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Strange Structures from Computable Model Theory

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Abstract Let L be a countable language, let $\mathcal{I}$ be an isomorphism-type of countable L-structures, and let $a \in 2^{\omega}$. We say that $\mathcal{I}$ is *a-strange* if it contains a computable-from-a structure and its Scott rank is exactly ω_1^a. For all a, a-strange structures exist. Theorem (AD): If $\mathcal{C}$ is a collection of $\aleph_1$ isomorphism-types of countable structures, then for a Turing cone of a's, no member of $\mathcal{C}$ is a-strange.

1 Introduction

Let L be a countable language, and let A be a countable L-structure. There is a countable ordinal associated with A, called the *Scott rank* of A, and denoted $\mathrm{SR}(A)$. There are, in fact, several different definitions of Scott rank in the literature, and they are not equivalent. But they are practically equivalent in that anything interesting is either true under all definitions or false under all definitions. This paper's official definition of Scott rank is that of Calvert, Goncharov, and Knight [2].

The Scott rank is invariant under isomorphism. So given an isomorphism-type, $\mathcal{I}$, there is a *Scott rank* of $\mathcal{I}$, denoted $\mathrm{SR}(\mathcal{I})$.

Let us now consider only computable languages and computable structures. A well-known theorem of Nadel states that the Scott rank of a computable structure can be at most $\omega_1^{CK}+1$. There are many concrete examples of computable structures with Scott rank less than ω_1^{CK}. For example, any countable ordinal, α, can be viewed as an isomorphism-type, namely, linear orderings of order-type α; if $\alpha < \omega_1^{CK}$, then $\alpha \leq \mathrm{SR}(\omega^{\alpha}) < \omega_1^{CK}$. Next, consider linear orderings of order-type $\omega_1^{CK}(1+\eta)$, where η is the order-type of the rationals. This is called the *Harrison linear ordering*, and, as Harrison proved, there is a computable linear ordering of this order-type. The isomorphism-type of the Harrison linear ordering has Scott rank $\omega_1^{CK}+1$. That leaves one remaining case.

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Question 1.1 Does there exist a computable structure whose Scott rank is exactly ω_1^{CK}?

Given any oracle $a \in 2^\omega$, we can do computability theory relativized to a. As usual, ω_1^a denotes the least ordinal not computable-from-a, that is, the relativized version of ω_1^{CK}.

Definition 1.2 Let $a \in 2^\omega$, let L be a computable-from-a language, and let $\mathcal{I}$ be an isomorphism-type of countable L-structures. We call $\mathcal{I}$ *a-strange* if it contains a computable-from-a structure and $\mathrm{SR}(\mathcal{I}) = \omega_1^a$.

In this terminology, Question 1.1 becomes: Do $\mathbf{0}$-strange structures exist?

For a long time, the existence of a-strange structures was an open question. The first result on the subject is due to Makkai [12], who proved that there is an arithmetic $a \in 2^\omega$ for which there exists an a-strange structure. Later, Knight and Millar [11] proved that $\mathbf{0}$-strange structures exist. Their proof, like every other proof in computability theory, relativizes to an arbitrary oracle. So for all $a \in 2^\omega$, a-strange structures exist. It is now known that there exist a-strange trees (see Calvert, Knight, and Millar [3]), a-strange graphs, fields, and linear orderings (see Calvert, Goncharov, and Knight [2]), a-strange Boolean algebras (see Fokina et al. [4]), and a-strange groups (see Morozov [14]).

We refer the reader to the references in the previous paragraph, as well as to Knight [10], for more information on this topic.

In this paper, "strange" is a technical mathematical term (see Definition 1.2). To avoid confusion, the word will only be used in that technical sense. When the author is engaging in vague, intuitive discussions, he will use the word "unnatural."

The purpose of this paper is to show that strange structures are unnatural. For the specific examples of strange structures that are produced in the previously cited references, the fact that they are unnatural is self-evident. The thesis of this paper is stronger: *all* strange structures are unnatural.

But what does that mean? Section 2 contains three different reasons for asserting that these structures are unnatural. (The reader who dislikes the word "unnatural" can view this as three reasons why the Scott rank ω_1^a case differs from the Scott rank $\omega_1^a + 1$ case.) For two of the three, the author is able to turn that reason into a precise mathematical statement. One of these two reasons is stronger than the other. The author is able to prove the weaker of the two statements, but not the stronger conjecture. The proof is given in Section 3.

2 Unnatural Structures from Computable Model Theory

Before giving the first reason that strange structures are unnatural, we briefly discuss a topic that is much better known than strange structures: Turing degrees. There is an analogous form of unnaturalness there.

All Turing degrees strictly between $\mathbf{0}$ and $\mathbf{0}'$ are unnatural. (There are natural classes of degrees, e.g., the c.e. degrees, but no natural degrees.) One reason for asserting that these degrees are unnatural is that none of them has a name. Some Turing degrees have names: $\mathbf{0}, \mathbf{0}', \mathbf{0}''$, the Turing degree of true arithmetic, the Turing degree of a universal Π_1^1 set, and so on. But no Turing degree strictly between $\mathbf{0}$ and $\mathbf{0}'$ has a name. And the fact that none of these degrees has a name is not a historical or sociological accident, but an intrinsic property of the mathematical object.

For the same reason, strange structures are unnatural: no strange structure has a name. (In Section 1, we stated that strange structures existed, but did not name any.) In this sense, the Scott rank ω_1^{CK} case is different than the Scott rank $\omega_1^{CK} + 1$ case. The "Harrison linear ordering" is a name, and "order-type $\omega_1^{CK}(1 + \eta)$" is another name; there is no such name for a $\mathbf{0}$-strange structure.

This is the first reason that strange structures are unnatural. The author does not have the slightest idea how to turn that reason into a precise mathematical statement. So that will be left as an open (and possibly unsolvable) problem.

The second reason that strange structures are unnatural is related to the first, and it, too, is analogous to something involving Turing degrees.

Let D denote the set of Turing degrees. For every $\mathbf{d}$ in D, there is a Turing degree strictly between $\mathbf{d}$ and $\mathbf{d}'$. Therefore, using the axiom of choice (AC), there exists a function $f : D \to D$ such that for all $\mathbf{d} \in D$, $\mathbf{d} <_T f(\mathbf{d}) <_T \mathbf{d}'$. But there does not appear to be a "definable" function f with the above property. That is, any such f is unnatural. This is related to the issue of names. Those Turing degrees which have names correspond to definable functions from D to D. For example, $\mathbf{0}$ corresponds to the identity function, $\mathbf{0}'$ corresponds to the Turing jump, the Turing degree of a universal Π_1^1 set corresponds to the hyperjump, and so on. But the degrees strictly between $\mathbf{0}$ and $\mathbf{0}'$ do not seem to correspond to such a definable function. We can turn this vague intuition about "definable" functions into a precise mathematical conjecture.

Conjecture 2.1 (ZF+DC+AD) *There does not exist a function $f : D \to D$ with the property that for all $\mathbf{d} \in D$, $\mathbf{d} <_T f(\mathbf{d}) <_T \mathbf{d}'$.*

Conjecture 2.1 is due to Martin, is approximately 40 years old, and is still open. (A much weaker version of this conjecture appeared earlier in Sacks [16].)

The axiom of determinacy (AD) is an axiom which contradicts AC. The axiom of dependent choice (DC) is a weak form of AC. (For information on these axioms, see Jech [8] or Moschovakis [15].) In Conjecture 2.1, we are using AD in a manner that is common in set theory: AD is frequently used to prove that pathologies produced by AC do not exist. An example of this is the famous theorem of Mycielski and Swierczkowski that AD implies every set of real numbers is Lebesgue measurable.

Assuming large cardinal axioms, it is true (in V, i.e., in the world of AC) that $L(\mathbb{R}) \models$ (ZF+DC+AD), where $L(\mathbb{R})$ is the smallest model of ZF containing all ordinals and all subsets of ω. Thus the way to interpret Conjecture 2.1 in the world of AC is as follows: any function $f : D \to D$ which is in $L(\mathbb{R})$ fails (in V) to have the property that for all $\mathbf{d} \in D$, $\mathbf{d} <_T f(\mathbf{d}) <_T \mathbf{d}'$. This includes any function explicitly or inductively definable using quantification over reals and over ordinals. That is, it includes any function that would ever be defined in practice (as opposed to asserted by AC to exist).

There is a stronger form of Conjecture 2.1, also due to Martin and also still open, which became one of the Victoria Delfino problems (see Kechris and Moschovakis [9, p. 281]). A recent expository paper by Marks, Slaman, and Steel [13] contains (among other things) a progress report on the problem, along with references.

Again, we have an analogy between natural isomorphism-types and natural Turing degrees. Whether a given isomorphism-type is a-strange depends only on the Turing degree of a; thus given a Turing degree $\mathbf{d}$, we have a concept of a $\mathbf{d}$-*strange* isomorphism-type. Just as natural Turing degrees correspond to definable functions from D to D, natural isomorphism-types correspond to definable functions with domain D which assign to each $\mathbf{d} \in D$ an isomorphism-type. For example, there is a definable function, $\mathbf{d} \mapsto \omega_1^{\mathbf{d}}(1+\eta)$, which assigns to each $\mathbf{d} \in D$ an isomorphism-type with a computable-from-$\mathbf{d}$ member and with Scott rank $\omega_1^{\mathbf{d}}+1$.

Conjecture 2.2, below, states that $\omega_1^{\mathbf{d}}$ is different from $\omega_1^{\mathbf{d}}+1$ in this respect. This conjecture, the analogue of Conjecture 2.1 for structures rather than Turing degrees, is (if true) the second reason that strange structures are unnatural.

Conjecture 2.2 (ZF+DC+AD) *There does not exist a function f with domain D with the property that for all $\mathbf{d} \in D$, $f(\mathbf{d})$ is the isomorphism-type of a $\mathbf{d}$-strange structure.*

There is a uniform procedure, for example, the procedure in [11], which assigns to each $a \in 2^\omega$ an a-strange structure. But if that procedure is applied to two different oracles in the same Turing degree, it does not produce isomorphic structures. Conjecture 2.2 says that no definable procedure avoids this defect: to produce an isomorphism-type which is $\mathbf{d}$-strange, we must choose a representative from the degree $\mathbf{d}$.

The third reason that strange structures are unnatural is not analogous to any fact about Turing degrees.

Let $\mathbf{d} \in D$, and let $\alpha = \omega_1^{\mathbf{d}}$. The isomorphism-type $\alpha(1+\eta)$ is not merely definable from the Turing degree $\mathbf{d}$, but is definable from the ordinal α; that is, to define $\alpha(1+\eta)$ we do *not* need to choose a $\mathbf{d} \in D$ such that $\omega_1^{\mathbf{d}} = \alpha$ (let alone choose a representative from $\mathbf{d}$). Being definable from the ordinal is thus a stronger form of regularity than being definable from the Turing degree; therefore, the failure to be definable from the ordinal is a weaker form of unnaturalness.

When the relevant isomorphism-types are all definable from countable ordinals, there is a definable set of $\aleph_1$ isomorphism-types which contains all the relevant isomorphism-types. For example, consider the following set of isomorphism-types:

$$S = \{\alpha(1+\eta) : \alpha \text{ a countable admissible ordinal}\}. \tag{2.3}$$

(By Friedman [5], under any reasonable coding of countable structures by elements of 2^ω, $S \subset 2^\omega$ is a Σ_1^1 set.) The set S obviously contains exactly $\aleph_1$ isomorphism-types, and for every $a \in 2^\omega$, there is an $\mathcal{I} \in S$ such that $\mathcal{I}$ contains a computable-from-a structure and $\mathrm{SR}(\mathcal{I}) = \omega_1^a + 1$. The following theorem tells us that ω_1^a is different from $\omega_1^a + 1$ in this respect. It is the third reason for asserting that strange structures are unnatural.

Theorem 2.4 (ZF+DC+AD) *Let L be a countable language, and let $\mathcal{C}$ be a collection of $\aleph_1$ isomorphism-types of countable L-structures. There exists an $a_0 \in 2^\omega$ such that for all $a \geq_T a_0$, no member of $\mathcal{C}$ is a-strange.*

This theorem will be proved in Section 3.

Harrington [7] proved from AD that any well-orderable collection of Borel sets has cardinality at most $\aleph_1$. Since the set of codes for any isomorphism-type is Borel, Theorem 2.4 actually holds for any well-orderable set of isomorphism-types. For example, if $\mathcal{C}$ is all ordinal-definable isomorphism-types, then for a Turing cone of a's, no member of $\mathcal{C}$ is a-strange.

We conclude Section 2 by considering versions of Theorem 2.4 in the context of ZFC. Trivially, ZFC+CH implies that this theorem is false. The following question is open.

Question 2.5 Is Theorem 2.4 consistent with ZFC?

Theorem 2.4 holds “locally”; that is, weak versions of AD (which are consistent with AC) imply weak versions of Theorem 2.4. For example, projective determinacy implies that Theorem 2.4 holds for all projective $\mathcal{C}$. Probably the most interesting case is when $\mathcal{C}$ is $\boldsymbol{\Sigma}^1_1$ (as is the set S of (2.3)). It is not known how much determinacy, if any, is needed to prove Theorem 2.4 for $\boldsymbol{\Sigma}^1_1$ sets $\mathcal{C}$. To prove it by using the methods of Section 3 requires a determinacy assumption a little stronger than $\boldsymbol{\Pi}^1_1$-determinacy and a lot weaker than $\boldsymbol{\Pi}^1_2$-determinacy.

3 The Proof

In this section, we prove Theorem 2.4. We work in ZF+DC until further notice. Fix a countable language, L. For simplicity of notation, we assume that L is computable. (If not, relativize the proof.)

We encode L-structures with universe ω by elements of 2^ω, using the coding of Gao [6, Sections 3.6, 11.3]. For $x \in 2^\omega$, A_x denotes the L-structure encoded by x. If $\mathcal{I}$ is an isomorphism-type of countably infinite L-structures, then $\mathcal{I}^* = \{x \in 2^\omega : A_x \in \mathcal{I}\}$. Note that—in contrast to the custom in computable model theory—$\mathcal{I}^*$ is not the set of computable structures (or computable-from-a structures) but the set of *all* structures with universe ω; it is an uncountable subset of 2^ω.

As mentioned in Section 1, our definition of *Scott rank* is that of [2].

Lemma 3.1 *Let $a \in 2^\omega$, and let $\mathcal{I}$ be the isomorphism-type of a countable L-structure. Suppose that $\mathcal{I}$ is a-strange.*

(a) For all $y \in \mathcal{I}^*$, $\omega_1^y \geq \omega_1^a$.
(b) There exists an $x \in \mathcal{I}^*$ such that $x \leq_T a$.
(c) $\mathcal{I}^*$ is $\boldsymbol{\Pi}^0_{(\omega_1^a+1)}$.

Proof (a) If $\omega_1^y < \omega_1^a$, then by a theorem of Nadel (see [2, Proposition 2.1]), $\mathrm{SR}(A_y) \leq \omega_1^y + 1 < \omega_1^a$; hence by the definition of strange (Definition 1.2), $A_y \notin \mathcal{I}$.

(b) This is due to the definition of strange.

(c) This can be proved directly from the Scott analysis. But we will avoid that here, and, instead, deduce this from some published theorems. Let $x \leq_T a$, $x \in \mathcal{I}^*$. Let $\mathcal{F}$ be the fragment of $L_{\omega_1\omega}$ consisting of all computable-from-a formulas. Let $t_{\mathcal{F}}$ be the topology on the space of codes for L-structures corresponding to $\mathcal{F}$ (see [6, Definition 11.3.1]). By the relativized version of [2, Proposition 2.2], since $\mathrm{SR}(A_x) = \omega_1^a$, for all $n \in \omega$, for all $\mathbf{i} = (i_0, \ldots, i_{n-1}) \in \omega^n$, the orbit of $\mathbf{i}$ (with respect to the automorphism group of A_x) is defined by a formula in $\mathcal{F}$. Therefore, A_x is an $\mathcal{F}$-atomic model. So by a theorem of Miller and Suzuki (see [6, Theorem 11.5.7]), $\mathcal{I}^*$ is a $t_{\mathcal{F}} - G_\delta$. Since the $t_{\mathcal{F}}$-basis consists of sets which are $\Delta^1_1(a)$, any set which is G_δ with respect to $t_{\mathcal{F}}$ is $\boldsymbol{\Pi}^0_{(\omega_1^a+1)}$ with respect to the original topology. □

Remark 3.2 The isomorphism-type of ω_1^a is also $\boldsymbol{\Pi}^0_{(\omega_1^a+1)}$. Hence it satisfies (a) and (c) of Lemma 3.1, but not (b). The isomorphism-type of $\omega_1^a(1+\eta)$ is $\boldsymbol{\Pi}^0_{(\omega_1^a+2)}$ and not $\boldsymbol{\Sigma}^0_{(\omega_1^a+2)}$. Hence it satisfies (a) and (b) of Lemma 3.1, but not (c).

The rest of Section 3 is descriptive set theory, and has nothing to do with languages or structures.

The proof of Theorem 2.4 uses Steel forcing (tagged tree forcing). It is used here only to give a Baire-category argument; sufficiently generic objects exist without enlarging the universe. So there is no need for a ground model to force over; of course, the reader who wants to have a ground model can have one. We give the information about Steel forcing which is needed to read the rest of the paper, but not much more. For the conventional (set-theoretic) forcing approach to Steel forcing, see Harrington [7] or Steel [17]; for a strictly topological approach, see Becker and Dougherty [1].

For α a countable ordinal, let $\mathbb{P}_\alpha$ denote the poset for Steel forcing with respect to α, and let $\Vdash_\alpha$ denote the corresponding forcing relation. We introduce a symbol, ∞, which is declared to be greater than any ordinal and greater than itself. A condition p of $\mathbb{P}_\alpha$ is a finite tree, T_p, on ω, with each node of T_p tagged by an element of $\alpha \cup \{\infty\}$, such that for $\sigma, \tau \in T_p$, if $\sigma \prec \tau$, then the tag on σ is greater than the tag on τ. For $p, q \in \mathbb{P}_\alpha$, q has more information than p if $T_p \subset T_q$ (T_q need not be an end extension of T_p) and the tags agree on T_p. (Note that if $\alpha < \beta$, then $\mathbb{P}_\alpha \subset \mathbb{P}_\beta$.) The generic object, G, is an infinite tagged tree on ω; by removing the tags, we get—modulo some sequence coding—an element of 2^ω; that element of 2^ω will be denoted G^*.

Definition 3.3 Let α, β, γ be countable ordinals, let $p \in \mathbb{P}_\alpha$, and let $p' \in \mathbb{P}_\beta$. We say that p is *γ-equivalent* to p' if $T_p = T_{p'}$ and for any $\sigma \in T_p$, for any $\delta < \gamma$, p tags σ with δ if and only if p' tags σ with δ.

A proof of the following lemma can be found in [1], [7], and [17].

Lemma 3.4 (Retagging lemma) *Let α, β, γ be countable ordinals, and let $p \in \mathbb{P}_\alpha$ and $p' \in \mathbb{P}_\beta$. Suppose that p is $(\omega\gamma)$-equivalent to p'. Then for any $\mathbf{\Pi}^0_\gamma$ set $P \subset 2^\omega$, ($p \Vdash_\alpha$ "$G^* \in P$") if and only if ($p' \Vdash_\beta$ "$G^* \in P$").*

Lemma 3.5 *Let α, β be countable ordinals such that $\alpha = \omega\alpha < \beta$. Let $p \in \mathbb{P}_\alpha \subset \mathbb{P}_\beta$. Let $Q \subset 2^\omega$ be $\mathbf{\Pi}^0_{\alpha+1}$. If ($p \Vdash_\alpha$ "$G^* \in Q$"), then ($p \Vdash_\beta$ "$G^* \in Q$").*

Proof Let $Q = \cap_i \cup_j P_i^j$, where each P_i^j is $\mathbf{\Pi}^0_{\alpha_i^j}$ for some $\alpha_i^j < \alpha$. Suppose it is not true that $p \Vdash_\beta$ "$G^* \in Q$". Then there is a $q \in \mathbb{P}_\beta$, q extending p, such that $q \Vdash_\beta$ "$G^* \notin Q$"; that is, $q \Vdash_\beta$ "$G^* \in \cup_i \cap_j (2^\omega \setminus P_i^j)$". So there is an $r \in \mathbb{P}_\beta$, r extending q, and there is a fixed $i \in \omega$ such that $r \Vdash_\beta$ "$G^* \in \cap_j (2^\omega \setminus P_i^j)$". Let r' be obtained from r by changing the tags as follows: all ordinals greater than or equal to α are changed to ∞. Then $r' \in \mathbb{P}_\alpha$, and, by the retagging lemma, $r' \Vdash_\alpha$ "$G^* \in \cap_j (2^\omega \setminus P_i^j)$". Hence $r' \Vdash_\alpha$ "$G^* \notin Q$". But r' extends p, so $r' \Vdash_\alpha$ "$G^* \in Q$". □

We need one more fact about Steel forcing, Lemma 3.6, below. This lemma is proved in Harrington [7, Theorem 2.9] and Steel [17, Theorem 1].

Lemma 3.6 *Let $s \in 2^\omega$, and let α be a countable s-admissible ordinal. For any G which is sufficiently generic for $\mathbb{P}_\alpha$, $\omega_1^{G^*} = \omega_1^{\langle G^*, s\rangle} = \alpha$.*

Lemmas 3.7 and 3.8, below, are proved in ZF+DC+AD. Lemma 3.7 (see Moschovakis [15, Theorem 7.D.4]) is well known. Lemma 3.8 is similar to a theorem of Steel [18, Theorem 3.1], and the proof is similar to the proof of Steel's theorem (a proof which is not given in the above reference). Lemma 3.8 is a result which is in the spirit of Vaught's conjecture.

Theorem 2.4 follows easily from Lemmas 3.1, 3.7, and 3.8.

Lemma 3.7 *Assume* AD. *There does not exist an injection from* 2^ω *into* $\aleph_1$.

Lemma 3.8 *Assume* AD. *Let* $C \subset 2^\omega$, *and let* E *be an equivalence relation on* C. *Suppose that for all* $b \in 2^\omega$ *there exists an* $a \geq_T b$ *and there exists an* E*-equivalence class* $Q \subset C$ *satisfying the following three properties.*

(a) For all $y \in Q$, $\omega_1^y \geq \omega_1^a$.
(b) There exists an $x \in Q$ *such that* $x \leq_T a$.
(c) Q *is* $\mathbf{\Pi}^0_{(\omega_1^a+1)}$.

Then there exists a function $f : 2^\omega \to C$ *with the property that* f *takes any two distinct elements of* 2^ω *to* E*-inequivalent elements of* C.

Proof Consider the following game, $\mathcal{G}$. On move n, Player I plays $b_n \in \{0, 1\}$ and Player II plays $a_n, x_n \in \{0, 1\}$, the players alternating moves in the usual way. After ω moves, Player I has played $b = (b_0, b_1, b_2, \ldots) \in 2^\omega$ and Player II has played $a = (a_0, a_1, a_2, \ldots) \in 2^\omega$ and $x = (x_0, x_1, x_2, \ldots) \in 2^\omega$. Player II wins the round of $\mathcal{G}$ if and only if [$b \leq_T a$ and $x \leq_T a$ and $x \in C$ and (if Q is the E-equivalence class of x, then a and Q satisfy properties (a) and (c))].

Claim 1 *Player I does not have a winning strategy for* $\mathcal{G}$.

Proof Suppose that Player I does have a winning strategy, s_{I}. Modulo some sequence coding, s_{I} is an element of 2^ω. By hypothesis, there is an $a \geq_T s_{\mathrm{I}}$ and an x, Q such that a, x and Q satisfy (a)–(c). Consider the round of $\mathcal{G}$ in which Player II plays this a, x and Player I follows s_{I} and plays b. Since $\langle s_{\mathrm{I}}, x\rangle \leq_T a$, $b \leq_T a$; hence Player II wins the round. This proves Claim 1. □

AD plus Claim 1 gives us a winning strategy, s_{II}, for Player II for $\mathcal{G}$. Again viewing s_{II} as an element of 2^ω, for all $z \in 2^\omega$, let $a(z)$ and $x(z)$ be Player II's moves in the round of $\mathcal{G}$ in which Player I plays $b = \langle z, s_{\mathrm{II}}\rangle$ and Player II follows the strategy s_{II}. Clearly,

$$\begin{aligned}&\text{(i) the function } z \mapsto x(z) \text{ is continuous;}\\ &\text{(ii) for all } z \in 2^\omega, a(z) \leq_T \langle z, s_{\mathrm{II}}\rangle.\end{aligned} \tag{3.9}$$

Since s_{II} is a *winning* strategy, by the definition of the game $\mathcal{G}$, the following five facts must be true for all $z \in 2^\omega$:

$$\begin{aligned}&\text{(i) } a(z) \geq_T \langle z, s_{\mathrm{II}}\rangle;\\ &\text{(ii) } x(z) \in C;\\ &\text{(iii) for all } y \in 2^\omega, \text{ if } yE(x(z)), \text{ then } \omega_1^y \geq \omega_1^{a(z)};\\ &\text{(iv) } x(z) \leq_T a(z);\\ &\text{(v) the } E\text{-equivalence class of } x(z) \text{ is } \mathbf{\Pi}^0_{(\omega_1^{a(z)}+1)}.\end{aligned} \tag{3.10}$$

Let us now consider only those $z \in 2^\omega$ which have the property that $\omega_1^z = \omega_1^{\langle z, s_{\mathrm{II}}\rangle}$. For any such z, part (ii) of (3.9) and part (i) of (3.10) imply that $\omega_1^z = \omega_1^{a(z)}$. Therefore, for any such z, parts (iii), (iv), and (v) of (3.10) imply that the following three facts must be true:

$$\begin{aligned}&\text{(i) for all } y \in 2^\omega, \text{ if } yE(x(z)), \text{ then } \omega_1^y \geq \omega_1^z;\\ &\text{(ii) } \omega_1^{x(z)} \leq \omega_1^z;\\ &\text{(iii) the } E\text{-equivalence class of } x(z) \text{ is } \mathbf{\Pi}^0_{(\omega_1^z+1)}.\end{aligned} \tag{3.11}$$

Let α, β be countable s_{II}-admissible ordinals such that $\alpha < \beta$. We now consider rounds of the game in which Player II follows s_{II} and Player I plays $b = \langle G^*, s_{\mathrm{II}}\rangle$, where G is sufficiently generic for $\mathbb{P}_\alpha$ (or $\mathbb{P}_\beta$).

Claim 2 *There do not exist a $p \in \mathbb{P}_\alpha$ and an E-equivalence class Q such that $p \Vdash_\alpha$ "$x(G^*) \in Q$".*

Assuming Claim 2, (3.10)(ii), together with standard forcing (or Baire-category) arguments, gives us a perfect set $P \subset 2^\omega$ such that for any two distinct members, G_1^* and G_2^*, of P, $x(G_1^*)$ and $x(G_2^*)$ are E-inequivalent elements of C. Therefore, the conclusion of this lemma follows from Claim 2. So all that remains to be proved is that claim.

Assume toward a contradiction that Claim 2 is false and that such a p and Q exist. Let G be compatible with p and sufficiently generic for $\mathbb{P}_\alpha$. Then $x(G^*) \in Q$. Taking $z = G^*$, Lemma 3.6 and (3.11)(ii) imply that $\omega_1^{x(G^*)} \leq \omega_1^{G^*} = \alpha < \beta$.

And (3.11)(iii) tells us that Q is $\mathbf{\Pi}^0_{\alpha+1}$. So by Lemma 3.5 and (3.9)(i), $p \Vdash_\beta$ "$x(G^*) \in Q$". Let H be compatible with p and sufficiently generic for $\mathbb{P}_\beta$. Then $x(H^*) \in Q$. Taking $z = H^*$, Lemma 3.6 and (3.11)(i) imply that for all $y \in 2^\omega$, if $y \in Q$, then $\omega_1^y \geq \beta$. Setting $y = x(G^*)$ gives a contradiction. □

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