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The Admissible Rules of BD_2 and GSc

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Abstract The Visser rules form a basis of admissibility for the intuitionistic propositional calculus. We show how one can characterize the existence of covers in certain models by means of formulae. Through this characterization, we provide a new proof of the admissibility of a weak form of the Visser rules. Finally, we use this observation, coupled with a description of a generalization of the disjunction property, to provide a basis of admissibility for the intermediate logics BD_2 and GSc.

1 Introduction

The admissible rules of a logic are those rules that can be added without making new theorems derivable. The intuitionistic propositional calculus (IPC) has many rules that are admissible, yet nonderivable. An example of an admissible rule of IPC is the following, shown to be both admissible and nonderivable by Mints [41]:

$$(\varphi \to \chi) \to \varphi \vee \psi \;/\; \big((\varphi \to \chi) \to \varphi\big) \vee \big((\varphi \to \chi) \to \psi\big).$$

Some rules are admissible in IPC as well as in its axiomatic extensions. An early example is the following rule, shown to be admissible in IPC by Harrop [25] and proven to be admissible in all intermediate logics by Prucnal [43]:

$$\neg\chi \to \varphi \vee \psi \;/\; (\neg\chi \to \varphi) \vee (\neg\chi \to \psi).$$

Some intermediate logics enjoy a nice characterization of their admissible rules. Independently, Iemhoff [28] and Rozière [47] proved that all admissible rules of IPC derive from the Visser rules, a scheme of rules that can be seen as a generalization of Mints's rule. The *Visser rules* are useful in describing the admissible rules of many an intermediate logic. When they are admissible in an intermediate logic, Iemhoff [29] showed that all other admissible rules must follow from them. The intermediate logic BD_2, the weakest intermediate logic of the second finite slice, however, is not

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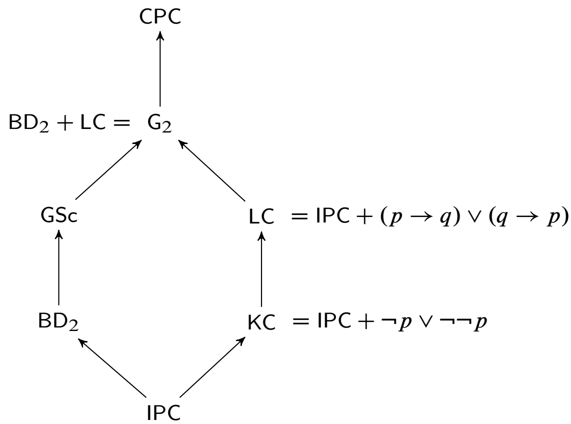

Figure 1 The intermediate logics with the interpolation property ordered by inclusion, as illustrated by Rothenberg [46, Figure 3.1].

amendable to this approach. Indeed, Citkin [11] showed that this intermediate logic does not admit the Visser rules.

The logic BD_2 was among the first intermediate logics to be studied. Jankov [32] introduced the logic under the name M (cf. Rose [45]) and proved it to be complete with respect to a particular class of Heyting algebras. McKay [40] proved that BD_2 derives the same implicationless formulae as IPC. The concept of finite slices was introduced by Hosoi [26], where BD_2 appeared in the guise of LP_2. The logic BD_2 also appears as one of the three pretabular intermediate logics and as one of the seven intermediate logics with interpolation, both proven by Maksimova [37], [38].

The seven logics with interpolation are ordered as in Figure 1. There is much known about the admissible rules of these logics. Per [29, Theorem 5.3], we know that the classical propositional calculus (CPC), the two-valued Gödel logic (or Smetanich's logic) G_2, and the Gödel–Dummett logic LC (see Dummett [16]) have no nontrivial admissible rules. The structural completeness of LC and G_2 was proven by Dzik and Wroński [17], and Citkin [10] showed that these logics are hereditarily structurally complete.[1] Both IPC and the Jankov–de Morgan logic KC have nontrivial admissible rules, and all admissible rules follow from the Visser rules by [29, Theorem 5.1] and [28]. It is known that BD_2 admits nontrivial rules, but to the best of our knowledge, no axiomatization of admissibility is known. We are unaware of any admissibility results on GSc of Avellone, Ferrari, and Miglioli [2], which is the intermediate logic defined by

$$\mathsf{GSc} := \mathsf{BD}_2 + \big((p \to q) \vee (p \to q) \vee (p \equiv \neg q)\big).$$

Since Jeřábek [33], there has been interest in a notion of admissibility concerning rules with multiple conclusions, as already suggested by Kracht [36]. This notion encompasses the disjunction property, and as such, it offers a convenient setting to formulate bases of admissibility.[2] For instance, Cintula and Metcalfe [9] gives a basis of multiconclusion admissibility for the implication-negation fragment of IPC. Similarly, Goudsmit and Iemhoff [23] provided bases of multiconclusion admissibility for the logics T_n with $n \geq 2$.

In this paper we introduce a scheme of multiconclusion rules, called $\mathsf{D}_n^{\neg\neg}$, inspired by Skura [53]. This scheme can be seen as a weakened version of the Visser rules. We prove that all admissible rules of BD_2 follow from the scheme $\mathsf{D}_n^{\neg\neg}$, and that all admissible rules of GSc follow from $\mathsf{D}_2^{\neg\neg}$. This provides a positive answer to the last two questions stated in Iemhoff [30].

The bulk of this paper is spent on developing the machinery to smoothly tackle these problems. Of central importance to our end goal is the notion of projective unification, as developed by Ghilardi [20], [21]. Using Jankov–de Jongh formulae and the universal model, we semantically characterize the admissibility of a variant of the Visser rules D_n. With this characterization, we prove that the rules D_n are admissible for all subframe logics. As a particular consequence, this proves that the restricted Visser rules of [29] are admissible for all subframe logics. This includes the logics IPC, BD_n, G_n, LC, M_n, KC, and Sm, all discussed in the aforementioned paper.

In Section 2, we provide the basic definitions and notation we work with. Most importantly, we define what we mean by a *basis of admissibility* in terms of (multiconclusion) consequence relations. Providing a basis of admissibility will be our formal codification of the intuitive statement that all admissible rules of BD_2 follow from $\mathsf{D}_n^{\neg\neg}$.

Section 3 describes the universal model. This model allows us to comfortably provide a connection between syntax and semantics for the intermediate logics at hand. In Section 4, we lay the groundwork for characterizing exactly in which situations $\mathsf{D}_n^{\neg\neg}$ is admissible. Moreover, we provide the scheme of rules D_n and show it to be admissible for all subframe logics. We introduce all the relevant admissible rules in Section 5. In Section 6, we finally obtain the bases of admissibility.

2 Preliminaries

We are concerned with propositional statements. Often, it will be useful to restrict the propositional variables to a given set, say, X. Typically, this set will be finite or countably infinite. The propositional language over these variables is defined through the following Backus–Naur form:

$$\mathcal{L}(X) := \top \mid \bot \mid X \mid \mathcal{L}(X) \wedge \mathcal{L}(X) \mid \mathcal{L}(X) \vee \mathcal{L}(X) \mid \mathcal{L}(X) \to \mathcal{L}(X).$$

We say that φ is a formula when $\varphi \in \mathcal{L}(X)$ for some X. For clarity, we reserve φ, ψ, χ for formulae and Γ, Π, Δ for sets of formulae. As abbreviations, we write $\neg\varphi$ to mean $\varphi \to \bot$ and write $\varphi \equiv \psi$ to mean $(\varphi \to \psi) \wedge (\psi \to \varphi)$. By a *substitution* we mean a function on formulae that commutes with all connectives.

The intuitionistic propositional calculus, from here onward abbreviated as IPC, has many equivalent definitions. For us, it is most convenient to see it as a Hilbert-style system, that is, a collection of theorems closed under modus ponens. We assume its definition to be known (for details, refer to Troelstra and van Dalen [57]). Intermediate logics are consistent axiomatic extensions of IPC. Let us give a formal definition.

Definition 1 (Intermediate logic) An *intermediate logic* L is given by a set of formulae containing the theorems of IPC, satisfying:

(i) if σ is a substitution and $\varphi \in \mathsf{L}$, then $\sigma(\varphi) \in \mathsf{L}$;
(ii) if $\varphi \to \psi \in \mathsf{L}$ and $\varphi \in \mathsf{L}$, then $\psi \in \mathsf{L}$;
(iii) $\bot \notin \mathsf{L}$.

We will often write $\mathsf{L} + \varphi$ to mean the least intermediate logic extending $\mathsf{L} \cup \{\varphi\}$.

To reason semantically, we use Kripke models. We repeat the definition below (for details, see, e.g., [57] or Chagrov and Zakharyaschev [8]).

Definition 2 (Kripke model) A *Kripke model*, on a set of variables X, is a monotone map $v : K \to \mathbf{P}X$, where K is a partially ordered set and $\mathbf{P}X$ denotes the set of subsets of X ordered by inclusion. We define *truth at a point* inductively in the usual manner:

$$\begin{array}{lll} k \Vdash \top & \text{iff} & \top, \\ k \Vdash \bot & \text{iff} & \bot, \\ k \Vdash x & \text{iff} & x \in v(k), \\ k \Vdash \varphi \wedge \psi & \text{iff} & k \Vdash \varphi \text{ and } k \Vdash \psi, \\ k \Vdash \varphi \vee \psi & \text{iff} & k \Vdash \varphi \text{ or } k \Vdash \psi, \\ k \Vdash \varphi \to \psi & \text{iff} & l \nVdash \varphi \text{ or } l \Vdash \psi \text{ for all } l \geq k. \end{array}$$

We often omit reference to the monotone map and refer to the model by its underlying partial order for the sake of brevity when little confusion is possible. Given a set $W \subseteq K$ we define

$$W\uparrow := \{k \in K \mid \text{there is a } w \in W \text{ with } w \leq k\}.$$

Such a set is called an *upset* when $W\uparrow = W$. We write $W \upuparrows$ for $W\uparrow - W$, where $-$ denotes set difference. When W is a singleton set, we will often omit braces, so $\{k\}\uparrow$ will be written as $k\uparrow$. An upset U is said to be *principal* when there is a $u \in U$ such that $u\uparrow = U$. We say that a model is *rooted* when K itself is principal and denote the *root*, the smallest element in K, by ρ_K. A model L is said to be a generated submodel of K when $L = K \upharpoonright U$ for some upset $U \subseteq K$. The model K is said to be *image-finite* when all principal upsets are finite.[3]

Given a model $v : K \to \mathbf{P}X$ and a node $k \in K$ we write $\mathrm{Th}(k)$ for the *theory* of that node, defined as

$$\mathrm{Th}(k) := \left\{\varphi \in \mathscr{L}(X) \;\middle|\; k \Vdash \varphi\right\}.$$

For convenience, we often write $W \Vdash \varphi$ to mean that $w \Vdash \varphi$ for all $w \in W$. We will also write $W \Vdash \Gamma$ to mean that $W \Vdash \varphi$ for all $\varphi \in \Gamma$.

Maps of Kripke models are commutative triangles, where the maps involved are understood to be continuous and open. That is to say, a map between Kripke models $v : K \to \mathbf{P}X$ and $u : L \to \mathbf{P}Y$ is a monotone function $f : K \to L$ such that $u \circ f = v$, and for all upsets $U \subseteq K$ the set $f(U)$ is an upset. Such a function is often called a p-morphism or bounded morphism; we will simply call it a *map*. We write $f(W)$ to mean the direct image of f under W, that is, $\{f(w) \mid w \in W\}$.

Given a not necessarily rooted model K, we can adjoin a new root to K. There is a choice of valuation to this new root. The operation of adjoining a root and selecting a suitable valuation will play an important role, so let us define it here. Note that $(-)/\emptyset$, in the notation of the following definition, is the same as the Smoryński operator $(-)'$ of [54].[4]

Definition 3 (Extension) Let $v : K \to \mathbf{P}X$ be a model, and let $Y \subseteq X$ be a set of variables such that $K \Vdash Y$. Write K_+ for the partial order of K adjoined with a smallest element denoted $*$. We define the *extension of K with Y*, denoted K/Y, to be the model

$$v/Y : K_+ \to \mathbf{P}X, \quad k \in K_+ \mapsto \text{if } k \in K, \text{then } v(k), \text{else } Y.$$

A rule is a pair of finite sets of formulae, written Γ/Δ. We say that such a rule is *single-conclusion* when $|\Delta| \leq 1$. To abstract away from all matters relating to axiomatizations, we use consequence relations, or rather, a generalization of the concept that also allows for nonsingle-conclusion rules. The definition we use below follows that of [9]. For more information on consequence relations per se we refer to Wójcicki [58] (see Scott [50] and Shoesmith and Smiley [51] for background on multiconclusion consequence relations)).

Definition 4 (Multiconclusion consequence relation) Let φ be a formula, and let $\Gamma, \Pi, \Delta, \Theta$ be finite sets of formulae. A *multiconclusion consequence relation* is a relation between finite sets of formulae, denoted $\vdash$, subject to the following axioms:[5]

reflexivity: $\varphi \vdash \varphi$;
monotonicity: if $\Gamma \vdash \Delta$, then $\Gamma, \Pi \vdash \Delta, \Theta$;
transitivity: if $\Gamma \vdash \Delta, \varphi$ and $\varphi, \Pi \vdash \Theta$, then $\Gamma, \Pi \vdash \Delta, \Theta$;
structurality: if $\Gamma \vdash \Pi$, then $\sigma(\Gamma) \vdash \sigma(\Pi)$ for all substitutions σ.

Given an intermediate logic L, we work with the multiconclusion relation $\vdash_{\mathsf{L}}$ defined by

$$\Gamma \vdash_{\mathsf{L}} \Delta \qquad \text{iff} \qquad \bigwedge \Gamma \to \bigvee \Delta \in \mathsf{L}.$$

We say that a rule Γ/Δ is *derivable* whenever $\Gamma \vdash \Delta$ holds.

Definition 5 (Admissible) A rule Γ/Δ is said to be *admissible* for $\vdash$, written $\Gamma \mid\!\sim \Delta$, when for all substitutions σ the following holds:

$$\text{if } \vdash \sigma(\varphi) \text{ for all } \varphi \in \Gamma, \text{ then } \vdash \sigma(\chi) \text{ for some } \chi \in \Delta.$$

Note that $\mid\!\sim$ is a multiconclusion consequence relation such that $\vdash \subseteq \mid\!\sim$. Given a set of rules $\mathcal{R}$ we write $\vdash^{\mathcal{R}}$ to mean the least consequence relation extending both $\vdash$ and $\mathcal{R}$. We say that $\mathcal{R}$ forms a *basis of admissibility* when $\vdash^{\mathcal{R}} = \mid\!\sim$.

3 The Universal Model

In this section we explicate some machinery convenient in discussing the universal model. Moreover, we introduce Jankov–de Jongh formulae. The main results of this section are well established within folklore. Some of the definitions and techniques are (slightly) novel though. In particular, Definition 8 appears to be absent from the literature, but it seems to smoothen some arguments, such as Theorem 2.

Bellissima [4] describes free Heyting algebras in terms of definable upsets of particular Kripke models (cf. Darnière and Junker [13] and Elageili and Truss [18]). Rybakov [48] considered a similar model, under the name "characterizing model," to prove results about admissibility. The central property of his model is that it is complete for all formulae on a specific set of variables. When considering intermediate logics with the finite model property, one can intuitively see that any model which contains all finite models satisfies this property. We use this to define what it means to be a "universal model" in Definition 9. From Theorem 1, it is clear that the common construction, as given for instance by [7], is a universal model in our sense.

Definition 6 (Cover) Let K be a Kripke frame. We say that $W \subseteq K$ *covers* $k \in K$, denoted $W \kappa k$, precisely if $k{\uparrow} = W{\uparrow} \cup \{k\}$.

The above definition is equivalent to the one given by Ghilardi [22]. Let us first note that $\emptyset \kappa k$ precisely if k is maximal. The relation κ is reflexive in the sense that $\{k\} \kappa k$. We also have that $(k{\Uparrow}) \kappa k$. Not every set $W \subseteq K$ need have a node k such that $W \kappa k$ holds.

A set W covers a node k precisely if k is a *tight predecessor* of W in the sense of Iemhoff [27]. When K is the canonical model on a given set of variables, one can see that W covers k precisely if k is a tight predecessor of $\bigcap W$ in the sense of [23]. Jeřábek [33] also has a notion of being a tight predecessor, but this notion is irreflexive. That is to say, W covers k and $k \notin W$ precisely if k is a tight predecessor of W in his sense. Bezhanishvili [7] calls W a *total cover* of k in precisely the same situation.

There is good reason to allow this reflexivity in the notion of covering. The following lemma shows that covers are preserved by maps, which would not be the case were we to impose irreflexivity.

Lemma 1 ([22]) *Let K and L be Kripke models, and let $f : K \to L$ be a monotone map respecting the underlying valuations. The statement (i) entails (ii), and the converse holds whenever K is conversely well founded.*

(i) *f is a map of Kripke models.*
(ii) *For all $k \in K$ and $W \subseteq K$ such that $W \kappa k$ we have $f(W) \kappa f(k)$.*

Proof The implication from (i) to (ii) follows from straightforward computation. Indeed, if $W \kappa k$, then $f(W) \kappa f(k)$ follows from the equation

$$f(k){\uparrow} = f(k{\uparrow}) = f\big(W{\uparrow} \cup \{k\}\big) = f(W{\uparrow}) \cup \big\{f(k)\big\} = f(W){\uparrow} \cup \big\{f(k)\big\}.$$

Suppose that (ii) holds. We prove, by well-founded induction, that for all $k \in K$ we have $f(k){\uparrow} = f(k{\uparrow})$. Consider k and $W := k{\Uparrow}$, and assume that $f(w){\uparrow} = f(w{\uparrow})$ for all $w \in W$. It follows that $f(W){\uparrow} = f(W{\uparrow})$. We know that $W \kappa k$, and thus $f(W) \kappa f(k)$ holds by assumption. From here we compute

$$f(k{\uparrow}) = f\big(\{k\}\big) \cup f(W{\uparrow}) = f\big(\{k\}\big) \cup f(W){\uparrow} = f(k){\uparrow},$$

proving (i) as desired. □

The theory of a node is determined by its valuation and by the nodes it covers, as illustrated by the following lemma. We will later use this property to pinpoint the existence of nodes covered by a specific set of nodes.

Lemma 2 *Let K be model, let $W \subseteq K$ be a set, and let $k \in K$ be such that $W \kappa k$. We now have*

$$k \Vdash \varphi \to \psi \text{ iff } W \Vdash \varphi \to \psi \text{ and } (k \nVdash \varphi \text{ or } k \Vdash \psi). \tag{1}$$

Proof By definition we know that $k \Vdash \varphi \to \psi$ if and only if $l \nVdash \varphi$ or $l \Vdash \psi$ for all $l \geq k$. Now because $W \kappa k$ the latter is equivalent to the statement that $l \nVdash \varphi$ or $l \Vdash \psi$ holds for $l \in K$ satisfying $l = k$ or $l \in W{\uparrow}$. □

In the canonical model, order is fully determined by the theory of the nodes. This can be the case in many more models, in particular, in submodels of the canonical model. Many consequences can be drawn from this definability of order alone, so let us give it a name.

Definition 7 (Refined model) A model K is said to be *refined* when, for all $k, l \in K$ such that $k \not\leq l$, there is a φ such that $k \Vdash \varphi$ yet $l \nVdash \varphi$.

Lemma 3 *Let K be a refined model on X, and let $W \subseteq K$ be a finite set of nodes. If $k \in K$ is such that it satisfies the equivalence (1) and $W\uparrow \subseteq k\uparrow$ holds, then $W \; \kappa \; k$.*

Proof We need to show that $k\uparrow = W\uparrow \cup \{k\}$. The inclusion from right to left holds by assumption. We proceed by contradiction, so assume the existence of a node $l \in K$ with $k > l$ and $l \notin W\uparrow$. The former, combined with the refinedness of K, ensures that there is a $\varphi \in \mathcal{L}(X)$ such that $k \nVdash \varphi$ and $l \Vdash \varphi$. Through the latter and refinedness we get $\psi_w \in \mathcal{L}(X)$ such that $w \Vdash \psi_w$ and $l \nVdash \psi_w$.

We note that $\psi := \bigvee_{w \in W} \psi$ is such that $W \Vdash \psi$, and thus, $W \Vdash \varphi \to \psi$. By the equivalence of Lemma 2, we know that $k \Vdash \varphi \to \psi$, and so $l \Vdash \varphi \to \psi$ follows by the preservation of truth. But $l \Vdash \varphi$, so this proves $l \Vdash \psi$. By definition, this gives a $w \in W$ such that $l \Vdash \psi_w$, a contradiction, as desired. □

Lemma 4 *Let L be a refined model, and let $f, g : K \to L$ be arbitrary maps. It follows that $f = g$.*

Proof If $l_1, l_2 \in L$ are such that $\mathrm{Th}(l_1) = \mathrm{Th}(l_2)$, then $l_1 = l_2$. This is immediate from the refinedness of L. Pick $k \in K$, and see that

$$\mathrm{Th}\big(f(k)\big) = \mathrm{Th}(k) = \mathrm{Th}\big(g(k)\big).$$

Consequently $f(k) = g(k)$ for all $k \in K$, proving the desired result. □

Every image-finite model on a set of variables has a unique map to the canonical model on the same set of variables. Below, we show this, making use of the existence criterion given by Lemma 3. We write $\mathsf{can}(X)$ to denote the canonical model on X.

Lemma 5 *Let $K \to \mathbf{P}X$ be an image-finite model. There is a unique map* $\mathrm{Th}_K(-) : K \to \mathsf{can}(X)$.

Proof The map is defined as

$$\mathrm{Th}_K(-) : K \to \mathsf{can}(X), \quad k \in K \mapsto \left\{ \varphi \in \mathcal{L}(X) \;\middle|\; k \Vdash \varphi \right\}.$$

When we can show that this is a map, we are done, because uniqueness is immediate through Lemma 4.

The monotonicity of $\mathrm{Th}_K(-)$ is clear by the preservation of truth. Let $W \subseteq K$ be arbitrary, and let $k \in K$ be such that $W \; \kappa \; k$. By Lemma 1 we need and prove that $\mathrm{Th}_K(W) \; \kappa \; \mathrm{Th}_K(k)$. First note that $W \subseteq k\uparrow$, and so W is finite as K is image-finite. Now also observe that $\mathrm{Th}_K(W)$ and $\mathrm{Th}_K(k)$ satisfy the equivalence as given in Lemma 2. The proof is now immediate through Lemma 3. □

The direct image of any image-finite model must be image-finite. Consequently, the above proves the following theorem. Note that the image-finite part is not a priori equal to the upper part in the sense of Bezhanishvili [7]. Bezhanishvili [7, Theorem 3.1.10] shows that, when considering finitely many variables, these two notions do coincide.

Theorem 1 *Let X be a finite set. The image-finite part of* $\mathsf{can}(X)$ *is the terminal object in the category of image-finite models on X.*

We introduce an auxiliary notion, which we will show to be a special case of being refined. This notion is not essentially new. It is, in fact, the disjunction of two notions well established within the literature on Kripke models.

Consider a surjective map $f : K \to L$ such that there are distinct $k_1, k_2 \in K$ with $f(k_1) = f(k_2)$ and $f(k) = k$ for all $k \in K - \{k_1, k_2\}$. In [14], such a map is said to be an α-reduction whenever $k_2 \,\kappa\, k_1$ or $k_1 \,\kappa\, k_2$, and it is called a β-reduction when $k_1{\upuparrows} = k_2{\upuparrows}$. Let us, for convenience, call such pairs of nodes k_1, k_2 α-redexes and β-redexes, respectively. Odintsov and Rybakov [42] call these redexes *twins* and *duplicates*, respectively. Similar configurations are described by others (see, e.g., [4, Lemmas 2.1, 2.0] and Anderson [1, Operations 1, 2]). We forego the distinction between these settings and call the nodes k_1 and k_2 analogous in both cases. It is easy to see that comparable analogous nodes form an α-redex, and incomparable analogous nodes form a β-redex.

Definition 8 (Analogous nodes) Let $v : K \to \mathbf{P}X$ be a model, and let $a, b \in K$ be nodes. We say that a and b are *analogous*, written $a \equiv b$, whenever $v(a) = v(b)$ and

$$a \leq k \text{ if and only if } b \leq k \text{ for all } k \in K - \{a, b\}.$$

A model is said to be *concrete* when all analogous nodes are equal.

We first make the connection with our motivating example in the following lemma. We call a map $f : K \to L$ a *reduction* when there exists a unique doubleton $\{a, b\} \subseteq K$ with $a \equiv b$ such that $f(k_1) = f(k_2)$ if and only if $k_1 = k_2$ or $\{k_1, k_2\} = \{a, b\}$. Consider any model K, and suppose that $a, b \in K$ are such that $a \equiv b$. The smallest equivalence relation R such that $a \; R \; b$ holds is a congruence relation with respect to the order on K. That is to say, if $a \leq b$, $a \; R \; a'$, and $b \; R \; b'$, then $a' \leq b'$ holds as well. Consequently, we can define a model K/R on the equivalence classes of R, and the quotient function $K \to K/R$ is a reduction.

Lemma 6 ([14]) *Let K be a finite model. For every proper map $f : K \to L$, there exists a chain of reductions $f_1, \ldots, f_n$ such that $f_n \cdots f_1 = f$.*

Proof We proceed by induction on the size of the model K. Let $f : K \to L$ be given, and consider the set

$$E := \big\{ \langle a, b \rangle \in K \times K \;\big|\; a \neq b \text{ and } f(a) = f(b) \big\}.$$

Order E by $\langle a_1, b_1 \rangle \leq \langle a_2, b_2 \rangle$ if and only if $a_1 \leq a_2$ and $b_1 \leq b_2$. Because f is proper we know E to be nonempty, and as K is finite we can pick a maximal $\langle a, b \rangle \in E$. We claim that $a \equiv b$. Indeed, if $k \in K - \{a, b\}$ is given and $a \leq k$, then $f(b) = f(a) \leq f(k)$, and so there must be a $k' \geq b$ such that $f(k) = f(k')$. Now $k = k'$ must hold; otherwise $\langle k', k \rangle > \langle a, b \rangle$, contradicting the maximality of $\langle a, b \rangle$. This proves that $b \leq k' = k$, as desired. The other direction can be proven similarly.

Now consider the smallest equivalence relation R such that $a \; R \; b$. Define the map $f_1 : K \to K/R$ to be the quotient map, and let $f' : K/R \to L$ be defined on representatives by f. It follows that f' is a well-defined map and $f' f_1 = f$. Also note that the size of K/R is smaller than that of K. Induction yields maps $f_2, \ldots, f_n$ such that $f_n \cdots f_2 = f'$. This proves that $f_n \cdots f_1 = f$, as desired. □

It is important to note that the relation $\equiv$ is reflexive and symmetric, but in general it is *not* transitive. We generalize the definition of analogous from the binary setting to the finitary setting and say that a set $W \subseteq K$ is analogous whenever two conditions are met: (i) if $v(w_1) = v(w_2)$ for all $w_1, w_2 \in W$ and (ii) if for all $k \in K - W$ one has $w < k$ for some $w \in W$ if $w < k$ for all $w \in W$. It is easy to see that doubleton sets are analogous when their constituents are analogous nodes, though the converse need not hold. We entertain this digression for a bit longer and define a generalization of analogous based on the above notion.

Lemma 7 *Let $v : K \to \mathbf{P}X$ be a model. Define the relations $\equiv$ and $\sqsubseteq$ on K as follows:*

$$a \cong b \text{ if and only if there is an analogous set } W \subseteq K \text{ with } x, y \in W,$$
$$a \sqsubseteq b \text{ if and only if there are } a', b' \in K \text{ such that } a \cong a' \leq b' \cong b.$$

The relation $\cong$ is an equivalence relation congruent with $\leq$. The relation $\sqsubseteq$ is the least reflexive, transitive relation extending both $\cong$ and $\leq$ such that $x \sqsubseteq y$ and $y \sqsubseteq x$ entail $x \equiv y$.

Proof The reflexivity and symmetry of $\cong$ are both evident. To prove transitivity, assume $a \cong b \cong c$. This gives us analogous sets $W_{ab} \ni a, b$ and $W_{bc} \ni b, c$. Note that $W_{ab} \cup W_{bc}$ is an analogous set, whence the transitivity follows.

It is clear that $\sqsubseteq$ extends $\leq$ and $\equiv$. We need to prove reflexivity, transitivity, and antisymmetry. The former is immediate from the reflexivity of $\leq$ and $\equiv$.

To prove transitivity, assume $a \sqsubseteq b \sqsubseteq c$. This yields $k_{ab}, k_{ba}, k_{bc}, k_{cb} \in K$ such that

$$a \cong k_{ab} \leq k_{ba} \cong b \cong k_{bc} \leq k_{cb} \cong c.$$

Let W be an analogous set such that $k_{ba}, b, k_{bc} \in W$. If $k_{cb} \in W$, then $k_{ba} \cong k_{cb}$, whence the desired result is immediate. Assume the contrary. Then we know from $k_{bc} \leq k_{cb}$ that $k_{ba} \leq k_{cb}$. But now $a \cong k_{ab} \leq k_{cb} \cong c$, as desired.

We now turn to antisymmetry, so assume $a \leq b$ and $b \leq a$. This yields $a_{ab}, b_{ab}, b_{ba}, a_{ba} \in K$ such that

$$a \cong a_{ab} \leq b_{ab} \cong b \qquad \text{and} \qquad b \cong b_{ba} \leq a_{ba} \cong a.$$

Consider analogous sets W_a and W_b such that $a, a_{ab}, a_{ba} \in W_a$ and $b, b_{ab}, b_{ba} \in W_b$. If these sets intersect, then we are done, so assume the contrary. It follows that $a_{ba} \leq b_{ab}$ because $a_{ab} \leq b_{ab}$ and $a_{ba}, a_{ba} \in W_a$. Similarly, $b_{ab} \leq a_{ba}$ because $b_{ba} \leq a_{ba}$ and $b_{ba}, b_{ab} \in W$. We now have, through the antisymmetry of $\leq$, that $a_{ba} = b_{ba}$, quod non. We leave minimality to the reader; the proof technique is similar to the above. □

Let $v : K \to \mathbf{P}X$ be a model, and let $\cong$ and $\sqsubseteq$ be the relations of Lemma 7. Define $\mathsf{C}K$ to be the set of $\cong$-equivalence classes, ordered by $\sqsubseteq$ on representatives, and define the model $\mathsf{C}v : \mathsf{C}K \to \mathbf{P}X$ on representatives. The canonical quotient function $p : K \to \mathsf{C}K$ can easily be seen to be a map of Kripke models. Moreover, to each map $f : K \to L$ such that $f(a) = f(b)$ when $a \cong b$ there is a unique map $g : \mathsf{C}K \to L$ such that $f = gp$. When we apply Lemma 6 to the map $p : K \to \mathsf{C}K$ it becomes apparent that $\cong$ is, intuitively, like a transitive closure of $\equiv$.

We do not explore this generalized notion any further and return to the binary case. Let us first tie the concept to that of coverings. Note again that the “nonstrictness” of the covering relation is quite essential.

Lemma 8 *Let $v : K \to \mathbf{P}X$ be a model. The following are equivalent, for all $k_1, k_2 \in K$:*

(i) *the nodes k_1 and k_2 are analogous;*
(ii) *there is a $W \subseteq K$ such that $W \kappa k_1, k_2$ and $v(k_1) = v(k_2)$.*

Proof Assume that (ii) holds, and let $k_1, k_2 \in K$ and $W \subseteq K$ be such that $W \kappa k_1, k_2$ and $v(k_1) = v(k_2)$. If $k \in K - \{k_1, k_2\}$ is such that $k_1 \leq k$, then $k \in W$ because $W \kappa k_1$. As $W \kappa k_2$, this proves $k_2 \leq k$. We can prove the converse through a similar argument, showing (i) to hold.

Conversely, suppose that (i) holds. We distinguish two cases: either k_1 and k_2 are comparable or they are not. In the latter case, we define $W_i := k_i\uparrow - \{k_1, k_2\}$. Observe that $W_1 = W_2$ because $k_1 \equiv k_2$. It is easy to see that $W_i \kappa k_i$ through the incomparability of k_1 and k_2, proving the desired result.

In the former case, we assume, for convenience, that $k_1 \leq k_2$. Now define $W := k_2\uparrow$, and see that $W \kappa k_2$ and $W \kappa k_1$. The first statement is trivial; the second holds because if $k \in K$ is such that $k_1 < k$, then $k_2 < k$ or $k_1 = k$. In both cases we derived (ii). □

The following can be shown by a straightforward computation, but is also an immediate corollary of Lemmas 1 and 8.

Corollary 1 *Let $f : K \to L$ be a morphism. If $a \equiv b$, then $f(a) \equiv f(b)$ for all $a, b \in K$. In particular, if f is bijective and L is concrete, then K is concrete too.*

Corollary 3 follows immediately from Corollary 2, and the former is a direct consequence of Lemmas 8 and 2. This shows, as promised, that concreteness is a special case of refinedness. In particular, this proves that the universal model, as we constructed it, is concrete. Because universal models are unique up to isomorphism, and analogousness is preserved through maps, it also follows that any universal model is concrete.

Corollary 2 *Let K be a model. For all $a, b \in K$ we have* $\mathrm{Th}(a) = \mathrm{Th}(b)$ *whenever $a \equiv b$.*

Corollary 3 *Any refined model is concrete.*

Note that the universal model (on a fixed set of variables), constructed for instance in de Jongh and Yang [15] or Bezhanishvili [7], is the terminal object in the category of image-finite models (again, on this same fixed set of variables). We use this property as the very definition of the universal model for arbitrary intermediate logics. In Theorem 1 we proved that such a model actually exists for IPC. Corollary 4 shows that universal models always exist.

Do note that here there is a difference between the established definition of a characterizing model, in the sense of Rybakov, and a universal model, in the sense defined below. A characterizing model is *complete*, whereas a universal model need only be complete when the logic at hand has the finite model property. This interpretation of what it means to be a universal model is not standard; for instance, Renardel de Lavalette, Hendriks, and de Jongh [44, Section 4] require a universal model to be large enough to distinguish between nonequivalent formulae, which entails completeness in particular.

In the case of IPC, a characterizing model needs to include the universal model, which follows immediately from Theorem 2 below.

Definition 9 Let L be an intermediate logic, and let X be a set of variables. The *universal model on* X, written $\mathsf{U}_{\mathsf{L}}(X)$, is a terminal object in the category of imagine-finite models on X satisfying L.

Corollary 4 *Let* L *be an intermediate logic. Now*

$$\mathsf{U}_{\mathsf{L}}(X) := \left\{k \in \mathsf{U}_{\mathsf{IPC}}(X) \;\middle|\; k \Vdash \varphi \text{ for all } \varphi \in \mathcal{L}(\varphi) \text{ with } \vdash_{\mathsf{L}}\varphi\right\}$$

is the universal model for L *over* X. *Moreover, if* L *has the finite model property, then the model* $\mathsf{U}_{\mathsf{L}}(X)$ *is complete with respect to* L *on* X. *That is to say, for all* $\varphi \in \mathcal{L}(X)$ *we have*

$$\vdash_{\mathsf{L}}\varphi \qquad \textit{iff} \qquad \mathsf{U}_{\mathsf{L}}(X) \Vdash \varphi.$$

Proof Let $v : K \to \mathbf{P}X$ be an image-finite model, and assume that $K \Vdash \varphi$ for all $\varphi \in \mathcal{L}(X)$ with $\vdash_{\mathsf{L}}\varphi$. There is a unique map $i : K \to \mathsf{U}_{\mathsf{IPC}}(X)$, and this map preserves the theory of K. This shows that $i(K) \subseteq \mathsf{U}_{\mathsf{L}}(X)$. Moreover, any map $f : K \to \mathsf{U}_{\mathsf{L}}(X)$ is such that $f(k) = i(k)$. Consequently, $\mathsf{U}_{\mathsf{L}}(X)$ truly is universal for L on X.

To show completeness, assume that $\not\vdash_{\mathsf{L}}\varphi$ for some $\varphi \in \mathcal{L}(X)$. By the finite model property, we know of a finite rooted model K of L on X such that $K \nVdash \varphi$. Universality ensures a map $K \to \mathsf{U}_{\mathsf{L}}(X)$, and so $\mathsf{U}_{\mathsf{L}}(X) \nVdash \varphi$, as desired. □

Let us now define the Jankov–de Jongh formulae. These formulae allow us to capture a principal upset in an image-finite concrete model as the upset satisfying a given formula. This definition is, in essence, the same as those given by [7] and [13]. We include it here for the sake of completeness.

Definition 10 (Characteristic formulae) Let $v : K \to \mathbf{P}X$ be a model, and let $k \in K$ be such that the upset it generates is finite. Make the following auxiliary definitions:

$$\begin{aligned} \mathsf{props}\,k &:= \{p \in X \mid k \Vdash p\}, \\ \mathsf{news}\,k &:= \{p \in X \mid k{\upuparrows} \Vdash p \text{ and } k \nVdash p\}. \end{aligned}$$

Let W denote the set of immediate successors of k. Now define maps $\mathsf{up}\,(-)$, $\mathsf{nd}\,(-) : k\!\uparrow \to \mathcal{L}(X)$ by well-founded recursion as follows:

$$\begin{aligned} \mathsf{up}\,k &:= \bigwedge \mathsf{props}\,k \wedge \Big(\Big(\bigvee \mathsf{news}\,k \vee \bigvee_{w \in W} \mathsf{nd}\,w\Big) \to \bigvee_{w \in W} \mathsf{up}\,w\Big), \\ \mathsf{nd}\,k &:= \mathsf{up}\,k \to \bigvee_{w \in W} \mathsf{up}\,w. \end{aligned}$$

In the above definition, it is understood that an empty disjunction stands for falsity ($\bot$), and an empty conjunction stands for truth ($\top$). Also remark that W is the minimal set such that $W \,\kappa\, k$. In particular, if $W = \emptyset$, that is to say, k is a maximal node, then the above specializes to

$$\mathsf{up}\,k = \bigwedge_{p \in X} (\text{if } k \Vdash p, \text{ then } p, \text{ else } p \to \bot) \qquad \text{and} \qquad \mathsf{nd}\,k = \mathsf{up}\,k \to \bot.$$

Theorem 2 (Characteristic formulae) *Let* $v : K \to \mathbf{P}X$ *be a concrete model, and let* $k \in K$ *be such that* $k\!\uparrow$ *is finite. The following hold for all* $l \in K$*:*

$$\begin{aligned} l \Vdash \mathsf{up}\,k \qquad &\textit{iff} \qquad k \le l, \\ l \nVdash \mathsf{nd}\,k \qquad &\textit{iff} \qquad l \le k. \end{aligned}$$

Proof We proceed by well-founded induction along k. For convenience, let W be the set of immediate successors of k.

By the induction hypothesis, one can see the upper statement to be equivalent to the following:

$$k \leq l \text{ iff } v(k) \subseteq v(l) \text{ and, for all } m \geq l, \\ (v(k) = v(m) \text{ and } W \subseteq m\uparrow) \text{ or } m \in W\uparrow. \tag{2}$$

The implication from left to right is straightforward. Let $l \geq k$ be arbitrary. Monotonicity guarantees $v(k) \subseteq v(l)$. Now consider any $m \geq l$, and note that as $l \geq k$ and $W \;\kappa\; k$ we know that either $k = m$ or $m \in W\uparrow$. In both cases the implication holds for trivial reasons.

To prove the other direction, assume that $k \not\leq l$ while $l \Vdash \mathsf{up}\, k$. By upward persistency and the finiteness of $l\uparrow$, we can, without loss of generality, assume l to be maximal with respect to $k \not\leq l$. We distinguish two cases, either $v(k) = v(l)$ and $W \subseteq l\uparrow$, or $l \in W\uparrow$. The latter case is clearly absurd, because then $l \in W\uparrow \subseteq k\uparrow$ would follow, contradicting $k \not\leq l$. In the former case, we know that $W \;\kappa\; l$ through the maximality of l. From Lemma 8 we learn that $l \equiv k$ and so $k = l$, quod non.

To finish our argument, we remark that $l \nVdash \mathsf{nd}\, k$ is equivalent to the existence of a node $m \geq l$ such that $m \Vdash \mathsf{up}\, k$ and $m \nVdash \mathsf{up}\, w$ for all $w \in W$. By the above, we know this to hold precisely if there is an $m \geq l$ such that $k \leq m$ and $w \not\leq m$ for all $w \in W$. Recall that $W \;\kappa\; k$, so if $k \leq m$ and $w \notin W\uparrow$, then we know that $k = m$. This shows that $l \nVdash \mathsf{nd}\, k$ holds precisely if $l \leq k$. □

Observe that, by the above, we know that to each finite $W \subseteq K$ with K concrete we have $k \in W\uparrow$ if and only if $k \Vdash \bigvee_{w \in W} \mathsf{up}\, w$. We will denote this disjunction by $\mathsf{up}\, W$ from now on. We close this section with the following corollary, relating concreteness and refinedness. The introduction of concreteness was motivated as an ostensible refinement of refinedness. In the setting of image-finite models, the two notions in fact coincide. The implication from left to right holds in general, per Corollary 3, and the converse holds through Theorem 2.

Corollary 5 *Every image-finite model is refined if and only if it is concrete.*

4 Existence of Covers

Recall that Lemma 5 proved that to each image-finite model there is a unique map into the canonical model. By Lemma 1, such a map must preserve covers. This suggests a close relation between the nodes covered by the theory of a model (in the universal model) and the possible extensions of this model. Observe that all statements in Corollary 6 still hold when replacing $\mathsf{can}(X)$ by $\mathsf{U}_{\mathsf{IPC}}(X)$.

Corollary 6 *Let $v : K \to \mathbf{P}X$ be a model, and let $W \subseteq \mathsf{can}(X)$ be a set of nodes such that* $\mathrm{Th}(W) = \mathrm{Th}(K)$. *For all $Y \subseteq X$ with $K \Vdash Y$ we have that* $W \;\kappa\; \mathrm{Th}(K/Y)$. *Moreover, if $k \in \mathsf{can}(X)$ is such that $W \;\kappa\; k$, then* $\mathrm{Th}(K/Y) = k$ *for* $Y = \mathrm{Th}(k) \cap X$.

Proof The first statement is immediate from Lemma 3. Let $k \in \mathsf{can}(X)$ be such that $W \;\kappa\; k$. From the first statement we gather that $W \;\kappa\; \mathrm{Th}(K/Y)$. It is quite clear that $\mathrm{Th}(K/Y)$ and k make the same variables true. So Lemma 8 shows these nodes to be analogous. But, as the model is concrete through Corollary 3, we know these nodes to be equal, whence they have equal theories. □

Iemhoff [29], [30] showed that there is a correspondence between the admissibility of certain rules and the existence of certain extensions. Per the previous lemma, this amounts to finding out which sets of the canonical model have nodes that they cover. When restricting to logics with the finite model property, it suffices to restrict attention to the universal model. Instead of fixating on the universal model, we often consider an arbitrary image-finite concrete model. This gives us slightly greater flexibility, because this allows us to also consider submodels of the universal model in particular.

Let us first start with some notions approximating the existence of covers. In Lemma 10, these properties will all be related to one another. The following definition is a generalization of the set Δ of [27, page 288], as already investigated in [23]. Here we present some more general arguments, although the proofs have a similar flavor.

Definition 11 (Vacuous implications) Let K be a model over X. The set of *vacuous implications* is defined as

$$\mathsf{I}(K) := \left\{\varphi \to \psi \in \mathcal{L}(X) \;\middle|\; K \Vdash \varphi \to \psi \text{ and } K \nVdash \varphi\right\}.$$

Definition 12 Let K be a model, let $W \subseteq K$ be a subset, and let $k \in K$ be a node. We say that W is *comparable above* k when for all $l \geq k$ one has $l\uparrow \subseteq W\uparrow$ or $W\uparrow \subseteq l\uparrow$.

Lemma 9 *Let K be a model, let $W \subseteq K$ be a subset, and let $k \in K$ be a node. If W is comparable above k and k is maximal with respect to $W\uparrow \subseteq k\uparrow$, then $W \kappa k$.*

Proof We need to prove that $k\uparrow = W\uparrow \cup \{k\}$. The inclusion from right to left holds by assumption. To prove the opposite, let $l \geq k$ be given. If $l = k$, then we are done, so assume $k < l$. This ensures that $W\uparrow \not\subseteq l\uparrow$. But we also know that $W\uparrow \subseteq l\uparrow$ or $l\uparrow \subseteq W\uparrow$, so $l\uparrow \subseteq W\uparrow$ must follow. This proves that $l \in W\uparrow$, as desired. □

The following lemma illustrates the partial internalizability of being comparable above. That is to say, W is comparable above some node in K precisely when the subtheory $\mathsf{I}(W)$ of $\mathrm{Th}(W)$ holds on K. We thus capture a property of the model in propositional language. We speak of partial internalization because the theory need not be finite in general, so the property is not fully expressed in one propositional statement. This can, however, be done when the model K is assumed to be image-finite.

Lemma 10 *Let K be a refined model, let $W \subseteq K$ be finite, and let $k \in K$ be such that $W\uparrow \subseteq k\uparrow$. The items (i) and (ii) are equivalent. If K is image-finite, then all the following are equivalent.*

(i) $k \Vdash \mathsf{I}(W)$.
(ii) *W is comparable above k.*
(iii) $k \Vdash \bigvee_{w \in W} \mathsf{nd}\, w \to \mathsf{up}\, W$.

Proof Assume that (i) holds. We proceed by contradiction, so we assume there is some $l \geq k$ such that $l\uparrow \not\subseteq W\uparrow$ and $W\uparrow \not\subseteq l\uparrow$. The former ensures that for all $w \in W$ we know that $w \not\leq l$, and the latter proves that $l \not\leq w$ for some $w \in W$. By refinedness, we thus know of $\varphi_w \in \mathcal{L}(X)$ such that $w \Vdash \varphi_w$ yet $l \nVdash \varphi_w$ per $w \in W$. Again through refinedness, we know of a $\psi \in \mathcal{L}(X)$ such that $l \Vdash \psi$ and

$w \nVdash \psi$ for some $w \in W$. Note that $\varphi := \bigvee_{w \in W} \varphi_w$ is a proper formula because W is finite. It follows that $W \Vdash \varphi$ and $k \nVdash \varphi$. Moreover, $W \nVdash \psi$ and $k \Vdash \psi$. As a consequence $\varphi \to \psi \in \mathsf{I}(W)$, and so $W \Vdash \varphi \to \psi$. But now $W \Vdash \psi$ follows, a clear contradiction. This proves (ii).

To prove the other direction, assume that (ii) holds. Suppose that $k \nVdash \varphi \to \psi$ for some $\varphi \to \psi \in \mathsf{I}(W)$. This gives us an $l \geq k$ such that $l \Vdash \varphi$ yet $l \nVdash \psi$. We distinguish two cases: either $l\uparrow \subseteq W\uparrow$ or $W\uparrow \subseteq l\uparrow$. In both cases we immediately arrive at a contradiction through upward persistency, proving (i).

Now suppose that K is image-finite. Because K is refined, we know it to be concrete by Corollary 3. We prove that (iii) is equivalent to (ii). By definition, (iii) holds if and only if for all $l \geq k$ one has $l \Vdash \mathsf{up}\, W$ whenever $l \Vdash \bigvee_{w \in W} \mathsf{nd}\, W$. This is equivalent to the statement that, for all $l \geq k$, we have $l \nVdash \mathsf{nd}\, w$ for all $w \in W$ or $l \Vdash \mathsf{up}\, W$. Through Theorem 2, we see that the former disjunct is equivalent to $W\uparrow \subseteq l\uparrow$, whereas the latter is equivalent to $l\uparrow \subseteq W\uparrow$. This is precisely (ii), as desired. □

Corollary 7 *Let $v : K \to \mathbf{P}X$ be a concrete, image-finite model, and let $W \subseteq K$ be finite. The following are equivalent:*

(i) *there exists a node $k \in K$ such that $W \kappa k$;*
(ii) *there exists a node $k \in K$ with $k \Vdash \mathsf{I}(W)$ and $W\uparrow \subseteq k\uparrow$;*
(iii) *$K \nVdash ((\bigvee_{w \in W} \mathsf{nd}\, w) \to \mathsf{up}\, W) \to \bigvee_{w \in W} \mathsf{nd}\, w$;*
(iv) *there exists a node $k \in K$ such that $W\uparrow \subseteq k\uparrow$ and W is comparable above k.*

Proof Suppose that (i) holds. Note that if $W \kappa k$, then $k \Vdash \mathsf{I}(W)$ by Lemma 2. From here (ii) is clear.

See that each of (ii), (iii), and (iv) ensure $W\uparrow \subseteq k\uparrow$, per Theorem 2 in the case of (iii). Their equivalence thus follows immediately from Lemma 10.

Finally, suppose that (iv) holds. Because K is image-finite, we know $k\uparrow$ to be finite. As such we can pick an $l \in k\uparrow$ maximal with respect to $W\uparrow \subseteq l\uparrow$. Through Lemma 9 we know that $W \kappa l$, proving (i) as desired. □

Theorem 3 *Let $v : K \to \mathbf{P}X$ be a concrete, image-finite model, and let $n \in \mathbb{N}$ be natural. The following are equivalent:*

(i) *for all $k \in K$ and all $W \subseteq k\uparrow$ with $|W| \leq n$ there exists a node $l \in K$ such that $W \kappa l$;*
(ii) *for all $\Delta \subseteq \mathcal{L}(X)$ with $|\Delta| \leq n$ and $\varphi \in \mathcal{L}(X)$ we have*

$$K \Vdash \Big(\bigvee \Delta \to \varphi\Big) \to \bigvee \Delta \quad \textit{implies} \quad K \Vdash \bigvee_{\chi \in \Delta} \Big(\bigvee \Delta \to \varphi\Big) \to \chi;$$

(iii) *for all $k \in K$ and all $W \subseteq k\uparrow$ with $|W| \leq n$ we have that*

$$K \Vdash \Big(\bigvee_{w \in W} \mathsf{nd}\, w \to \mathsf{up}\, W\Big) \to \bigvee_{w \in W} \mathsf{nd}\, w \quad \textit{implies}$$
$$K \Vdash \bigvee_{a \in W} \Big(\bigvee_{w \in W} \mathsf{nd}\, w \to \mathsf{up}\, W\Big) \to \mathsf{nd}\, a.$$

Proof Suppose that (i) holds, and let $\Delta \subseteq \mathcal{L}(X)$ and $\varphi \in \mathcal{L}(X)$ be such that $|\Delta| \leq n$ and $K \Vdash (\bigvee \Delta \to \varphi) \to \bigvee \Delta$. We proceed by contraposition, so assume that $K \nVdash \bigvee_{\chi \in \Delta}(\bigvee \Delta \to \varphi) \to \chi$. This gives us some $k \in K$ such that

$$k \nVdash \left(\bigvee \Delta \to \varphi\right) \to \chi,$$

for all $\chi \in \Delta$. From this we obtain, per $\chi \in \Delta$, a node $w_\chi \geq k$ such that $w_\chi \Vdash \bigvee \Delta \to \varphi$ and $w_\chi \nVdash \chi$. Define $W := \{w_\chi \mid \chi \in \Delta\}$, and observe that $|W| \leq n$ and $W\uparrow \subseteq k\uparrow$. By assumption, this yields an $l \in K$ such that $W \kappa k$. Upward persistency ensures that $l \nVdash \bigvee \Delta$, so from Lemma 2 it readily follows that $l \Vdash \bigvee \Delta \to \varphi$. This yields $l \nVdash (\bigvee \Delta \to \varphi) \to \bigvee \Delta$, proving that (ii) holds.

It is quite clear that (ii) entails (iii). Now assume that (iii) holds. We distinguish two cases: either the assumption is false or the conclusion holds. In the former case, the desired result is immediate from Corollary 7. Suppose we are in the latter case, that is, the conclusion holds. In particular, this means that the conclusion holds in k. As a consequence, we can pick a node $a \in W$ such that

$$k \Vdash \Big(\bigvee_{w \in W} \mathsf{nd}\, w \to \mathsf{up}\, W\Big) \to \mathsf{nd}\, a.$$

Fix this a, and see that the same formula holds at a by the preservation of truth and $k \leq a$. Because $a \in W\uparrow$, we, through Theorem 2, know that $a \Vdash \mathsf{up}\, W$. This yields $a \Vdash \bigvee_{w \in W} \mathsf{nd}\, w \to \mathsf{up}\, W$, and so $a \Vdash \mathsf{nd}\, a$ must follow. Yet we can now derive $a \nleq a$ through Theorem 2, which is blatantly false. This proves (i), as desired. □

5 Admissible Rules

Iemhoff [29] investigated the admissibility of the Visser rules in intermediate logics. In particular, she semantically characterized when the following rules V_n^-, known as the *restricted Visser rules*, are admissible for all $n \in \mathbb{N}$ by means of the weak extension property:

$$\frac{\bigwedge_{i=1}^{n}(p_i \to q_i) \to p_{n+1} \vee p_{n+2}}{\bigvee_{j=1}^{n+2} \bigwedge_{i=1}^{n}(p_i \to q_i) \to p_j}\ \mathsf{V}_n^-.$$

Unfortunately, this result does not nicely stratify over the index n. The rule D_n as given below, however, does stratify satisfactorily, hence our interest in this rule scheme. Intuitively, the mismatch between D_n and the rules V_n^- can be felt for instance in Jeřábek [34, Lemma 3.2]. It should be noted that, for logics with the finite model property, we know *all* restricted Visser rules to be admissible precisely when *all* rules D_n are admissible, due to Corollary 8 and the characterization of [29, Theorem 4.7]. We remark that the rule $\mathbf{r}_n$ of Skura [52] can, informally, be seen as a contrapositive formulation of the rule D_n:

$$\frac{(\bigvee_{i=1}^{n} p_i \to q) \to \bigvee_{j=1}^{n} p_j}{\bigvee_{j=1}^{n}(\bigvee_{i=1}^{n} p_i \to q) \to p_j}\ \mathsf{D}_n.$$

In Corollary 8 below, we show that the admissibility of D_n has semantic counterparts, making heavy use of the theory developed in the previous section. The property (i) of that corollary is, in essence, a stratification of the weak extension property, restricted to the finite models of an intermediate logic.

Corollary 8 *Let* L *be an intermediate logic with the finite model property. The following are equivalent:*

(i) *for every finite rooted* $K \Vdash \mathsf{L}$ *and every* $W \subseteq K$ *with* $|W| \leq n$ *there is an extension of* W *forcing* L*;*
(ii) *for all* X*, all* $k \in \mathsf{U}_{\mathsf{L}}(X)$*, and* $W \subseteq k{\uparrow}$ *with* $|W| \leq n$ *there is a node covered by* W*;*
(iii) L *admits* D_n.

Proof The equivalence between (i) and (ii) is immediate through Corollary 6. By Corollary 4 and Theorem 3, it is clear that (iii) and (ii) are equivalent too. □

An intermediate logic L is said to be a *subframe logic* when for every model $v : K \to \mathbf{P}X$ of L and every subset $W \subseteq K$ we have that $v \upharpoonright W : W \to \mathbf{P}X$ is a model of L, too. For details on subframe logics in general, we refer to Bezhanishvili and Ghilardi [6], Yang [59], and Zakharyaschev [60]. Let us note again that the logics BD_n, as described in Section 6, are known examples of subframe logics.

Theorem 4 *Each subframe logic admits the rules* D_n *for all* $n \in \mathbb{N}$.

Proof By Zakharyaschev [61, Theorem 4.1], we know L to have the finite model property. We proceed via Corollary 8, so let $K \Vdash \mathsf{L}$ be a finite rooted model, and let $W \subseteq K$ be arbitrary. See that $K \upharpoonright (W \cup \{\rho_K\})$ is an extension of W. But as L is a subframe logic and this is a subframe of K, we know this to be a model of L. This proves the desired result. □

The above can intuitively be understood as saying that, in subframe logics, all finite models can be built in an inductive manner by means of extensions. From here, it seems plausible enough that if every finite model is contained within a rooted model, then all models can be built. More formally, [29] showed that the weak extension property and the disjunction property together entail the extension property. From this it is clear that IPC is the sole subframe logic with the disjunction property. In order to fully characterize admissibility for subframe logics, it thus makes sense to look for generalizations of the disjunction property.

The following lemma is a first attempt at internalizing the existence of nodes below certain sets of nodes. At first reading one can fix $n = 2$, think of K as any universal model, and take $L = K$. The lemma then gives a semantic characterization of the disjunction property, much like Maksimova [39, Theorem 1] and Gabbay and de Jongh [19, Lemma 14]. Corollary 10 investigates what happens when we let L be the set of maximal nodes in K.

Lemma 11 *Let* $v : K \to \mathbf{P}X$ *be an image-finite, concrete model, let* $L \subseteq K$ *be an arbitrary subset, and let n be natural. The following are equivalent:*

(i) *for all* $\Delta \subseteq \mathcal{L}(X)$ *with* $|\Delta| \leq n$ *we have*

$$K \Vdash \bigvee \Delta \qquad \textit{implies} \qquad L \Vdash \chi \quad \textit{for some } \chi \in \Delta;$$

(ii) *for all* $W \subseteq L$ *with* $|W| \leq n$ *we have a* $k \in K$ *such that* $W{\uparrow} \subseteq k{\uparrow}$.

Proof Assume that (i) holds, and take $W \subseteq L$ with $|W| \leq n$. Define $\chi_w := \mathsf{nd}\, w$ and $\Delta := \{\chi_w \mid w \in W\}$, and note that $|\Delta| \leq n$. See that $w \nVdash \mathsf{nd}\, w$ through Theorem 2, and so $L \nVdash \chi$ for all $\chi \in \Delta$. This proves that $K \nVdash \bigvee \Delta$. As a consequence, we know of a $k \in K$ such that $k \nVdash \chi$ for all $\chi \in \Delta$. By Theorem 2, this proves that $k \leq w$ for all $w \in W$, and so (ii) follows.

Suppose that (ii) holds. Let $\Delta \subseteq \mathcal{L}(X)$ with $|\Delta| \leq n$ be given. If $L \nVdash \chi$ for all $\chi \in \Delta$, then this yields $w_\chi \in L$ such that $w_\chi \nVdash \chi$ for each $\chi \in \Delta$. Consequently, there is a $k \in K$ such that $W\uparrow \subseteq k\uparrow$, where W is defined as $\{w_\chi \mid \chi \in \Delta\}$. It is easy to see that $k \nVdash \bigvee \Delta$, and so (i) follows. □

Lemma 11 leads to several interesting results, in particular, after applying completeness with respect to universal models. Observe that Corollary 9 below is simply a dual formulation of [39, Theorem 1] restricted to intermediate logics with the finite model property.

Corollary 9 *Any intermediate logic with the finite model property has the disjunction property precisely if every pair of finite rooted models is contained in a finite rooted model.*

Corollary 10 *Let $v : K \to \mathbf{P}X$ be an image-finite, concrete model. The following are equivalent for all $n \in \mathbb{N}$:*

(i) *for all $\Delta \subseteq \mathcal{L}(X)$ with $|\Delta| \leq n$ we have that $K \Vdash \bigvee \Delta$ entails $K \Vdash \neg\neg\chi$ for some $\chi \in \Delta$;*
(ii) *for all $W \subseteq K$ consisting of maximal nodes with $|W| \leq n$ there is a $k \in K$ such that $W\uparrow \subseteq k\uparrow$.*

Proof This is immediate from Lemma 11 and the observation that a formula φ holds at all maximal nodes if and only if $\neg\neg\varphi$ holds in the entire model. □

Corollary 11 (*n*th doubly negated disjunction property) *Let L be an intermediate logic with the finite model property, and let $n \in \mathbb{N}$ be a natural number. The following are equivalent:*

(i) *for all sets of formulae Δ with $|\Delta| \leq n$ we have that $\vdash \bigvee \Delta$ implies $\vdash \neg\neg\chi$ for some $\chi \in \Delta$;*
(ii) *given one-point models $K_1, \ldots, K_n$ there exists a rooted finite model K of L which contains $K_1, \ldots, K_n$ as generated submodels.*

The 0th doubly negated disjunction property states that $\nvdash \bot$. Written as a multiconclusion rule this amounts to $\bot/\emptyset$, which is admissible in every intermediate logic. Let us say that a model K satisfies a rule Γ/Δ whenever it holds that if $K \Vdash \Gamma$, then $K \Vdash \chi$ for some $\chi \in \Gamma$. It is clear that for the empty model K we have $K \Vdash \bot$, so the empty model does not satisfy the rule $\bot/\emptyset$. As a consequence, any model that satisfies the multiconclusion rules of an intermediate logic must be nonempty.

6 Logics of Bounded Depth

Equipped with the above developed machinery, we are ready to tackle the problem of admissibility for BD_2. Let us start with a formal definition, as adapted from [8].

Definition 13 (Logic of bounded depth) Define, by induction, the formula $\mathsf{bd}_n \in \mathcal{L}(p_1, \ldots, p_n)$ by

$$\mathsf{bd}_0 := \bot,$$
$$\mathsf{bd}_{n+1} := p_{n+1} \vee (p_{n+1} \to \mathsf{bd}_n).$$

For any $n \geq 1$ we define the *intermediate logic of bounded depth n*, denoted BD_n, as the least intermediate logic containing the axiom bd_n.

The logics BD_n are the intermediate logics complete with respect to finite Kripke models of height at most n, as for instance proven by [37, Assertion 4.1]. Note that

BD_1 is simply equal to CPC. We also make use of the logic T_n (see [8]), which is complete with respect to finite Kripke trees that branch at most n times.[6]

The logic T_{n+1} is also known as the nth Gabbay–de Jongh logic, as described by [19]. For convenience we write T_ω for the logic IPC, and we write $n \leq \omega$ to mean $n \in \mathbb{N}$ or $n = \omega$. The following lemma characterizes the absence of covers in the universal model of T_n. The proof is a minor adaptation of the original proof of [19, Lemmas 17, 19]. Note that the implication from (ii) to (i) is similar to the proof of [8, Proposition 2.41], but the setting is slightly different.

Lemma 12 *Let K be a rooted, concrete, image-finite model. The following are equivalent:*

(i) *for all $\varphi_0, \ldots, \varphi_n$, the model K satisfies*

$$\bigwedge_{i=0}^{n}\Big(\Big(\varphi_i \to \bigvee_{j \neq i} \varphi_j\Big) \to \bigvee_{j \neq i} \varphi_j\Big) \to \bigvee_{i=0}^{n} \varphi_i;$$

(ii) *for each finite antichain $W \subseteq K$ there is a $k \in K$ such that $W \kappa k$ only if $|W| \leq n$.*

Proof Assume that (i) holds, and suppose there is some finite $W \subseteq K$ such that $|W| > n$ and $W \kappa k$. Pick some $\mathcal{U}$ which partitions W into $n+1$ disjoint sets. We know that $k \notin k\Uparrow$, and so $k \nVdash \mathsf{up}\, k\Uparrow$ through Theorem 2. To each $U \in \mathcal{U}$ we assign $\varphi_U := \mathsf{up}\, U$, and we claim that the following holds. Assuming this claim, we immediately obtain a contradiction through (i):

$$k \Vdash \Big(\mathsf{up}\, U \to \bigvee_{U \neq V \in \mathcal{U}} \mathsf{up}\, V\Big) \to \bigvee_{U \neq V \in \mathcal{U}} \mathsf{up}\, V.$$

We proceed via Lemma 2, which amounts to proving that the above implication holds on W and that if k forces the antecedent, then it forces the succedent. To see the former, assume that $l \in W\uparrow$ is given. When $l \Vdash \mathsf{up}\, U$ we are done, so assume the contrary. This ensures that $l \notin U\uparrow$ through Theorem 2. Pick some $V \in \mathcal{U}$ such that $l \in U\uparrow$, which we know to exist, as $W = \bigcup \mathcal{U}$ and $l \in W\uparrow$. It follows that both $V \neq U$ and $l \Vdash \mathsf{up}\, V$ hold, so we are done.

We finish the argument by proving that the antecedent does not hold at W. Pick any $w \in U$, and suppose that $w \in V\uparrow$ for some $V \in \mathcal{U} - \{U\}$. This would give some $v \in V$ with $v \leq w$, violating the assumption that W is an antichain. Consequently, we know by Theorem 2 that $w \Vdash \mathsf{up}\, U$, yet $w \nVdash \bigvee_{U \neq V \in \mathcal{U}} \mathsf{up}\, V$. We thus know (ii) has to hold.

Now suppose that (ii) holds, whereas (i) does not. The latter yields a $k \in K$ such that

$$k \Vdash \bigwedge_{i=0}^{n}\Big(\Big(\varphi_i \to \bigvee_{j \neq i} \varphi_j\Big) \to \bigvee_{j \neq i} \varphi_j\Big) \qquad \text{and} \qquad k \nVdash \bigvee_{i=0}^{n} \varphi_i,$$

yet the implication does hold on $k\Uparrow$. We know that $k \nVdash \varphi_i$ for all $i = 0, \ldots, n$, so $k \nVdash \varphi_i \to \bigvee_{j \neq i} \varphi_j$ follows. This entails the existence of $w_i \geq k$ such that $w_i \Vdash \varphi_i$ but $w_i \nVdash \bigvee_{j \neq i} \varphi_j$. One can readily see that $W := \{w_0, \ldots, w_n\}$ is an antichain and $k \notin W$. We have that $W \not\kappa\, k$ by assumption, so there must be some $l > k$ and $I \subseteq \{0, \ldots, n\}$ with $|I| \geq 2$ and $l < w_i$ for all $i \in I$. By the choice of k we know that $l \Vdash \varphi_i$ for some i. The preservation of truth ensures that $w_j \Vdash \varphi_i$ for all $j \in I$.

But there is some $j \in I$ with $j \neq i$, contradicting $w_j \nVdash \bigvee_{i \neq j} \varphi_i$. This proves that (ii) implies (i). □

We include the following lemma for the sake of completeness, although it is a well-established fact.

Lemma 13 *Let $v : K \to \mathbf{P}X$ be a refined model of BD_n. It follows that any chain $W \subseteq K$ satisfies $|W| \leq n$.*

Proof Suppose we have $w_n < w_{n-1} < \cdots < w_0 \in W$. Through refinedness, we know of $\varphi_i \in \mathcal{L}(X)$ such that $w_i \Vdash \varphi_i$ but $w_{i+1} \nVdash \varphi_i$ per $0 \leq i < n$. Define a substitution

$$\sigma : \mathcal{L}(p_1, \ldots, p_n) \to \mathcal{L}(X), \quad p_i \mapsto \varphi_{i-1}.$$

We prove, by induction along m, that $w_m \nVdash \sigma(\mathsf{bd}_m)$. The base case is clear because $w_0 \nVdash \bot$. Now suppose $w_m \nVdash \sigma(\mathsf{bd}_m)$ and

$$w_{m+1} \Vdash \sigma(\mathsf{bd}_{m+1}) = \sigma\big(p_{m+1} \vee (p_{m+1} \to \mathsf{bd}_m)\big) = \varphi_m \vee \big(\varphi_m \to \sigma(\mathsf{bd}_m)\big).$$

As a consequence, at least one of $w_{m+1} \Vdash \varphi_m$ and $w_{m+1} \Vdash \varphi_m \to \sigma(\mathsf{bd}_m)$ must hold. The former case contradicts the choice of φ_m. In the latter case, because $w_m \Vdash \varphi_m$, we know $w_m \Vdash \sigma(\mathsf{bd}_m)$, which is false by induction. This finishes the proof. □

Corollary 12 *For all $k \in \mathsf{U}_{\mathsf{BD}_2+\mathsf{T}_n}(X)$ we have that $k\upuparrows$ is a set of maximal nodes of size at most n.*

Proof Write $W := k\upuparrows$, and note that $W \; \kappa \; k$. Maximality is immediate through Lemma 13. We claim that W is an antichain. Indeed, if $a, b \in W$ are such that $a \leq b$, then $k < a \leq b$, so by Lemma 13 it follows that $a = b$. By Lemma 12, we now know that $|W| \leq n$, proving the desired result. □

The multiconclusion rule below is a combination of the nth doubly negated disjunction property, per Corollary 11(i), and the rule D_n. We spend a few words explaining why these rules are admissible. Note that the rule $\mathsf{D}_n^{\neg\neg}$ is similar to the rule $\mathbf{y}_n$ of [53, Theorem 4.1], with the proviso that the rule below is multiconclusion whereas the rule $\mathbf{y}_n$ ought to correspond to a single-conclusion rule:

$$\frac{(\bigvee_{i=1}^{n} \chi_i \to \varphi) \to \bigvee_{j=1}^{n} \chi_j}{\{\neg\neg((\bigvee_{i=1}^{n} \chi_i \to \varphi) \to \chi_j) \mid j = 1, \ldots, n\}} \; \mathsf{D}_n^{\neg\neg}.$$

Lemma 14 *The rule $\mathsf{D}_n^{\neg\neg}$ is admissible for $\mathsf{L} := \mathsf{BD}_2 + \mathsf{T}_n$ for all $n \leq \omega$.*

Proof Consider the rules

$$\bigvee_{i=1}^{n} x_i \big/ \{\neg\neg x_i \mid 1 \leq i \leq n\},$$

$$\Big(\bigvee_{i=1}^{n} x_i \to y\Big) \to \bigvee_{j=1}^{n} x_j \Big/ \bigvee_{j=1}^{n} \Big(\bigvee_{i=1}^{n} x_i \to y\Big) \to x_j.$$

If both are admissible, then their composition is as well, because $\mathrel{|\!\sim}$ is closed under transitivity. By Lemmas 12 and 13, we know that any set of n many one-point models has an extension satisfying L. Corollary 11 thus proves that the first rule is admissible. Via Corollary 8 and, essentially, the argument of Theorem 4, the second rule can be seen to be admissible. □

Lemma 15 *The rule* $\mathsf{D}_n^{\neg\neg}$ *is* not *derivable in* $\mathsf{L} := \mathsf{BD}_2 + \mathsf{T}_n$ *for all* $2 \leq n \leq \omega$.

Proof Let X be any set of cardinality n. We need to prove that the rule $\mathsf{D}_n^{\neg\neg}$ is not derivable in L. Recall that a rule Γ/Δ is derivable whenever the implication $\bigwedge \Gamma \to \bigvee \Delta$ holds in the logic, so we will construct a rooted model on which the conjunction of the assumptions of the rule is confirmed, yet the disjunction of the conclusions is falsified. In this particular case there is but one assumption, and there are n conclusions.

Pick a maximal $w_x \in \mathsf{U}_\mathsf{L}(X)$ per $x \in X$ such that $w_y \Vdash z$ if and only if $y = z$. Write $W := \{w_x \mid x \in X\}$. There exists a node $k \in \mathsf{U}_\mathsf{L}(X)$ with $W \, \kappa \, k$, and note that $k \nVdash x$ for all $x \in X$. One can see that

$$k \Vdash \Big(\bigvee_{x \in X} \neg\neg x \to \bigvee X\Big) \to \bigvee_{x \in X} \neg\neg x,$$

because the conclusion holds at W, and the assumption of the assumption does not hold at k. Consider the following, for any $y \in x$:

$$\neg\neg\Big(\bigvee_{x \in X} \neg\neg x \to \bigvee X\Big) \to \neg\neg y.$$

If this formula were to hold at k, then it would also hold at $W - \{w_y\}$. As this set is nonempty, this cannot be. This proves that $k\uparrow$ is the desired countermodel. □

The remainder of this paper is devoted to showing that the rule $\mathsf{D}_n^{\neg\neg}$ is enough to derive all admissible rules of $\mathsf{BD}_2 + \mathsf{T}_n$ for all $n \leq \omega$. Goudsmit and Iemhoff [23] proved a similar result for T_n; the approach taken there works in this setting as well. We proceed in a more general fashion than strictly necessary, in the hope that greater generality leads to more intrinsic arguments. In the following we fix an intermediate logic L and the corresponding provability and (multiconclusion) admissibility relation by $\vdash$ and $\mid\!\sim$, respectively.

We first introduce the concept of an admissible approximation.[7] The definition captures the properties of a "projective approximation" in the sense of [21] that we use to obtain a basis of admissibility, as shown in Lemma 17.

Definition 14 (Admissible approximation) An *admissible approximation* of a formula $\varphi \in \mathcal{L}(X)$ is a formula $\psi \in \mathcal{L}(X)$ such that the following holds for all $Y \supseteq X$ and finite $\Delta \subseteq \mathcal{L}(Y)$:

$$\varphi \mid\!\sim \Delta \text{ if and only if } \psi \vdash \chi \text{ for some } \chi \in \Delta.$$

Such an approximation is *anchored* by a set of rules R if $\varphi \vdash^{\mathcal{R}} \psi$.

The following lemma shows that admissible approximations are unique up to provable equivalence. In the future we will write $\mathsf{A}\varphi$ for an admissible approximation of φ, given that it exists. This makes sense when its use only depends on the approximation up to provable equivalence.

Lemma 16 *For all* $\varphi \in \mathcal{L}(\varphi)$ *and all* ψ_1, ψ_2 *that admissibly approximate* φ*, we have* $\psi_1 \vdash \psi_2$.

Proof We know that $\varphi \mid\!\sim \psi_2$ from $\psi_2 \vdash \psi_2$, because ψ_2 admissibly approximates φ. For the same reason we derive $\psi_1 \vdash \psi_2$, proving the desired result. □

Lemma 17 *Let* $\mathsf{R} \subseteq \mid\!\sim$ *be a set of rules. If each formula has an admissible approximation anchored by* $\mathcal{R}$*, then* $\vdash^{\mathcal{R}} = \mid\!\sim$.

Proof The inclusion from left to right holds by assumption. To prove the other direction, consider $\varphi, \psi \in \mathcal{L}(X)$, and assume $\varphi \mid\!\sim \psi$. We know that $\mathsf{A}\varphi$ exists, and $\mathsf{A}\varphi \vdash \psi$. Note that $\varphi \vdash^{\mathcal{R}} \mathsf{A}\varphi \vdash \psi$, whence the desired result follows from the transitivity of $\vdash^{\mathcal{R}}$ and $\vdash \subseteq \vdash^{\mathcal{R}}$. $\square$

Definition 15 A formula φ is said to be *closed* under a set of rules $\mathcal{R}$ if $\varphi \vdash^{\mathcal{R}} \Delta$ implies that $\varphi \vdash \chi$ for some $\chi \in \Delta$.

To obtain an admissible approximation, we first consider an ostensibly stronger notion, namely, that of projectivity. It is easy to prove that every projective formula is closed under all admissible rules (see Iemhoff and Metcalfe [31, Lemma 6]).

Definition 16 (Projective) Let L be an intermediate logic, and let $\varphi \in \mathcal{L}(X)$ be a formula. We say that φ is L-*projective* whenever there is a substitution $\sigma : \mathcal{L}(X) \to \mathcal{L}(X)$ such that $\vdash_{\mathsf{L}} \sigma(\varphi)$ and $\varphi \vdash_{\mathsf{L}} \sigma(\psi) \equiv \psi$ for all $\psi \in \mathcal{L}(X)$. The substitution σ is said to be the *projective unifier* of φ.

The following theorem is a straightforward generalization of [21, Theorem 5]. The equivalence between the first two items follows from the same argument as is given there. Equivalence between the latter two items is a direct consequence of Corollary 6. With the machinery developed so far, we can readily characterize those formulas that satisfy (iii), thus describing the L-projective formulae.

Theorem 5 *Let L be an intermediate logic with the finite model property, and let $\varphi \in \mathcal{L}(X)$ be a formula. The following are equivalent:*

(i) *φ is L-projective;*
(ii) *for all finite models $v : K \to \mathbf{P}X$ with $K \Vdash \mathsf{L}$ and $K \upharpoonright (\rho_K \Uparrow) \Vdash \varphi$ there is an extension of $K \upharpoonright (\rho_K \Uparrow)$ that forces φ;*
(iii) *for all finite antichains $W \subseteq \mathsf{U}_{\mathsf{L}}(X)$ with an $l \in \mathsf{U}_{\mathsf{L}}(X)$ such that $W \,\kappa\, l$ and $W \Vdash \varphi$, there is a $k \in \mathsf{U}_{\mathsf{L}}(X)$ such that $k \Vdash \varphi$.*

From now on, fix $2 \leq n \leq \omega$, and let the intermediate logic at hand be $\mathsf{L} := \mathsf{BD}_2 + \mathsf{T}_n$. We will construct an admissible approximation anchored by $\mathsf{D}_n^{\neg\neg}$ to each formula φ. Let us first, in very broad brushstrokes, illustrate how we are about to proceed. If $\varphi \mid\!\sim \Delta$, then $\mathsf{A}\varphi \vdash \Delta$ has to hold by its very definition, so in particular, if $\varphi \vdash_{\mathsf{D}_n^{\neg\neg}} \Delta$, then $\mathsf{A}\varphi \vdash \Delta$ must hold. In Lemma 18, we show that a formula which is closed under $\mathsf{D}_n^{\neg\neg}$ in a suitable sense (see Lemma 18(ii)) is in fact projective. Using this observation, we obtain admissible approximations through iteratively closing formulae under $\mathsf{D}_n^{\neg\neg}$ in Lemma 20, keeping in mind that this terminates, as there are but finitely many formulae modulo L-equivalence on any finite set of variables.

Lemma 18 *The following are equivalent for each $\varphi \subseteq \mathcal{L}(X)$:*

(i) *φ is L-projective;*
(ii) *for all $\Delta \subseteq \mathcal{L}(X)$ and $\chi \in \mathcal{L}(\chi)$ with $|\Delta| \leq n$ we have*

$$\varphi \vdash \left(\bigvee \Delta \to \varphi\right) \to \bigvee \Delta \qquad \textit{implies}$$

$$\varphi \vdash \neg\neg\left(\left(\bigvee \Delta \to \varphi\right) \to \chi\right) \quad \textit{for some } \chi \in \Delta;$$

(iii) *for all sets of maximal nodes $W \subseteq \mathsf{U}_{\mathsf{L}}(X)$ with $W \Vdash \varphi$ and $1 \neq |W| \leq n$ we have a $k \in \mathsf{U}_{\mathsf{L}}(X)$ such that $W \,\kappa\, k$.*

Proof The implication from (i) to (ii) is immediate. Indeed, every projective formula is closed under all admissible rules. The rules $\mathsf{D}_n^{\neg\neg}$ are admissible by Lemma 14, so (ii) follows.

Suppose that (ii) holds, and let $W \subseteq \mathsf{U}_\mathsf{L}(X)$ be such that $W \Vdash \varphi$ and $1 \neq |W| \leq n$. By Corollary 8 we are done when we can find some $l \in \mathsf{U}_\mathsf{L}(X)$ such that $W \subseteq l\uparrow$. This we obtain immediately through Corollary 10, proving (iii).[8]

Suppose that (iii) holds. Let $W \subseteq \mathsf{U}_\mathsf{L}(X)$ be such that $W \,\kappa\, l$ for some $l \in \mathsf{U}_\mathsf{L}(X)$ and $W \Vdash \varphi$. By Theorem 5 we know that it suffices to find a $k \in \mathsf{U}_\mathsf{L}(X)$ such that $W \,\kappa\, k$ and $k \Vdash \varphi$. Because $W \subseteq l\upuparrows$, we know that W is an antichain of maximal elements and $|W| \leq n$. If $|W| = 1$, then the desired result is immediate, because W covers itself. All requirements of (iii) are met, whence (i) follows. □

We can apply the above theorem to prove that the intermediate logics $\mathsf{BD}_2 + \mathsf{T}_n$ have different admissible rules. Note that the corollary does not apply to $\mathsf{BD}_2 + \mathsf{T}_n$ for $n = 0, 1$. Indeed, if $n = 0$, then this is CPC, and if $n = 1$, then it equals the greatest nonclassical intermediate logic, known as Smetanich's logic Sm. In both of these logics, all admissible rules are derivable, as proven by [29, Theorem 5.3].

Corollary 13 *The rule* $\mathsf{D}_{n+1}^{\neg\neg}$ *is* not *admissible in* $\mathsf{BD}_2 + \mathsf{T}_n$ *for all* $2 \leq n \leq \omega$.

Proof Suppose the contrary. Let X be a set of cardinality $n+1$. There exists a set of maximal nodes $W \subseteq \mathsf{U}_{\mathsf{BD}_2+\mathsf{T}_n}(X)$ with $|W| = n+1$. Instantiating Lemma 18 to $\varphi = \top$ now proves that there is a $k \in \mathsf{U}_{\mathsf{BD}_2+\mathsf{T}_n}(X)$ such that $W \,\kappa\, k$. But this contradicts Corollary 12. □

Lemma 19 *If* $W \subseteq \mathsf{U}_{\mathsf{IPC}}(X)$ *is a set of maximal nodes of size at least* 2 *and* $l \in W$, *then* $\mathsf{nd}\, l$ *and the formula below are provably equivalent:*

$$\neg\neg\Big(\Big(\bigvee_{w\in W} \mathsf{nd}\, w \to \mathsf{up}\, W\Big) \to \mathsf{nd}\, l\Big).$$

Proof The implication from left to right is clear. The other implication we prove semantically through Corollary 4. Now assume a node k forces the above implication, but $k \nVdash \mathsf{nd}\, l$. This proves that $k \leq l$ by Theorem 2. Note that $l \Vdash \mathsf{up}\, W$ by Theorem 2 and $l \in W$. By upward persistency and the fact that $l \Vdash \varphi$ if and only if $l \Vdash \neg\neg\varphi$, we now obtain $l \Vdash \mathsf{nd}\, l$. Yet now $l \nleq l$ by Theorem 2, a clear contradiction. □

Take X to be some fixed and finite set of variables. For convenience, we will write Uuniv and Muniv for the set of upsets and the set of maximal nodes in $\mathsf{U}_\mathsf{L}(X)$, respectively. It follows immediately from Lemma 12 that $\mathsf{U}_\mathsf{L}(X)$ is finite, and so there are but finitely many upsets.

Fix some $U \in \mathsf{Uuniv}$ and $W \in \mathsf{Muniv}$ such that $W \subseteq U$. Recall from Corollary 7 that there is no cover of W within U precisely if

$$U \Vdash \Big(\Big(\bigvee_{w\in W} \mathsf{nd}\, w\Big) \to \mathsf{up}\, W\Big) \to \bigvee_{w\in W} \mathsf{nd}\, w.$$

So when W does not have a cover within U, we obtain, from the above, the completeness of the universal model, and Theorem 2 that

$$\mathsf{up}\, U \vdash^{\mathsf{D}_n^{\neg\neg}} \Big\{\mathsf{up}\, U \wedge \neg\neg\Big(\Big(\bigvee_{w\in W} \mathsf{nd}\, w \to \mathsf{up}\, W\Big) \to \mathsf{nd}\, a\Big) \;\Big|\; a \in W\Big\}. \tag{3}$$

Below we define a map Approx meant to be such that the above right-hand side equals $\{\mathsf{up}\,V \mid V \in \mathsf{Approx}(U, W)\}$. One can verify that this indeed holds through a short computation. We define

$$\mathsf{Approx} : \mathsf{Uuniv} \times \mathsf{Muniv} \to \mathbf{P}\mathsf{Uuniv},$$
$$\langle U, W\rangle \mapsto \left\{\left\{k \in U \;\middle|\; k \Vdash \neg\neg\Big(\Big(\bigvee_{w \in W} \mathsf{nd}\,w \to \mathsf{up}\,W\Big) \to \mathsf{nd}\,a\Big)\right\} \;\middle|\; a \in W\right\}.$$

Note that each $V \in \mathsf{Approx}(U, W)$ is an upset such that $V \subset U$. It is important that this inclusion be strict, that is to say, $U \notin \mathsf{Approx}(U, W)$. Suppose that $U \in \mathsf{Approx}(U, W)$ is true. There must be some $a \in W$ such that

$$U = \left\{k \in U \;\middle|\; k \Vdash \neg\neg\Big(\Big(\bigvee_{w \in W} \mathsf{nd}\,w \to \mathsf{up}\,W\Big) \to \mathsf{nd}\,a\Big)\right\}.$$

Because $a \in W \subseteq U$ holds, the above ensures that $a \Vdash \mathsf{nd}\,a$, a contradiction by Theorem 2.

Another important observation to make is that U is empty precisely if there exists no $k \in U$ such that $\emptyset\ \kappa\ k$. Indeed, $\emptyset\ \kappa\ k$ simply means that k is a maximal node, and as U is finite, it has a maximal node precisely if it has any node at all.

In the lemma below we employ the above mapping to construct an order on the set of sets of upsets in $\mathsf{U}_\mathsf{L}(X)$. Naturally, each upset corresponds to a formula in L modulo derivability. We think of a set of upsets as corresponding to a disjunction of formulae modulo derivability. The order will be such that the smallest elements, called *normal forms* in the language of rewrite systems,[9] correspond to disjunctions of projective formulae. Moreover, the order will be such that for each element there is a smallest element below it.

Lemma 20 *Let $2 \le n \le \omega$ be given, and consider $\mathsf{L} := \mathsf{BD}_2 + \mathsf{T}_n$. Every formula has an admissible approximation in* L.

Proof Let φ be a formula, and take X to be a finite set such that $\varphi \in \mathcal{L}(X)$. Realize that there are but finitely many sets of maximal nodes in $\mathsf{U}_\mathsf{L}(X)$. From here onward, let Uuniv denote for the set of all upsets in $\mathsf{U}_\mathsf{L}(X)$. Note that this set is finite.

Let $\preceq$ be the least reflexive transitive relation on $\mathbf{P}$Uuniv such that

$$\mathcal{U} \preceq \mathcal{U} - \{U\} \cup \mathsf{Approx}(U, W)$$

holds for all sets $\mathcal{U} \subseteq \mathsf{Uuniv}$, all upsets $U \in \mathcal{U}$, and all sets of maximal nodes $W \subseteq U$ without covers in U. A straightforward inductive argument, using the reasoning above, shows that for all $\mathcal{U} \preceq \mathcal{V}$

$$\mathsf{up}\Big(\bigcup \mathcal{U}\Big) \vdash^{\mathsf{D}_n^{\neg\neg}} \{\mathsf{up}\,V \mid V \in \mathcal{V}\} \qquad \text{and} \qquad \mathsf{up}\Big(\bigcup \mathcal{V}\Big) \vdash \mathsf{up}\Big(\bigcup \mathcal{U}\Big).$$

Because $\mathbf{P}$Uuniv is finite, we know every sequence on $\preceq$ will eventually stabilize. We say that $\mathcal{U}$ is a normal form whenever $\mathcal{U} \preceq \mathcal{U}'$ entails $\mathcal{U} = \mathcal{U}'$. By the previous remark, it is clear that to each $\mathcal{U}$ there is a normal form.

We claim that every normal form $\mathcal{U}$ is such that for all $U \in \mathcal{U}$ the formula $\mathsf{up}\,U$ is projective. This follows from Lemma 18 and the discussion above. Indeed, if $\mathsf{up}\,U$ were to not be projective, then Lemma 18 ensures the existence of a set of maximal nodes $W \subseteq \mathsf{U}_\mathsf{L}(X)$ such that $W \Vdash \mathsf{up}\,U$ and $1 \neq |W| \le n$, yet W does not cover anything forcing $\mathsf{up}\,U$. Note that $W \subseteq U$ holds by Theorem 2. As a consequence,

$$\mathcal{U} \preceq \mathcal{U} - \{U\} \cup \mathsf{Approx}(U, W).$$

This inequality is strict, violating the assumption that $\mathcal{U}$ is a normal form. Hence, $\mathsf{up}\,U$ must be projective. Define $U := \{k \in \mathsf{U_L}(X) \mid k \Vdash \varphi\}$, and let $\mathcal{V}$ be a normal form associated to $\mathcal{U}$. We simply set $\mathsf{A}\varphi := \bigvee_{V \in \mathcal{V}} \mathsf{up}\,V$, which satisfies all the desired properties. □

Theorem 6 *Let $2 \leq n \leq \omega$ be given. The rules $\mathsf{D}_m^{\neg\neg}$ for all $m \leq n$ form a basis of admissibility for $\mathsf{BD}_2 + \mathsf{T}_n$.*

Proof This is an immediate consequence of Lemmas 20 and 17. □

Theorem 7 *The rule $\mathsf{D}_2^{\neg\neg}$ is a basis of admissibility for*

$$\mathsf{GSc} := \mathsf{BD}_2 + \big((p \to q) \vee (p \to q) \vee (p \equiv \neg q)\big).$$

Proof This follows immediately from Theorem 6 whenever $\mathsf{GSc} = \mathsf{BD}_2 + \mathsf{T}_2$ holds. Let us first prove $\mathsf{GSc} \subseteq \mathsf{BD}_2 + \mathsf{T}_2$. Take some $k \in \mathsf{U}_{\mathsf{BD}_2 + \mathsf{T}_2}(X)$. We want to prove that $k \Vdash \mathsf{GSc}$, from which the desired result is entailed by the completeness of the universal model, as proven in Corollary 4. Assume the contrary, that is, suppose there are $\varphi_1, \varphi_2 \in \mathcal{L}(X)$ such that

$$k \nVdash \varphi_1 \to \varphi_2, \qquad k \nVdash \varphi_2 \to \varphi_1, \qquad k \nVdash \varphi_1 \equiv \neg\varphi_2.$$

The first two conjuncts give $w_i \geq k$ with $w_i \Vdash \varphi_i$ and $w_i \nVdash \varphi_{3-i}$ for $i = 1, 2$, and so w_1, w_2 must be incomparable. By Corollary 12 we know that $k \Uparrow$ is an antichain of size at most 2. Now note that $w_i \Vdash \varphi_1 \equiv \neg\varphi_2$ and $k \nVdash \varphi_i$ for $i = 1, 2$. We obtain $k \Vdash \varphi_1 \equiv \neg\varphi_2$ per Lemma 2, a contradiction with the third conjunct.

We now prove the other inclusion. To this end, take $k \in \mathsf{U}_{\mathsf{BD}_2}(X)$, and suppose that $k \Vdash \mathsf{GSc}$. From Lemma 13 it readily follows that $W := k \Uparrow$ consists of maximal nodes. We are done if $|W| < 2$, so suppose $a \neq b \in W$ are given. Note that

$$k \Vdash (\mathsf{up}\,a \to \mathsf{up}\,b) \vee (\mathsf{up}\,a \to \mathsf{up}\,b) \vee (\mathsf{up}\,a \equiv \neg\mathsf{up}\,b)$$

must hold by assumption. Due to Theorem 2, one can see that the first two disjuncts are false, because a and b are incomparable. Note that if $w \in W$ and $w \neq b$, then $w \nVdash \mathsf{up}\,b$ and so $w \Vdash \mathsf{up}\,a$, which proves $w = a$. This proves that $W = \{a, b\}$. Consequently, $k \Vdash \mathsf{T}_2$ follows, proving the desired result through the completeness of the universal model. □

Notes

1. For more on structural completeness from the perspective of admissibility, we refer to [49, Chapter 5].

2. The difference between single-conclusion and multiconclusion rules can be felt in formulating the Visser rules. In the terminology of [11], V_n^- as given at the start of Section 5 is the join-extension of the rule V_n given by [29].

3. For details on the general theory of image-finite models and their duals, see Bezhanishvili and Bezhanishvili [5].

4. Through the duality between finite Kripke frames and finite Heyting algebras as given by [14], this operation is known as the Troelstra sum ([56]), star sum ([3]), vertical sum ([7]), concatenation ([12]), and glued sum ([24]).

5. We use a comma to denote set-union and omit braces around singleton sets for improved readability.

6. As we only consider $\mathsf{BD}_2 + \mathsf{T}_n$ in the following, we could also have omitted T_n altogether. Indeed, with Corollary 12 it can be shown that $\mathsf{T}_2 + \mathsf{BW}_n = \mathsf{T}_2 + \mathsf{T}_n$, where BW_n is the intermediate logic of bounded width, as given by [8]. We prefer the detour through T_n due to the connection between the admissibility of D_n and T_n studied in [23], which makes the logic a nice conceptual fit for this setting.

7. Our use of the term "admissible approximation" is slightly different from earlier forms such as [23, Definition 19]. Typically, one would define an admissible approximation of φ to be a set of formulae Δ such that $\bigvee \Delta$ is an admissible approximation in our sense, together with the constraint that all formulae in Δ be projective. Even though this additional constraint will be satisfied below, we deem it unnecessary to include it in the definition. Definition 14 only appeals to the relation between derivability and admissibility, and this is all the information we need. See also [35, Definition 3.6].

8. Observe that, when $W = \emptyset$, the statement $W \ \kappa \ k$ simply means that k is maximal. In this case, one can also immediately see the proof, because instantiating $\Delta = \emptyset$ in (ii) immediately proves that $\varphi \nvdash \bot$.

9. See Terese [55] for background on rewriting systems.

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