{C}$  be a finitely complete category. The *filter category*  $\mathfrak{F}\mathbf{C}$  over  $\mathbf{C}$  is described by the following data.

- Objects are pairs  $(C, \mathcal{F}_I)$ , where  $C$  is an object of  $\mathbf{C}$ , and  $\mathcal{F}_I$  is an  $I$ -indexed filter base on  $\text{Sub}(C)$ .

We will usually write  $\mathcal{F}$  for  $(C, \mathcal{F}_I)$ , when the underlying object and indexing set are not relevant, and just call it a *filter*. We say that the  $\mathcal{F}_i, i \in I$ , are the *base objects* of the filter.

- Morphisms are “germs of continuous morphisms.” A continuous morphism  $\alpha : (C, \mathcal{F}_I) \rightarrow (D, \mathcal{G}_J)$  is a partial morphism



in  $\mathbf{C}$ , defined on some base object  $\mathcal{F}_i$ , such that for all  $j \in J$ , there exists  $i' \in I$  such that  $\mathcal{F}_{i'} \leq \alpha^* \mathcal{G}_j$  in  $\text{Sub}(C)$ .

We declare two such morphisms  $\alpha : \mathcal{F}_i \rightarrow D, \alpha' : \mathcal{F}_j \rightarrow D$  *equivalent* if there exists  $k \in I$  such that  $\mathcal{F}_k \leq \mathcal{F}_i \wedge \mathcal{F}_j$  and  $\alpha|_{\mathcal{F}_k} = \alpha'|_{\mathcal{F}_k}$ ; that is, the

following pullback square commutes:

$$\begin{array}{ccc} \mathcal{F}_k & \xrightarrow{\quad} & \mathcal{F}_i \\ \downarrow & & \downarrow \alpha \\ \mathcal{F}_j & \xrightarrow{\alpha'} & D \end{array}$$

We have an embedding of  $\mathbf{C}$  into  $\mathfrak{F}\mathbf{C}$ , where an object  $C$  of  $\mathbf{C}$  is identified with the “simple” filter  $(C, \{C\})$ . We will usually still denote the latter with  $C$ .

We will not be overly pedantic about distinguishing between morphisms and their germs and will write both in the same style.

**Lemma 3.4** *The category  $\mathfrak{F}\mathbf{C}$  is finitely complete.*

**Proof** It is sufficient that  $\mathfrak{F}\mathbf{C}$  has a terminal object, binary products, and equalizers. We give their construction, and omit the proof of the universal properties.

The terminal object is the filter  $(1, \{1\})$ . The product of  $(C, \mathcal{F}_I)$  and of  $(D, \mathcal{G}_J)$  is the filter  $(C \times D, (\mathcal{F} \times \mathcal{G})_{I \times J})$ , where  $(\mathcal{F} \times \mathcal{G})_{(i,j)} := \mathcal{F}_i \times \mathcal{G}_j$ , for all  $i \in I, j \in J$ .

The equalizer of two morphisms  $\alpha, \beta : (C, \mathcal{F}_I) \rightarrow (D, \mathcal{G}_J)$ , represented by  $\alpha : \mathcal{F}_i \rightarrow D$  and  $\beta : \mathcal{F}_j \rightarrow D$ , is the inclusion  $(C', (\mathcal{F} \wedge C')_I) \rightarrow (C, \mathcal{F}_I)$ , where  $C'$  is the equalizer of  $\alpha$  and  $\beta$  in  $\mathbf{C}$ , and  $(\mathcal{F} \wedge C')_i := \mathcal{F}_i \wedge C'$  for all  $i \in I$ .  $\square$

**Lemma 3.5** *A morphism  $\alpha : \mathcal{F} \rightarrow \mathcal{G}$  of  $\mathfrak{F}\mathbf{C}$ , defined on a base object  $\mathcal{F}_i$ , is a monomorphism if and only if there exists a base object  $\mathcal{F}_j \leq \mathcal{F}_i$  such that  $\alpha|_{\mathcal{F}_j}$  is a monomorphism in  $\mathbf{C}$ .*

**Proof** See [4, Lemma 2.2].  $\square$

**Proposition 3.6** *For all filters  $\mathcal{F}$  in  $\mathfrak{F}\mathbf{C}$ ,  $\text{Sub}(\mathcal{F})$  is a meet-complete semilattice, and, for all  $\alpha : \mathcal{F} \rightarrow \mathcal{G}$ , the change of base functor  $\alpha^*$  preserves all meets.*

**Proof** By the previous lemma, if  $\alpha : (C, \mathcal{F}_I) \rightarrow (D, \mathcal{G}_J)$  is a monomorphism, there is some base object  $\mathcal{F}_i$  such that  $\alpha|_{\mathcal{F}_i} : \mathcal{F}_i \rightarrow D$  is a monomorphism in  $\mathbf{C}$ . Then  $(C, \mathcal{F}_I)$  is isomorphic to the filter  $(D, (\mathcal{G} \wedge \mathcal{F}_i)_J)$ . It follows that subobjects of  $(D, \mathcal{G}_J)$  are in one-to-one correspondence to objects  $(D, \mathcal{G}'_J)$ , such that the base  $\mathcal{G}'_J$  generates a filter larger than  $\mathcal{G}_J$ .

Given an arbitrary family of subobjects  $\{(D, \mathcal{G}_{J(i)}^{(i)})\}_{i \in I}$ , let  $\mathcal{H}$  be the filter generated by finite meets of the form

$$\mathcal{G}_{j^{(i_1)}}^{(i_1)} \wedge \dots \wedge \mathcal{G}_{j^{(i_n)}}^{(i_n)},$$

for  $(i_1, \dots, i_n)$  an arbitrary finite sequence in  $I$ , and  $j^{(i_k)} \in J^{(i_k)}, k = 1, \dots, n$ . Then

$$\bigwedge_{i \in I} (D, \mathcal{G}_{J(i)}^{(i)}) \simeq (D, \mathcal{H}).$$

That this is preserved by change of base can be easily verified by the explicit construction of pullbacks in  $\mathfrak{F}\mathbf{C}$ .  $\square$

An important feature of the filter construction is that it preserves some of the additional properties that  $\mathbf{C}$  may have.

**Proposition 3.7** *Let  $\mathbf{C}$  be a finitely complete category.*

- (a) If  $\mathbf{C}$  is regular, then  $\mathfrak{F}\mathbf{C}$  is also regular.
- (b) If  $\mathbf{C}$  is coherent, then  $\mathfrak{F}\mathbf{C}$  is also coherent.

**Proof** See [4, Proposition 3.1] and [4, Proposition 3.2], respectively. □

Moreover, if  $\mathbf{C}$  has all finite coproducts, then  $\mathfrak{F}\mathbf{C}$  has them too. In this case, the initial object of  $\mathfrak{F}\mathbf{C}$  is the simple filter  $(0, \{0\})$ ; this is isomorphic to any nonproper filter  $(C, \mathcal{F}_I)$ , where  $\mathcal{F}_i = 0$  for some  $i \in I$ . Given two filters  $(C, \mathcal{F}_I)$  and  $(D, \mathcal{G}_J)$ , their coproduct in  $\mathfrak{F}\mathbf{C}$  is the filter  $(C + D, (\mathcal{F} + \mathcal{G})_{I \times J})$ , where  $(\mathcal{F} + \mathcal{G})_{(i,j)} := \mathcal{F}_i + \mathcal{G}_j$ , for all  $i \in I, j \in J$ .

It is *not*, however, the case that  $\mathfrak{F}\mathbf{C}$  is necessarily a Heyting category, when  $\mathbf{C}$  is: binary meets do not, in general, distribute over infinite joins in the (complete) subobject lattices of  $\mathfrak{F}\mathbf{C}$ , as shown in [4, Example 1.6]. But this is not a problem, since we really only need  $\mathfrak{F}\mathbf{Set}$  to be a coherent category.

As it happens, coherent categories admit a “natural” Grothendieck topology, sometimes called the *precanonical* topology: for all objects  $C$  of  $\mathbf{C}$ , a  $K$ -cover of  $C$  is a *finite* family  $\{f_i : C_i \rightarrow C\}_{i=1}^n$  such that the union of the images of the  $f_i$  is the whole of  $C$ .

As shown in [9, Example C2.1.12(d)],  $K$  is *subcanonical*; that is, representable presheaves, of the form  $\mathbf{y}C$ , for  $C$  an object of  $\mathbf{C}$ , are  $K$ -sheaves.

Explicitly, for a filter category  $\mathfrak{F}\mathbf{C}$ , that  $\{\beta_k : \mathcal{G}_k \rightarrow \mathcal{F}\}_{k=1}^n$  is a  $K$ -cover means that, for all choices of base objects  $\mathcal{G}_{k,j_k}$  of  $\mathcal{G}_k, k = 1, \dots, n$ , there exists a base object  $\mathcal{F}_i$  of  $\mathcal{F}$  such that

$$\mathcal{F}_i \leq \beta_1 \mathcal{G}_{1,j_1} \vee \dots \vee \beta_n \mathcal{G}_{n,j_n}.$$

**Definition 3.8** We will denote the topos  $\text{Sh}(\mathfrak{F}\mathbf{Set}, K)$  by  $\mathcal{N}$ , for a *nonstandard* universe.

As for all Grothendieck topoi, the *global sections* functor

$$\Gamma : \mathcal{N} \rightarrow \mathbf{Set},$$

sending a sheaf  $F$  to the set  $\text{Hom}(1, F)$ , has a left adjoint  $\Delta : \mathbf{Set} \rightarrow \mathcal{N}$  called the *constant objects* functor. This can be explicitly characterized as follows: for all sets  $S$ , at all filters  $\mathcal{F}$  of  $\mathfrak{F}\mathbf{Set}$ ,

$$(\Delta S)\mathcal{F} = \{\alpha : \mathcal{F} \rightarrow S \mid \alpha \text{ takes a finite number of values}\}.$$

Here,  $S$  is identified with the simple filter  $(S, \{S\})$ . It follows that the Yoneda embedding preserves all coproducts of a finite number of copies of  $1$ , but *not* the natural numbers object.

Let  $\mathcal{L}$  be a many-sorted first-order language, and suppose that we have fixed an interpretation of  $\mathcal{L}$  in  $\mathbf{Set}$ . We call formulae of  $\mathcal{L}$  *internal*, and denote them with small Greek letters. We also want the types of  $\mathcal{L}$  to be closed under the clause

- ▷ if  $S$  is a type, then  $S^*$  is a type,

where  $S^*$  is meant to denote the type of finite sequences of elements of type  $S$ . We will borrow all the notation from the first section in handling finite sequences.

We will identify types and function and relation symbols of  $\mathcal{L}$  with their interpretation in  $\mathbf{Set}$ , and use the standard double square bracket notation for the derived interpretations that we are now going to define. We will take advantage of this semantic overload, and say, for instance, that the type  $S$  is inhabited, or that it is infinite, if its interpretation in  $\mathbf{Set}$  is; also a formula  $\varphi$  is *true* if its interpretation is true in  $\mathbf{Set}$ .

Let  $\mathcal{L}_{\text{st}}$  be the extension of  $\mathcal{L}$  with a unary predicate symbol  $\text{st}_S \subseteq S$  for each type  $S$ . We denote formulae of  $\mathcal{L}_{\text{st}}$  with *capital* Greek letters. We will use abbreviations

$$\begin{aligned}\forall^{\text{st}}x : S \Phi(x) &:= \forall x : S (\text{st}_S(x) \rightarrow \Phi(x)), \\ \exists^{\text{st}}x : S \Phi(x) &:= \exists x : S (\text{st}_S(x) \wedge \Phi(x)),\end{aligned}$$

as well as the defined predicate

$$\text{hyper}_S(s) := \forall^{\text{st}}x : S (x \in s),$$

for  $s : S^*$ , with the relative quantifiers

$$\begin{aligned}\forall^{\text{hyp}}_S : S^* \Phi(s) &:= \forall s : S^* (\text{hyper}_S(s) \rightarrow \Phi(s)), \\ \exists^{\text{hyp}}_S : S^* \Phi(s) &:= \exists s : S^* (\text{hyper}_S(s) \wedge \Phi(s)).\end{aligned}$$

We will often drop the subscript and just write  $\text{st}(x)$  or  $\text{hyper}(s)$ .

We define an interpretation of  $\mathcal{L}_{\text{st}}$  in  $\mathcal{N}$  as follows ( $\mathbf{y}$  denotes the Yoneda embedding):

- (i) for each type  $S$ ,  $\llbracket S \rrbracket := \mathbf{y}S$ ;
- (ii) for each constant  $c : S$ ,  $\llbracket c \rrbracket := \mathbf{y}c : 1 \rightarrow \mathbf{y}S$ ;
- (iii) for each function symbol  $f : S_1, \dots, S_n \rightarrow S$ ,  $\llbracket f \rrbracket := \mathbf{y}f : \mathbf{y}(S_1 \times \dots \times S_n) \rightarrow \mathbf{y}S$ ;
- (iv) for each relation symbol  $R \subseteq S_1, \dots, S_n$  of  $\mathcal{L}$ ,  $\llbracket R \rrbracket := \mathbf{y}R \rhd \mathbf{y}(S_1 \times \dots \times S_n)$ ;
- (v) for each type  $S$ ,  $\llbracket \text{st}_S \rrbracket := \Delta S$ .

In particular,  $\llbracket \text{st}_{\mathbb{N}} \rrbracket$  is the natural numbers object in  $\mathcal{N}$ , and the larger sheaf  $\llbracket \mathbb{N} \rrbracket$  is a nonstandard model of arithmetic.

The following, fundamental theorem connects the forcing semantics of internal formulae in  $\mathcal{N}$  with truth in the metatheory. It is found as [22, Theorem 1] and is an extension of [16, Lemma 2.1].

**Theorem 3.9** *Let  $\varphi(x)$  be an internal formula, with free variable  $x$  of type  $S$ , and let  $(C, \mathcal{F}_I)$  be a filter. For all  $\alpha \in \llbracket S \rrbracket \mathcal{F}$ ,*

$$\mathcal{F} \Vdash \varphi(\alpha)$$

*if and only if there exists  $i \in I$  such that, for all  $u \in \mathcal{F}_i$ , it holds that  $\varphi(\alpha(u))$ .*

**Corollary 3.10 (Transfer theorem)** *Let  $\varphi$  be an internal sentence. Then  $\varphi$  is true if and only if  $\Vdash \varphi$ .*

Theorem 3.9 says everything there is to know about internal formulae; we move on to the semantics of the standardness predicate.

**Lemma 3.11** *Let  $\mathcal{F}$  be a filter, let  $S$  be a type of  $\mathcal{L}$ , and let  $\alpha \in \llbracket S \rrbracket \mathcal{F}$ . Then,  $\mathcal{F} \Vdash \text{st}_S(\alpha)$  if and only if there exist a  $K$ -cover  $\{\beta_k : \mathcal{G}_k \rightarrow \mathcal{F}\}_{k=1}^n$ , and elements  $x_1, \dots, x_n \in S$  such that the diagrams*

$$\begin{array}{ccc} \mathcal{G}_k & \xrightarrow{\beta_k} & \mathcal{F} \\ \downarrow ! & & \downarrow \alpha \\ 1 & \xrightarrow{x_k} & S \end{array}$$

*commute in  $\mathfrak{S}\mathbf{Set}$ , that is,  $\alpha\beta_k = x_k!$ , for  $k = 1, \dots, n$ .*

**Proof** This follows immediately from the interpretation chosen for the standardness predicate and the description of  $\Delta S$ .  $\square$

**Lemma 3.12** *Let  $\Phi(x, y)$  be an external formula, with free variables  $x : S$  and  $y : T$ , let  $\mathcal{F}$  be a filter, and let  $\alpha \in \llbracket S \rrbracket \mathcal{F}$ . Then*

- (a)  $\mathcal{F} \Vdash \forall^{\text{st}} y : T \Phi(\alpha, y)$  if and only if, for all  $y \in T$ ,  $\mathcal{F} \Vdash \Phi(\alpha, y!)$ ;
- (b)  $\mathcal{F} \Vdash \exists^{\text{st}} y : T \Phi(\alpha, y)$  if and only if there exist a  $K$ -cover  $\{\beta_k : \mathcal{G}_k \rightarrow \mathcal{F}\}_{k=1}^n$ , and elements  $y_1, \dots, y_n \in T$  such that

$$\mathcal{G}_k \Vdash \Phi(\alpha \beta_k, y_k!), \quad k = 1, \dots, n,$$

or, equivalently, there exists  $t \in T^*$  such that  $\mathcal{F} \Vdash \exists y \in t! \Phi(\alpha, y)$ .

**Proof** See [21, Lemma 3.3].  $\square$

With these lemmas, we are able to prove that the simple axioms that we imposed on the standardness predicate in the system  $\text{E-HA}_{\text{st}}^{\omega*}$  hold in  $\mathcal{N}$ . That the predicate respects equality is immediate; that closed terms are standard, in this context, is due to the fact that, for all types  $S$  and elements  $\alpha \in \llbracket S \rrbracket 1$ ,

$$\Vdash \text{st}_S(\alpha),$$

as any morphism  $\alpha : 1 \rightarrow S$  is obviously constant.

**Proposition 3.13** *For all types  $S, T$ , the following statement is true in  $\mathcal{N}$ :*

$$\forall^{\text{st}} f : S \rightarrow T \quad \forall^{\text{st}} x : S \quad \text{Sst}_T(f(x)).$$

**Proof** By Lemma 3.12,  $\Vdash \forall^{\text{st}} f : S \rightarrow T \quad \forall^{\text{st}} x : S \quad \text{Sst}_T(f(x))$  if and only if, for all  $f \in (S \rightarrow T)$  and  $x \in S$ ,

$$\Vdash \text{st}_T(f!(x!)).$$

But  $\llbracket f!(x!) \rrbracket = \llbracket f(x!) \rrbracket$ , and the latter is clearly standard.  $\square$

**Proposition 3.14** *The external induction schema  $\text{IA}^{\text{st}}$  holds in  $\mathcal{N}$ .*

**Proof** Let  $\Phi(x, n)$  be an external formula, with  $x : S$  and  $n : \mathbb{N}$ , let  $\mathcal{F}$  be a filter, and let  $\alpha \in \llbracket S \rrbracket \mathcal{F}$ . Suppose that

$$\mathcal{F} \Vdash \Phi(\alpha, 0!) \wedge \forall^{\text{st}} n : \mathbb{N} (\Phi(\alpha, n) \rightarrow \Phi(\alpha, Sn)).$$

Then, by Lemma 3.12, we have that  $\mathcal{F} \Vdash \Phi(\alpha, 0!)$  and that, for all  $n \in \mathbb{N}$ ,  $\mathcal{F} \Vdash \Phi(\alpha, n!)$  implies  $\mathcal{F} \Vdash \Phi(\alpha, Sn!)$ . By induction in the metatheory, we obtain that, for all  $n \in \mathbb{N}$ ,

$$\mathcal{F} \Vdash \Phi(\alpha, n!),$$

so, again by the semantics of the external quantifiers,  $\mathcal{F} \Vdash \forall^{\text{st}} n : \mathbb{N} \Phi(\alpha, n)$ .  $\square$

Lemma 3.12 also has the following easy consequence.

**Corollary 3.15** *Let  $\varphi$  be an internal formula. Then  $\Vdash \forall^{\text{st}} x : S \exists^{\text{st}} y : T \varphi(x, y)$  if and only if it is true that  $\forall x \in S \exists y \in T \varphi(x, y)$ . Equivalently, the rule*

$$\text{TR}_{\forall\exists} : \frac{\forall x : \sigma \exists y : \tau \varphi(x, y)}{\forall^{\text{st}} x : \sigma \exists^{\text{st}} y : \tau \varphi(x, y)}$$

holds in  $\mathcal{N}$ .

We have, by now, a good picture of the semantics of first-order logic in the filter topos  $\mathcal{N}$ . In the next section, we will deal with the characteristic principles of non-standard Dialectica.

**3.2 Characteristic principles** For the results in this section, we cannot take much credit, since a characterization of first-order logic in the topoi  $\text{Sh}(\mathfrak{F}\mathbf{C}, K)$ , with  $\mathbf{C}$  coherent, has already been provided by Butz [4, Proposition 4.5], albeit with a different aim and formalism. The choice of principles, however, is different, due to our focus on nonstandard arithmetic; moreover, it will allow us to see *herbrandization* “in action,” once we decide to “de-herbrandise” in the following section.

We start from the truly nonstandard principles: sequence overspill and underspill.

**Proposition 3.16** *The principle  $\text{OS}^*$  holds in  $\mathcal{N}$ .*

**Proof** Let  $\varphi(y, s)$  be an internal formula, with variables  $y : T$  and  $s : S^*$ , let  $(C, \mathcal{F}_I)$  be an arbitrary filter, let  $\alpha \in \llbracket T \rrbracket \mathcal{F}$ , and assume that

$$\mathcal{F} \Vdash \forall^{\text{st}} s : S^* \varphi(\alpha, s).$$

By Lemma 3.12, for all  $s \in S^*$ ,  $\mathcal{F} \Vdash \varphi(\alpha, s!)$ ; by transfer (Theorem 3.9), for all  $s \in S^*$ , there exists  $i \in I$  such that, for all  $u \in \mathcal{F}_i$ ,

$$\varphi(\alpha(u), s).$$

Define a filter  $(C \times S^*, \mathcal{G}_{I \times S^*})$  as follows: for all  $i \in I, t \in S^*$ ,

$$\mathcal{G}_{(i,t)} := \{(u, s) \mid u \in \mathcal{F}_i \wedge t \subseteq s \wedge \varphi(\alpha(u), s)\}.$$

The filter condition is easily checked: given  $\mathcal{G}_{(i,t)}$ ,  $\mathcal{G}_{(j,t')}$ , pick  $k \in I$  such that  $\mathcal{F}_k \subseteq \mathcal{F}_i \cap \mathcal{F}_j$ , and  $t'' := t \cdot t'$ ; then,  $\mathcal{G}_{(k,t'')} \subseteq \mathcal{G}_{(i,t)} \cap \mathcal{G}_{(j,t')}$ .

The projections  $\pi_1 : \mathcal{G} \rightarrow \mathcal{F}$  and  $\pi_2 : \mathcal{G} \rightarrow S^*$  are clearly continuous. We now check

$$\mathcal{G} \Vdash \text{hyper}(\pi_2).$$

By definition, this means  $\mathcal{G} \Vdash \forall^{\text{st}} x : S (x \in \pi_2)$ ; equivalently, for all  $x \in S$ ,  $\mathcal{G} \Vdash x! \in \pi_2$ . By transfer, it suffices to prove that, for all  $x \in S$ , there exists  $(i, t) \in I \times S^*$  such that, for all  $u \in \mathcal{F}_i$  and  $s \supseteq t$ , it holds that  $x \in s$ ; so we can take  $t := \langle x \rangle$  and  $i \in I$  arbitrary.

Furthermore,  $\mathcal{G} \Vdash \varphi(\alpha\pi_1, \pi_2)$  holds by construction. Hence, to derive that

$$\mathcal{F} \Vdash \exists s : S^* (\text{hyper}(s) \wedge \varphi(\alpha, s)),$$

it remains to show that  $\pi_1$  is covering. Let  $\mathcal{G}_{(i,t)}$  be an arbitrary base set of  $\mathcal{G}$ . By the assumption, we can find  $j \in I$  such that, for all  $u \in \mathcal{F}_j$ ,  $\varphi(\alpha(u), t)$ ; then, if we choose  $k \in I$  such that  $\mathcal{F}_k \subseteq \mathcal{F}_i \cap \mathcal{F}_j$ , we have that

$$\mathcal{F}_k \subseteq \pi_1 \mathcal{G}_{(i,t)}.$$

This concludes the proof. □

**Lemma 3.17** *Let  $\Phi(x)$  be an external formula,  $x : S$ , such that*

$$\Vdash \exists x : S \Phi(x). \tag{2}$$

*Then*

$$\forall y : T (\forall x : S (\Phi(x) \rightarrow \varphi(y, x)) \rightarrow \exists^{\text{st}} x : S \varphi(y, x))$$

*holds in  $\mathcal{N}$  for all internal formulae  $\varphi$ .*

**Proof** Let  $\mathcal{F}$  be any filter, let  $\varphi(y, x)$  be an internal formula, with a variable  $y : T$ , and let  $\alpha \in \llbracket T \rrbracket \mathcal{F}$ . Suppose  $\mathcal{F} \Vdash \forall x : S (\Phi(x) \rightarrow \varphi(\alpha, x))$ ; equivalently,

$$\mathcal{F} \times S \Vdash \Phi(\pi_2) \rightarrow \varphi(\alpha\pi_1, \pi_2). \tag{3}$$

Assume (2). Then, there exist a cover  $\{\mathcal{G}_k \rightarrow 1\}_{k=1}^n$  and elements  $\sigma_k \in \llbracket S \rrbracket \mathcal{G}_k$ ,  $k = 1, \dots, n$ , such that

$$\mathcal{G}_k \Vdash \Phi(\sigma_k), \quad k = 1, \dots, n.$$

By our interpretation of the type  $S$ , the  $\sigma_k$ 's correspond to morphisms  $\sigma_k : \mathcal{G}_k \rightarrow S$  in  $\mathfrak{F}\mathbf{Set}$ ; by the monotonicity of the forcing relation, we obtain

$$\mathcal{F} \times \mathcal{G}_k \Vdash \Phi(\sigma_k\pi_2),$$

which, by the commutativity of the diagrams

$$\begin{array}{ccc} \mathcal{F} \times \mathcal{G}_k & \xrightarrow{\pi_2} & \mathcal{G}_k \\ \downarrow \text{id} \times \sigma_k & & \downarrow \sigma_k \\ \mathcal{F} \times S & \xrightarrow{\pi_2} & S \end{array}$$

for  $k = 1, \dots, n$ , is the same as  $\mathcal{F} \times \mathcal{G}_k \Vdash \Phi(\pi_2(\text{id} \times \sigma_k))$ .

Therefore, from (3), it follows, by monotonicity, that

$$\mathcal{F} \times \mathcal{G}_k \Vdash \varphi(\alpha\pi_1, \sigma_k\pi_2);$$

by transfer, for all  $k = 1, \dots, n$ , there exist base sets  $\mathcal{F}_{i_k}$  of  $\mathcal{F}$ ,  $\mathcal{G}_{k,j_k}$  of  $\mathcal{G}_k$ , such that, for all  $u \in \mathcal{F}_{i_k}$  and  $v \in \mathcal{G}_{k,j_k}$ , it holds that  $\varphi(\alpha(u), \sigma_k(v))$ .

Now, since the  $\mathcal{G}_k$ 's cover 1, there exists some  $x \in \sigma_1\mathcal{G}_{1,j_1} \cup \dots \cup \sigma_n\mathcal{G}_{n,j_n}$ . For such an  $x$ , taking  $\mathcal{F}_i \subseteq \mathcal{F}_{i_1} \cap \dots \cap \mathcal{F}_{i_n}$  and using transfer, we have

$$\mathcal{F} \Vdash \varphi(\alpha, x!);$$

hence  $\mathcal{F} \Vdash \exists^{\text{st}} x : S \varphi(\alpha, x)$ , which was to be proved. □

**Proposition 3.18** *The principle  $\text{US}^*$  holds in  $\mathcal{N}$ .*

**Proof** This follows from Lemma 3.17, by taking  $\Phi(s) := \text{hyper}(s)$ , and using for condition (2) the fact that, by sequence overspill, hyperfinite enumerations of any type exist in  $\mathcal{N}$ . □

Given sequence overspill and underspill, one can adapt the proofs of the first section to show that other principles, including idealization and the herbrandized generalized Markov's principle, hold in  $\mathcal{N}$ . However, one should pay attention to the fact that, while finite types were all inhabited and actually had infinitely many elements, in this context a type  $S$  can be finite or even empty. So, for instance, the implication  $\text{OS}^* \rightarrow \text{OS}$  only holds for types with infinitely many elements: by the definition of standardness, a finite set has only standard elements.

Next, we deal with two characteristic principles of nonstandard Dialectica, whose validity in the filter topoi is independent of the metatheory.

**Proposition 3.19** *The principle  $\text{NCR}$  holds in  $\mathcal{N}$ .*

**Proof** Let  $\mathcal{F}$  be any filter, let  $\Phi(z, x, y)$  be an external formula, with  $x : S, y : T, z : U$ , and let  $\alpha \in \llbracket U \rrbracket \mathcal{F}$ . Assume that  $\mathcal{F} \Vdash \forall y : T \exists^{\text{st}} x : S \Phi(\alpha, x, y)$ , or, equivalently,

$$\mathcal{F} \times T \Vdash \exists^{\text{st}} x : S \Phi(\alpha\pi_1, x, \pi_2).$$

By the semantics of the  $\exists^{\text{st}}$  quantifier in  $\mathcal{N}$ , this means that there exists  $s \in S^*$  such that

$$\mathcal{F} \times T \Vdash \exists x \in s! \Phi(\alpha\pi_1, x, \pi_2);$$

equivalently, since  $s! = s!\pi_1$ ,  $\mathcal{F} \Vdash \forall y : T \exists x \in s! \Phi(\alpha, x, y)$ . Therefore,

$$\mathcal{F} \Vdash \exists^{\text{st}} s : S^* \forall y : T \exists x \in s! \Phi(\alpha, x, y). \quad \square$$

The next proof is a variant of [4, Lemma 5.6(vi)]. It utilizes the following general result about Grothendieck topoi. Here,  $\mathbf{a}$  is the sheafification functor.

**Lemma 3.20** *Let  $(\mathbf{C}, J)$  be a site. A set  $\{f_i : C_i \rightarrow C\}_{i \in I}$  of morphisms of  $\mathbf{C}$  is  $J$ -covering if and only if the set  $\{\mathbf{a}y f_i : \mathbf{a}y C_i \rightarrow \mathbf{a}y C\}_{i \in I}$  is jointly epimorphic in  $\text{Sh}(\mathbf{C}, J)$ .*

**Proof** See [14, Corollary III.7.7].  $\square$

**Proposition 3.21** *The principle  $\text{HIP}_{\forall}^{\text{st}}$  holds in  $\mathcal{N}$ .*

**Proof** Let  $\mathcal{F}$  be any filter, let  $\varphi(z, x)$  be an internal formula, let  $\Psi(z, y)$  be an external formula, with  $x : S$ ,  $y : T$ ,  $z : U$ , and let  $\alpha \in \llbracket U \rrbracket \mathcal{F}$ . Suppose

$$\mathcal{F} \Vdash \forall^{\text{st}} x : S \varphi(\alpha, x) \rightarrow \exists^{\text{st}} y : T \Psi(\alpha, y).$$

By the semantics of first-order logic in a Heyting category, this is equivalent to

$$\alpha^* \llbracket \forall^{\text{st}} x \varphi(z, x) \rrbracket \leq \alpha^* \llbracket \exists^{\text{st}} y : T \Psi(z, y) \rrbracket$$

in  $\text{Sub}(\mathbf{y}\mathcal{F})$ . By the semantics of the  $\forall^{\text{st}}$  predicate, we can write

$$\alpha^* \llbracket \forall^{\text{st}} x \varphi(z, x) \rrbracket = \alpha^* \bigwedge_{x \in S} \llbracket \varphi(z, x!) \rrbracket,$$

and, by the suitable transfer theorem, for all  $x \in S$ ,

$$\llbracket \varphi(z, x!) \rrbracket = \mathbf{y}\{z \in U \mid \varphi(z, x)\}.$$

Since the Yoneda embedding preserves and reflects all limits, we obtain

$$\alpha^* \llbracket \forall^{\text{st}} x \varphi(z, x) \rrbracket = \mathbf{y}\left(\alpha^* \bigwedge_{x \in S} \{z \in U \mid \varphi(z, x)\}\right) =: \mathbf{y}\mathcal{H}.$$

For the consequence, we have, by the semantics of  $\exists^{\text{st}}$  in  $\mathcal{N}$ , that

$$\begin{aligned} \alpha^* \llbracket \exists^{\text{st}} y : T \Psi(z, y) \rrbracket &= \alpha^* \bigvee_{t \in T^*} \llbracket \exists y \in t! \Psi(z, y) \rrbracket = \\ &= \bigvee_{t \in T^*} \alpha^* \llbracket \exists y \in t! \Psi(z, y) \rrbracket =: \bigvee_{t \in T^*} F_t, \end{aligned}$$

where we also used that unions are stable under pullback. Thus, there is a monomorphism  $m : \mathbf{y}\mathcal{H} \hookrightarrow \bigvee_{t \in T^*} F_t$ .

Let  $t_t : F_t \hookrightarrow \bigvee_{t \in T^*} F_t$  be the inclusions of the  $F_t$  in their union, for all  $t \in T^*$ , and consider the pullback diagrams

$$\begin{array}{ccc} m^* F_t & \xrightarrow{\quad} & F_t \\ \downarrow & & \downarrow t_t \\ \mathbf{y}\mathcal{H} & \xrightarrow{m} & \bigvee_{t \in T^*} F_t \end{array}$$

Now, we use the fact that each  $m^* F_t$  can be covered with a family of representable sheaves, to obtain a family  $\{f_t : \mathbf{y}\mathcal{G}_t \rightarrow \mathbf{y}\mathcal{H}\}_{t \in T^*}$  of morphisms, such that each  $m f_t$  factors through a *single*  $F_t$ .

Moreover, since the  $\{t_t : F_t \twoheadrightarrow \bigvee_{t \in T^*} F_t\}_{y \in T}$  jointly cover  $\bigvee_{t \in T^*} F_t$ , and in a Heyting pretopos all epimorphisms are stable under pullback (see [14, Proposition IV.7.3]), the family  $\{f_t : \mathbf{y}\mathcal{G}_t \rightarrow \mathbf{y}\mathcal{H}\}_{t \in T^*}$  is jointly epimorphic over  $\mathbf{y}\mathcal{H}$ .

By the previous lemma, we can extract a family of the form  $\{\mathbf{y}\beta_k : \mathbf{y}\mathcal{G}_k \rightarrow \mathbf{y}\mathcal{H}\}_{k=1}^n$ , where  $\{\beta_k : \mathcal{G}_k \rightarrow \mathcal{H}\}_{k=1}^n$  is a  $K$ -cover in  $\mathfrak{F}\mathbf{Set}$ . Let  $t := t_1 \dots t_n$ , such that  $\mathbf{y}\beta_k$  factors through  $F_{t_k}$ ,  $k = 1, \dots, n$ . Then,

$$\mathbf{y}\mathcal{H} = \alpha^* \llbracket \forall^{\text{st}} x \varphi(z, x) \rrbracket \leq \alpha^* \bigvee_{k=1}^n \llbracket \exists y \in t_k ! \Psi(z, y) \rrbracket = \alpha^* \llbracket \exists y \in t ! \Psi(z, y) \rrbracket.$$

Translating back to forcing semantics, this is precisely the statement that

$$\mathcal{F} \Vdash \forall^{\text{st}} x : S \varphi(\alpha, x) \rightarrow \exists y \in t ! \Psi(\alpha, y),$$

from which it follows that

$$\mathcal{F} \Vdash \exists^{\text{st}} t : T^* (\forall^{\text{st}} x : S \varphi(\alpha, x) \rightarrow \exists^{\text{st}} y \in t \Psi(\alpha, y)). \quad \square$$

So far, we have used no principles whose constructive status is controversial, neither in the construction of the model, nor in our proofs. However, for our last pair of characteristic principles to hold, we must require that the *axiom of choice* holds in the metatheory.

**Proposition 3.22** *Suppose that the axiom of choice holds in the metatheory. Then the principle  $\text{HAC}^{\text{st}}$  holds in  $\mathcal{N}$ .*

**Proof** Let  $\mathcal{F}$  be any filter, let  $\Phi(z, x, y)$  be an external formula, with  $x : S, y : T, z : U$ , and let  $\alpha \in \llbracket U \rrbracket \mathcal{F}$ . Assume that

$$\mathcal{F} \Vdash \forall^{\text{st}} x : S \exists^{\text{st}} y : T \Phi(\alpha, x, y);$$

by Lemma 3.12, this means in  $\mathcal{N}$  that, for all  $x \in S$ , there exists  $t \in T^*$  such that

$$\mathcal{F} \Vdash \exists y \in t ! \Phi(\alpha, x!, y).$$

With the axiom of choice, we can find a function  $f \in S \rightarrow T^*$  such that, for all  $x \in S$ ,

$$\mathcal{F} \Vdash \exists y \in f(x) ! \Phi(\alpha, x!, y).$$

Since  $\llbracket f(x)! \rrbracket = \llbracket f!(x!) \rrbracket$ , it follows that

$$\mathcal{F} \Vdash \exists^{\text{st}} f : S \rightarrow T^* \forall^{\text{st}} x : S \exists y \in f(x) \Phi(\alpha, x, y). \quad \square$$

In fact, a herbrandized version of the axiom of choice would suffice; but that would be a strange axiom to have in one's metatheory. The condition is necessary to a certain extent, for  $\text{HAC}^{\text{st}}$  implies a herbrandized axiom of choice—call it  $\text{HAC}$ —in  $\mathbf{Set}$ . Suppose that

$$\forall x \in S \exists y \in T \varphi(x, y).$$

By Corollary 3.15, it follows that  $\Vdash \forall^{\text{st}} x : S \exists^{\text{st}} y : T \varphi(x, y)$ . If  $\text{HAC}^{\text{st}}$  holds in  $\mathcal{N}$ , we can deduce

$$\Vdash \exists^{\text{st}} f : S \rightarrow T^* \forall^{\text{st}} x : S \exists y \in f(x) \varphi(x, y);$$

applying the transfer theorem again, we obtain

$$\exists f \in S \rightarrow T^* \forall x \in S \exists y \in f(x) \varphi(x, y)$$

in **Set**.

In the same way, the transfer rules can be used to rule out the unconstrained validity of other principles in  $\mathcal{N}$ , as in the following example.

**Example 3.23** Let  $T(s)$  be a binary tree, that is, an internal formula on binary sequences such that

- (1)  $T(\langle \rangle)$  holds, and
- (2)  $\forall n, m \in \mathbb{N} \forall s \in 2^{\mathbb{N}} (T(\bar{s}m) \wedge n \leq m) \rightarrow T(\bar{s}n)$ , where  $\bar{s}n := \langle s_0, \dots, s_{n-1} \rangle$ .

The *fan theorem* is the statement that for any such  $T$ , if, for all sequences  $s \in 2^{\mathbb{N}}$ , there exists  $n \in \mathbb{N}$  such that  $\neg T(\bar{s}n)$ , then there exists some  $n \in \mathbb{N}$  such that  $\neg T(\bar{s}n)$  holds for all  $s \in 2^{\mathbb{N}}$ .

We consider the following, external version of the fan theorem:

$$\text{FAN}^{\text{st}} : \forall^{\text{st}} s : 2^{\mathbb{N}} \exists^{\text{st}} n : \mathbb{N} \neg T(\bar{s}n) \rightarrow \exists^{\text{st}} n : \mathbb{N} \forall^{\text{st}} s : 2^{\mathbb{N}} \neg T(\bar{s}n).$$

We claim that, if  $\text{FAN}^{\text{st}}$  holds in  $\mathcal{N}$ , then the fan theorem holds in the metatheory. For suppose that, for all  $s \in 2^{\mathbb{N}}$ , there exists  $n \in \mathbb{N}$  such that  $\neg T(\bar{s}n)$ . By transfer,

$$\Vdash \forall^{\text{st}} s : 2^{\mathbb{N}} \exists^{\text{st}} n : \mathbb{N} \neg T(\bar{s}n);$$

if  $\text{FAN}^{\text{st}}$  holds, we deduce

$$\Vdash \exists^{\text{st}} n : \mathbb{N} \forall^{\text{st}} s : 2^{\mathbb{N}} \neg T(\bar{s}n).$$

This means that there exists a finite sequence  $t$  of natural numbers such that

$$\Vdash \exists n \in t! \forall^{\text{st}} s : 2^{\mathbb{N}} \neg T(\bar{s}n).$$

By condition 2 on binary trees, we have that, if  $\neg T(\bar{s}n)$  and  $m \geq n$ , then also  $\neg T(\bar{s}m)$ ; therefore, picking  $\tilde{n} := \max\{t_0, \dots, t_{|t|-1}\}$ , we are sure that

$$\Vdash \forall^{\text{st}} s : 2^{\mathbb{N}} \neg T(\bar{s}\tilde{n}).$$

By transfer, for all  $s \in 2^{\mathbb{N}}$ ,  $\neg T(\bar{s}\tilde{n})$ , and we have proved the fan theorem.

## 4 The Uniform Diller–Nahm Interpretation

In this section, we take a step back, forgetting about nonstandard arithmetic for a while; a reconsideration of ideas from Lifschitz, Berger, and Hernest leads us to a new functional interpretation—uniform Diller–Nahm—of which nonstandard Dialectica can be seen a posteriori as a *herbrandized* version.

**4.1 Calculability and herbrandization** In [13], Lifschitz made the suggestion to see constructive mathematics as an *extension* of classical mathematics.

Lifschitz’s proposal is to enrich the language of Heyting arithmetic with a predicate  $\mathsf{K}(n)$ , “ $n$  is calculable,” and then extend Kleene’s recursive realizability relation, written it  $xr_{\mathsf{K}}\varphi$ , with the clause

$$\triangleright xr_{\mathsf{K}}\mathsf{K}(n) \text{ if and only if } x = n,$$

all the while interpreting quantifiers *uniformly*:

- $\triangleright xr_{\mathsf{K}}\forall n \varphi(n)$  if and only if  $\forall n (xr_{\mathsf{K}}\varphi(n))$ ,
- $\triangleright xr_{\mathsf{K}}\exists n \varphi(n)$  if and only if  $\exists n (xr_{\mathsf{K}}\varphi(n))$ .

In this definition, quantifiers are, by themselves, completely void of any computational meaning; it is by invoking quantifiers *restricted to calculable numbers*,  $\forall n (K(n) \rightarrow \dots)$  and  $\exists n (K(n) \wedge \dots)$ , that one restores it.

A couple of decades later, Lifschitz’s demand was rediscovered, from a completely different perspective, in the area of proof mining. Rather than the foundational issue of injecting a certain “modular constructiveness” into classical reasoning, it was the practical problem of more efficient *program extraction* from proofs that was addressed.

Even in fully intuitionistic proofs, a fine-grained analysis reveals instances of formulae with quantifiers that are *computationally redundant*; that is, the constructive content that is encoded in the quantifiers is never used in the program extracted with the aid of a functional interpretation.

One would want a way to flag such quantifiers, telling the extraction program to just “pass through” them. This is the function performed by Berger’s *uniform* quantifiers in [2] and by Hernest’s quantifiers *without computational meaning* in [7]. But, realizability being a rudimentary functional interpretation—this is clear, in particular, for Kreisel’s *modified* brand (see [18])—this is also what Lifschitz’s calculability predicate achieved!

One possibly unexpected consequence of Lifschitz’s ideas is that there will be two types of disjunction as well. One is a computationally empty disjunction  $\vee$ , with

$$\Phi \vee \Psi \leftrightarrow \exists z : 0 (z = 0 \rightarrow \Phi \wedge \neg z = 0 \rightarrow \Psi),$$

while there is also a computationally relevant disjunction  $\vee_K$ , with

$$\Phi \vee_K \Psi \leftrightarrow \exists z : 0 (K(z) \wedge z = 0 \rightarrow \Phi \wedge \neg z = 0 \rightarrow \Psi).$$

These are not equivalent: in fact, only the second computationally relevant disjunction will act as a disjunction with respect to all the formulae in the language; the computationally empty disjunction only acts as a disjunction with respect to “internal” formulae (i.e., those not containing the  $K$ -predicate).

*Herbrandization* can be seen as a way of repairing this schism. The idea is to weaken the computational meaning of the  $K$ -predicate and define instead

- ▷  $x r_K K(n)$  if and only if  $x$  codes a sequence and  $n$  is one of the components of the sequence coded by  $x$ .

This is reminiscent of *Herbrand disjunctions* in classical logic—whence the name. There are some technical difficulties to overcome and this idea works especially well in the context of modified realizability, leading to *Herbrand realizability* as introduced in [30].

This process of herbrandization is reflected in many of the characteristic principles of Herbrand realizability. While the axiom of choice for finite types

$$\text{AC} : \quad \forall x : \sigma \exists y : \tau \Phi(x, y) \rightarrow \exists f : \sigma \rightarrow \tau \forall x : \sigma \Phi(x, f x)$$

is a characteristic principle of modified realizability, the herbrandized axiom of choice

$$\begin{aligned} \text{HAC}^{\text{st}} : \quad & \forall^{\text{st}} x : \sigma \exists^{\text{st}} y : \tau \Phi(x, y) \\ & \rightarrow \exists^{\text{st}} f : (\sigma \rightarrow \tau^*)^* \forall^{\text{st}} x : \sigma \exists y \in f[x] \Phi(x, y) \end{aligned}$$

(writing *st* again instead of  $K$ ) is a characteristic principle of Herbrand realizability. It seems natural to regard many of the characteristic principles of nonstandard

Dialectica as herbrandizations of other, unherbrandized, principles, suggesting that also nonstandard Dialectica can be obtained by a process of herbrandization from a functional interpretation which incorporates many of Lifschitz's ideas. The aim of this section is to show that this is indeed the case.

**Remark 4.1** Note that it is an immediate consequence of herbrandization that disjunction loses any constructive meaning. In fact, a good way to think about herbrandization is as a way of weakening the computational meaning of the  $K$ -predicate in such a way that  $\vee_K$  collapses to the ordinary, computationally empty, disjunction.

Something like this is presumably unavoidable when one wants to interpret nonstandard systems: indeed, there seems to be a clash between the computational meaning of disjunction and nonstandard arithmetic. One way in which this manifests itself is that systems for nonstandard arithmetic often do not have the disjunction property: for example,  $E\text{-HA}_{\text{st}}^{\omega*} + \text{OS}_0$  does not have the disjunction property, as proved in Avigad and Helzner [1]. Another manifestation is the incompatibility of Church's thesis for disjunctions

$$\begin{aligned} \forall x : 0 (\varphi(x) \vee \psi(x)) &\rightarrow \exists f : 0 \rightarrow 0 (f \text{ is computable} \wedge \\ \forall x : 0 (f(x) = 0 &\rightarrow \varphi(x) \wedge f(x) \neq 0 \rightarrow \psi(x))) \end{aligned}$$

with the existence of nonstandard models for arithmetic (see [15]), showing, for instance, that there are no nonstandard models of arithmetic in the effective topos. This should be compared with the Herbrand topos from van den Berg [29], where nonstandard models of arithmetic do exist.

Before defining our new functional interpretation, we should first “de-herbrandize” our system. By the previous discussion, we can already guess that “internal formula” has to be replaced by the next best thing—“internal *and*  $\vee$ -free formula.”

**Notation** If  $P$  is an axiom schema where certain schematic variables range over *internal* formulae of  $E\text{-HA}_{\text{st}}^{\omega*}$ , we write  $P_{\vee}$  for the same axiom schema, where “internal” is replaced by “internal and  $\vee$ -free.”

Our tentative characteristic system is then  $E\text{-HA}_{\text{st}\vee}^{\omega*}$ , that is, the system  $E\text{-HA}_{\text{st}}^{\omega*}$  with  $IA_{\vee}$  in place of  $IA$ , plus the characteristic principles  $\text{OS}_{\vee}^*$ ,  $\text{US}_{\vee}^*$ ,  $\text{AC}_{\vee}^{\text{st}}$ ,

$$\text{IP}_{\vee}^{\text{st}} : (\forall^{\text{st}} x : \sigma \varphi(x) \rightarrow \exists^{\text{st}} y : \tau \Psi(y)) \rightarrow \exists^{\text{st}} y : \tau (\forall^{\text{st}} x : \sigma \varphi(x) \rightarrow \Psi(y)),$$

$$\text{NU} : \forall y : \tau \exists^{\text{st}} x : \sigma \Phi(x, y) \rightarrow \exists^{\text{st}} x : \sigma \forall y : \tau \Phi(x, y).$$

Since the restriction to  $\vee$ -free formulae also applies to the internal induction schema  $IA$ , we do not get a proper system of arithmetic. This would actually be inconsistent with the nonstandard uniformity principle  $\text{NU}$ .

**Proposition 4.2** *In  $E\text{-HA}_{\text{st}\vee}^{\omega*}$ , the principle  $\text{NU}$  implies*

$$\neg \forall n : 0 (n = 0 \vee \neg n = 0).$$

**Proof** Suppose  $\forall n : 0 (n = 0 \vee \neg n = 0)$ . This is equivalent to

$$\forall n : 0 \exists^{\text{st}} z : 0 (z = 0 \rightarrow n = 0 \wedge \neg z = 0 \rightarrow \neg n = 0),$$

which, by nonstandard uniformity, implies

$$\exists^{\text{st}} z : 0 \forall n : 0 (z = 0 \rightarrow n = 0 \wedge \neg z = 0 \rightarrow \neg n = 0),$$

the statement that all natural numbers are zero, or all are nonzero, which is a contradiction.  $\square$

With the interpretation of  $\text{st}(n)$  as “ $n$  is calculable,” this is not unexpected; for how could we know whether a noncalculable number is zero or nonzero? Notice that  $\forall^{\text{st}} n : 0 (n = 0 \vee \neg n = 0)$  is still provable, thanks to the external induction schema.

The reason why we called NU a *uniformity* principle is the similarity of

$$\forall s : 0^* \exists^{\text{st}} n : 0 \Phi(s, n) \rightarrow \exists^{\text{st}} n : 0 \forall s : 0^* \Phi(s, n)$$

to Troelstra’s *uniformity principle* (see [28, Proposition 8.21])

$$\text{UP} : \quad \forall S \subseteq \mathbb{N} \exists n \in \mathbb{N} \Phi(S, n) \rightarrow \exists n \in \mathbb{N} \forall S \subseteq \mathbb{N} \Phi(S, n),$$

a second-order principle that is validated by higher-order versions of recursive realizability, and which also has nonclassical consequences.

We can now define our de-herbrandized functional interpretation, prove that it is sound, and show that it is characterized by the desired proof system.

## 4.2 The $U$ -translation

**Definition 4.3** To every formula  $\Phi(\underline{a})$  of the language of  $\text{E-HA}_{\text{st}\vee}^{\omega^*}$ , with free variables  $\underline{a}$ , we associate inductively its *uniform Diller–Nahm* translation  $\Phi(\underline{a})^U = \exists^{\text{st}} \underline{x} \forall^{\text{st}} \underline{y} \varphi_U(\underline{x}, \underline{y}, \underline{a})$ , where  $\varphi_U$  is internal and  $\vee$ -free:

- ▷  $\varphi(\underline{a})^U := \varphi_U(\underline{a}) := \varphi(\underline{a})$ , for  $\varphi$  internal atomic;
- ▷  $\text{st}_\sigma(x)^U := \exists^{\text{st}} y : \sigma (y = x)$ .

Let  $\Phi(\underline{a})^U = \exists^{\text{st}} \underline{x} \forall^{\text{st}} \underline{y} \varphi_U(\underline{x}, \underline{y}, \underline{a})$ ,  $\Psi(\underline{b})^U = \exists^{\text{st}} \underline{u} \forall^{\text{st}} \underline{v} \psi_U(\underline{u}, \underline{v}, \underline{b})$ :

- ▷  $(\Phi(\underline{a}) \wedge \Psi(\underline{b}))^U := \exists^{\text{st}} \underline{x}, \underline{u} \forall^{\text{st}} \underline{y}, \underline{v} (\varphi_U(\underline{x}, \underline{y}, \underline{a}) \wedge \psi_U(\underline{u}, \underline{v}, \underline{b}))$ ;
- ▷  $(\Phi(\underline{a}) \vee \Psi(\underline{b}))^U := \exists^{\text{st}} z : 0, \underline{x}, \underline{u} \forall^{\text{st}} \underline{y}, \underline{v} (z = 0 \rightarrow \varphi_U(\underline{x}, \underline{y}, \underline{a}) \wedge \neg z = 0 \rightarrow \psi_U(\underline{u}, \underline{v}, \underline{b}))$ ;
- ▷  $(\Phi(\underline{a}) \rightarrow \Psi(\underline{b}))^U := \exists^{\text{st}} \underline{U}, \underline{Y} \forall^{\text{st}} \underline{x}, \underline{v} (\forall \underline{y} \in \underline{Y} \underline{x} \underline{v} \varphi_U(\underline{x}, \underline{y}, \underline{a}) \rightarrow \psi_U(\underline{U} \underline{x}, \underline{v}, \underline{b}))$ ;
- ▷  $(\exists z \Phi(z, \underline{a}))^U := \exists^{\text{st}} \underline{x} \forall^{\text{st}} \underline{y} \exists z \forall \underline{y}' \in \underline{y} \varphi_U(\underline{x}, \underline{y}', z, \underline{a})$ ;
- ▷  $(\forall z \Phi(z, \underline{a}))^U := \exists^{\text{st}} \underline{x} \forall^{\text{st}} \underline{y} \forall z \varphi_U(\underline{x}, \underline{y}, z, \underline{a})$ ;
- ▷  $(\exists^{\text{st}} z \Phi(z, \underline{a}))^U := \exists^{\text{st}} z, \underline{x} \forall^{\text{st}} \underline{y} \varphi_U(\underline{x}, \underline{y}, z, \underline{a})$ ;
- ▷  $(\forall^{\text{st}} z \Phi(z, \underline{a}))^U := \exists^{\text{st}} \underline{X} \forall^{\text{st}} \underline{y}, z \varphi_U(\underline{X} z, \underline{y}, z, \underline{a})$ .

**Remark 4.4** In the  $D_{\text{st}}$ -translation, finite sequence application and abstraction were used to obtain upward closed formulas, which is useful—but not essential—for the soundness proof. Since this is no longer required in the  $U$ -translation, we revert to the usual application and abstraction. There is, however, no substantial difference between the two choices, for the two versions of  $\text{HAC}^{\text{st}}$  with the different kinds of application are equivalent.

The first thing to notice is that, if this interpretation is restricted to formulae that contain only external quantifiers—or, if you prefer, everything is declared standard—it is the same as the usual Diller–Nahm translation. In fact, except for a minor change in the interpretation of the uniform existential quantifier, it is to the Diller–Nahm variant precisely what Herneš’s light Dialectica interpretation is to Dialectica.

Second, the interpretation is *idempotent*: formulae of the form

$$\exists^{\text{st}} \underline{x} \forall^{\text{st}} \underline{y} \varphi(\underline{x}, \underline{y}, \underline{a})$$

with  $\varphi$  internal and  $\forall$ -free are interpreted as themselves, as shown by an easy induction on their structure. This is a feature that the  $D_{\text{st}}$ -translation lacked, due to the clause for the  $\exists^{\text{st}}$  quantifier.

We will now prove the soundness of the interpretation. We will not handle everything explicitly, though: except those concerning the quantifiers, all the logical axioms and rules admit the same realizers as those for the Diller–Nahm interpretation.

We write  $\text{E-HA}_{\forall}^{\omega*}$  for the system  $\text{E-HA}^{\omega*}$  with  $\text{IA}_{\forall}$  in place of  $\text{IA}$ .

**Theorem 4.5 (Soundness of uniform Diller–Nahm)** *Suppose that*

$$\text{E-HA}_{\text{st}\forall}^{\omega*} + \text{OS}_{\forall}^* + \text{US}_{\forall}^* + \text{NU} + \text{AC}^{\text{st}} + \text{IP}_{\forall}^{\text{st}} + \Delta_{\forall} \vdash \Phi(\underline{a}),$$

where  $\Delta_{\forall}$  is a set of internal,  $\forall$ -free sentences. Let  $\Phi(\underline{a})^U = \exists^{\text{st}} \underline{x} \forall^{\text{st}} \underline{y} \varphi_U(\underline{x}, \underline{y}, \underline{a})$ . Then, from the proof we can extract a tuple of closed terms  $\underline{t}$  such that

$$\text{E-HA}_{\forall}^{\omega*} + \Delta_{\forall} \vdash \forall \underline{y} \varphi_U(\underline{t}, \underline{y}, \underline{a}).$$

**Proof** We proceed by induction on the length of the derivation.

(1) *The logical axioms and rules of intuitionistic first-order predicate logic.* We consider the quantifier axioms and rules, and give another couple of examples, referring again to [27, Theorem 3.5.4] for the rest.

(i) Example—weakening :  $A \rightarrow A \vee B$ .

Suppose that  $A^U = \exists^{\text{st}} \underline{x} \forall^{\text{st}} \underline{y} \varphi(\underline{x}, \underline{y}, \underline{a})$ ,  $B^U = \exists^{\text{st}} \underline{u} \forall^{\text{st}} \underline{v} \psi(\underline{u}, \underline{v}, \underline{b})$ .

Then

$$(A \rightarrow A \vee B)^U = \exists^{\text{st}} \underline{Z}, \underline{X}', \underline{U}, \underline{S} \forall^{\text{st}} \underline{x}, \underline{y}', \underline{v} (\forall \underline{y} \in \underline{S} \underline{x} \underline{y}' \underline{v} \varphi(\underline{x}, \underline{y}, \underline{a}) \rightarrow (\underline{Z} \underline{x} = 0 \rightarrow \varphi(\underline{X}' \underline{x}, \underline{y}', \underline{a}) \wedge \neg \underline{Z} \underline{x} = 0 \rightarrow \psi(\underline{U} \underline{x}, \underline{v}, \underline{b}))),$$

and we can take

$$\begin{aligned} \underline{Z} &:= \lambda \underline{x}. 0, & \underline{X}' &:= \lambda \underline{x}. \underline{x}, \\ \underline{U} &\text{arbitrary}, & \underline{S} &:= \lambda \underline{x}, \underline{y}', \underline{v}. \langle \underline{y}' \rangle. \end{aligned}$$

(ii)  $\forall z A \rightarrow A[b/z]$ .

Suppose that  $A^U = \exists^{\text{st}} \underline{x} \forall^{\text{st}} \underline{y} \varphi(\underline{x}, \underline{y}, z, \underline{a})$ . Then

$$\begin{aligned} (\forall z A \rightarrow A[b/z])^U &= \exists^{\text{st}} \underline{X}', \underline{S} \forall^{\text{st}} \underline{x}, \underline{y}' \\ (\forall \underline{y} \in \underline{S} \underline{x} \underline{y}' \forall z \varphi(\underline{x}, \underline{y}, z, \underline{a}) \rightarrow \varphi(\underline{X}' \underline{x}, \underline{y}', b, \underline{a})), \end{aligned}$$

so we can take

$$\underline{X}' := \lambda \underline{x}. \underline{x}, \quad \underline{S} := \lambda \underline{x}, \underline{y}'. \langle \underline{y}' \rangle.$$

(iii)  $A[b/z] \rightarrow \exists z A$ .

Suppose that  $A^U = \exists^{\text{st}} \underline{x} \forall^{\text{st}} \underline{y} \varphi(\underline{x}, \underline{y}, z, \underline{a})$ . Then

$$\begin{aligned} (A[b/z] \rightarrow \exists z A)^U &= \exists^{\text{st}} \underline{X}', \underline{S} \forall^{\text{st}} \underline{x}, \underline{t} \\ (\forall \underline{y} \in \underline{S} \underline{x} \underline{t} \varphi(\underline{x}, \underline{y}, b, \underline{a}) \rightarrow \exists z \forall \underline{y}' \in \underline{t} \varphi(\underline{X}' \underline{x}, \underline{y}', z, \underline{a})), \end{aligned}$$

and we can take

$$\underline{X}' := \lambda \underline{x}. \underline{x}, \quad \underline{S} := \lambda \underline{x}, \underline{t}. \underline{t}.$$

(iv) Example—modus ponens.

Suppose that  $A^U = \exists^{\text{st}} \underline{x} \forall^{\text{st}} \underline{y} \varphi(\underline{x}, \underline{y}, \underline{a})$ ,  $B^U = \exists^{\text{st}} \underline{u} \forall^{\text{st}} \underline{v} \psi(\underline{u}, \underline{v}, \underline{b})$  and that we have terms  $\underline{t}_1$  realizing the interpretation of  $A^U$  and  $\underline{T}_2, \underline{T}_3$  realizing the interpretation of  $(A \rightarrow B)^U$ .

This means that we have

$$\text{E-HA}_{\forall}^{\omega*} + \Delta_{\forall} \vdash \forall \underline{y} \varphi(\underline{t}_1, \underline{y}, \underline{a}),$$

and

$$\text{E-HA}_{\forall}^{\omega*} + \Delta_{\forall} \vdash \forall \underline{x}, \underline{v} (\forall \underline{y} \in \underline{T}_3 \underline{x} \underline{v} \varphi(\underline{x}, \underline{y}, \underline{a}) \rightarrow \psi(\underline{T}_2 \underline{x}, \underline{v}, \underline{b})).$$

Taking  $\underline{t}_4 := \underline{T}_2 \underline{t}_1$ , we obtain

$$\text{E-HA}_{\forall}^{\omega*} + \Delta_{\forall} \vdash \forall \underline{v} \psi(\underline{t}_4, \underline{v}, \underline{b}),$$

as desired.

$$(v) \frac{B \rightarrow A}{B \rightarrow \forall z A}.$$

Suppose that  $A^U = \exists^{\text{st}} \underline{x} \forall^{\text{st}} \underline{y} \varphi(\underline{x}, \underline{y}, \underline{z}, \underline{a})$ ,  $B^U = \exists^{\text{st}} \underline{u} \forall^{\text{st}} \underline{v} \psi(\underline{u}, \underline{v}, \underline{b})$ , where  $z$  is not free in  $\psi$ , and that we have terms  $\underline{T}_1, \underline{T}_2$  realizing  $(B \rightarrow A)^U$ . Then,

$$\text{E-HA}_{\forall}^{\omega*} + \Delta_{\forall} \vdash \forall \underline{u}, \underline{y} (\forall \underline{v} \in \underline{T}_2 \underline{u} \underline{y} \psi(\underline{u}, \underline{v}, \underline{b}) \rightarrow \varphi(\underline{T}_1 \underline{u}, \underline{y}, \underline{z}, \underline{a})).$$

Then  $\underline{T}_3 := \underline{T}_1$  and  $\underline{T}_4 := \underline{T}_2$  realize the interpretation of  $B \rightarrow \forall z A$ .

$$(vi) \frac{A \rightarrow B}{\exists z A \rightarrow B}.$$

Suppose that  $A^U = \exists^{\text{st}} \underline{x} \forall^{\text{st}} \underline{y} \varphi(\underline{x}, \underline{y}, \underline{z}, \underline{a})$ ,  $B^U = \exists^{\text{st}} \underline{u} \forall^{\text{st}} \underline{v} \psi(\underline{u}, \underline{v}, \underline{b})$ , where  $z$  is not free in  $\psi$ , and that we have terms  $\underline{T}_1, \underline{T}_2$  realizing  $(A \rightarrow B)^U$ . Then,

$$\text{E-HA}_{\forall}^{\omega*} + \Delta_{\forall} \vdash \forall \underline{x}, \underline{v} (\forall \underline{y} \in \underline{T}_2 \underline{x} \underline{v} \varphi(\underline{x}, \underline{y}, \underline{z}, \underline{a}) \rightarrow \psi(\underline{T}_1 \underline{x}, \underline{v}, \underline{b})).$$

We have

$$\begin{aligned} (\exists z A \rightarrow B)^U &= \exists^{\text{st}} \underline{U}, \underline{S} \forall^{\text{st}} \underline{x}, \underline{v} \\ &(\forall \underline{s} \in \underline{S} \underline{x} \underline{v} \exists z \forall \underline{y} \in \underline{s} \varphi(\underline{x}, \underline{y}, \underline{z}, \underline{a}) \rightarrow \psi(\underline{U} \underline{x}, \underline{v}, \underline{b})); \end{aligned}$$

so we can take  $\underline{T}_3 := \underline{T}_1$  and  $\underline{T}_4 := \lambda \underline{x}, \underline{v}. \langle \underline{T}_2 \underline{x} \underline{v} \rangle$  to obtain

$$\text{E-HA}_{\forall}^{\omega*} + \Delta_{\forall} \vdash \forall \underline{x}, \underline{v} (\forall \underline{s} \in \underline{T}_4 \underline{x} \underline{v} \exists z \forall \underline{y} \in \underline{s} \varphi(\underline{x}, \underline{y}, \underline{z}, \underline{a}) \rightarrow \psi(\underline{T}_3 \underline{x}, \underline{v}, \underline{b})).$$

(2) *The nonlogical axioms of extensional Heyting arithmetic in all finite types* (with the restricted induction schema  $\text{IA}_{\forall}$ ). These are all internal and  $\forall$ -free and, hence, are realized by the empty tuple.

(3) *The defining axioms of the external quantifiers.* Let  $\Phi(x)^U := \exists^{\text{st}} \underline{u} \forall^{\text{st}} \underline{v} \varphi(\underline{u}, \underline{v}, x)$  here.

$$(i) \forall^{\text{st}} x \Phi(x) \leftrightarrow \forall x (\text{st}(x) \rightarrow \Phi(x)).$$

The interpretation of  $\forall^{\text{st}} x \Phi(x) \rightarrow \forall x (\text{st}(x) \rightarrow \Phi(x))$  is

$$\begin{aligned} \exists^{\text{st}} \underline{U}', \underline{S}, \underline{T} \forall^{\text{st}} \underline{U}, \underline{y}, \underline{v}' (\forall \underline{x} \in \underline{S} \underline{U} \underline{y} \underline{v}' \forall \underline{v} \in \underline{T} \underline{U} \underline{y} \underline{v}' \varphi(\underline{U} \underline{x}, \underline{v}, \underline{x}) \rightarrow \\ \forall \underline{x} (x = \underline{y} \rightarrow \varphi(\underline{U}' \underline{U} \underline{y}, \underline{v}', x))); \end{aligned}$$

so we can take

$$\begin{aligned} \underline{U}' &:= \lambda \underline{U}, y. \underline{U}y, & S &:= \lambda \underline{U}, y, \underline{v}'. \langle y \rangle, \\ \underline{T} &:= \lambda \underline{U}, y, \underline{v}'. \langle \underline{v}' \rangle. \end{aligned}$$

On the other hand, the interpretation of  $\forall x (\text{st}(x) \rightarrow \Phi(x)) \rightarrow \forall^{\text{st}}x \Phi(x)$  is

$$\begin{aligned} \exists^{\text{st}} \underline{U}', S, \underline{T} \forall^{\text{st}} x', \underline{U}, \underline{v}' (\forall y \in Sx' \underline{U}v' \forall v \in \underline{T}x' \underline{U}v' \forall x \\ (x = y \rightarrow \varphi(\underline{U}y, v, x)) \rightarrow \varphi(\underline{U}'\underline{U}x', \underline{v}', x')), \end{aligned}$$

and we can take

$$\begin{aligned} \underline{U}' &:= \lambda \underline{U}, x'. \underline{U}x', & S &:= \lambda x', \underline{U}, \underline{v}'. \langle x' \rangle, \\ \underline{T} &:= \lambda x', \underline{U}, \underline{v}'. \langle \underline{v}' \rangle. \end{aligned}$$

(ii)  $\exists^{\text{st}}x \Phi(x) \leftrightarrow \exists x (\text{st}(x) \wedge \Phi(x))$ .

The interpretation of  $\exists^{\text{st}}x \Phi(x) \rightarrow \exists x (\text{st}(x) \wedge \Phi(x))$  is

$$\begin{aligned} \exists^{\text{st}} Y, \underline{U}', \underline{T} \forall^{\text{st}} x, \underline{u}, \underline{s} (\forall v \in \underline{T}x \underline{u} \underline{s} \varphi(\underline{u}, v, x) \rightarrow \\ \exists x' \forall v' \in \underline{s} (Yx \underline{u} = x' \wedge \varphi(\underline{U}'x \underline{u}, v', x'))); \end{aligned}$$

so we can take

$$\begin{aligned} Y &:= \lambda x, \underline{u}. x, & \underline{U}' &:= \lambda x, \underline{u}. \underline{u}, \\ \underline{T} &:= \lambda x, \underline{u}, \underline{s}. \underline{s}. \end{aligned}$$

The interpretation of its converse  $\exists x (\text{st}(x) \wedge \Phi(x)) \rightarrow \exists^{\text{st}}x \Phi(x)$  is

$$\begin{aligned} \exists^{\text{st}} X, \underline{U}, \underline{S} \forall^{\text{st}} y, \underline{u}', \underline{v} (\forall \underline{s} \in \underline{S}y \underline{u}' \underline{v} \exists x' \forall v' \in \underline{s} \\ (y = x' \wedge \varphi(\underline{u}', \underline{v}', x')) \rightarrow \varphi(\underline{U}y \underline{u}', \underline{v}, Xy \underline{u}')), \end{aligned}$$

and we can take

$$\begin{aligned} X &:= \lambda y, \underline{u}'. y, & \underline{U} &:= \lambda y, \underline{u}'. \underline{u}', \\ \underline{S} &:= \lambda y, \underline{u}', \underline{v}. \langle \underline{v} \rangle. \end{aligned}$$

(4) *The axioms for the standardness predicate.*

(i)  $\text{st}(x) \wedge x = y \rightarrow \text{st}(y)$ .

The interpretation of this axiom is

$$\exists^{\text{st}} Y' \forall^{\text{st}} x' (x = x' \wedge x = y \rightarrow y = Y'x'),$$

so we can take  $Y' := \lambda x'. x'$ .

(ii)  $\text{st}(a)$  for all closed terms  $a$ .

We have  $(\text{st}(a))^U = \exists^{\text{st}}x (a = x)$ , so we can take  $x := a$ .

(iii)  $\text{st}(f) \wedge \text{st}(x) \rightarrow \text{st}(fx)$ .

The interpretation of this axiom is

$$\exists^{\text{st}} Y \forall^{\text{st}} f', x' (f = f' \wedge x = x' \rightarrow fx = Yf'x'),$$

so we can take  $Y := \lambda f', x'. f'x'$ .

(5) *The external induction schema.*

As in [30], we consider the equivalent external induction rule

$$\text{IR}^{\text{st}} : \frac{\Phi(0) \quad \forall^{\text{st}}n : 0(\Phi(n) \rightarrow \Phi(n+1))}{\forall^{\text{st}}n : 0 \Phi(n)},$$

from which the external induction schema is obtained by taking

$$\Phi(m) := \Psi(0) \wedge \forall^{\text{st}}n : 0(\Psi(n) \rightarrow \Psi(n+1)) \rightarrow \Psi(m).$$

So, suppose that  $(\Phi(n))^U = \exists^{\text{st}} \underline{x} \forall^{\text{st}} \underline{y} \varphi(\underline{x}, \underline{y}, n, \underline{a})$ , and that we have realizers  $\underline{t}_1$ , and  $\underline{T}_2, \underline{T}_3$  for the premises; that is,

$$\text{E-HA}_{\forall}^{\omega*} + \Delta_{\forall} \vdash \forall \underline{y} \varphi(\underline{t}_1, \underline{y}, 0, \underline{a}),$$

and

$$\text{E-HA}_{\forall}^{\omega*} + \Delta_{\forall} \vdash \forall n, \underline{x}, \underline{y}' (\forall \underline{y} \in \underline{T}_3 n \underline{x} \underline{y}' \varphi(\underline{x}, \underline{y}, n, \underline{a}) \rightarrow \varphi(\underline{T}_2 n \underline{x}, \underline{y}', n + 1, \underline{a})).$$

By taking  $\underline{T}_4 := \lambda n. \underline{R} \underline{t}_1 \underline{T}_2 n$ , we obtain, by induction for  $\forall$ -free formulae in  $\text{E-HA}_{\forall}^{\omega*}$ , that

$$\text{E-HA}_{\forall}^{\omega*} + \Delta_{\forall} \vdash \forall n, \underline{y} \varphi(\underline{T}_4 n, \underline{y}, n, \underline{a}),$$

which was to be proved.

(6) *The principles*  $\text{OS}_{\forall}^*$ ,  $\text{US}_{\forall}^*$ ,  $\text{NU}$ ,  $\text{AC}^{\text{st}}$ ,  $\text{IP}_{\forall}^{\text{st}}$ .

(i)  $\text{OS}_{\forall}^*$ :  $\forall^{\text{st}} s \varphi(s) \rightarrow \exists s (\forall^{\text{st}} x (x \in s) \wedge \varphi(s))$ , with  $\varphi$  internal and  $\forall$ -free.

This is interpreted as

$$\exists^{\text{st}} S \forall^{\text{st}} s' (\forall s \in S s' \varphi(s) \rightarrow \exists s (s' \subseteq s \wedge \varphi(s))),$$

and we can take  $S := \lambda s'. \{s'\}$ .

(ii)  $\text{US}_{\forall}^*$ :  $\forall s (\forall^{\text{st}} x (x \in s) \rightarrow \varphi(s)) \rightarrow \exists^{\text{st}} s \varphi(s)$ , with  $\varphi$  internal and  $\forall$ -free.

The interpretation of this axiom schema is

$$\exists^{\text{st}} S \forall^{\text{st}} s' (\forall s (s' \subseteq s \rightarrow \varphi(s)) \rightarrow \varphi(S s'));$$

so we can take  $S := \lambda s'. s'$ .

For the principles  $\text{NU}$ ,  $\text{AC}^{\text{st}}$ ,  $\text{IP}_{\forall}^{\text{st}}$ , we can just observe that the premise and the conclusion have identical interpretations, so it is trivial to find realizers for the implication. We do the first as an example.

(iii)  $\text{NU}$ :  $\forall y \exists^{\text{st}} x \Phi(x, y) \rightarrow \exists^{\text{st}} x \forall y \Phi(x, y)$ .

Let  $\Phi(x, y)^U := \exists^{\text{st}} \underline{u} \forall^{\text{st}} \underline{v} \varphi(\underline{u}, \underline{v}, x, y)$ . Both the premise and the conclusion are interpreted as

$$\exists^{\text{st}} x, \underline{u} \forall^{\text{st}} \underline{v} \forall y \varphi(\underline{u}, \underline{v}, x, y);$$

so the implication is interpreted as

$$\begin{aligned} & \exists^{\text{st}} X', \underline{U}', \underline{S} \forall^{\text{st}} x, \underline{u}, \underline{v}' \\ & (\forall \underline{v} \in \underline{S} x \underline{u} \underline{v}' \forall y \varphi(\underline{u}, \underline{v}, x, y) \rightarrow \forall y \varphi(\underline{U}' x \underline{u}, \underline{v}', X' x \underline{u}, y)), \end{aligned}$$

and we can take

$$\begin{aligned} X' & := \lambda x, \underline{u}. x, & \underline{U}' & := \lambda x, \underline{u}. \underline{u}, \\ \underline{S} & := \lambda x, \underline{u}, \underline{v}'. \{\underline{v}'\}. \end{aligned}$$

This concludes the proof.  $\square$

As usual, the soundness of an interpretation leads to a conservation result.

**Corollary 4.6** *The system*

$$\text{E-HA}_{\text{st}\forall}^{\omega*} + \text{OS}_{\forall}^* + \text{US}_{\forall}^* + \text{NU} + \text{AC}^{\text{st}} + \text{IP}_{\forall}^{\text{st}}$$

*is conservative with respect to  $\forall$ -free formulae of  $\text{E-HA}_{\forall}^{\omega*}$ .*

**Proof** This follows immediately from the previous theorem.  $\square$

Now, let

$$\text{H} := \text{E-HA}_{\text{st}\forall}^{\omega*} + \text{OS}_{\forall}^* + \text{US}_{\forall}^* + \text{NU} + \text{AC}^{\text{st}} + \text{IP}_{\forall}^{\text{st}}.$$

**Theorem 4.7 (Characterization of uniform Diller–Nahm)** *Let  $\Phi$  be a formula in the language of  $\text{E-HA}_{\text{st}\forall}^{\omega*}$ .*

(a) *We have  $\text{H} \vdash \Phi \leftrightarrow \Phi^U$ .*

(b) *If for all formulae  $\Psi$  of the language of  $\text{E-HA}_{\text{st}\forall}^{\omega*}$ , with  $\Psi^U = \exists^{\text{st}} \underline{x} \forall^{\text{st}} \underline{y} \psi(\underline{x}, \underline{y})$ ,*

$$\text{H} + \Phi \vdash \Psi$$

*implies that there exist closed terms  $\underline{t}$  such that*

$$\text{E-HA}_{\forall}^{\omega*} \vdash \forall \underline{y} \psi(\underline{t}, \underline{y})$$

*holds, then  $\text{H} \vdash \Phi$ .*

**Proof** We prove (a) by induction on the logical structure of  $\Phi$ . For  $\Phi \equiv \varphi$  internal atomic, obviously  $\text{H} \vdash \varphi \leftrightarrow \varphi^U$ .

Let  $\Phi \equiv \text{st}(x)$ . If  $x$  is standard, it follows that  $\exists^{\text{st}} y (x = y)$ , by taking  $y := x$ . Conversely, if  $\exists^{\text{st}} y (x = y)$ , by the first axiom for the standardness predicate it follows that  $x$  is standard. Hence,

$$\text{H} \vdash \text{st}(x) \leftrightarrow \exists^{\text{st}} y (x = y).$$

For the induction hypothesis, using an appropriate embedding of tuples of types into higher types and a compatible coding of tuples of terms (see [27, Section 1.6.17]), we can assume, given formulae  $\Phi$  and  $\Psi$ , that there exist internal,  $\forall$ -free formulae  $\varphi, \psi$  such that

$$\text{H} \vdash \Phi(x) \leftrightarrow \exists^{\text{st}} x \forall^{\text{st}} y \varphi(x, y),$$

$$\text{H} \vdash \Psi(x) \leftrightarrow \exists^{\text{st}} u \forall^{\text{st}} v \psi(u, v).$$

(i) For  $\wedge$ , by intuitionistic logic,

$$\exists^{\text{st}} x \forall^{\text{st}} y \varphi(x, y) \wedge \exists^{\text{st}} u \forall^{\text{st}} v \psi(u, v)$$

is equivalent to

$$\exists^{\text{st}} x, u \forall^{\text{st}} y, v (\varphi(x, y) \wedge \psi(u, v)).$$

(ii) For  $\vee$ ,

$$\exists^{\text{st}} x \forall^{\text{st}} y \varphi(x, y) \vee \exists^{\text{st}} u \forall^{\text{st}} v \psi(u, v)$$

is equivalent in  $\text{H}$  to

$$\exists^{\text{st}} z : 0 (z = 0 \rightarrow \exists^{\text{st}} x \forall^{\text{st}} y \varphi(x, y) \wedge \neg z = 0 \rightarrow \exists^{\text{st}} u \forall^{\text{st}} v \psi(u, v)).$$

By  $\text{IP}_{\forall}^{\text{st}}$ , this is equivalent to

$$\exists^{\text{st}} z : 0 (\exists^{\text{st}} x \forall^{\text{st}} y (z = 0 \rightarrow \varphi(x, y)) \wedge \exists^{\text{st}} u \forall^{\text{st}} v (\neg z = 0 \rightarrow \psi(u, v))),$$

and we are back to the case of conjunction.

(iii) For  $\rightarrow$ , we proceed as with the Diller–Nahm implication. By intuitionistic logic and the principle  $\text{IP}_{\forall}^{\text{st}}$ ,

$$\exists^{\text{st}} x \forall^{\text{st}} y \varphi(x, y) \rightarrow \exists^{\text{st}} u \forall^{\text{st}} v \psi(u, v)$$

is equivalent to

$$\forall^{\text{st}} x \exists^{\text{st}} u \forall^{\text{st}} v (\forall^{\text{st}} y \varphi(x, y) \rightarrow \psi(u, v)).$$

Now, adapting Proposition 2.23, we see that  $\text{E-HA}_{\text{st}\forall}^{\omega*} + \text{US}_{\forall}^* \vdash \text{HGMP}_{\forall}^{\text{st}}$ , so this is equivalent to

$$\forall^{\text{st}}x \exists^{\text{st}}u \forall^{\text{st}}v \exists^{\text{st}}s (\forall y \in s \varphi(x, y) \rightarrow \psi(u, v)).$$

Two applications of  $\text{AC}^{\text{st}}$  then lead to

$$\exists^{\text{st}}U, S \forall^{\text{st}}x, v (\forall y \in Sxv \varphi(x, y) \rightarrow \psi(Ux, v)).$$

(iv) For  $\exists z$ , adapting Proposition 2.20, we see that  $\text{E-HA}_{\text{st}\forall}^{\omega*} + \text{OS}_{\forall}^* \vdash \text{I}_{\forall}$ ; therefore

$$\exists z \exists^{\text{st}}x \forall^{\text{st}}y \varphi(x, y, z)$$

is equivalent to

$$\exists^{\text{st}}x \forall^{\text{st}}s \exists z \forall y \in s \varphi(x, y, z).$$

(v) For  $\forall z$ , we use that by NU

$$\forall z \exists^{\text{st}}x \forall^{\text{st}}y \varphi(x, y, z)$$

is equivalent to

$$\exists^{\text{st}}x \forall^{\text{st}}y \forall z \varphi(x, y, z).$$

(vi) For  $\exists^{\text{st}}z$ , nothing really needs to be done.

(vii) For  $\forall^{\text{st}}z$ , we just use  $\text{AC}^{\text{st}}$  once to obtain that

$$\forall^{\text{st}}z \exists^{\text{st}}x \forall^{\text{st}}y \varphi(x, y, z)$$

is equivalent to

$$\exists^{\text{st}}X \forall^{\text{st}}y, z \varphi(Xz, y, z).$$

This proves item (a).

For (b), suppose that  $\Phi$  satisfies the condition, and  $\Phi^U = \exists^{\text{st}}\underline{x} \forall^{\text{st}}\underline{y} \varphi(\underline{x}, \underline{y})$ . Then, from

$$\text{H} + \Phi \vdash \Phi$$

it follows that there exist closed terms  $\underline{t}$  such that

$$\text{E-HA}_{\forall}^{\omega*} \vdash \forall \underline{y} \varphi(\underline{t}, \underline{y}).$$

From this, we obtain  $\text{E-HA}_{\text{st}\forall}^{\omega*} \vdash \forall^{\text{st}}\underline{y} \varphi(\underline{t}, \underline{y})$ , whence

$$\text{E-HA}_{\text{st}\forall}^{\omega*} \vdash \exists^{\text{st}}\underline{x} \forall^{\text{st}}\underline{y} \varphi(\underline{x}, \underline{y});$$

so  $\text{H} \vdash \exists^{\text{st}}\underline{x} \forall^{\text{st}}\underline{y} \varphi(\underline{x}, \underline{y})$  as well. But then, by the equivalence of (a),

$$\text{H} \vdash \Phi,$$

which was to be proved.  $\square$

We now show how the uniform Diller–Nahm interpretation may be used to extract programs from proofs, and eliminate instances of its characteristic principles.

**Theorem 4.8 (Program extraction by the  $U$ -interpretation)** *Let  $\forall^{\text{st}}x \exists^{\text{st}}y \varphi(x, y)$  be a sentence of  $\text{E-HA}_{\text{st}\forall}^{\omega*}$ , with  $\varphi$  internal and  $\forall$ -free, and let  $\Delta_{\forall}$  be a set of internal,  $\forall$ -free sentences. If*

$$\text{E-HA}_{\text{st}\forall}^{\omega*} + \text{OS}_{\forall}^* + \text{US}_{\forall}^* + \text{NU} + \text{AC}^{\text{st}} + \text{IP}_{\forall}^{\text{st}} + \Delta_{\forall} \vdash \forall^{\text{st}}x \exists^{\text{st}}y \varphi(x, y),$$

then from the proof we can extract a closed term  $T$  such that

$$\text{E-HA}_{\forall}^{\omega*} + \Delta_{\forall} \vdash \forall x \varphi(x, Tx).$$

**Proof** The  $U$ -translation of  $\forall^{\text{st}}x \exists^{\text{st}}y \varphi(x, y)$  is

$$\exists^{\text{st}}f \forall^{\text{st}}x \varphi(x, fx),$$

so the thesis immediately follows from the soundness theorem.  $\square$

Finally, we derive a few properties of the system  $H$ , which follow from the properties of the uniform Diller–Nahm interpretation.

**Proposition 4.9** *The system  $E\text{-HA}_{\text{st}\forall}^{\omega*} + \text{OS}_{\forall}^* + \text{US}_{\forall}^* + \text{NU} + \text{AC}^{\text{st}} + \text{IP}_{\forall}^{\text{st}}$  is closed under the restricted transfer rules*

$$\text{TR}_{\forall\forall} : \frac{\forall^{\text{st}}x : \sigma \varphi(x)}{\forall x : \sigma \varphi(x)},$$

$$\text{TR}_{\exists\forall} : \frac{\exists x : \sigma \varphi(x)}{\exists^{\text{st}}x : \sigma \varphi(x)},$$

where  $\varphi$  ranges over internal  $\forall$ -free formulae.

**Proof** This is an adaptation of [30, Proposition 5.12]. Suppose

$$H \vdash \forall^{\text{st}}x \varphi(x).$$

By the soundness theorem, it follows that

$$E\text{-HA}_{\forall}^{\omega*} \vdash \forall x \varphi(x),$$

which, since  $H$  is an extension of  $E\text{-HA}_{\forall}^{\omega*}$ , implies  $H \vdash \forall^{\text{st}}x \varphi(x)$ .

Now, suppose that

$$H \vdash \exists x \varphi(x);$$

by conservativity, this implies  $E\text{-HA}_{\forall}^{\omega*} \vdash \exists x \varphi(x)$ . Being a subsystem of  $E\text{-HA}^{\omega*}$ ,  $E\text{-HA}_{\forall}^{\omega*}$  inherits the existence property; so we can find a closed term  $t$  such that

$$E\text{-HA}_{\forall}^{\omega*} \vdash \varphi(t).$$

Since  $t$  is provably standard in  $H$ , this implies  $H \vdash \exists^{\text{st}}x \varphi(x)$ .  $\square$

**Proposition 4.10** *The system  $H := E\text{-HA}_{\text{st}\forall}^{\omega*} + \text{OS}_{\forall}^* + \text{US}_{\forall}^* + \text{NU} + \text{AC}^{\text{st}} + \text{IP}_{\forall}^{\text{st}}$  has the following form of the existence property: if*

$$H \vdash \exists^{\text{st}}x \Phi(x),$$

then there exists a closed term  $t$  such that  $H \vdash \Phi(t)$ .

**Proof** Let  $\Phi(x)^U = \exists^{\text{st}}\underline{u} \forall^{\text{st}}\underline{v} \varphi(x, \underline{u}, \underline{v})$ . By the characterization theorem,  $H$  proves that  $\Phi$  is equivalent to its  $U$ -translation; so, if  $H \vdash \exists^{\text{st}}x \Phi(x)$ , then

$$H \vdash \exists^{\text{st}}x, \underline{u} \forall^{\text{st}}\underline{v} \varphi(x, \underline{u}, \underline{v}).$$

By the soundness of uniform Diller–Nahm, we can extract closed terms  $t, \underline{T}$  such that

$$E\text{-HA}_{\forall}^{\omega*} \vdash \forall \underline{v} \varphi(t, \underline{T}, \underline{v}),$$

which, by conservativity and weakening the quantifier, implies

$$H \vdash \forall^{\text{st}}\underline{v} \varphi(t, \underline{T}, \underline{v}).$$

Since the terms in  $\underline{T}$  are provably standard in  $H$ , we obtain

$$H \vdash \exists^{\text{st}}\underline{u} \forall^{\text{st}}\underline{v} \varphi(t, \underline{u}, \underline{v}),$$

which, again by the characterization theorem, implies  $H \vdash \Phi(t)$ .  $\square$

**Corollary 4.11** *The system  $\text{E-HA}_{\text{st}\vee}^{\omega*} + \text{OS}_{\vee}^* + \text{US}_{\vee}^* + \text{NU} + \text{AC}^{\text{st}} + \text{IP}_{\vee}^{\text{st}}$  has the disjunction property.*

**Proof** This follows from the validity of  $\Phi \vee \Psi \leftrightarrow \exists^{\text{st}} z : 0(z = 0 \rightarrow \Phi \wedge \neg z = 0 \rightarrow \Psi)$  in  $\text{H}$  and the previous proposition.  $\square$

**4.3 De-herbrandization and the topos  $\mathcal{U}$**  Let us examine how de-herbrandization is reflected in the topos-theoretic analysis. By looking at Lemma 3.12(b), we see that herbrandization has a direct categorical analogue in the choice of finite families as  $K$ -covers. The obvious choice for uniform Diller–Nahm, then, is to consider the smaller topology  $K_1$ , where covers of an object  $C$  are *single* covering morphisms  $\{f : D \twoheadrightarrow C\}$ . A fortiori,  $K_1$  is also subcanonical for coherent categories.

**Definition 4.12** The Grothendieck topos  $\text{Sh}(\mathfrak{F}\mathbf{Set}, K_1)$  will be called  $\mathcal{U}$ .

The constant objects functor  $\Delta_1$  of  $\mathcal{U}$  is defined, for all sets  $S$ , at all filters  $\mathcal{F}$  of  $\mathfrak{F}\mathbf{Set}$ , by

$$(\Delta_1 S)_{\mathcal{F}} = \{\alpha : \mathcal{F} \rightarrow S \mid \alpha \text{ is constant}\}.$$

We have  $\Delta_1 2 \simeq 1 + 1 =: 2$  in  $\mathcal{U}$ , for  $\Delta_1$  preserves coproducts; unlike in  $\mathcal{N}$ , there is a proper monomorphism  $m : 2 \hookrightarrow \mathbf{y}2$ . Moreover, since the sheafification functor  $\mathbf{a}$  of  $\mathcal{N}$  is itself left adjoint to the inclusion of  $\mathcal{N}$  in  $\mathcal{U}$ , we have that  $\mathbf{a}\Delta_1 2 \simeq \mathbf{y}2$ . We say that  $m$  is a  $K$ -dense morphism.

Indeed, this fact alone characterizes the topology of  $\mathcal{N}$  with respect to  $\mathcal{U}$ . We recall a general result about elementary topoi.

**Proposition 4.13** *Let  $m : A \hookrightarrow X$  be a monomorphism in a topos  $\mathcal{E}$ . Then there exists a smallest local operator  $j$  on  $\mathcal{E}$  such that  $m$  is  $j$ -dense.*

**Proof** See [9, Example A4.5.14(b)].  $\square$

We can now provide a characterization of  $\mathcal{N}$  as a subtopos of  $\mathcal{U}$ .

**Proposition 4.14** *Let  $j$  be the smallest local operator on  $\mathcal{U}$  such that  $m : 2 \hookrightarrow \mathbf{y}2$  is  $j$ -dense. Then  $\text{sh}_j(\mathcal{U}) \simeq \mathcal{N}$ .*

**Proof** First, observe that the relevant definitions imply that the  $K$ -covering families are precisely those finite families  $\{\mathcal{G}_k \rightarrow \mathcal{F}\}_{k=1}^n$  such that  $\mathcal{G}_1 + \mathcal{G}_2 + \dots + \mathcal{G}_k \rightarrow \mathcal{F}$  is  $K_1$ -covering. From this description, it follows that  $K$  is the smallest topology extending  $K_1$  for which also families consisting of two sum inclusions  $\{\mathcal{F}_1 \rightarrow \mathcal{F}_1 + \mathcal{F}_2, \mathcal{F}_2 \rightarrow \mathcal{F}_1 + \mathcal{F}_2\}$  are covering.

So to prove the proposition, it suffices to show that, for any local operator for which  $m : 2 \hookrightarrow \mathbf{y}2$  is  $j$ -dense, we must have that families consisting of two sum inclusions  $\{\mathcal{F}_1 \rightarrow \mathcal{F}_1 + \mathcal{F}_2, \mathcal{F}_2 \rightarrow \mathcal{F}_1 + \mathcal{F}_2\}$  are  $j$ -covering. To show this, consider the pullback square

$$\begin{array}{ccc} \mathbf{y}\mathcal{F}_1 + \mathbf{y}\mathcal{F}_2 & \longrightarrow & 2 \\ \downarrow & & \downarrow m \\ \mathbf{y}(\mathcal{F}_1 + \mathcal{F}_2) & \longrightarrow & \mathbf{y}2 \end{array}$$

where the map on the bottom of the square is obtained by applying the Yoneda embedding to the coproduct of the maps  $\mathcal{F}_1 \rightarrow 1$  and  $\mathcal{F}_2 \rightarrow 1$ . So if  $m$  is  $j$ -dense, then so must be the map on the left of the square. But then it follows from Lemma 3.20 that  $\{\mathcal{F}_1 \rightarrow \mathcal{F}_1 + \mathcal{F}_2, \mathcal{F}_2 \rightarrow \mathcal{F}_1 + \mathcal{F}_2\}$  is  $j$ -covering.  $\square$

Let  $\mathcal{L}_{\text{st}}$  be a first-order language enriched with a standardness predicate, as in Section 3; we interpret  $\mathcal{L}_{\text{st}}$  in  $\mathcal{U}$  just as we did in  $\mathcal{N}$ , except that we take

$$(v') \text{ for each type } S, \llbracket \text{st}_S \rrbracket := \Delta_1 S,$$

so that  $\llbracket \text{st}_{\mathbb{N}} \rrbracket$  is again the natural numbers object.

As we foretold, “internal” becomes “internal and  $\vee$ -free” in this larger topos. Let  $\Vdash_1$  be the forcing relation in  $\mathcal{U}$ .

**Theorem 4.15** *Let  $\varphi(x)$  be an internal,  $\vee$ -free formula, with free variable  $x$  of type  $S$ , and let  $(C, \mathcal{F}_I)$  be a filter. For all  $\alpha \in \llbracket S \rrbracket \mathcal{F}$ ,*

$$\mathcal{F} \Vdash_1 \varphi(\alpha)$$

*if and only if there exists  $i \in I$  such that, for all  $u \in \mathcal{F}_i$ , it holds that  $\varphi(\alpha(u))$ .*

**Corollary 4.16 (Transfer theorem)** *Let  $\varphi$  be an internal and  $\vee$ -free sentence. Then  $\varphi$  is true if and only if  $\Vdash_1 \varphi$ .*

The standardness predicate, and consequently the existential quantifier, are deherbrandized, as we wished.

**Lemma 4.17** *Let  $\mathcal{F}$  be a filter, let  $S$  be a type of  $\mathcal{L}$ , and let  $\alpha \in \llbracket S \rrbracket \mathcal{F}$ . Then:*

- (a)  $\mathcal{F} \Vdash_1 \text{st}_S(\alpha)$  if and only if there exist a covering map  $\beta : \mathcal{G} \twoheadrightarrow \mathcal{F}$ , and an element  $x \in S$  such that  $\alpha\beta = x!$  in  $\mathfrak{F}\mathbf{Set}$ ;
- (b)  $\mathcal{F} \Vdash_1 \forall^{\text{st}} y : T \Phi(\alpha, y)$  if and only if, for all  $y \in T$ ,  $\mathcal{F} \Vdash_1 \Phi(\alpha, y!)$ ;
- (c)  $\mathcal{F} \Vdash_1 \exists^{\text{st}} y : T \Phi(\alpha, y)$  if and only if there exists  $y \in T$  such that  $\mathcal{F} \Vdash_1 \Phi(\alpha, y!)$ .

**Proof** This is an easy variation on the proof of Lemma 3.12.  $\square$

**Proposition 4.18** *The following principles all hold in  $\mathcal{U}$ :  $\text{OS}_{\vee}^*$ ,  $\text{US}_{\vee}^*$ ,  $\text{IP}_{\vee}^{\text{st}}$ ,  $\text{NU}$ , and, if the axiom of choice holds in the metatheory,  $\text{AC}^{\text{st}}$ .*

**Proof** The proof is the same as for the corresponding principles in  $\mathcal{N}$ , with only minor adjustments required. Pick single covering maps instead of finite families.  $\square$

Before moving on to the conclusions, we want to remark that, irrespective of any interest in nonstandard arithmetic and with the caveat about  $\text{AC}^{\text{st}}$  and the axiom of choice,  $\mathcal{U}$  also provides a model for the logic of the “standard” Diller–Nahm translation, under the interpretation  $\llbracket 0 \rrbracket_{\wedge} := \Delta_1 \mathbb{N}$ ,  $\llbracket 0 \rightarrow 0 \rrbracket_{\wedge} := \Delta_1 (\mathbb{N} \rightarrow \mathbb{N})$ , and so on. In this case, we obtain a weaker transfer theorem for  $\vee$ -free formulae whose quantifiers are all bounded, that is, they range over some finite sequence.

## 5 Conclusions and Directions for Future Work

If we are to sum up what we believe are *conceptually* interesting points of this article, it may come down to the following.

- (1) *Sequence overspill and underspill.* These are two principles, which are dual to each other, that not only seem to be useful and constructively acceptable generalizations of overspill and underspill to higher types, but they are also linked to well-known nonconstructive principles, the lesser limited principle of omniscience, and Markov’s principle, suggesting that classical modes of reasoning can be recovered in a constructive nonstandard calculus.

- (2) *The significance of  $\mathcal{U}$ .* The filter topos  $\mathcal{U}$  doubles as a model of the logic of the Diller–Nahm interpretation and as a cue to its extension with uniform quantifiers—the uniform Diller–Nahm interpretation. This was previously unknown, and might lead to an improved understanding of the underlying, *geometric* structure of Diller–Nahm logic.
- (3) *A better view on herbrandization.* The comparison of  $\mathcal{N}$  with  $\mathcal{U}$  provided a categorical counterpart to herbrandization, and allowed for a refined analysis of its effects. This includes the re-contextualization of NCR as a herbrandized uniformity principle; on the contrary, the role of finite sequences in HGMP<sup>st</sup> appeared to be a by-product of their role in US\*, rather than the result of herbrandization.

A final consideration: the introduction of the nonstandard Dialectica interpretation in [30] had nonstandard analysis as its main motivation; the benchmark to meet was eliminating overspill and underspill from proofs, retrieving what computational content they may have. But our analysis of uniform Diller–Nahm suggests an alternative route: one where we start with the Dialectica interpretation, and progressively apply small “patches,” fixing whichever shortcomings might arise.

So Dialectica requires decidability of atomic formulae: we may be unhappy with that, and turn to Diller–Nahm. Then we could add uniform quantifiers à la Herness, with optimization of program extraction in mind, which would lead us to uniform Diller–Nahm. Then, we notice that the system we obtain—E-HA<sub>stV</sub><sup>ω\*</sup> and characteristic principles—is just one connective away from being a system of intuitionistic arithmetic. Also, fixing that may require that we weaken the existence property; in this way, we may come up with herbrandization, and obtain, in principle, the  $D_{st}$ -interpretation, *without ever actually thinking* of nonstandard arithmetic.

In fact, we can take the “equation”

$$\text{standardness} \simeq \text{herbrandized calculability}$$

as a definition of sorts; one that replaces the intuition of nonstandard natural numbers as having a separate existence, lying, somewhere beyond reach, on a line together with the finite ones, with an “operational” interpretation: a nonstandard number is badly incalculable—so badly, that it cannot even be narrowed down to a finite selection of candidates.

We conclude with a review of new questions that our results raise.

Concerning the proof theory of nonstandard arithmetic, we would like to know how independent the principles OS\* and US\* are. We know that the Herbrand realizability interpretation vacuously accepts the former, yet does *not* have a realizer for the latter (see [30]); so the Herbrand topos from van den Berg [29] provides a model of nonstandard arithmetic with full transfer, but no underspill principle. We do not know, however, of nonstandard models where overspill holds, and underspill does not.

Moreover, we defined a new functional interpretation, but ignore, so far, how useful it is for applications. Its similarity to light Dialectica is encouraging; on the other hand, the use of functional interpretations has been most successful in program extraction from *classical* proofs, and we have not investigated yet how well uniform Diller–Nahm composes with negative translations, such as Kuroda’s in [12].

In light of the results, Palmgren’s work on the topos  $\mathcal{N}$  indicates that the characteristic principles of nonstandard Dialectica lead to a useful calculus for nonstandard

analysis. We conjectured that the characteristic principles of uniform Diller–Nahm may be a good axiomatization of Lifschitz’s calculability arithmetic (see [13]); is this correct, and could this also be a useful calculus by itself?

On a more speculative note, Oliva [19] provided a unified view of the Dialectica, Diller–Nahm, and modified realizability interpretations, through *linear logic*. Is there an equivalent of herbrandization in linear logic—connected, perhaps, to the bang (!) modalizer—such that nonstandard Dialectica and Herbrand realizability, too, would be amenable to such a treatment?

We hope that these and related questions can be answered in future work.

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