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Selective and Ramsey Ultrafilters on G-spaces

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Abstract Let G be a group, and let X be an infinite transitive G-space. A free ultrafilter $\mathcal{U}$ on X is called *G-selective* if, for any G-invariant partition $\mathcal{P}$ of X, either one cell of $\mathcal{P}$ is a member of $\mathcal{U}$, or there is a member of $\mathcal{U}$ which meets each cell of $\mathcal{P}$ in at most one point. We show that in ZFC with no additional set-theoretical assumptions there exists a G-selective ultrafilter on X. We describe all G-spaces X such that each free ultrafilter on X is G-selective, and we prove that a free ultrafilter $\mathcal{U}$ on ω is selective if and only if $\mathcal{U}$ is G-selective with respect to the action of any countable group G of permutations of ω.

A free ultrafilter $\mathcal{U}$ on X is called *G-Ramsey* if, for any G-invariant coloring $\chi : [X]^2 \to \{0, 1\}$, there is $U \in \mathcal{U}$ such that $[U]^2$ is χ-monochromatic. We show that each G-Ramsey ultrafilter on X is G-selective. Additional theorems give a lot of examples of ultrafilters on $\mathbb{Z}$ that are $\mathbb{Z}$-selective but not $\mathbb{Z}$-Ramsey.

0 Introduction

A free ultrafilter $\mathcal{U}$ on an infinite set X is said to be *selective* if, for any partition $\mathcal{P}$ of X, either one cell of $\mathcal{P}$ is a member of $\mathcal{U}$, or some member of $\mathcal{U}$ meets each cell of $\mathcal{P}$ in at most one point. The selective ultrafilters on $\omega = \{0, 1, \dots\}$ are also known under the name *Ramsey ultrafilters* (see, e.g., [1]), because $\mathcal{U}$ is selective if and only if, for each coloring $\chi : [\omega]^2 \to \{0, 1\}$ of 2-element subsets of ω, there exists $U \in \mathcal{U}$ such that the restriction $\chi|_{[U]^2} \equiv \text{const}$.

Let G be a group, and let X be a G-space with the action $G \times X \to X$, $(g, x) \mapsto gx$. All G-spaces under consideration are supposed to be *transitive*: for any $x, y \in X$, there exists $g \in G$ such that $gx = y$. The nontransitive case needs some extra investigation. If $G = X$ and gx is the product of g and x in G, then X is called a *regular G-space*. A partition $\mathcal{P}$ of a G-space X is *G-invariant* if $gP \in \mathcal{P}$ for all $g \in G$, $P \in \mathcal{P}$.

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Now let X be an infinite G-space. We say that a free ultrafilter $\mathcal{U}$ on X is *G-selective* if, for any G-invariant partition $\mathcal{P}$ of X, either some cell of $\mathcal{P}$ is a member of $\mathcal{U}$, or there exists $U \in \mathcal{U}$ such that $|P \cap U| \le 1$ for each $P \in \mathcal{P}$. Clearly, each selective ultrafilter on X is G-selective.

Selective ultrafilters on ω exist under some set-theoretical assumptions additional to ZFC (say, the continuum hypothesis CH), but there are models of ZFC with no selective ultrafilters (see [1]). In contrast to these facts, we show (Theorem 1.1) that a G-selective ultrafilter exists on any infinite G-space X. Then we characterize (Theorem 1.2) all G-spaces X such that each free ultrafilter on X is G-selective, and we show (Theorem 1.3) that a free ultrafilter $\mathcal{U}$ on ω is G-selective for any transitive group G of permutations on ω if and only if $\mathcal{U}$ is selective.

For a G-space X and $n \ge 2$, a coloring $\chi : [X]^n \to \{0, 1\}$ is said to be *G-invariant* if, for any $\{x_1, \dots, x_n\} \in [X]^n$ and $g \in G$, $\chi(\{x_1, \dots, x_n\}) = \chi(\{gx_1, \dots, gx_n\})$. We say that a free ultrafilter $\mathcal{U}$ on X is *(G, n)-Ramsey* if, for every G-invariant coloring $\chi : [X]^n \to \{0, 1\}$, there exists $U \in \mathcal{U}$ such that $\chi|_{[U]^n} \equiv$ const. In the case in which $n = 2$, we write "G-Ramsey" instead of "$(G, 2)$-Ramsey."

We show (Theorem 2.1) that every G-Ramsey ultrafilter is G-selective, but the converse statement is very far from the truth. Theorems 2.2 and 2.6 give us plenty of ultrafilters on $\mathbb{Z}$ that are $\mathbb{Z}$-selective but not $\mathbb{Z}$-Ramsey. Moreover, we conjecture that each $\mathbb{Z}$-Ramsey ultrafilter on $\mathbb{Z}$ is selective. By Corollary 2.8, each $(\mathbb{Z}, 4)$-Ramsey ultrafilter is selective.

A B-Ramsey ultrafilter on the countable Boolean group $B = \bigoplus_\omega \mathbb{Z}_2$ needs not be selective, but a B-Ramsey ultrafilter cannot be constructed in ZFC without additional assumptions.

1 Selective Ultrafilters

Let X be a G-space, and let $x_0 \in X$. We put $St(x_0) = \{g \in G : gx_0 = x_0\}$ and identify X with the left coset space $G/St(x_0)$ of G by $St(x_0)$. If $\mathcal{P}$ is a G-invariant partition of $X = G/S$, $S = St(x_0)$, we take $P_0 \in \mathcal{P}$ such that $S \in P_0$, put $H = \{g \in G : gS \in P_0\}$, and note that the subgroup H completely determines that $\mathcal{P}$: $xS, yS \in G/S$ are in the same cell of $\mathcal{P}$ if and only if $y^{-1}x \in H$. Thus, $\mathcal{P} = \{x(H/S) : x \in L\}$, where L is a set of representatives of the left cosets of G by H.

Theorem 1.1 *For every infinite G-space X, there exists a G-selective ultrafilter $\mathcal{U}$ on X.*

Proof We take $x_0 \in X$, put $S = St(x_0)$, and identify X with G/S. We choose a maximal filter $\mathcal{F}$ on G/S having a base consisting of subsets of the form A/S, where A is a subgroup of G such that $S \subset A$ and $|A : S| = \infty$. Then we take an arbitrary ultrafilter $\mathcal{U}$ on G/S such that $\mathcal{F} \subseteq \mathcal{U}$. To show that $\mathcal{U}$ is G-selective, we take an arbitrary subgroup H of G such that $S \subseteq H$ and consider a partition $\mathcal{P}_H$ of G/S determined by H.

If $|H \cap A : S| = \infty$ for each subgroup A of G such that $A/S \in \mathcal{F}$, then by the maximality of $\mathcal{F}$ we have $H/S \in \mathcal{F}$. Hence, $H/S \in \mathcal{U}$. Otherwise, there exists a subgroup A of G such that $A/S \in \mathcal{F}$ and $|H \cap A : S|$ is finite, $|H \cap A : S| = n$. We take an arbitrary $g \in G$ and denote $gH \cap A = T_g$. If $a \in T_g$, then $a^{-1}T_g \subseteq A$ and $a^{-1}T_g \subseteq H$. Hence, $a^{-1}T_g/S \subseteq A \cap H/S$ so $|T_g/S| \le n$. If x and y determine the same coset by H, then they determine the same set T. Then we choose $U \in \mathcal{U}$

such that $|U \cap x(H \cap A/S)| \leq 1$ for each $x \in G$. Thus, $|U \cap P| \leq 1$ for each cell P of the partition $\mathcal{P}_H$. □

Theorem 1.2 *Let G be a group, let S be a subgroup of G such that $|G : S| = \infty$, and let $X = G/S$. Each free ultrafilter on X is G-selective if and only if, for each subgroup T of G such that $S \subset T \subset G$, either $|T : S|$ is finite or $|G : T|$ is finite.*

Proof We suppose that there exists a subgroup T of G such that $S \subset T \subset G$ and $|T : S| = \infty$, $|G : T| = \infty$. We pick a family $\{g_n T : n \in \omega\}$ of distinct cosets of G by T and, using the Zorn lemma, choose a maximal family $\mathcal{U}$ of subsets of G/S such that, for each $U \in \mathcal{U}$,

$$\{n \in \omega : U \cap g_n(T/S) \text{ is infinite}\}$$

is infinite. Clearly, $\mathcal{U}$ is an ultrafilter, and by the construction, each $U \in \mathcal{U}$ meets infinitely many members of the G-invariant partition $\mathcal{P}$ determined by T in infinitely many points, so $\mathcal{U}$ is not G-selective.

On the other hand, if $|T : S| < \infty$, then the G-invariant partition $\mathcal{P}$ determined by T consists of finite sets of cardinality $|T : S|$. If $|G : T| < \infty$, then $\mathcal{P}$ is a finite partition. Therefore, each free ultrafilter of G/S is G-selective. □

Let G be an infinite abelian group such that, for each subgroup S of G, either S is finite or G/S is finite. If G has an element of infinite order, then G is isomorphic to $\mathbb{Z} \times F$, where F is finite. If G is a torsion group, then G is isomorphic to $\mathbb{Z}_{p^\infty} \times F$, where $\mathbb{Z}_{p^\infty}$ is the Prüfer p-group (see [3, Section 3]) and F is finite. This is an elementary exercise on abelian groups. Thus, the class of abelian groups G such that each ultrafilter on G is G-selective is very narrow.

Theorem 1.3 *If a free ultrafilter $\mathcal{U}$ on ω is G-selective with respect to the action of any transitive group G of permutations of ω, then $\mathcal{U}$ is selective.*

Proof Let $\mathcal{P}$ be a partition of ω such that each member of $\mathcal{P}$ is not a member of $\mathcal{U}$.

Claim. The partition $\mathcal{P}$ can be partitioned $\mathcal{P} = \bigcup_{n\in\omega} \mathcal{P}_n$ so that, for each $n \in \omega$, $\bigcup \mathcal{P}_n$ is infinite and is not a member of $\mathcal{U}$. If the set $\mathcal{P}'$ of all finite blocks of $\mathcal{P}$ is finite, then we take an arbitrary infinite block P_0, put $\mathcal{P}_0 = \{\mathcal{P}', \{P_0\}\}$, and enumerate all remaining infinite blocks of $\mathcal{P}$ as $\mathcal{P}_1, \mathcal{P}_2, \dots$. If $\mathcal{P}'$ is infinite, then we partition $\mathcal{P}' = \mathcal{P}'_0 \cup \mathcal{P}'_1$ such that $\mathcal{P}'_0$ and $\mathcal{P}'_1$ are infinite. We take $i \in \{0, 1\}$ (say, $i = 0$) such that $\bigcup \mathcal{P}'_0 \notin \mathcal{U}$. Then we repeat this procedure for $\mathcal{P}'_1$ and so on. After ω steps, we get a desired partition of $\mathcal{P}'$. Such partition of $\mathcal{P}'$ together with $\mathcal{P} \setminus \mathcal{P}'$ gives us the desired partition of $\mathcal{P}$.

For each $n \in \omega$, we put $Q_n = \bigcup \mathcal{P}_n$, take an arbitrary countable group $G = \{g_n : n \in \omega\}$, and identify ω with $G \times G$, so that $Q_n = \{g_n\} \times G$, $n \in \omega$. We consider $G \times G$ as a regular $(G \times G)$-space and note that the partition $\{Q_n : n \in \omega\}$ of $G \times G$ is $(G \times G)$-invariant. Since $\mathcal{U}$ is $(G \times G)$-selective, there exists $U \in \mathcal{U}$ such that $|U \cap Q_n| \leq 1$ for each $n \in \omega$. By the construction of Q_n, $|U \cap P| \leq 1$ for each $P \in \mathcal{P}$. Hence, $\mathcal{U}$ is selective. □

2 Ramsey Ultrafilters

Theorem 2.1 *For a G-space X, each G-Ramsey ultrafilter on X is G-selective.*

Proof Let $\mathcal{P}$ be a G-invariant partition of X. We define a coloring $\chi : [X]^2 \to \{0,1\}$ by the following rule: $\chi(\{x,y\}) = 0$ if and only if x, y are in the same cell of the partition $\mathcal{P}$. Since $\mathcal{P}$ is G-invariant, χ is also G-invariant. We take $U \in \mathcal{U}$ such that $\chi|_{[U]^2} \equiv i$ for some $i \in \{0,1\}$. If $i = 0$ and $x \in U$, then U is contained in the block P of $\mathcal{P}$ such that $x \in P$. If $i = 1$, then U meets each block of $\mathcal{P}$ in at most one point. Hence, $\mathcal{U}$ is G-selective. □

Let G be a group with the identity e. Each G-invariant 2-coloring of the regular G-space can be described as follows. We say that a coloring $\chi' : G \setminus \{e\} \to \{0,1\}$ is *symmetric* if $\chi'(x) = \chi'(x^{-1})$ for each $x \in G \setminus \{e\}$. Then we put $\chi(\{x,y\}) = \chi'(x^{-1}y)$ and note that $\chi(\{gx,gy\}) = \chi(\{x,y\})$ for all $\{x,y\} \in [G]^2$ and $g \in G$. On the other hand, if a coloring $\chi : [G]^2 \to \{0,1\}$ is G-invariant, then the coloring $\chi' : G \setminus \{e\} \to \{0,1\}$, $\chi'(x) = \chi(\{e,x\})$ is symmetric and uniquely determines χ.

We fix an arbitrary linear ordering $\leq$ of G and, for each subset U of G, put $D(U) = \{x^{-1}y : x, y \in U, x < y\}$. For an ultrafilter $\mathcal{U}$ on G, we define a family $D(\mathcal{U})$ of subsets of G by

$$V \in D(\mathcal{U}) \Leftrightarrow \exists U \in \mathcal{U} : D(U) \subseteq V.$$

We also use the product $\mathcal{V}\mathcal{U}$ of ultrafilters on G defined as follows (see [4, Chapter 4]). We take an arbitrary $V \in \mathcal{V}$ and, for each $g \in V$, pick $U_g \in \mathcal{U}$. Then $\bigcup_{g \in V} gU_g$ is a member of $\mathcal{V}\mathcal{U}$, and each member of the ultrafilter $\mathcal{V}\mathcal{U}$ contains a subset of this form. We denote $\mathcal{U}^{-1} = \{U^{-1} : U \in \mathcal{U}\}$, $U^{-1} = \{g^{-1} : g \in U\}$.

Theorem 2.2 *Let $\leq$ be the natural linear ordering of $\mathbb{Z}$, let $\mathbb{Z}^+ = \{z \in \mathbb{Z} : z > 0\}$, and let $\mathcal{U}$ be a free ultrafilter on $\mathbb{Z}$ such that $\mathbb{Z}^+ \in \mathcal{U}$. Then the following statements hold:*

(i) $D(\mathcal{U}) \subseteq (-\mathcal{U}) + \mathcal{U}$;

(ii) $\mathcal{U}$ is $\mathbb{Z}$-Ramsey if and only if $D(\mathcal{U}) = (-\mathcal{U}) + \mathcal{U}$ and if and only if $D(\mathcal{U})$ is an ultrafilter.

Proof (i) We take an arbitrary $U \in \mathcal{U}$ such that $U \subseteq \mathbb{Z}^+$. For each $z \in U$, put $U(z) = \{x \in U : x > z\}$. Then $D(U) = \bigcup_{z \in U}(-z + U(z))$. Since $U(z) \in \mathcal{U}$, by the definitions of $-\mathcal{U}$ and $(-\mathcal{U}) + \mathcal{U}$, we have $D(\mathcal{U}) \subseteq (-\mathcal{U}) + \mathcal{U}$.

(ii) We assume that $\mathcal{U}$ is $\mathbb{Z}$-Ramsey and take $U \in \mathcal{U}$, $U \subseteq \mathbb{Z}^+$. For each $z \in U$, we pick an arbitrary $U_z \in \mathcal{U}$ such that $z < x$ for each $x \in U$. Then we put $W = \bigcup_{z \in U}(-z + U_z)$ and define a symmetric coloring $\chi' : \mathbb{Z} \setminus \{0\} \to \{0,1\}$. If $x \in W \cup (-W)$, then we put $\chi'(x) = 0$; otherwise, $\chi'(x) = 1$. We take a coloring $\chi : [\mathbb{Z}]^2 \to \{0,1\}$ determined by χ'. Since $\mathcal{U}$ is $\mathbb{Z}$-Ramsey, there is $V \in \mathcal{U}$, $V \subseteq U$, such that $\chi|_{[V]^2} \equiv i$ for some $i \in \{0,1\}$. By the definition of χ', $i = 0$ and $D(V) \subseteq W$. Hence, $W \in D(\mathcal{U})$ so $(-\mathcal{U}) + \mathcal{U} \subseteq D(\mathcal{U})$. By part (i), $D(\mathcal{U}) \subseteq (-\mathcal{U}) + \mathcal{U}$ so $D(\mathcal{U}) = (-\mathcal{U}) + \mathcal{U}$.

On the other hand, let $D(\mathcal{U}) = (-\mathcal{U}) + \mathcal{U}$. We consider an arbitrary symmetric coloring $\chi' : \mathbb{Z} \setminus \{0\} \to \{0,1\}$ and denote by χ the corresponding coloring of $[\mathbb{Z}]^2$. Since $(-\mathcal{U}) + \mathcal{U}$ is an ultrafilter, there is $W \in (-\mathcal{U}) + \mathcal{U}$, $W \subseteq \mathbb{Z}^+$, such that $\chi'|_W \equiv i$, $i \in \{0,1\}$. We take $V \in \mathcal{U}$ such that $D(V) \subseteq W$. Then $\chi|_{[V]^2} \equiv i$ so $\mathcal{U}$ is $\mathbb{Z}$-Ramsey. □

Let G be a discrete group. The Stone–Čech compactification βG of G can be identified with the set of all ultrafilters on G, and βG with the above-defined multiplication is a semigroup which has the minimal ideal $K(\beta G)$ (see [4, Chapter 6]).

Corollary 2.3 *Each ultrafilter $\mathcal{U}$ from the closure* $\mathrm{cl}\, K(\beta\mathbb{Z})$ *is not $\mathbb{Z}$-Ramsey.*

Proof On the contrary, we suppose that some ultrafilter $\mathcal{U} \in \mathrm{cl}\, K(\beta\mathbb{Z})$ is $\mathbb{Z}$-Ramsey. Since $\mathcal{U} \in \mathrm{cl}\, K(\beta\mathbb{Z})$, by [2, Corollary 5.0.28], for every $U \in \mathcal{U}$, there exists a finite subset K of $\mathbb{Z}$ such that $\mathbb{Z} = K + U - U$. We note that $U - U = D(U) \cup (-D(U)) \cup \{0\}$. Now we partition $\mathbb{Z}^+ = Z_0 \cup Z_1$,

$$Z_0 = \bigcup_{n\in\omega}[2^{2n}, 2^{2n+1}), \qquad Z_1 = \mathbb{Z}^+ \setminus Z_0,$$

and applying Theorem 2.2(ii), choose $U \in \mathcal{U}$ and $i \in \{0, 1\}$ such that $D(U) \subseteq Z_i$. Clearly, $F + U - U \neq \mathbb{Z}$ for each finite subset F of $\mathbb{Z}$. Hence, $\mathcal{U} \notin K(\beta\mathbb{Z})$ and we get a contradiction. □

We say that a free ultrafilter $\mathcal{U}$ on an abelian group G is a *Schur ultrafilter* if, for any $U \in \mathcal{U}$, there are distinct $x, y \in U$ such that $x + y \in U$. We note that each idempotent of $\beta\mathbb{Z} \setminus \mathbb{Z}$ is a Schur ultrafilter.

Corollary 2.4 *Each Schur ultrafilter $\mathcal{U}$ on $\mathbb{Z}$ is not $\mathbb{Z}$-Ramsey.*

Proof On the contrary, we suppose that $\mathcal{U}$ is $\mathbb{Z}$-Ramsey and $\mathbb{Z}^+ \in \mathcal{U}$. Since $\mathcal{U}$ is a Schur ultrafilter, by Theorem 2.2, $D(\mathcal{U}) = \mathcal{U} = -\mathcal{U} + \mathcal{U}$. By [4, Corollary 13.19], $(-\mathcal{U}) + \mathcal{U} \neq \mathcal{U}$ for every free ultrafilter $\mathcal{U}$ on $\mathbb{Z}$. □

A free ultrafilter $\mathcal{U}$ on $\mathbb{Z}$ is called *prime* if $\mathcal{U}$ cannot be represented as a sum of two free ultrafilters.

Corollary 2.5 *Every $\mathbb{Z}$-Ramsey ultrafilter on $\mathbb{Z}$ is prime.*

Proof We need two auxiliary claims.

Claim 1. If $\mathcal{U}, \mathcal{V}$ are free ultrafilters and $\mathcal{U} + \mathcal{V}$ is $\mathbb{Z}$-Ramsey, then $D(\mathcal{U} + \mathcal{V}) = D(\mathcal{U}) = D(\mathcal{V})$; in particular (see Theorem 2.2), $\mathcal{U}$ and $\mathcal{V}$ are $\mathbb{Z}$-Ramsey.

Let $\mathcal{W} = \mathcal{U} + \mathcal{V}$, $U \in \mathcal{U}$, $V_x \in \mathcal{V}$, $x \in U$, and $W = \bigcup_{x\in U} x + V_x$. To see that $D(\mathcal{V}) = D(\mathcal{W})$, we fix $x \in U$ and put $V_x' = \{y \in V : y > x\}$. If $y_1, y_2 \in V_x$ and $y_2 > y_1$, then $y_2 - y_1 = (x + y_2) - (x + y_1)$, so $D(V_x) \subseteq D(W)$ and $D(\mathcal{W}) = D(\mathcal{V})$, because $D(\mathcal{W})$ is an ultrafilter.

To show that $D(\mathcal{U}) = D(\mathcal{W})$, we take $x_1, x_2 \in U$, $x_1 < x_2$, and pick an arbitrary $y \in V_{x_1} \cap V_{x_2}$. Since $x_2 - x_1 = (x_2 + y) - (x_1 + y)$ and $x_1 + y, x_2 + y \in W$, $D(U) \subseteq D(W)$ so $D(\mathcal{W}) = D(\mathcal{U})$.

Claim 2. If $\mathcal{W}$ is $\mathbb{Z}$-Ramsey, then $\mathcal{W}$ is a right cancellable element of the semigroup $\beta\mathbb{Z}$.

If not, by [4, Theorem 8.18], $\mathcal{W} = \mathcal{U} + \mathcal{W}$ for some idempotent $\mathcal{U}$. By Claim 1, $\mathcal{U}$ is $\mathbb{Z}$-Ramsey, which contradicts Corollary 2.4.

Lastly, suppose that some $\mathbb{Z}$-Ramsey ultrafilter $\mathcal{W}$ is represented as $\mathcal{W} = \mathcal{U} + \mathcal{V}$. Applying Theorem 2.2 and Claim 1, we get $D(\mathcal{W}) = D(\mathcal{U}) = D(\mathcal{V})$ and

$$D(\mathcal{W}) = (-\mathcal{U}) + (-\mathcal{V}) + \mathcal{U} + \mathcal{V}, \qquad D(\mathcal{V}) = (-\mathcal{V}) + \mathcal{V},$$
$$D(\mathcal{U}) = (-\mathcal{U}) + \mathcal{U}.$$

By Claim 2, $(-\mathcal{U}) + (-\mathcal{V}) + \mathcal{U} = (-\mathcal{V})$. It follows that $\mathbb{Z}^+ \in \mathcal{U}$ if and only if $\mathbb{Z}^+ \notin \mathcal{V}$. On the other hand, $(-\mathcal{U}) + \mathcal{U} = (-\mathcal{V}) + \mathcal{V}$. So, $\mathbb{Z}^+ \in \mathcal{U}$ if and only if $\mathbb{Z}^+ \in \mathcal{V}$. Hence, $\mathcal{W}$ is prime. □

We do not know whether every $\mathbb{Z}$-Ramsey ultrafilter $\mathcal{U}$ is strongly prime, that is, $\mathcal{U}$ does not lie in the closure of the set $\mathbb{Z}^* + \mathbb{Z}^*$. A free ultrafilter $\mathcal{U}$ on a group G is *strongly prime* if and only if some member of $\mathcal{U}$ is sparse. A subset S of an infinite group G is called *sparse* (see [5]) if, for every infinite subset X of G, there exists a finite subset $F \subset X$ such that $\bigcap_{g \in F} gS$ is finite.

Following [6], we say that a subset A of a group G is k-*thin*, $k \in \mathbb{N}$, if

$$|gA \cap A| \leq k$$

for each $g \in G \setminus \{e\}$. Clearly, each k-thin subset is sparse.

Theorem 2.6 *Let $\mathcal{U}$ be a $\mathbb{Z}$-Ramsey ultrafilter on $\mathbb{Z}$, $\mathbb{Z}^+ \in \mathcal{U}$. If there exists a* 1*-thin subset A of G such that $A \in \mathcal{U}$, then $\mathcal{U}$ is selective.*

Proof We fix an arbitrary coloring $\varphi : [\mathbb{Z}]^2 \to \{0, 1\}$ and define a symmetric coloring $\chi' : \mathbb{Z} \setminus \{0\} \to \{0, 1\}$ as follows. If $g \in \mathbb{Z} \setminus \{0\}$ and there are $a, b \in A$, $a < b$, such that $g = b - a$, then we put $\chi'(g) = \chi'(-g) = \varphi(\{a, b\})$. Otherwise, $\chi'(g) = \chi'(-g) = 1$. There is at most one such pair, because A is 1-thin. Then we consider the coloring $\chi : [\mathbb{Z}]^2 \to \{0, 1\}$ determined by χ'. Since $\mathcal{U}$ is $\mathbb{Z}$-Ramsey, there exists $U \in \mathcal{U}$, $U \subseteq A$, such that $\chi|_{[U]^2} \equiv$ const. By the construction of χ, we have $\chi|_{[U]^2} \equiv \varphi|_{[U]^2}$. Thus, $\varphi|_{[U]^2} \equiv$ const and $\mathcal{U}$ is selective. □

We recall that a free ultrafilter $\mathcal{U}$ on $\mathbb{Z}$ is a *Q-point* if, for every partition $\mathcal{P}$ of $\mathbb{Z}$ into finite cells, there is a member of $\mathcal{P}$ which meets each cell in at most one point.

Corollary 2.7 *If a free ultrafilter $\mathcal{U}$ on $\mathbb{Z}$ is $\mathbb{Z}$-Ramsey and a Q-point, then $\mathcal{U}$ is selective.*

Proof To apply Theorem 2.6, it suffices to show that every Q-point $\mathcal{U}$ has a 1-thin set. We suppose that $\mathbb{Z}^+ \in \mathcal{U}$, use the partition $\mathbb{Z}^+ = Z_0 \cup Z_1$ from Corollary 2.3, and take $i \in \{1, 2\}$ and $U \in \mathcal{U}$ such that U meets each cell $[2^m, 2^{m+1})$ of Z_i in at most one point. Clearly, U is 1-thin. □

We do not know if each P-point in $\mathbb{Z}^*$ is $\mathbb{Z}$-Ramsey. Recall that $\mathcal{U}$ is a P-point if, for every partition $\mathcal{P}$ of $\mathbb{Z}$, either some cell of $\mathcal{P}$ is a member of $\mathcal{U}$, or there exists $U \in \mathcal{U}$ such that $U \cap P$ is finite for each $P \in \mathcal{P}$.

In the proof of the next corollary, we use the following observation: if $\mathcal{U}$ is $(\mathbb{Z}, n)$-Ramsey and $m < n$, then $\mathcal{U}$ is $(\mathbb{Z}, m)$-Ramsey. Indeed, every $\mathbb{Z}$-invariant coloring $\chi : [\mathbb{Z}]^m \to \{0, 1\}$ defines a $\mathbb{Z}$-invariant coloring $\chi' : [\mathbb{Z}]^n \to \{0, 1\}$ by the following rule: $\chi'(\{x_1, \ldots, x_n\}) = \chi(\{x_1, \ldots, x_m\})$.

Corollary 2.8 *Each $(\mathbb{Z}, 4)$-Ramsey ultrafilter $\mathcal{U}$ on $\mathbb{Z}$ is selective.*

Proof Since $\mathcal{U}$ is $(\mathbb{Z}, 2)$-Ramsey, to apply Theorem 2.6, it suffices to find a 1-thin member of $\mathcal{U}$.

We define a coloring $\chi_1 : [\mathbb{Z}]^4 \to \{0, 1\}$ by the following rule: $\chi_1(F) = 0$ if and only if there is a numeration $F = \{x, y, z, t\}$ such that $x + y = z + t$. Since χ_1 is $\mathbb{Z}$-invariant, there is $Y \in \mathcal{U}$ such that $\chi_1|_{[Y]^4} \equiv i$. Since A is infinite, $i = 1$.

Then we define a coloring $\chi_2 : [\mathbb{Z}]^3 \to \{0, 1\}$ by the following rule: $\chi_2(F) = 0$ if and only if F is an arithmetic progression. Since χ_2 is $\mathbb{Z}$-invariant and $\mathcal{U}$ is $(\mathbb{Z}, 3)$-Ramsey, there is $Z \in \mathcal{U}$ such that $Z \subset Y$ and $\chi_2|_{[Z]^3} \equiv i$. Clearly, $i = 1$.

Lastly, $\chi_1|_{[Z]^4} \equiv 1$ and $\chi_2|_{[Z]^3} \equiv 1$ imply that Z is 1-thin. □

A free ultrafilter $\mathcal{U}$ on an abelian group G is said to be a *PS-ultrafilter* if, for any coloring $\chi : G \to \{0, 1\}$, there exists $U \in \mathcal{U}$ such that the set $PS(U)$ is χ-monochromatic, where $PS(U) = \{a + b : a, b \in U, a \neq b\}$. Clearly, each selective ultrafilter on G is a PS-ultrafilter. We denote by $PS(\mathcal{U})$ a filter with the base $\{PS(U) : U \in \mathcal{U}\}$. The following statements were proven in [6] (see also [2, Chapter 10]). If there exists a PS-ultrafilter on some countable abelian group, then there is a P-point in ω^*. If G has no elements of order 2, then each PS-ultrafilter on G is selective. A strongly summable ultrafilter on the countable Boolean group B is a PS-ultrafilter but not selective. It is easy to see that an ultrafilter $\mathcal{U}$ on a countable Boolean group B is a PS-ultrafilter if and only if $\mathcal{U}$ is B-Ramsey. Thus, a B-Ramsey ultrafilter need not be selective, but these ultrafilters cannot be constructed in ZFC without additional assumptions.

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