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An Abelian Rule for BCI—and Variations

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Abstract We show the admissibility for BCI of a rule form of the characteristic implicational axiom of abelian logic, this rule taking us from $(\alpha \to \beta) \to \beta$ to α. This is done in Section 8, with surrounding sections exploring the admissibility and derivability of various related rules in several extensions of BCI.

1 Introduction

On the basis of a Hilbert system comprising a set of axioms and a set of primitive rules, a rule is said to be *derivable* if for any application of the rule, the conclusion of the rule can be derived from the premises for that application, together with the axioms of the system, by means of the primitive rules. A rule is said to be *admissible* for the system if the set of theorems of the system (i.e., the formulas derivable from the axioms by means of the primitive rules) is closed under the rule: for any application of the rule in which all the premises are theorems, so is the conclusion. Evidently, derivability implies admissibility for any rule, relative to any system; when the converse holds for a given system, that system is said to be *structurally complete*. More precisely, defining a rule to be *substitution-invariant* if any substitution instance of an application of the rule is in turn an application of the rule, the system is structurally complete when every admissible substitution-invariant rule is derivable. The notion of structural completeness was first introduced by Pogorzelski [19].

Our main goal is to prove that the following *abelian rule*

$$\frac{(\alpha \to \beta) \to \beta}{\alpha} \tag{1}$$

is admissible, though not derivable, in BCI. We will also discuss a handful of related rules, proving (or re-proving) structural incompleteness of certain logics in the process. An ulterior motive—there always is one—is to illustrate by example a proof-theoretical method that seems well suited to deal with pure implication logics.

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2 Structural Completeness

For the Hilbert systems under consideration here only substitution-invariant rules will be employed. Further, we consider only rules with finitely many premises, so we will be concerned only with structural completeness "in the finitary sense."[1]

An alternative way of isolating the concepts of admissibility, derivability, and structural completeness invokes consequence relations. An axiomatic system gives rise to a consequence relation, $\vdash$, say—a standard consequence relation in the terminology of Wójcicki [22]—defined by setting $\Gamma \vdash \alpha$ to hold precisely when there is a sequence of formulas each of which is an axiom or an element of Γ, or is obtained from earlier members of the sequence by application of one of the primitive rules. Then an n-premise rule is derivable according to the above definition just in case for any application of the rule, passing from premises $\alpha_1, \ldots, \alpha_n$ to conclusion β, we have $\alpha_1, \ldots, \alpha_n \vdash \beta$, while admissibility means that for any such application, if $\varnothing \vdash \sigma(\alpha_i)$ for $i = 1, \ldots, n$ and some substitution σ, then $\varnothing \vdash \sigma(\beta)$. Whether or not a consequence relation $\vdash$ has been obtained from a Hilbert system in the manner just described, we can apply the concept of structural completeness to $\vdash$ itself, calling $\vdash$ structurally complete when the following condition is satisfied; here we omit the $\varnothing$ on the left, as is customary, and understand the condition as prefaced by "for all n and all formulas $\alpha_1, \ldots, \alpha_n, \beta$:"

<u>If</u> for all σ, $\vdash \sigma(\alpha_i)$ for each i $(1 \leq i \leq n)$ implies $\vdash \sigma(\beta)$, <u>then</u> $\alpha_1, \ldots, \alpha_n \vdash \beta$.

We will mostly conduct the discussion in the terms given in the preceding paragraph, though with occasional references to the consequence relations induced by Hilbert systems, as here.

Our attention will be on logics in a language with binary $\to$ (implication) as its sole primitive connective, formulas being freely generated with its aid in the usual manner, from propositional variables (or "sentence letters") $v_1, \ldots v_n, \ldots$ taken for definiteness here as denumerably many in number. In what follows x, y, z, sometimes subscripted, and also v, will be used for arbitrary such variables. As is evident from the preceding paragraph, $\alpha, \beta, \ldots$ are used as metalinguistic variables ("schematic letters") for formulas.

3 The Logics We Consider

The logic we are mainly concerned with here is BCI. It was isolated under that name by C. A. Meredith in the 1950s and subsequently attained prominence as the implicational fragment of Girard's linear logic; for the rationale (from combinatory logic) behind the labeling of the axioms, see Bunder [3], and for further historical background on these logics, see Došen [6]. BCI can be presented as a Hilbert system, with all formulas of the following three forms as axioms:

(B) $(\alpha \to \beta) \to ((\gamma \to \alpha) \to (\gamma \to \beta))$,
(C) $(\alpha \to (\beta \to \gamma)) \to (\beta \to (\alpha \to \gamma))$,
(I) $\alpha \to \alpha$

and as the sole primitive rule, *modus ponens*:

$$\frac{\alpha \quad \alpha \to \beta}{\beta}.$$

In the presence of C, the schema B could equivalently be replaced with

(B′) $(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$.

BCI has a particularly pleasing Gentzen-style presentation, which we will consider in Section 5. In fact, a sequent calculus for BCI will be one of our main tools in the paper.

We choose four *axiomatic* extensions of BCI as the logics we focus on. We will also consider further three nonaxiomatic extensions, arising naturally from considerations of *algebraizability*, but let us first introduce the axiomatic extensions briefly.

The smallest extension of BCI in which all theorems are provably equivalent (cf. Humberstone [8], Kowalski and Butchart [13]) is *monothetic* BCI. Up to logical equivalence then, this logic has only one theorem (*thesis*); hence the name. Monothetic BCI is axiomatized by replacing I with

(I*) $(\alpha \to \alpha) \to (\beta \to \beta)$.

We will use BCI* as a shorthand for monothetic BCI. In BCI* the *truth constant* t can be defined by putting $t = v \to v$ for some selected variable v.

Another extension is BCK, which can be obtained from BCI by replacing I with

(K) $\alpha \to (\beta \to \alpha)$.

Again, [6] supplies some useful background on this logic (also isolated by Meredith, though considered earlier by Tarski, as [6] recalls). BCK has a straightforward sequent calculus, which is a natural extension of the sequent calculus for BCI. It also has a very well behaved algebraic semantics, namely the quasivariety of BCK-*algebras*. In more detail: the consequence relation $\vdash_{\mathsf{BCK}}$ induced (à la Section 1) by the present axiomatization is algebraizable, the class of BCK-algebras providing an *equivalent quasivariety semantics*, while $\vdash_{\mathsf{BCI}}$ is not algebraizable (see Blok and Pigozzi [1] for these observations as well as an explanation of the terminology just employed in formulating them). In BCK, the formula $(x \to x) \to (y \to y)$ is provable, so the truth constant t can be defined just as before.

Yet another extension of BCI, incomparable with BCK, and indeed inconsistent with it (in the sense that all formulas are provable if K is added as a further axiom), is the implicational fragment of *abelian logic*, introduced under that name in Meyer and Slaney [14] (though this fragment was considered earlier by Meredith and then by J. A. Kalman; see Humberstone [9, p. 1122] for these and further references, and 7.25 of the same work for an extended discussion of the fragment, under the name BCIA[2]). Since the full version of abelian logic does not concern us here, from now on by abelian logic we mean its implicational fragment, which, following [15] and Butchart and Rogerson [4], we denote by $\mathsf{A}_\to$ (the "full" system being called A). With this convention in place, our abelian logic $\mathsf{A}_\to$ can be presented as an extension of BCI by the axiom (or more accurately, axiom-schema)

(A) $((\alpha \to \beta) \to \beta) \to \alpha$.

In $\mathsf{A}_\to$, the formula $(x \to x) \to (y \to y)$ is provable as well, so the truth constant t becomes definable, just as in BCK. Moreover, we can define a negation-like connective, putting $-x = x \to t$, and an addition-like connective, putting $x + y = -x \to y = (x \to t) \to y$.[3] With these, we have $-x + x \equiv t$, $t \equiv x + -x$, $x + t \equiv x$, $x \equiv t + x$, as well as $(x + y) + z \equiv x + (y + z)$ and $x + y \equiv y + x$, where $\equiv$ stands for provable equivalence (i.e., the provability of the two implications, from left to right and conversely; note that in all of our logics, this suffices for the interreplaceability of the formulas concerned in all contexts). An algebraic semantics for the *theorems* of $\mathsf{A}_\to$ is provided by the class of abelian groups, hence the grouplike notation. There is a matrix semantics for $\mathsf{A}_{\rightarrow}$ as a whole (encompassing rules as well as theorems) which involves ordered abelian groups, described in Section 4.

Although BCK is incomparable with $\mathsf{A}_{\rightarrow}$, the grouplike connectives can be defined in BCK as well, but the addition connective $+$ should not be confused with *fusion* (or "multiplicative conjunction"), $\circ$, often considered in substructural logics (for which one has $(\alpha \circ \beta) \rightarrow \gamma$ and $\alpha \rightarrow (\beta \rightarrow \gamma)$ provably equivalent). In BCK, $\alpha + \beta$ does not coincide with the fusion of α and β, even if that fusion exists. Also, most of the abelian group properties fail, for example, commutativity: in BCK we have $x + y \equiv y$. Remarkably, however, associativity of addition survives, and as it turns out, it defines precisely the intersection of BCK and $\mathsf{A}_{\rightarrow}$, considered as sets of theorems. We will call this logic BC(K $\vee$ A): a piece of *ad hoc* notation meant to indicate that BC(K $\vee$ A) is the extension of monothetic BCI by something behaving like a disjunction of K and A.[4]

4 Algebraizable Companions

To introduce the three remaining logics (consequence relations, to be precise), we will take a detour through algebraizability. First, a piece of notation: for an axiomatic extension L of BCI, we write L^+ for the further extension of L by the rule

$$\frac{\alpha \quad \beta}{\alpha \rightarrow \beta}. \tag{2}$$

The corresponding consequence relations, we denote by $\vdash_L$ and $\vdash_{L^+}$. The reader unfamiliar with the notion of an algebraizable logic may safely skip the rest of this section: except for the L^+ notation, nothing essential depends on it.

The reader familiar with algebraizability will notice that L^+ is an algebraizable logic (by [1, Corollary 4.8]) for any axiomatic extension L of BCI. The set of theorems of L may or may not be equal to that of L^+. If they are equal, then (2) is admissible in L. If moreover, (2) is not derivable in L, that is, if $L \neq L^+$ as consequence relations, then this constitutes an easy proof of structural incompleteness of L.

It is well known, and very easy to prove, that $\mathsf{BCK}^+ = \mathsf{BCK}$. Butchart and Rogerson [4] distinguish $\mathsf{A}_{\rightarrow}$ from $\mathsf{A}^+_{\rightarrow}$; in fact our L^+ notation follows their example.

For the relations between BCI, BCI*, and $(\mathsf{BCI}^*)^+$, we note that (2) is clearly admissible for monothetic BCI: its admissibility follows from what "monothetic" means, or, if monothetic BCI is simply understood by in terms of its axiomatic description as BCI*, from the findings of Kowalski and Butchart [13]. Its non-derivability can be shown by an appeal to the well-known local deduction theorem for BCI (as deployed, e.g., in [4]), which remains intact for monothetic BCI. Adding (2) to BCI itself, as was done in Kabziński [10], is nonconservative, in fact, $\mathsf{BCI}^+ = (\mathsf{BCI}^*)^+$ was shown in [8].

For BC(K $\vee$ A) we have the following. Raftery and van Alten [20], [21] show that the intersection of the quasiequational theories of BCK-algebras and abelian groups is axiomatized relative to the quasiequational theory of BCI-algebras by the equation $(x + y) + z = x + (y + z)$. The intrinsic *assertional* logic of the quasivariety defined by these quasiequations (see Blok and Raftery [2] for the definition and details) is precisely $\mathsf{BC}(\mathsf{K} \vee \mathsf{A})^+$. It follows that $\mathsf{BC}(\mathsf{K} \vee \mathsf{A})^+$ is axiomatized by $(x + y) + z \equiv x + (y + z)$, relative to $(\mathsf{BCI}^*)^+ = \mathsf{BCI}^+$. But now it is easy to

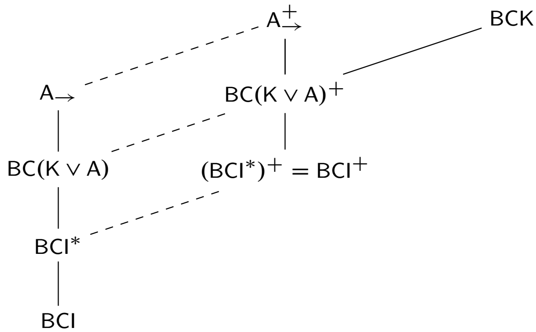

Figure 1 Inclusion relations among our logics. Dashed lines indicate inclusions between logics with the same set of theorems.

show that BC(K ∨ A)$^+$ has the same theorems as BC(K ∨ A). Namely, if φ is a theorem of BC(K ∨ A)$^+$, then by algebraizability $\varphi = 1$ holds in the algebraic models of BC(K ∨ A)$^+$. In particular, $\varphi = 1$ holds in all BCK-algebras and in all abelian groups, proving that φ is a theorem of BCK and (by the completeness theorem from [14]) of A$_\to$; hence φ is a theorem of BC(K ∨ A).[5]

Thus, BC(K∨A) is the least extension of monothetic BCI in which, for all formulas α, β, γ, we have $(\alpha + \beta) + \gamma \equiv \alpha + (\beta + \gamma)$. It suffices to add to monothetic BCI the $\to$-direction of this equivalence as a new axiom, since the converse direction is already BCI-provable (for an arbitrary formula in place of the occurrences of t which are hidden in the $+$ notation).

Figure 1 depicts all the logics considered here, viewed as consequence relations. They are different as sets of theorems, unless they are joined by a dashed line.

Analogues of the systems we denote by L and L^+, but for the full rather than the purely implicational language of abelian logic, are also distinguished (and contrasted with a third consequence relation) in Paoli, Spinks, and Veroff [18], whose axiomatic description via Hilbert systems of the two consequence relations adds (2) to *modus ponens*, to obtain the stronger system. The distinction between A$_\to$ and A$^+_\to$ is already evident in [14]; in fact, along with a simple matrix-based characterization of the contrast: in both cases we can use the integers, interpreting $\alpha \to \beta$ as $v(\beta) - v(\alpha)$ for an evaluation v, but for A$_\to$ we take the nonnegative integers as designated elements, whereas for A$^+_\to$ we take 0 as the sole designated value.[6] As all the references cited in this paragraph observe, there is nothing special about the additive group of the integers here, and we may consider a semantic account in terms of all (partially) ordered abelian groups (for full A, all abelian ℓ-groups), with the positive cones as the sets of designated elements in the one case and the singleton of the unit element in the other. The designated element being term-definable (or "formula-definable")—by t (or indeed any $\alpha \to \alpha$)—in the latter case, A$^+_\to$ turns out to be algebraizable, with the variety of abelian groups serving as its equivalent quasivariety semantics. Note, however, that the order induced by $x \leq y$ if and only if $x \to y$ is designated, is in this case the discrete order, not the lattice order. The details of the argument are given in [4], where it is also shown that, by contrast, A$_\to$ (or, more explicitly, $\vdash_{\mathsf{A}_\to}$) is not algebraizable.

5 Sequent Systems for BCI and BCK

We begin with some terminological and notational preliminaries. In Section 1, Γ was used for a set of formulas (in the discussion of consequence relations). From now on, Γ, Δ, ... stand for *multisets* rather than sets (of formulas), and set-theoretic notation and terminology is to be given a suitable multiset interpretation. In particular: for a multiset Γ of formulas, the notation $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ is understood as a numbering of *occurrences* of formulas, not formulas, and $|\Gamma|$ is n. A formula occurring j times in Γ and k times in Δ occurs $j + k$ times in the multiset union $\Gamma \cup \Delta$ (written as "Γ, Δ" when a rule is being displayed, or as "Γ, δ" when $\Delta = \{\delta\}$). We use the notation $\Gamma \smallsetminus \{\alpha\}$ only when α occurs at least once in Γ and the notation denotes the result of removing exactly one such occurrence from Γ. A multiset of multisets $\Gamma_1, \ldots, \Gamma_n$ is a *partition* of Δ when Δ is the multiset union of $\Gamma_1, \ldots, \Gamma_n$. We will sometimes write such partitions as $\{\Gamma_i\}_{i=1}^{n}$, and we allow $n = 0$ for the case of the empty multiset partitioned into zero parts. This will simplify the statement of Lemma 6.1 below.

By a *sequent* we mean a pair (Γ, α), where Γ is a possibly empty multiset of formulas and α a formula. We write sequents in the usual form $\Gamma \Rightarrow \alpha$, with the *separator* $\Rightarrow$ replacing the comma.

We are now in a position to give a sequent calculus ("Gentzen system") presentation for each of BCI and BCK, selected so as to have the same operational rules,[7] namely:

$$\frac{\Gamma \Rightarrow \alpha \quad \Delta, \beta \Rightarrow \gamma}{\Gamma, \Delta, \alpha \to \beta \Rightarrow \gamma}\ (\to\Rightarrow) \qquad \frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \alpha \to \beta}\ (\Rightarrow\to)$$

and the same structural rule of *cut*:

$$\frac{\Gamma \Rightarrow \alpha \quad \Delta, \alpha \Rightarrow \beta}{\Gamma, \Delta \Rightarrow \beta}.$$

They differ, however, in their initial sequents. The system for BCI has initial sequents

$$x \Rightarrow x$$

for any variable x, whereas the system for BCK has initial sequents

$$\Gamma, x \Rightarrow x$$

for any multiset of formulas Γ and any variable x. In both systems, cut is eliminable and the rule $(\Rightarrow\to)$ is invertible.

Lemma 5.1 *Let $L \in \{\mathsf{BCI}, \mathsf{BCK}\}$. If a sequent $\Gamma \Rightarrow \alpha$ is provable in L, then it is provable without cut in L. If a sequent $\Gamma \Rightarrow \alpha \to \beta$ is provable in L, then the sequent $\Gamma, \alpha \Rightarrow \beta$ is provable in L.*

From now on, our official sequent systems for BCI and BCK will be cut-free, but following the usual practice we will make use of cut in the proofs whenever convenient. In the case of BCK, we can also help ourselves to the rule of *weakening*:

$$\frac{\Gamma \Rightarrow \alpha}{\beta, \Gamma \Rightarrow \alpha}$$

in the sense made precise in the next lemma.

Lemma 5.2 *If a sequent $\Gamma \Rightarrow \alpha$ is provable in the sequent system for* BCK *extended by the rule of weakening, then $\Gamma \Rightarrow \alpha$ is provable in* BCK *without weakening.*

The sense in which these sequent calculi are Gentzen systems for BCI and BCK is that a formula α is provable in BCI (resp., BCK) as presented in Section 3 if and only if the sequent $\Rightarrow \alpha$ (if one prefers to make the left-hand side explicit, the sequent $\varnothing \Rightarrow \alpha$) is provable in the sequent calculus for BCI (resp., for BCK). We have no hesitation in referring to the formula α as a theorem in this case, even when discussing the sequent system.

6 Link Formulas

In this section, we combine Kowalski [11, Lemma 3.1] and [12, Lemma 3.1], following our strategy of giving BCI and BCK a uniform treatment. Here, and elsewhere below, we use the following notational device: for a nonempty multiset $\Delta = \{\delta_1, \ldots, \delta_k\}$ and formula ε we write $\delta_1 \cdots \delta_k \to \varepsilon$ or simply $\Delta \to \varepsilon$ to denote the formula

$$\delta_1 \to \big(\delta_2 \to \cdots \to (\delta_k \to \varepsilon)\ldots\big).$$

It is convenient to use the notation above with possibly empty Δ; to that end, we declare $\varnothing \to \varepsilon$ to denote the formula ε. To simplify the notation even further, we write $\Gamma\Delta \to \varepsilon$ for $(\Gamma \cup \Delta) \to \varepsilon$, where $\Gamma \cup \Delta$ is the multiset union of Γ and Δ. Because we consider only extensions of BCI, the order in which the δ_i are selected here does not make a difference to the provability of a sequent involving such an implication. Note also that despite the notation, the "$\delta_1 \cdots \delta_k$" part of the abbreviation does not denote a subformula of the formula represented by the notation.[8]

Lemma 6.1 *Let $\Gamma \Rightarrow v$ be a sequent, with v a variable. The following hold.*

1. *The sequent $\Gamma \Rightarrow v$ is provable in* BCI *if and only if there exists a formula $\gamma = \gamma_n\gamma_{n-1}\cdots\gamma_1 \to \gamma_0 \in \Gamma$ and a partition $\{\Gamma_i\}_{i=1}^n$ of $\Gamma \smallsetminus \{\gamma\}$ such that*
   (a) $\gamma_0 = v$,
   (b) for every $i \in \{1, \ldots, n\}$ the sequent $\Gamma_i \Rightarrow \gamma_i$ is provable in BCI.
2. *If $v \notin \Gamma$, then $\Gamma \Rightarrow v$ is provable in* BCK *if and only if there exists a formula $\gamma = \gamma_n\gamma_{n-1}\cdots\gamma_1 \to \gamma_0 \in \Gamma$ and a partition $\{\Gamma_i\}_{i=1}^n$ of $\Gamma \smallsetminus \{\gamma\}$ such that*
   (a) $\gamma_0 = v$,
   (b) for every $i \in \{1, \ldots, n\}$ the sequent $\Gamma_i \Rightarrow \gamma_i$ is provable in BCK.

Proof We begin with (2). For the forward direction, we argue by induction on the length of the cut-free proof of $\Gamma \Rightarrow v$. If this is an initial sequent, the claim holds vacuously. For the inductive step, the last rule in a cut-free proof of $\Gamma \Rightarrow v$ must be

$$\frac{\Pi \Rightarrow \alpha \quad \Delta, \beta \Rightarrow v}{\Gamma \Rightarrow v}$$

with $\alpha \to \beta \in \Gamma$ and $\Pi, \Delta = \Gamma \smallsetminus \{\alpha \to \beta\}$. Notice that $v \notin \Delta$, because otherwise $\Gamma \Rightarrow v$ would be an initial sequent. Arguing case by case, we will show that the requirements of the lemma are satisfied. (1) If $\Delta, \beta \Rightarrow v$ is an initial sequent, then $\beta = v$ and so $\Gamma = \Pi, \Delta, \alpha \to v$. Since $\Pi \Rightarrow \alpha$ is provable, by weakening we get $\Pi, \Delta \Rightarrow \alpha$ and thus $\alpha \to \beta$ satisfies the requirements. (2) If $\Delta, \beta \Rightarrow v$ is not initial, the inductive hypothesis applies to $\Delta, \beta \Rightarrow v$ and thus there is a formula $\delta = \delta_n\delta_{n-1}\cdots\delta_1 \to v \in \Delta \cup \{\beta\}$ and a partition of $(\Delta \cup \{\beta\}) \smallsetminus \{\delta\}$ into $\Delta_1, \ldots, \Delta_n$ such that the sequents $\Delta_i \Rightarrow \delta_i$ are provable. Now there are two cases again. (2.1) If $\beta = \delta$, then $\alpha \to \beta$ satisfies all requirements of the lemma. (2.2) If $\beta \neq \delta$, then

$\beta \in \Delta_j$ for some $j \in \{1, \ldots, n\}$, and the sequent $\Pi, \Delta_j \smallsetminus \beta, \alpha \to \beta \Rightarrow \delta_j$ is provable, by application of

$$\frac{\Pi \Rightarrow \alpha \quad \Delta_j \Rightarrow \delta_j}{\Pi, \Delta_j \smallsetminus \{\beta\}, \alpha \to \beta \Rightarrow \delta_j}.$$

Then, we obtain provable sequents

$$\begin{array}{r}
\Delta_1 \Rightarrow \delta_1 \\
\vdots \quad \vdots \\
\Delta_{j-1} \Rightarrow \delta_{j-1} \\
\Pi, \Delta_j \smallsetminus \{\beta\}, \alpha \to \beta \Rightarrow \delta_j \\
\Delta_{j+1} \Rightarrow \delta_{j+1} \\
\vdots \quad \vdots \\
\Delta_n \Rightarrow \delta_n,
\end{array}$$

where

$$\Delta_1, \ldots, \Delta_{j-1}, \Pi, \Delta_j \smallsetminus \{\beta\}, \alpha \to \beta, \Delta_{j+1}, \ldots, \Delta_n = \Gamma \smallsetminus \{\delta\}$$

and the requirements of the lemma are satisfied by δ.

For the backward direction, suppose the sequents

$$\begin{array}{c}
\Lambda_1 \Rightarrow \gamma_1 \\
\vdots \quad \vdots \\
\Lambda_n \Rightarrow \gamma_n
\end{array}$$

are provable, and $\Lambda_1, \ldots, \Lambda_n = \Gamma \smallsetminus \{\gamma\}$. Then, since $v \Rightarrow v$ is an initial sequent, we can apply $(\to\Rightarrow)$ successively, beginning with

$$\frac{\Lambda_2 \Rightarrow \gamma_2 \quad \dfrac{\Lambda_1 \Rightarrow \gamma_1 \quad v \Rightarrow v}{\Lambda_1, \gamma_1 \to v \Rightarrow v}}{\Lambda_2, \Lambda_1, \gamma_2 \to (\gamma_1 \to v) \Rightarrow v}.$$

After n such applications, we get a proof of the sequent

$$\Lambda_n, \ldots, \Lambda_1, \gamma_n \cdots \gamma_1 \to v \Rightarrow v$$

which, since $\gamma_n \cdots \gamma_1 \to v = \gamma$, is precisely $\Gamma \Rightarrow v$. This finishes the proof of (2).

The proof of (1) can now easily be obtained from the proof of (2) by modifying (in fact, ignoring) the cases dealing with initial sequents. We leave the details for the reader. □

Turning to terminology now, for a provable sequent $\Gamma \Rightarrow v$, any formula γ satisfying the lemma above will be called a *link* formula. It should be clear by now how γ links subproofs together. Note that a link formula may not be unique, and so for a given sequent $\Gamma \Rightarrow v$ different cases of link formulas often need to be considered. To indicate a particular link formula γ, we will say that $\Gamma \Rightarrow v$ is provable *with link formula* γ, or that γ is a *link* in (the proof of) $\Gamma \Rightarrow v$.

In [11] and [12] what we have just called link formulas were called *split formulas* and *split terms*, respectively. We changed the terminology to make it more descriptive: as we have seen, the role of a link formula is to join together several subproofs.

We also believe that the new name will help to reduce interference with other technical uses of "split" and "splitting" of which there are quite a number.

The next two lemmas will form an "induction pump" which will be used to prove our main results. The first will serve to simplify certain provable sequents, the second dealing with those that cannot be further simplified. The statements of the lemmas as well as their proofs are precisely the same for BCI and BCK, so until the end of this section, by "provable" we mean "provable in (the sequent calculus for) BCI" or "provable in (...) BCK" in a systematically ambiguous way.

Lemma 6.2 *Let $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ be a nonempty multiset of formulas, and let Δ be a multiset of formulas. Suppose that the sequent $\Gamma, \Delta\Gamma \to x \Rightarrow x$ is provable, with link formula $\gamma_1 = \beta_1 \cdots \beta_k \to x$. Then, for some $\Gamma' \subseteq \Gamma \smallsetminus \{\gamma_1\}$ and some $i \in \{1, \ldots, k\}$, the sequent $\Delta\Gamma' \to \beta_i \Rightarrow \Gamma' \to \beta_i$ is provable.*

Proof As $\Gamma, \Delta\Gamma \to x \Rightarrow x$ is provable with link formula γ_1, using Lemma 6.1 and renumbering, if necessary, we obtain the following provable sequents

$$\begin{aligned} \gamma_2, \ldots, \gamma_p, \Delta\Gamma \to x &\Rightarrow \beta_1 \\ \Lambda_2 &\Rightarrow \beta_2 \\ \vdots\ & \quad \vdots \\ \Lambda_k &\Rightarrow \beta_k, \end{aligned}$$

where $\Lambda_2, \ldots, \Lambda_k = \gamma_{p+1}, \ldots, \gamma_n$ and $1 \le p \le n$. We claim that the sequent

$$\Delta\gamma_2 \cdots \gamma_p \to \beta_1 \Rightarrow \Delta\Gamma \to x \tag{3}$$

is provable as well. To prove the claim it suffices to show that the sequent

$$\Delta, \Gamma, \Delta\gamma_2 \cdots \gamma_p \to \beta_1 \Rightarrow x \tag{4}$$

is provable, and to show that, in turn, it suffices to find a suitable link formula. Consider $\gamma_1 = \beta_1 \cdots \beta_k \to x$. The sequents below are clearly provable:

$$\begin{aligned} \Delta, \gamma_2, \ldots, \gamma_p, \Delta\gamma_2 \cdots \gamma_p \to \beta_1 &\Rightarrow \beta_1 \\ \Lambda_2 &\Rightarrow \beta_2 \\ \vdots\ & \quad \vdots \\ \Lambda_k &\Rightarrow \beta_k \end{aligned}$$

and moreover

$$\Delta, \gamma_2, \ldots, \gamma_p, \Delta\gamma_2 \cdots \gamma_p \to \beta_1, \Lambda_2, \ldots, \Lambda_k = \Delta, \Gamma \smallsetminus \{\gamma_1\}, \Delta\gamma_2 \cdots \gamma_p \to \beta_1.$$

This shows that the sequent (4) is indeed provable, with link formula γ_1. Therefore, the sequent (3) is provable, as claimed. Now, applying cut to (3) and the sequent

$$\gamma_2, \ldots, \gamma_p, \Delta\Gamma \to x \Rightarrow \beta_1$$

which is provable by assumption, we get that

$$\gamma_2, \ldots, \gamma_p, \Delta\gamma_2 \cdots \gamma_p \to \beta_1 \Rightarrow \beta_1$$

is provable, and therefore, so is

$$\Delta\gamma_2 \cdots \gamma_p \to \beta_1 \Rightarrow \gamma_2 \cdots \gamma_p \to \beta_1.$$

Now, putting $\Gamma' = \{\gamma_2, \ldots, \gamma_p\}$ finishes the proof. □

Lemma 6.3 *Suppose that* $\Gamma, \Delta\Gamma \to x \Rightarrow x$ *is provable, with link formula* $\Delta\Gamma \to x$*. Then, all members of* Δ *are theorems.*

Proof The proof is by induction on $|\Gamma|$. If Γ is empty, the claim follows immediately by properties of the link formula $\Delta \to x$. Suppose that the claim holds for all $|\Gamma| < n$. Let $\Gamma = \{\gamma_1, \dots, \gamma_n\}$. We can assume that Δ is nonempty, so let $\Delta = \{\delta_1, \dots, \delta_m\}$. Since $\Delta\Gamma \to x$ is a link formula, we have provable sequents

$$\begin{aligned} \Lambda_1 &\Rightarrow \gamma_1 \\ \vdots\ &\quad \vdots \\ \Lambda_n &\Rightarrow \gamma_n \\ \Lambda_{n+1} &\Rightarrow \delta_1 \\ \vdots\ &\quad \vdots \\ \Lambda_{n+m} &\Rightarrow \delta_m. \end{aligned}$$

As $\Lambda_1, \dots, \Lambda_{n+m} = \Gamma$, at least m of the multisets on the left-hand side must be empty, so at least m of the formulas on the right-hand side are theorems. Suppose that γ_i is among the theorems. Since γ_j also occurs on the left-hand side, say, $\gamma_i \in \Lambda_j$ for some $j \in \{1, \dots, n+m\}$, we first apply cut once

$$\frac{\Rightarrow \gamma_i \quad \Lambda_j \Rightarrow \varphi}{\Lambda_j \smallsetminus \{\gamma_i\} \Rightarrow \varphi}$$

to remove one occurrence of γ_i from the left-hand side. Then, we obtain provable sequents

$$\begin{aligned} \Lambda'_1 &\Rightarrow \gamma_1 \\ \vdots\ &\quad \vdots \\ \Lambda'_n &\Rightarrow \gamma_n \\ \Lambda'_{n+1} &\Rightarrow \delta_1 \\ \vdots\ &\quad \vdots \\ \Lambda'_{n+m} &\Rightarrow \delta_m, \end{aligned}$$

where $\Lambda'_\ell = \Lambda_\ell$ for all $\ell \in \{1, \dots, j-1, j+1, \dots, n+m\}$ and $\Lambda'_j = \Lambda_j \smallsetminus \{\gamma_i\}$. Now, applying ($\to\Rightarrow$) $n - 1 + m$ times, as in the proof of Lemma 6.1, to $x \Rightarrow x$ and all sequents above except $\Rightarrow \gamma_i$, we obtain a provable sequent

$$\Delta\big(\Gamma \smallsetminus \{\gamma_i\}\big) \to x \Rightarrow \big(\Gamma \smallsetminus \{\gamma_i\}\big) \to x$$

which, by invertibility of ($\Rightarrow\to$), yields a provable sequent

$$\Gamma \smallsetminus \{\gamma_i\}, \Delta\big(\Gamma \smallsetminus \{\gamma_i\}\big) \to x \Rightarrow x$$

to which in turn the inductive hypothesis applies, proving that all members of Δ are theorems. The remaining case is that no γ_i is a theorem. But then, $\Lambda_1, \dots, \Lambda_n$ are all nonempty and therefore, $\Lambda_{n+1} = \cdots = \Lambda_{n+m} = \varnothing$. So, $\delta_1, \dots, \delta_m$ are theorems, as claimed. □

7 Admissibility of Two Nonstandard Rules

In this section we state our admissibility results in their most general form. The rules we prove to be admissible are somewhat nonstandard, but the gain in generality offsets the loss in standardness. Admissibility of more standard rules will be shown in a series of corollaries in the following section.

Theorem 7.1 *Let $\alpha_1, \dots, \alpha_n, \beta$ be formulas. If $(\alpha_1 \to \cdots (\alpha_n \to \beta) \dots) \to \beta$ is a theorem of* BCI*, then α_i is a theorem of* BCI *for every $1 \leq i \leq n$.*

Proof We argue by contradiction. Let $(\alpha_1 \to (\alpha_2 \to \cdots (\alpha_n \to \beta) \dots)) \to \beta$ be the shortest theorem of BCI such that α_i is not a theorem of BCI for some $1 \leq i \leq n$. Using our shorthand notation, this means that $(\alpha_1 \alpha_2 \cdots \alpha_n \to \beta) \to \beta$ is a theorem of BCI, but α_i is not. Since $\beta = \Gamma \to x$, for some multiset Γ and a variable x, we get that the sequent $\Gamma, \alpha_1 \alpha_2 \cdots \alpha_n \Gamma \to x \Rightarrow x$ is provable in BCI. We have two cases to consider.

Case 1. Suppose that $\gamma_1 = \beta_1 \cdots \beta_k \to x \in \Gamma$ is a link formula. Then, applying Lemma 6.2 we get that $\alpha_1 \alpha_2 \cdots \alpha_n \Gamma' \to \beta_j \Rightarrow \Gamma' \to \beta_j$ is provable, for some $j \in \{1, \dots, k\}$, where $\Gamma' \subseteq \Gamma \smallsetminus \{\gamma_1\}$. So, the formula $\Gamma' \to \beta_j$ is strictly shorter than $\Gamma \to x$. But then, since α_i is not a theorem, we have constructed a formula that is shorter than the shortest one with the required properties, which is a contradiction.

Case 2. Suppose that $\alpha_1 \alpha_2 \cdots \alpha_n \Gamma \to x$ is a link formula. Then, applying Lemma 6.3 we get that $\alpha_1, \dots, \alpha_n$ are all theorems. In particular, α_i is a theorem, which is a contradiction. □

Theorem 7.2 *Let $\alpha_1, \dots, \alpha_n, \beta$ be formulas. If $(\alpha_1 \to \cdots (\alpha_n \to \beta) \dots) \to \beta$ is a theorem of* BCK*, then α_i is a theorem of* BCK *for every $1 \leq i \leq n$, or β is a theorem of* BCK.

Proof Let the formula $(\alpha_1 \to (\alpha_2 \to \cdots (\alpha_n \to \beta) \dots)) \to \beta$ be the shortest BCK theorem such that α_i is not a BCK theorem for some $1 \leq i \leq n$, and β is not a BCK theorem either. Thus, letting $\beta = \Gamma \to x$, we get that the sequent $\Gamma, \alpha_1 \alpha_2 \cdots \alpha_n \Gamma \to x \Rightarrow x$ is provable in BCK, but some α_i is not a theorem, and β is not a theorem. We have now *three* cases to consider. Cases 1 and 2 proceed exactly as in the proof of Theorem 7.1, so we omit them.

Case 3. Suppose that $\Gamma, \alpha_1 \alpha_2 \cdots \alpha_n \Gamma \to x \Rightarrow x$ is an initial sequent. Then, $x \in \Gamma$ and therefore $\beta = \Gamma \to x$ is a theorem. This is a contradiction. □

Observe that if we had fusion ($\circ$, from Section 3) in the language, Theorem 7.1 could be reformulated as stating that the rule

$$\frac{(\alpha_1 \to (\alpha_2 \to \cdots (\alpha_n \to \beta) \dots)) \to \beta}{\alpha_1 \circ \cdots \circ \alpha_n}$$

is admissible for BCI. But, although adding fusion to BCI results in a conservative extension, the results about link formulas do not transfer to that setting, at least not as they were formulated above.

8 An Abelian Rule

This section will be devoted to the rule (1), introduced in Section 1. We call it abelian because it is a “rule form” of the abelian axiom-schema A from Section 3. We will also consider one of its generalizations and one of its particularizations. Let us begin with the generalization. For every $n, i \in \mathbb{N}$ such that $1 \leq i \leq n$, consider the rule

$$\frac{(\alpha_1 \to (\alpha_2 \to \cdots (\alpha_n \to \beta)\ldots)) \to \beta}{\alpha_i}. \tag{5}$$

Observe that for $i = n = 1$, we obtain (1). Further, for arbitrary n, in the presence of the "C" of "BCI," we could without loss take the conclusion of (5) to be α_1, since any α_i can then be permuted to the front of the antecedent of the premise of the rule.

Theorem 8.1 *For every $n, i \in \mathbb{N}$ such that $1 \leq i \leq n$, the rule (5) is admissible in* BCI. *In particular, the abelian rule (1) is admissible in* BCI. *These rules are not derivable in any logic that has classical logic among its extensions.*

Proof Admissibility is immediate from Theorem 7.1.

For nonderivability, let L be any logic that has the two-element Boolean algebra as a model. Taking the valuation $v(x_i) = 0$ for all i and $v(y) = 1$ in this algebra, we get $v((x_1 \to (x_2 \to \cdots (x_n \to y)\ldots)) \to y) = 1$, so (5) is not derivable in L for any $1 \leq i \leq n$. □

Somewhat more surprisingly, admissibility of (1) breaks down rather quickly above BCI, even if we stay below abelian logic.

Theorem 8.2 *Let L be an extension of monothetic* BCI *such that the formula $((x \to y) \to y) \to x$ is not a theorem of L. Then, (1) is not admissible for L.*

Proof Let $\alpha = ((x \to y) \to y) \to x$ and $\beta = x \to x$. Note that $\alpha \to \beta$ is provable already in BCI, in view of the following instance of B′:

$$\big(x \to ((x \to y) \to y)\big) \to \big(\big(((x \to y) \to y\big) \to x\big) \to (x \to x)\big),$$

whose antecedent is a well-known BCI theorem (often used in its schematic form as an alternative to the axiom-schema C in axiomatizing BCI). Thus *modus ponens* gives us its consequent, our formula $\alpha \to \beta$. As β is also BCI-provable, it follows that $(\alpha \to \beta) \to \beta$ is a theorem of monothetic BCI and hence of L. But, by assumption, α is not a theorem of L, as the admissibility of (1) would require. □

In particular, (1) is not admissible in monothetic BCI, BC(K ∨ A), or BCK. It was shown in [12] that a rule weaker than (1), namely,

$$\frac{(\beta \to \alpha) \to \alpha}{(\alpha \to \beta) \to \beta} \tag{6}$$

is admissible, but not derivable, in BCK.[9] Now, consider the rule

$$\frac{(((\alpha \to \beta) \to \beta) \to \alpha) \to \alpha}{(\alpha \to \beta) \to \beta}, \tag{7}$$

which is a slight strengthening of (6), having a weaker premise, but also a substitution instance of (1).

Lemma 8.3 *The rule (7) is admissible but not derivable in* BCK.

Proof To show admissibility, consider any formula $(((\alpha \to \beta) \to \beta) \to \alpha) \to \alpha$ that is provable in BCK. By Theorem 7.2, either $(\alpha \to \beta) \to \beta$ is a BCK theorem, or α is a BCK theorem. But, if α is a BCK theorem, so is $(\alpha \to \beta) \to \beta$, proving the claim.

To show nonderivability, take the algebra $\mathbf{H}_3 = (\{1, a, 0\}, \to, 1)$, where

$\to$	1	a	0
1	1	a	0
a	1	1	0
0	1	1	1

This algebra is the $\to$-reduct of the three-element Heyting algebra, so in particular a BCK-algebra; it is also easy to check directly that any evaluation maps the axioms of BCK to the value 1, and that for any evaluation, *modus ponens* preserves the property of being mapped to 1. Then, evaluating $v(x) = 0$ and $v(y) = a$, we obtain $v((((x \to y) \to y) \to x) \to x) = 1$ but $v((x \to y) \to y) = a$. □

The rule (7), unlike any other rule considered so far, has the property of being admissible in both BCK and $\mathsf{A}_\to$. Any such rule will be admissible in BC(K ∨ A), since BC(K ∨ A) = BCK ∩ $\mathsf{A}_\to$; if it is not derivable in BCK or in $\mathsf{A}_\to$, then it is clearly not derivable in BC(K ∨ A). Thus, we have the following.

Theorem 8.4 *Any rule admissible in* BCK *and in* $\mathsf{A}_\to$*, but not derivable in at least one of them, is admissible but not derivable in* BC(K ∨ A). *In particular, the rule (7) is such.*

In fact, the rule (7) is not only admissible, but even derivable in $\mathsf{A}_\to$, since both the premise and the conclusion are provably equivalent to α in that logic. This observation provides background for the following generalization of Theorem 8.4, for which we are grateful to an anonymous referee.

Theorem 8.5 *Any rule admissible in* BCK *and derivable in* $\mathsf{A}^+_\to$*, is admissible in* BC(K ∨ A)$^+$. *In particular, the rule (7) is such, and therefore* BC(K ∨ A)$^+$ *is not structurally complete.*

Proof Consider an inference rule

$$\frac{\psi}{\varphi_1, \ldots, \varphi_n}. \qquad (\dagger)$$

As BC(K ∨ A)$^+$ is algebraizable, we can use the algebraic criterion for admissibility in [16, Theorem 7.11(i)],[10] where τ is the defining equation $x = 1$. By [20], an arbitrary algebraic model of BC(K ∨ A)$^+$ has the form $\mathbf{B} \times \mathbf{G}$, where $\mathbf{B}$ is a BCK-algebra and $\mathbf{G}$ is an abelian group. Since (†) is admissible in BCK, [16, Theorem 7.11(i)] gives that $\mathbf{B}$ is a homomorphic image of a BCK-algebra $\mathbf{B}'$ that satisfies the quasi-identity corresponding to (†), namely,

$$\varphi_1 = 1 \;\&\; \cdots \;\&\; \varphi_n = 1 \Longrightarrow \psi = 1. \qquad (\ddagger)$$

Since (†) is derivable in $\mathsf{A}^+_\to$, all abelian groups satisfy (‡) as well. Therefore, (‡) holds in the algebra $\mathbf{B}' \times \mathbf{G}$ that is still an algebraic model of BC(K ∨ A)$^+$. Thus, by [16, Theorem 7.11(i)] again, (†) is admissible in BC(K ∨ A)$^+$ because $\mathbf{B} \times \mathbf{G}$ is a homomorphic image of $\mathbf{B}' \times \mathbf{G}$.

As (7) is not derivable in BCK, it is not derivable in BC(K ∨ A)$^+$ and so BC(K ∨ A)$^+$ is not structurally complete. □

Of the eight logics appearing in Figure 1, BCI*, BC(K ∨ A), and $\mathsf{A}_\to$ are structurally incomplete by the discussion at the end of Section 3. Of the remaining ones, BCI was shown to be structurally incomplete in [16], BCK in [12], and BC(K ∨ A)$^+$ by

Theorem 8.5 above. Our next result settles the question of structural completeness for $\mathsf{A}^+_{\rightarrow}$.

Theorem 8.6 *The logic* $\mathsf{A}^+_{\rightarrow}$ *is structurally incomplete.*

Proof Consider the rule

$$\frac{\alpha \rightarrow (\alpha \rightarrow (\beta \rightarrow \beta))}{\alpha \rightarrow (\beta \rightarrow \beta)}, \tag{8}$$

and suppose that $\alpha \rightarrow (\alpha \rightarrow (\beta \rightarrow \beta))$ is a theorem of $\mathsf{A}^+_{\rightarrow}$. By algebraizability, $\alpha \rightarrow (\beta \rightarrow \beta) = 1$ holds in the free countably generated abelian group $\mathbf{F}_\omega$. Further, as $\mathbf{F}_\omega \models \beta \rightarrow \beta = 0$, we have $\mathbf{F}_\omega \models \alpha + \alpha = 0$. But $\mathbf{F}_\omega$ is torsion-free, so $\alpha = 0$ in $\mathbf{F}_\omega$. By algebraizability again, α is then a theorem of $\mathsf{A}^+_{\rightarrow}$, and therefore so is $\alpha \rightarrow (\beta \rightarrow \beta)$. Thus, (8) is admissible in $\mathsf{A}^+_{\rightarrow}$. To see that it is not derivable, take $\mathbb{Z}_2$ and evaluate $v(\alpha) = 1$.[11] We get $v(\alpha \rightarrow (\alpha \rightarrow (\beta \rightarrow \beta))) = 0$, but $v(\alpha \rightarrow (\beta \rightarrow \beta)) = 1$. □

Thus all logics from Figure 1 turn out to be structurally incomplete, except BCI^+, for which the answer is not known.

Question 8.7 *Is* BCI^+ *structurally complete?*

The question would be answered negatively, if it turned out that the rule (8) were admissible in BCI^+, or indeed in BCI^*.

9 Closing Comments

To round out the discussion, we provide a BCK generalization of (6), analogous to the BCI generalization of (1) to (5). For every $n, i \in \mathbb{N}$ such that $1 \le i \le n$, consider the rule

$$\frac{(\alpha_1 \rightarrow (\alpha_2 \rightarrow \cdots (\alpha_n \rightarrow \beta) \ldots)) \rightarrow \beta}{(\beta \rightarrow \alpha_i) \rightarrow \alpha_i}, \tag{9}$$

which for $i = n = 1$ is just (6).

Theorem 9.1 *For every* $n, i \in \mathbb{N}$ *such that* $1 \le i \le n$*, the rule (9) is admissible but not derivable in* BCK.

Proof Admissibility is immediate from Theorem 7.2. For nonderivability evaluate $v(x_i) = a$ and $v(y) = 0$ in the algebra $\mathbf{H}_3$. □

The underivability of the BCI-admissible abelian rule (1), or more generally (5), shows that BCI is not structurally complete, something already known from [16, p. 28]. There, Olson, Raftery, and van Alten demonstrate, by an elaborate finite model property argument, that for this logic the rule (10) below is admissible though not derivable; note that the second and third premises just assert the equivalence of β with $\gamma \rightarrow (\alpha \rightarrow \beta)$:

$$\frac{(\alpha \rightarrow \beta) \rightarrow \beta \quad (\gamma \rightarrow (\alpha \rightarrow \beta)) \rightarrow \beta \quad \beta \rightarrow (\gamma \rightarrow (\alpha \rightarrow \beta))}{\beta \rightarrow (\alpha \rightarrow \beta)}. \tag{10}$$

Using our (5) we can simplify this rule considerably, dropping its first premise altogether. The resulting rule is of course underivable given that (10) is, and its admissibility is shown as follows. Suppose that (a) $(\gamma \rightarrow (\alpha \rightarrow \beta)) \rightarrow \beta$ and (b) $\beta \rightarrow (\gamma \rightarrow (\alpha \rightarrow \beta))$ are theorems of BCI. We can permute antecedents in (b) to get (b)′: $\gamma \rightarrow (\beta \rightarrow (\alpha \rightarrow \beta))$. From (a), the generalized abelian rule (5) delivers

γ as BCI-provable. But from this and (b)$'$, we have the conclusion of (10) by *modus ponens*.

Another natural question is whether the admissible but underivable rule (5) forms a basis for the admissible rules of BCI. Since (5) is a rule-schema rather than a single rule, to remain standard we will define it as an infinite family of rules (5_n), one for each positive integer n.

Question 9.2 *If* $(5_n)_{n \in \mathbb{N}}$ *are taken as additional primitive rules alongside modus ponens and the axioms of* BCI*, do all* BCI*-admissible rules then become derivable?*

We conclude our discussion with a remark about the status of the rules just labeled as (5_n). The $n = 1$ case, alias (1), our abelian rule, distinguishes itself among the rules considered here as the only one which is $\mathsf{A}_{\to}$-derivable, and we can put this into a wider perspective by considering not the consequence relation $\vdash_{\mathsf{A}_{\to}}$ but rather the associated generalized (or multiple-conclusion) consequence relation, which we may call $\Vdash_{\mathsf{A}_{\to}}$. We modify the definition (from note 6) of what it is for a consequence relation to be determined by a matrix in the obvious way: $\Vdash$ is determined by a matrix when for all sets of formulas Γ, Δ,[12] we have $\Gamma \Vdash \Delta$ if and only if on every evaluation on which every formula in Γ has a designated value, some formula in Δ has a designated value. Then, taking our cue from the fact that $\vdash_{\mathsf{A}_{\to}}$ is determined by the integer matrix with nonnegative integers as designated, let us just stipulate that $\Vdash_{\mathsf{A}_{\to}}$ is to be the generalized consequence relation determined by this same matrix. (This leaves open the question of how most conveniently to characterize $\Vdash_{\mathsf{A}_{\to}}$ proof-theoretically—perhaps with the aid of the generalized rules alluded to in Section 7.) Note that the generalized consequence relation coincides with the consequence relation when there is exactly one formula on the right; that is, $\Gamma \Vdash_{\mathsf{A}_{\to}} \alpha$ (more fastidiously: $\Gamma \Vdash_{\mathsf{A}_{\to}} \{\alpha\}$) if and only if $\Gamma \vdash_{\mathsf{A}_{\to}} \alpha$. The salient fact about abelian logic underlying the rule (5_n), making use of the abbreviative "$\alpha_1 \cdots \alpha_n \to$ __" notation from Section 6, is that for all formulas $\alpha_1, \ldots, \alpha_n, \beta$, we have

$$(\alpha_1 \cdots \alpha_n \to \beta) \to \beta \Vdash_{\mathsf{A}_{\to}} \alpha_1, \ldots, \alpha_n. \tag{11}$$

The justification for this claim is that where v is an evaluation into the matrix described above for $\mathsf{A}_{\to}$, put a_i for $v(\alpha_i)$ and b for $v(\beta)$. Suppose that v gives the formula on the left of the $\Vdash$ in (11) a designated value. Then, $b-(b-(a_1+\cdots+a_n)) = a_1 + \cdots + a_n \geq 0$. In that case, a_i must be nonnegative for at least one i, so at least one of the formulas on the right of the $\Vdash$ receives a designated value.

In the case in which $n = 1$, the multiplicity on the right of (11) disappears, giving us our abelian rule (1). But it is worth pausing to note that some cases of the more general version of that rule, (5), also arise as standard rather than generalized rules, when we have $\alpha_i = \alpha_j$ for distinct i, j, on the left of (11), the simplest of which would be the following:

$$\frac{(\alpha \to (\alpha \to \beta)) \to \beta}{\alpha}.$$

So these also deserve to be regarded as abelian rules admissible for BCI. Of course our principal point remains: whether abelian in this sense or not, all the sequential rules subsumed under (5) are BCI-admissible.

Notes

1. References and comparative terminological information on this may be found in Humberstone [7, note 11]. Section 2 of the same paper provides general historical information and conceptual clarification on the notion of structural completeness. Numerous papers have been devoted to the application of this notion among the "substructural" logics such as those of concern to us below; let us mention in particular Olson, Raftery, and van Alten [16] and Cintula and Metcalfe [5].

2. We are about to adopt the label $\mathsf{A}_{\to}$ for this logic; the "BCIA" nomenclature was chosen in [9] to emphasize that the logic in question was an extension of BCI, though the label involves some redundancy: as is reported in Meyer and Slaney [15], the C and I axioms are not independent, each being provable from B and A with the aid of *modus ponens*.

3. Since $\alpha \to t$ behaves like a ("De Morgan") negation, $\neg\alpha$, of α, and $\neg\alpha \to \beta$ is a standard definition of the connective known variously as "fission" or "multiplicative disjunction," dual to the fusion connective mentioned below, we should here recall (from [14]) that in abelian logic these connectives are equivalent (form equivalent compounds from any given pair of components, that is).

4. If $\vee$ were available as a connective, we could write this as a schema $\mathsf{K} \vee \mathsf{A}'$, where the schematic letters in A' have been changed from those in A above, so as not to overlap with those in K.

5. We owe the argument from this paragraph to an anonymous referee.

6. The consequence relations $\vdash_{\mathsf{A}_{\to}}$ and $\vdash_{\mathsf{A}^{+}_{\to}}$ are determined by these two matrices, respectively, where saying that $\vdash$ is *determined by* a matrix means that for all sets of formulas Γ and formulas α: $\Gamma \vdash \alpha$ if and only every matrix evaluation on which every $\gamma \in \Gamma$ receives a designated value is an evaluation on which α has a designated value.

7. A sequent calculus for abelian logic (in $\to$ and $\neg$) can be found in Paoli [17, p. 112] with a very different ($\to\Rightarrow$) rule (as well as different initial sequents), but we do not need this for present purposes, since we are interested in the current systems only to facilitate reasoning about what is provable in BCI and BCK as introduced in Section 3.

8. One may think of the role of fusion (mentioned briefly in Section 3) as being able to provide just such genuine subformulas as antecedents.

9. Nonderivability is shown in [12] algebraically. An alternative, more syntactical argument is implicit in [9, Exercise 4.22.11(i)]. If (6) were derivable in BCK, it would be admissible for all extensions of BCK, such as the implicational fragment of intuitionistic logic. To obtain that fragment, we may add the contraction schema $(\alpha \to (\alpha \to \beta)) \to (\alpha \to \beta)$ to BCK as axiomatized in Section 3: but (6) takes us from an instance of this schema, $(x \to (x \to y)) \to (x \to y)$ to $((x \to y) \to x) \to x$, which is Peirce's law, a well-known nontheorem of intuitionistic logic. In fact in [9], the discussion of (6) is put in terms of the commutativity of a binary connective $\ddot{\vee}$, where $\alpha \mathbin{\ddot{\vee}} \beta$ is defined as—or taken as primitive but stipulated to behave like—$(\alpha \to \beta) \to \beta$.

10. Discussed in greater generality in J. G. Raftery, *Admissible rules and the Leibniz hierarchy*, to appear in this journal.

11. For a (propositional) logical version of this nonderivability argument: consider the extension of $\vdash_{A^+_\to}$ to the consequence relation of the biconditional fragment (interpreting $\to$ as $\leftrightarrow$) of classical logic. The rule is no longer admissible here—take α, β, as x, y—and is therefore not derivable for $\vdash_{A^+_\to}$.

12. Note that Γ, Δ are back to being sets rather than multisets of formulas here.

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