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Algebraic Logic Perspective on Prucnal's Substitution

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Abstract A term $td(p,q,r)$ is called a *ternary deductive* (TD) term for a variety of algebras $\mathcal{V}$ if the identity $td(p,p,r) \approx r$ holds in $\mathcal{V}$ and $(\mathsf{c},\mathsf{d}) \in \theta(\mathsf{a},\mathsf{b})$ yields $td(\mathsf{a},\mathsf{b},\mathsf{c}) \approx td(\mathsf{a},\mathsf{b},\mathsf{d})$ for any $\mathcal{A} \in \mathcal{V}$ and any principal congruence θ on $\mathcal{A}$. A connective $f(p_1,\ldots,p_n)$ is called *td-distributive* if $td(p,q,f(r_1,\ldots,r_n)) \approx f(td(p,q,r_1),\ldots,td(p,q,r_n))$. If L is a propositional logic and $\mathcal{V}$ is a corresponding variety (algebraic semantic) that has a TD term td, then any admissible in L rule, the premises of which contain only td-distributive operations, is derivable, and the substitution $r \mapsto td(p,q,r)$ is a projective L-unifier for any formula containing only td-distributive connectives. The above substitution is a generalization of the substitution introduced by T. Prucnal to prove structural completeness of the implication fragment of intuitionistic propositional logic.

1 Introduction

In this paper, we study admissibility of structural inference rules in algebraizable (propositional) logics. A (structural inference) *rule* is an expression of the form $A_1,\ldots,A_n/B$, where $A_1,\ldots,A_n$ are (propositional) formulas called *premises* (of the rule) and B is a formula called the *conclusion* (of the rule). We recall that given a (propositional) logic L, a rule $A_1,\ldots,A_n/B$ is *admissible* in L if the logic L is closed under this rule; that is, for any substitution σ (of formulas for propositional variables), formula $\sigma(B)$ is valid in L as long as all formulas $\sigma(A_1),\ldots,\sigma(A_n)$ are valid in L. A substitution that simultaneously makes all formulas $A_1,\ldots,A_n$ valid in L is known as an L-*unifier* of formulas $A_1,\ldots,A_n$. An L-unifier of a single formula is a substitution that makes this formula valid in L. Thus, a rule is admissible in a

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given logic L if every L-unifier of its premises is an L-unifier of its conclusion. So, studying the admissibility of the rules in a logic L is, in a way, studying L-unifiers.

Let us also recall that a logic L has an algebraic semantic if there is a variety $\mathcal{V}(\mathsf{L})$ of algebras in the same signature as formulas of L and there is a translation ϵ which for each formula A gives a finite set of identities $\epsilon(A)$ in such a way that A is valid in L if and only if all identities from $\epsilon(A)$ hold in $\mathcal{V}(\mathsf{L})$. Thus, using ϵ we can translate a rule $A_1, \dots, A_n/B$ into a finite set of quasi-identities: $\epsilon(A_1), \dots, \epsilon(A_n) \Rightarrow \mathrm{i}; \mathrm{i} \in \epsilon(B)$.

In 1971, Pogorzelski [27] introduced a notion of a structurally complete logic, which is a logic where every admissible rule is derivable. In the same paper he noted that Prucnal had observed that even though intuitionistic propositional logic (IPL) is not structurally complete (e.g., the rule $\neg p \to (q \vee r)/(\neg p \to q) \vee (\neg p \to r)$ is admissible but not derivable in IPL; see Harrop [19]), all admissible in IPL rules containing formulas with implication as the only connective are derivable. In 1976, Mints [25, Theorems 1, 2] published the proof that any admissible in IPL (structural) rule that does not have occurrences of $\to$ or $\vee$ is derivable. Then in 1973, Prucnal extended his result about implications fragments of IPL to five different classes of logics (see [28]) and proved that implicational fragments of theses logics are structurally complete. To prove the claim, Prucnal used a substitution of the formulas of some special form. It turned out that this kind of substitution can be used for different classes of logics and nowadays it is called *Prucnal's substitution* or *Prucnal's trick*[1] (see Prucnal [28], [29], Slaney and Meyer [33], Olson, Raftery, and van Alten [26], Wojtylak [36], Dzik [13], Cintula and Metcalfe [9] to name a few).

In many cases, the variety $\mathcal{V}$ that is an algebraic semantic for a logic L has a ternary deductive (TD) term introduced in Blok and Pigozzi [5], that is, a term $td(p, q, r)$ such that for any algebra $\mathcal{A} \in \mathcal{V}$

$$td(\mathsf{a}, \mathsf{a}, \mathsf{b}) = \mathsf{b},$$
$$td(\mathsf{a}, \mathsf{b}, \mathsf{c}) = td(\mathsf{a}, \mathsf{b}, \mathsf{d}) \quad \text{if } (\mathsf{c}, \mathsf{d}) \in \theta(\mathsf{a}, \mathsf{b})$$

for any $\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}$ from $\mathcal{A}$. The distributive connectives (td-distributive for short) relative to the TD term td, that is, the td-distributive connectives for which

$$td\big(p, q, f(r_1, \dots, r_n)\big) \approx f\big(td(p, q, r_1), \dots, td(p, q, r_n)\big)$$

holds, play a special role. In this paper we show that any admissible in L rule, the premises of which contain only connectives distributive relative to the TD term, is derivable. To prove this, we are generalizing the idea used by Prucnal and we are using the following substitutions:

$$\sigma : r \mapsto td(p, q, r).$$

For many types of logics, Dzik obtained the results about structural completeness by using Prucnal's substitution in algebraic context. These results were presented at some conferences, but unfortunately they were never published. Dzik had informed the author about the handwritten manuscript by Wroński [37] that was circulating among logicians.

Generally speaking, there are two approaches to determine whether a logic L is structurally complete:

(a) algebraic: to show that a quasivariety generated by a Lindenbaum–Tarski algebra of L forms a variety;

(b) using the unifiers: to show that each formula A has a projective unifier;[2] that is, for each A there is a substitution σ such that $A \vdash_{\mathsf{L}} p \leftrightarrow \sigma(p)$.

We consider both approaches. First, we show that Prucnal's substitution is related to projectivity of reducts of algebras and this leads to admissibility of the rules of certain form. Second, we show that Prucnal's substitution is a projective unifier for formulas containing only *td*-distributive connectives (in [14], Dzik is using, for a similar purpose, a discriminator term which coincides with a TD term in the case of semisimple congruence permutable varieties).

2 Basic Definitions

2.1 Deductive systems and logics We consider a (propositional) language consisting of a set of (propositional) formulas Fm constructed in a regular way of (propositional) variables $\mathcal{P}$, where $\mathcal{P}$ is a countable set, and connectives $\mathcal{C} = \{f_1, \ldots, f_k\}$. Fm can be regarded as an absolutely free algebra. We will call the elements of Fm *formulas* or *terms* interchangeably. If $\mathcal{C}' \subseteq \mathcal{C}$, we will say that a formula A is $\mathcal{C}'$*-formula* if A contains connectives only from $\mathcal{C}'$.

A *substitution* is a mapping $\sigma : \mathcal{P} \mapsto \mathsf{Fm}$ that can be extended by $\sigma(A(p_1, \ldots, p_n)) = A(\sigma(p_1), \ldots, \sigma(p_n))$ to any formula A. Since Fm can be viewed as an absolutely free algebra, every substitution σ is an endomorphism of Fm, which we will also denote by σ. If Γ is a set of formulas, then $\sigma(\Gamma) := \{\sigma(A); A \in \Gamma\}$.

A *deductive system* is a couple $\mathsf{S} = \langle \mathsf{Fm}; \vdash \rangle$, where $\vdash$ is a binary (consequence) relation defined on finite sets of formulas and formulas, and $\vdash$ satisfies the following conditions: for any finite sets of formulas Γ, Δ and any formula A

(R) $A \vdash A$;
(M) if $\Gamma \vdash A$, then $\Gamma \cup \Delta \vdash A$;
(T) if $\Gamma \vdash A$ and $\Delta \vdash B$ for every $B \in \Gamma$, then $\Delta \vdash A$;
(S) if $\Gamma \vdash A$, then $\sigma(\Gamma) \vdash \sigma(A)$.

(Above and later we use the notation $\Gamma \vdash A$ instead of $\langle \Gamma; \{A\} \rangle \in \vdash$.)

If $\mathsf{S} = \langle \mathsf{Fm}; \vdash \rangle$ is a deductive system by $\mathsf{Th}(\mathsf{S})$ (or by $\mathsf{Th}(\vdash_{\mathsf{S}})$), we denote a set of *theorems*

$$\mathsf{Th}(\vdash) := \{A; \vdash A, A \in \mathsf{Fm}\}$$

(where $\vdash A$ means $\emptyset \vdash A$). If $\mathsf{S}_1 = \langle \mathsf{Fm}; \vdash_1 \rangle$ and $\mathsf{S}_2 = \langle \mathsf{Fm}; \vdash_2 \rangle$ are deductive systems and $\vdash_1 \subseteq \vdash_2$ (i.e., $\Gamma \vdash_1 A$ yields $\Gamma \vdash_2 A$), we will say that S_2 is an *extension* of S_1.

If S is a deductive system, then the set of its theorems $\mathsf{Th}(\mathsf{S})$ will be called *logic defined by* S.

A *structural inference rule* (*rule* for short) is an expression of type Γ/A, where Γ is a finite set of formulas and A is a formula. A rule Γ/A is called *admissible in a logic* $\mathsf{L} = \mathsf{Th}(\mathsf{S})$ if $\sigma(\Gamma) \subseteq \mathsf{L}$ yields $\sigma(A) \in \mathsf{L}$ for every substitution σ.

A deductive system S is said to be *structurally complete* (see, e.g., [27], Rybakov [32]) if any proper extension of S contains theorems not belonging to $\mathsf{Th}(\mathsf{S})$; that is, S is a maximal deductive system among the deductive systems defining the same logic. By adding to a given deductive system S all admissible in $\mathsf{Th}(\mathsf{S})$ rules, S can be extended to a structurally complete deductive system with the same logic—admissible closure of S (see Rybakov [32, p. 89]) or structural completion of S (see Humberstone [21, p. 1440]; see also Bergman [2, Proposition 1.2]).

If $\mathsf{S} = \langle \mathsf{Fm}; \vdash \rangle$ is a deductive system and $\mathsf{r} = \Gamma/A$ is a rule, then the rule r is called *derivable in* S if $\Gamma \vdash A$. It is not hard to see that a deductive system S is structurally complete if and only if every admissible in S rule is derivable in S.

A logic $\mathsf{L} = \mathsf{Th}(\mathsf{S})$ is *structurally complete* if any admissible in L rule is valid in any model where all the theorems of S are valid. (We will make this statement more precise in the following section.) Note that structural completeness of a deductive system is a property of a consequence relation, while structural completeness of a logic is a property of the set of theorems. For example, as it is well known (see, e.g., [32]), the IPL defined by a set of axiom schemata and a single inference rule modus ponens (see, e.g., Kleene [23]) is not structurally complete. But if we extend this deductive system by adding to it all of Visser's rules (see Iemhoff [22]), we get a structurally complete version of IPL (which cannot be achieved by adding any finite sets of rules; see Rybakov [31]). Thus, IPL as logic is not structurally complete, although it has a structurally complete deductive system defining it.

The structural completeness of a logic L means that the deductive system induced by all (algebraic) models of L is structurally complete. Very often it is a deductive system that admits some version of deduction theorem (see, e.g., the notion of a general deduction theorem in [32, Definition 5.1.3], the notion of a uniform deduction theorem scheme in Czelakowski [12, p. 371], or the notion of a TD term in [5, p. 568]). In this paper, we will be using TD terms (see the definition in the next section).

From this point forward, we will be concerned only with structural completeness of logics.

Let $\mathcal{C}' \subseteq \mathcal{C}$ be a subset of connectives, and let $\mathsf{L} = \mathsf{Th}(\mathsf{S})$ be a logic. We will say that a logic L is $\mathcal{C}'$-*structurally complete* if every admissible in L rule containing only $\mathcal{C}'$-formulas is derivable. A logic L is $\mathcal{C}'$-structurally complete if and only if any admissible in L rule containing only $\mathcal{C}'$-formulas is valid in any model of L (see [25]).

It is worth noting that the definition of $\mathcal{C}'$-structural completeness does not impose any restrictions on substitutions. For instance, we did not require that only the theorems that are $\mathcal{C}'$-formulas be considered. Thus, $\mathcal{C}'$-structural completeness of a logic and structural completeness of its $\mathcal{C}'$-fragment are not the same. Indeed, $\mathcal{C}'$-structural completeness of a logic L simply means that any admissible in L rule of certain form is derivable in L. On the other hand, structural completeness of a $\mathcal{C}'$-fragment of L is concerned with admissibility and derivability in a different logic, namely, in the logic that arises from L if we consider only $\mathcal{C}'$-formulas. In other words, the $\mathcal{C}'$-structural completeness is a restriction on the set of rules, while completeness of $\mathcal{C}'$-fragment is a completeness in a different logic (see [21]). For example (see [21, Digression, p. 56]), the rule $\mathsf{r} := \neg\neg p/p$ is not admissible in IPL, but it is admissible in the $\{\wedge, \neg\}$-fragment of IPL. The reason is simple: in IPL the set of available substitutions is richer and, for instance, we can substitute p with $\neg\neg p \to p$ and show that r is not admissible in IPL. Another example is that IPL is $\{\to, \neg\}$-structurally complete (see [25]; see also [32, Paragraph 5.5]) while its $\{\to, \neg\}$-fragment is not (see Cintula and Metcalfe [10]): the rule $(p \to \neg q), ((\neg\neg p \to p) \to r), ((\neg\neg q \to q) \to r)/r$ is admissible but not derivable in the $\{\to, \neg\}$-fragment of IPL. (It is clear that the above rule is not derivable either in IPL or in its $\{\to, \neg\}$-fragment, but in IPL this rule is not admissible because the substitution $q \mapsto (\neg p \wedge q), r \mapsto ((\neg\neg p \to p) \vee (\neg\neg(\neg p \wedge q) \to (\neg p \wedge q)))$ makes all premises valid in IPL while the conclusion is not.)

A logic $\mathsf{L} = \mathsf{Th}(\mathsf{S})$ is said to be *hereditarily structurally complete* (see Citkin [11], Rybakov [32]) or *deductive* (see [2]) if L and all its extensions are structurally complete. Accordingly, a logic L is *hereditarily $\mathcal{C}'$-structurally complete* if L and all its extensions are $\mathcal{C}'$-structurally complete.

2.2 Algebraic semantic In this paper, we consider only deductive systems that have an *algebraic semantic* (see Blok and Pigozzi [4]); that is, we assume that with each deductive system $\mathsf{S} = \langle \mathsf{Fm}; \vdash \rangle$ we can associate a quasivariety of algebras $\mathcal{Q}(\mathsf{S})$ (in signature $\mathcal{C}$) by associating with each formula $A \in \mathsf{Fm}$ a finite set of identities $\epsilon(A) = \{\epsilon_1(A) \approx \delta_1(A), \dots, \epsilon_m(A) \approx \delta_m(A)\}$ in such a way that $A_1, \dots, A_n \vdash A$ if and only if all quasi-identities $\epsilon(A_1), \dots, \epsilon(A_n) \Rightarrow \epsilon(A)$ hold in $\mathcal{Q}(\mathsf{S})$. (Here and later, in order to simplify notation, if $\epsilon_1, \dots, \epsilon_n, \epsilon$ are the finite sets of identities by $\epsilon_1, \dots, \epsilon_n \Rightarrow \epsilon$, we denote the set of quasi-identities $\{\epsilon_1, \dots, \epsilon_n \Rightarrow \mathsf{i}; \mathsf{i} \in \epsilon\}$.) We will say that $\mathcal{Q}(\mathsf{S})$ is a *corresponding quasivariety of* S and that ϵ is a *translation*.

As usual, if an identity i holds in some algebra $\mathcal{A}$, we will denote this by $\mathcal{A} \models \mathsf{i}$. And if ϵ is a set of identities by $\mathcal{A} \models \epsilon$, we will denote that all the identities from ϵ are valid in $\mathcal{A}$. If $\mathcal{K}$ is a class of algebras, by $\mathcal{V}(\mathcal{K})$ and $\mathcal{Q}(\mathcal{K})$ we denote, respectively, a variety and a quasivariety generated by algebras of $\mathcal{K}$. If $\mathcal{K}$ consists of a single algebra $\mathcal{A}$, we will write $\mathcal{V}(\mathcal{A})$ and $\mathcal{Q}(\mathcal{A})$ instead of $\mathcal{V}(\{\mathcal{A}\})$ and $\mathcal{Q}(\{\mathcal{A}\})$.

Using the translation from formulas into identities with any logic $\mathsf{L} = \mathsf{Th}(\mathsf{S})$ that has an algebraic semantic, we can associate a variety

$$\mathsf{V}(\mathsf{L}) = \left\{\mathcal{A}; \mathcal{A} \models \epsilon(A), A \in \mathsf{L}\right\};$$

that is, $\mathsf{V}(\mathsf{L})$ is a variety of all algebras in which all the identities corresponding to the theorems from L hold. By $\mathsf{V}(\mathsf{S})$ we denote the variety corresponding to its logic; that is, $\mathsf{V}(\mathsf{S}) = \mathsf{V}(\mathrm{Th}(\mathsf{S}))$. It is clear that $\mathsf{V}(\mathsf{S}) \supseteq \mathcal{Q}(\mathsf{S})$. We will say that $\mathsf{V}(\mathsf{L})$ is a *corresponding variety of* L or of a deductive system S for that matter.

Let us observe that, from an algebraic semantic standpoint, a deductive system S is structurally complete if $\mathcal{Q}(\mathsf{S}) = \mathsf{V}(\mathsf{L})$ (Example 1 shows that the converse is not true). Using the terminology from [2], we say that a *quasivariety $\mathcal{Q}$ is structurally complete* if it is a variety. On the other hand, a logic L is structurally complete if there is a unique quasivariety $\mathcal{Q}$ for which $\mathcal{V}(\mathcal{Q}) = \mathcal{V}(\mathsf{L})$. Recall that every quasivariety contains free algebras; hence, a logic is structurally complete if $\mathcal{V}(\mathsf{L}) = \mathcal{Q}(\mathcal{F}_\omega)$, where $\mathcal{F}_\omega$ is free in $\mathcal{V}(\mathsf{L})$ (or in $\mathcal{Q}(\mathsf{L})$ for that matter) algebra of countable rank (see [2]). So, we will say that a *variety $\mathcal{V}$ is structurally complete* if $\mathcal{V} = \mathcal{Q}(\mathcal{F}_\omega)$. We will also say that a *quasi-identity* q *is admissible in a variety $\mathcal{V}$* (or *$\mathcal{V}$-admissible*) if it is valid in $\mathcal{F}_\omega$, and we will say that q is *derivable in $\mathcal{V}$* (or *$\mathcal{V}$-derivable*) if q is valid in every algebra of $\mathcal{V}$. Thus, a variety $\mathcal{V}$ is structurally complete if and only if every admissible in $\mathcal{V}$ quasi-identity is derivable. Accordingly, we will say that a variety $\mathcal{V}$ is *hereditarily structurally complete* (or *primitive* in [17] or *deductive* in [20]) if $\mathcal{V}$ and all subvarieties of $\mathcal{V}$ are structurally complete; that is, all subquasivarieties of $\mathcal{V}$ are varieties. Thus, a logic L is hereditarily structurally complete if the corresponding variety is primitive (see [26, Corollary 7.15]).

Let $\underline{p}$ be a list of variables $p_1, \dots, p_n$. Strings of distinct variables are indicated by $\underline{p}, \underline{q}, \dots$, and if a term t contains variables only from the list $\underline{p} = p_1, \dots, p_n$, we express this fact by the notation $t(\underline{p})$ or $t(p_1, \dots, p_n)$. Accordingly, if $\mathsf{a}_1, \dots, \mathsf{a}_n$ are elements of an algebra, sometimes we will write $t(\underline{a})$ instead of $t(a_1, \ldots, a_n)$. By $|\underline{p}|$, we denote the *length* of $\underline{p}$, that is, the number of elements in $\underline{p}$: if $\underline{p} := p_1, \ldots, p_n$, then $|\underline{p}| = n$.

Example 1 For IPL in the signature $\{\to, \wedge, \vee, \mathbf{0}\}$, the corresponding variety is the variety of Heyting algebras, and if $A(\underline{p})$ is a formula, then $\epsilon(A)$ consists of a single identity; namely, $A(\underline{p}) \approx \mathbf{1}$, where $\mathbf{1}$ is an abbreviation for $\mathbf{0} \to \mathbf{0}$. If $\mathsf{r} := A_1, \ldots, A_m/B$ is a rule, then $\epsilon(\mathsf{r})$ consists of a single quasi-identity $A_1(\underline{p}) \approx \mathbf{1}, \ldots, A_m(\underline{p}) \approx \mathbf{1} \Rightarrow B(\underline{p}) \approx \mathbf{1}$. Let us note that if we use a different translation, say, $\epsilon(A) := A \to \mathbf{0} = \mathbf{0}$, then, by Glivenko's theorem, the same variety becomes a corresponding variety for classical propositional logic (CPL). Thus, depending on the translation, the same variety can be an algebraic semantic for different logics, and, as we will see, one of the logics can be structurally complete and another not.

Let $\mathcal{C}' \subseteq \mathcal{C}$ be a subset of connectives (or operations—we will use the terms *connective* and *operation* interchangeably). Then algebra $\langle \mathsf{A}; \mathcal{C}' \rangle$ is said to be a *reduct* of an algebra $\langle \mathsf{A}; \mathcal{C} \rangle$. We can extend our definition of structural completeness to reducts in the following way. We will say that a variety $\mathcal{V}$ is $\mathcal{C}'$-*structurally complete* if any admissible in $\mathcal{V}$ quasi-identity containing operations only from $\mathcal{C}'$ is derivable in $\mathcal{V}$. In other words, a variety $\mathcal{V}$ is $\mathcal{C}'$-structurally complete if every valid in $\mathcal{F}_\omega$ quasi-identity containing operations only from $\mathcal{C}'$ is derivable. Clearly, if the translation contains only $\mathcal{C}'$-terms, a logic L is $\mathcal{C}'$-structurally complete if the corresponding variety $\mathcal{V}(\mathsf{L})$ is $\mathcal{C}'$-structurally complete.

Example 2 Even though IPL and, hence, a variety $\mathcal{H}$ of Heyting algebras are not structurally complete, they are $\{\to, \wedge\}$-structurally complete (see [25]).

Now we will be focusing on studying the links between structural completeness of varieties and TD terms.

2.3 Retracts of algebras In this section, we give a definition of $\mathcal{C}'$-retract of an algebra, and we study some properties of $\mathcal{C}'$-retracts that are used later in the paper.

If $\mathcal{A}$ and $\mathcal{B}$ are algebras and $\mathcal{C}' \subseteq \mathcal{C}$ is a subset of principal operations, we say (cf. Grätzer [18]) that $\mathcal{B}$ is a $\mathcal{C}'$-*retract* of $\mathcal{A}$ if there is a monomorphism $\mu : \mathcal{B} \to \mathcal{A}$ of $\mathcal{C}'$-reduct of $\mathcal{B}$ into $\mathcal{C}'$-subreduct of $\mathcal{A}$ and an epimorphism $\varphi : \mathcal{A} \to \mathcal{B}$ such that $\varphi \circ \mu : \mathcal{B} \to \mathcal{B}$ is the identity. If θ is a congruence on an algebra $\mathcal{A}$ and quotient algebra $\mathcal{A}/\theta$ is a $\mathcal{C}'$-retract of $\mathcal{A}$, we say that θ is $\mathcal{C}'$-*retractable*.

We will need the following rather simple property of $\mathcal{C}'$-reducts.

Proposition 2.1 *Let $\mathcal{B}$ be a $\mathcal{C}'$-retract of $\mathcal{A}$, and let*

$$\mathrm{q} := \bigwedge_{i=1}^{m} t_i(\underline{x}) \approx t_i'(\underline{x}) \Rightarrow t(\underline{x}) \approx t'(\underline{x})$$

be a quasi-identity such that $\mathcal{B} \not\models \mathrm{q}$ and terms $t_i, t_i'; i = 1, \ldots, m$ contain operations only from $\mathcal{C}'$. Then $\mathcal{A} \not\models \mathrm{q}$.

Proof Let φ and μ be the mappings from the definition of $\mathcal{C}'$-retract. If $\mathcal{B} \not\models \mathrm{q}$, there are such elements of $\mathcal{B}$ that

$$t_i(\underline{b}) = t_i'(\underline{b}); \quad i = 1, \ldots, n \quad \text{while } t(\underline{b}) \neq t'(\underline{b}).$$

Letting $\underline{a} = \mu(\underline{b})$, let us check that

$$t_i(\underline{a}) = t_i'(\underline{a}); \quad i = 1, \dots, n \qquad \text{and} \qquad t(\underline{a}) \neq t'(\underline{a}).$$

Indeed, since μ is a monomorphism of $\mathcal{C}'$-retract and all terms t_i, t_i' contain operations only from $\mathcal{C}'$, we can conclude that

$$t_i(\underline{a}) = t_i'(\underline{a}); \quad i = 1, \dots, n,$$

because for computation of values of these terms we use only the operations from $\mathcal{C}'$.

On the other hand,

$$\begin{aligned}\varphi\big(t(\underline{a})\big) &= t\big(\varphi(\underline{a})\big) = t\big(\varphi\big(\mu(\underline{b})\big)\big) = t(\underline{b}) \neq t'(\underline{b}) \\ &= t'\big(\varphi\big(\mu(\underline{b})\big)\big) = t'\big(\varphi(\underline{a})\big) = \varphi\big(t'(\underline{a})\big).\end{aligned}$$

Recall that φ is a homomorphism; therefore $\varphi(t(\underline{a})) \neq \varphi(t'(\underline{a}))$ yields $t(\underline{a}) \neq t'(\underline{a})$. □

2.4 Ternary deductive term The notion of TD term was introduced in [5]. All the definitions and statements of this section regarding TD term can be found in [5].

If $\mathcal{A}$ is an algebra, $a \in \mathcal{A}$ is an element, and θ is a congruence, by $[a]_\theta$ we will denote a congruence class containing element a; that is, $[a]_\theta = \{b \in \mathcal{A}; a \equiv b \mod(\theta)\}$. If $\mathcal{A}$ is an algebra and $a, b \in \mathcal{A}$, by $\theta(a, b)$ we denote a *principal congruence* (see [18]) induced by elements a, b; that is, θ is the smallest congruence such that $a \equiv b \mod(\theta)$. A congruence θ on an algebra $\mathcal{A}$ is called *compact* in [18] (or *finitely generated* in Burris and Sankappanavar [8]) if θ is a finite join of principal congruences. This means that there is a finite set of pairs of elements $(a_i, b_i); i = 1, \dots, m$ and that θ is the smallest congruence on $\mathcal{A}$ such that $a_i \equiv b_i \mod(\theta)$ for all $i = 1, \dots, m$. If $\underline{a}$ and $\underline{b}$ are, respectively, the lists of elements $a_1, \dots, a_m$ and $b_1, \dots, b_m$, by $\theta(\underline{a}, \underline{b})$ we will denote the compact congruence generated by pairs $(a_1, b_1), \dots, (a_m b_m)$; that is, $\theta(\underline{a}, \underline{b}) = \bigvee \theta(a_i, b_i); i = 1, \dots, m$.

Let us observe the following property of compact congruences.

Proposition 2.2 *If a quasi-identity* q *is refutable in some quotient algebra $\mathcal{A}/\theta$ of an algebra $\mathcal{A}$, then there is a compact congruence θ' on $\mathcal{A}$ such that* q *is refutable in $\mathcal{A}/\theta'$.*

Proof Let $q := \bigwedge_{i=1}^{m} t_i(\underline{x}) \approx t_i'(\underline{x}) \Rightarrow t(\underline{x}) \approx t'(\underline{x})$, and let $\mathcal{A}/\theta \not\models q$. Then for some elements $a_1, \dots, a_n \in \mathcal{A}$, for all $i = 1, \dots, m$,

$$t_i(\underline{a}) \equiv t_i'(\underline{a}) \mod(\theta) \qquad \text{and} \qquad t(\underline{a}) \not\equiv t'(\underline{a}) \mod(\theta).$$

That is,

$$\big(t_i(\underline{a}), t_i'(\underline{a})\big) \in \theta \quad \text{while } \big(t(\underline{a}), t'(\underline{a})\big) \notin \theta.$$

Let θ' be a compact congruence generated by pairs $(t_i(\underline{a}), t_i'(\underline{a})); i = 1, \dots, m$. By the definition of compact congruence, we have $\theta' \subseteq \theta$; hence

$$t(\underline{a}) \not\equiv t'(\underline{a}) \mod(\theta').$$

On the other hand,

$$\big(t_i(\underline{a}), t_i'(\underline{a})\big) \in \theta' \quad \text{for all } i = 1, \dots, m;$$

therefore, the quasi-identity q is refutable in $\mathcal{A}/\theta'$. □

The following definition introduces the notion of a TD term that is central for this paper.

Definition 2.1 ([5, p. 568]) A *TD term* for an algebra $\mathcal{A}$ is a ternary term $td(p,q,r)$ such that, for any $\mathsf{a},\mathsf{b},\mathsf{c},\mathsf{d} \in \mathcal{A}$

$$\begin{aligned} td(\mathsf{a},\mathsf{a},\mathsf{b}) &= \mathsf{b}, \\ td(\mathsf{a},\mathsf{b},\mathsf{c}) &= td(\mathsf{a},\mathsf{b},\mathsf{d}) \quad \text{if } (\mathsf{c},\mathsf{d}) \in \theta(\mathsf{a},\mathsf{b}). \end{aligned} \tag{1}$$

If $\mathcal{K}$ is a class of similar algebras, $td(p,q,r)$ is a TD term for $\mathcal{K}$ if $td(p,q,r)$ is a TD term for every algebra from $\mathcal{K}$.

Let us note that, for a given variety, a TD term is not uniquely defined: the same variety may have different deduction terms.

Example 3 ([5, p. 548]) The variety of Heyting algebras has two different TD terms:

$$\begin{aligned} td_{\to}(p,q,r) &:= (p \to q) \to \big((q \to p) \to r\big), \\ td_{\wedge}(p,q,r) &:= (p \to q) \wedge (q \to p) \wedge r. \end{aligned} \tag{2}$$

Any variety having a TD term has equationally definable principal congruences:

$$(\mathsf{c},\mathsf{d}) \in \theta(\mathsf{a},\mathsf{b}) \quad \text{iff} \quad td(\mathsf{a},\mathsf{b},\mathsf{c}) = td(\mathsf{a},\mathsf{b},\mathsf{d}). \tag{3}$$

Thus, a TD term gives us a uniform way to define the principal congruences. By iterating the TD term, a (3)-like characterization of principal congruences can be extended to the compact congruences (see [5, Theorem 2.6]): if $\underline{\mathsf{a}}, \underline{\mathsf{b}}$ are lists of elements of an algebra $\mathcal{A}$, then

$$(\mathsf{c},\mathsf{d}) \in \theta(\underline{\mathsf{a}},\underline{\mathsf{b}}) \quad \text{iff} \quad td(\underline{\mathsf{a}},\underline{\mathsf{b}},\mathsf{c}) = td(\underline{\mathsf{a}},\underline{\mathsf{b}},\mathsf{d}), \tag{4}$$

where $td(\underline{\mathsf{a}},\underline{\mathsf{b}},c) := td(\mathsf{a}_1,\mathsf{b}_1,td(\mathsf{a}_2,\mathsf{b}_2,\dots,td(\mathsf{a}_m,\mathsf{b}_m,\mathsf{c}))\dots)$.

It is easily seen that, by the definition of TD term,

$$td(\underline{\mathsf{a}},\underline{\mathsf{a}},\mathsf{c}) = \mathsf{c}.$$

Let us also recall from [5, Theorem 2.3] that if f is an n-ary operation (connective) from $\mathcal{C}$, then

$$td\big(\mathsf{a},\mathsf{b},f(\mathsf{c}_1,\dots,\mathsf{c}_n)\big) = td\Big(\mathsf{a},\mathsf{b},f\big(td(\mathsf{a},\mathsf{b},\mathsf{c}_1),\dots,td(\mathsf{a},\mathsf{b},\mathsf{c}_n)\big)\Big). \tag{5}$$

As we will see, structural completeness of a variety $\mathcal{V}$ is related to a stronger version of the above property: if $td(p,q,r)$ is a TD term for $\mathcal{V}$ and $f \in \mathcal{C}$, we will say that f is *td-distributive* if the following identity holds in $\mathcal{V}$:

$$td\big(p,q,f(r_1,\dots,r_n)\big) \approx f\big(td(p,q,r_1),\dots,td(p,q,r_n)\big). \tag{6}$$

If td is a TD term of an algebra $\mathcal{A}$ in signature $\mathcal{C}$, by $\mathcal{C}(td)$ we denote the subset of all td-distributive operations from $\mathcal{C}$.

Taking into account that $td(\underline{p},\underline{q},r)$ is just an iteration of $td(p,q,r)$, by using a simple induction one can prove the following.

Proposition 2.3 *Given a variety $\mathcal{V}$ with a TD term td, if $f(r_1,\dots,r_n)$ is a td-distributive operation in $\mathcal{V}$ and $\underline{p},\underline{q}$ are lists of variables of the same length, then*

$$td\big(\underline{p},\underline{q},f(r_1,\dots,r_n)\big) \approx f\big(td(\underline{p},\underline{q},r_1),\dots,td(\underline{p},\underline{q},r_n)\big) \tag{7}$$

holds in $\mathcal{V}$.

Recall from Blok and Pigozzi [6, Corollary 2.4(2-1)] that for each TD term $td(p,q,r)$ and each term $C(r_1,\ldots,r_m)$, the following identity holds:

$$td\big(p,q,C(r_1,\ldots,r_m)\big) \approx td\big(p,q,C\left(td(p,q,r_1),\ldots,td(p,q,r_m)\right)\big).$$

So, if we take $C := r$, we get that the identity

$$td(p,q,r) \approx td\big(p,q,td(p,q,r)\big) \tag{8}$$

holds for every TD term.

In the following section (see Theorem 3.6), we will demonstrate that the td-distributive connectives are closely related to hereditary structural completeness.

3 TD Term And Structural Completeness

To study the connections between TD terms and structural $\mathcal{C}'$-completeness, we will first establish the connections between td-distributive operations and $\mathcal{C}(td)$-retracts.

3.1 TD term and retraction In this section, we prove the important lemma which is needed for the proof of the main theorem.

Lemma 3.1 *Let $\mathcal{A}$ be an algebra that has a TD term $td(p,q,r)$. Then every compact congruence on $\mathcal{A}$ is $\mathcal{C}(td)$-retractable.*

Proof Let θ be a compact congruence on $\mathcal{A}$. Then for some lists $\underline{\mathsf{a}}$ and $\underline{\mathsf{b}}$ of elements of $\mathcal{A}$, we have $\theta = \theta(\underline{\mathsf{a}},\underline{\mathsf{b}})$. Let us define the mapping

$$\mu : [\mathsf{c}]_\theta \mapsto td(\underline{\mathsf{a}},\underline{\mathsf{b}},\mathsf{c}) \tag{9}$$

and verify that the mapping $\mu : \mathcal{A}/\theta \to \mathcal{A}$ is indeed a $\mathcal{C}(td)$-retraction. So, we need to show that

(a) μ is a monomorphism of $\mathcal{C}(td)$-reduct of $\mathcal{A}/\theta$ into $\mathcal{C}(td)$-subreduct of $\mathcal{A}$;
(b) $\varphi \circ \mu$, where $\varphi : \mathcal{A} \to \mathcal{A}/\theta$ is a natural homomorphism, that is, the identity relation.

First, let us observe that our definition of μ is consistent: if $\mathsf{d} \in [\mathsf{c}]_\theta$, then, by definition of td and (4), we have $td(\underline{\mathsf{a}},\underline{\mathsf{b}},\mathsf{c}) = td(\underline{\mathsf{a}},\underline{\mathsf{b}},\mathsf{d})$; that is, the value of φ does not depend on the selection of a particular member of a congruence class.

(a) Next, let us check that μ is a homomorphism. Indeed, assume that

$$f(p_1,\ldots,p_n) \in \mathcal{C}(td) \qquad \text{and} \qquad \mathsf{c}_1,\ldots,\mathsf{c}_n \in \mathcal{A}.$$

Then, by the definition of μ,

$$\mu\big(f\left([\mathsf{c}_1]_\theta,\ldots,[\mathsf{c}_n]_\theta\right)\big) = td(\underline{\mathsf{a}},\underline{\mathsf{b}},\mathsf{d}),$$

where d is an element from $[f(\mathsf{c}_1,\ldots,\mathsf{c}_n)]_\theta$. As we saw, in order to define the value of μ, we can take any element from $[f(\mathsf{c}_1,\ldots,\mathsf{c}_n)]_\theta$. Recall that θ is a congruence and, hence, if we take

$$\mathsf{d} = f(\mathsf{c}_1,\ldots,\mathsf{c}_n),$$

then

$$\mathsf{d} \in \big[f(\mathsf{c}_1,\ldots,\mathsf{c}_n)\big]_\theta.$$

Thus,

$$\mu\big(f\left([\mathsf{c}_1]_\theta,\ldots,[\mathsf{c}_n]_\theta\right)\big) = td(\underline{\mathsf{a}},\underline{\mathsf{b}},\mathsf{d}) = td\big(\underline{\mathsf{a}},\underline{\mathsf{b}},f(\mathsf{c}_1,\ldots,\mathsf{c}_n)\big).$$

Since f is td-distributive, by (7)

$$td(\underline{a}, \underline{b}, f(c_1, \dots, c_n)) = f\left(td(\underline{a}, \underline{b}, c_1), \dots, td(\underline{a}, \underline{b}, c_n)\right) = f\left(\mu([c_1]_\theta), \dots, \mu([c_n]_\theta)\right).$$

Therefore,

$$\mu(f([c_1]_\theta, \dots, [c_n]_\theta)) = f\left(\mu([c_1]_\theta), \dots, \mu([c_n]_\theta)\right).$$

Hence, μ is a homomorphism.

We also need to prove that μ is a one-to-one correspondence. Indeed, by (4) $[c]_\theta = [d]_\theta$ if and only if $td(\underline{a}, \underline{b}, c) = td(\underline{a}, \underline{b}, d)$; hence, $[c]_\theta = [d]_\theta$ if and only if $\mu(c) = \mu(d)$.

(b) Let us verify that $\varphi \circ \mu$ is the identity relation. Let $[c]_\theta$ be an element of $\mathcal{A}/\theta$. Then

$$\mu([c]_\theta) = td(\underline{a}, \underline{b}, c) \qquad \text{and} \qquad \varphi(\mu([c]_\theta)) = \left[td(\underline{a}, \underline{b}, c)\right]_\theta.$$

Thus, we need to verify that

$$[c]_\theta = \left[td(\underline{a}, \underline{b}, c)\right]_\theta. \tag{10}$$

Recall that θ is a congruence defined by $td(p, q, r)$; that is, by the definition of TD term, (10) is equivalent to

$$td(\underline{a}, \underline{b}, c) = td(\underline{a}, \underline{b}, td(\underline{a}, \underline{b}, c)),$$

and application of (8) completes the proof. □

3.2 $\mathcal{C}'$-structural completeness Now we can prove the main theorem.

Theorem 3.2 *Let $\mathcal{V}$ be a variety with TD term td. Then every $\mathcal{V}$-admissible quasi-identity whose premises contain only connectives from $\mathcal{C}(td)$ is $\mathcal{V}$-derivable.*

Proof Let $\mathcal{F}_\omega$ be a free algebra of $\mathcal{V}$ of countable rank. We need to demonstrate that any quasi-identity whose premises contain operations only from $\mathcal{C}(td)$ and is valid in $\mathcal{F}_\omega$ (i.e., is $\mathcal{V}$-admissible) is valid in any algebra of $\mathcal{V}$ (i.e., is $\mathcal{V}$-derivable).

Indeed, assume that q is a quasi-identity that is not $\mathcal{V}$-derivable. Then q is refutable in some countable algebra $\mathcal{A} \in \mathcal{V}$. Any countable algebra from $\mathcal{V}$ is a homomorphic image of the free algebra $\mathcal{F}_\omega$ of countable rank. Therefore, for some congruence θ on $\mathcal{F}_\omega$, we have $\mathcal{A} \cong \mathcal{F}_\omega/\theta$; that is, $\mathcal{F}_\omega/\theta \not\models$ q. Then, by virtue of Proposition 2.2, there is a compact congruence θ' on $\mathcal{F}_\omega$ such that $\mathcal{F}_\omega/\theta' \not\models$ q. In turn, by virtue of Lemma 3.1, $\mathcal{F}_\omega/\theta'$ is a $\mathcal{C}(td)$-retract of $\mathcal{F}_\omega$. Now we can apply Proposition 2.1 and conclude that $\mathcal{F}_\omega \not\models$ q; that is, quasi-identity q is not $\mathcal{V}$-admissible. □

As an immediate consequence, we obtain the following.

Corollary 3.3 *Every variety with TD term td is $\mathcal{C}(td)$-structurally complete.*

Proof By the definition of $\mathcal{C}(td)$-structural completeness, the premises of quasi-identities that we are considering contain only td-distributive operations, and we can apply Theorem 3.2. □

Since any TD term for a variety is a TD term for each subvariety, we can repeat the proof of the theorem by using a free algebra of a subvariety and obtain the following corollary.

Corollary 3.4 *Every variety with TD term td is $\mathcal{C}(td)$-primitive.*

3.3 $\mathcal{C}'$-structural completeness for logics For logics, Theorem 3.2 can be rephrased in the following way.

Theorem 3.5 *Let* L *be a logic whose corresponding variety has a TD term td. Given an admissible in* L *rule* r*, if the premises of the translation* $\epsilon(\mathsf{r})$ *contain only connectives from* $\mathcal{C}(td)$*, then* r *is derivable.*

Corollary 3.6 *Let* L *be a logic whose corresponding variety has a TD term td. If the translations of any formula contain only td-distributive connectives, then logic* L *is hereditarily* $\mathcal{C}(td)$*-structurally complete. Particularly, if all connectives are td-distributive, then* L *is hereditarily structurally complete.*

As we saw in Example 3, a variety may have more than one TD term. Each of these terms may have different sets of *td*-distributive connectives.

Let us look at some applications of Theorem 3.2.

- Let us consider IPL in the signature $\to, \wedge, \vee, \mathbf{0}$. It is easy to see that connectives $\to$ and $\wedge$ are $td_{\to}$-distributive (where $td_{\to}$ is defined in (2)); hence, by virtue of Theorem 3.2, IPL is hereditarily $\{\to, \wedge\}$-structurally complete (see [25]).
- Brouwerian semilattices $\langle \mathcal{A}; \cdot, \to, \mathbf{1}\rangle$ have the same TD term as Hilbert algebras (see [5]):
$$td(p, q, r) := (p \to q) \to \big((q \to p) \to r\big).$$
It is easy to see that all three operations are compatible with this TD term; hence the varieties of Brouwerian semilattices and Hilbert algebras are hereditarily structurally complete.
- Recall from [5, Theorem 2.8] that any discriminator[3] variety has a TD term; in fact, the discriminator is also a TD term. Thus, for any set $\mathcal{C}'$ of *td*-distributive operations, discriminator varieties are hereditarily $\mathcal{C}'$-structurally complete (see [14]).

Remark 3.1 Let us note that the first example states hereditary $\{\wedge, \to\}$-completeness of IPL (the structural $\{\wedge, \to\}$-completeness of IPL was first observed by Mints [25]), while the second example states the hereditary structural completeness of the $\{\wedge, \to\}$-fragment of IPL that was first observed by Prucnal [28].

It is worth noting that, for structural completeness, it is irrelevant whether all the connectives from TD term are *td*-distributive, while it is crucial that all the connectives occurring in translation are *td*-distributive. For instance, in IPL the connectives $\wedge, \vee, \mathbf{0}$ are $td_{\wedge}$-distributive (where $td_{\wedge}$ was defined in (2)), but we cannot claim the hereditary $\{\wedge, \vee, \mathbf{0}\}$-structural completeness because the translation $\epsilon(A)$ is $A = \mathbf{1}$ and the constant $\mathbf{1}$ is not $td_{\wedge}$-distributive.

3.4 Application to hoops Hoops were introduced in a manuscript by Büchi and Owens [7] in the 1970s. Later, hoops were extensively studied by Blok and Pigozzi [5], Blok and Ferreirim [3], and Ferreirim [15]. Hoops capture a common $\{\wedge, \to\}$-fragment of many logics including all fuzzy logics. Many of the familiar logics have enriched hoops as an algebraic semantic. For instance, the following algebras can be regarded as hoops enriched with additional operations: modal algebras, cylindric algebras, relation algebras, Heyting algebras, Wajsberg algebras, De Morgan algebras, and so on.

Let us consider the deductive system in the signature $\to, \cdot, \mathbf{1}$ and defined by the following axiom schemata (see Raftery and van Alten [30])

(a) $(p \to q) \to ((r \to p) \to (r \to q))$,
(b) $(p \to (q \to r)) \to (q \to (p \to r))$,
(c) $p \to (q \to p)$,
(d) $p \to (q \to (p \cdot q))$,
(e) $(p \to (q \to r)) \to ((p \cdot q) \to r)$

and the only inference rule modus ponens: (MP) $p, p \to q/q$. The variety corresponding to this deductive system is a variety of hoops (see [5]) defined by identities

(i) $\mathbf{1} \cdot p \approx p \cdot \mathbf{1} \approx p$,
(ii) $p \cdot q \approx q \cdot p$,
(iii) $p \to p \approx \mathbf{1}$,
(iv) $(p \to q) \cdot p \approx (q \to p) \cdot q$,
(v) $p \to (q \to r) \approx (p \cdot q) \to r$.

We will use the following abbreviations defined by induction:

$$p^1 := p \quad \text{and} \quad p^n := p^{n-1} \cdot p \quad \text{for each } n > 1;$$
$$p \xrightarrow{1} q := p \to q \quad \text{and}$$
$$p \xrightarrow{n} q := p \to (p \xrightarrow{n-1} q) \quad \text{for each } n > 1.$$

Let us note that the following identity holds in every hoop:

$$p \xrightarrow{n} q \approx p^n \to q.$$

An element a of a hoop is called n-*potent* if $\mathsf{a}^{n+1} = \mathsf{a}^n$. A 1-potent element is referred to as *idempotent*. A hoop is said to be n-*potent* (see, e.g., [3]) if for a given n it satisfies the following identity:

$$p^{n+1} \approx p^n. \tag{11}$$

The above identity is equivalent to

$$p \xrightarrow{n+1} q \approx p \xrightarrow{n} q. \tag{12}$$

Let us recall (see [5]) that the set of all n-potent hoops forms a variety with a TD term

$$td(p, q, r) := (p \to q) \xrightarrow{n} \big((q \to p) \xrightarrow{n} r\big).$$

Theorem 3.7 *In each n-potent hoop operations $\to, \cdot$ and $\mathbf{1}$ are td-distributive with respect to the above TD-term.*

Proof Let us start by showing that $\mathbf{1}$ is *td*-distributive. Indeed, in each hoop the following identity holds (see [3]):

$$p \to \mathbf{1} \approx \mathbf{1}.$$

Hence,

$$td(p, q, \mathbf{1}) \approx (p \to q) \xrightarrow{n} \big((q \to p) \xrightarrow{n} \mathbf{1}\big) \approx (p \to q) \xrightarrow{n} \mathbf{1} = \mathbf{1},$$

and, therefore, $\mathbf{1}$ is *td*-distributive.

Now, let us note that

$$\begin{aligned} td(p,q,r) &\approx (p \to q) \xrightarrow{n} \big((q \to p) \xrightarrow{n} r\big) \\ &\approx (p \to q)^n \to \big((q \to p)^n \to r\big) \\ &\approx \big((p \to q)^n \cdot (q \to p)^n\big) \to r. \end{aligned} \tag{13}$$

Observe that in every n-potent hoop, any element a^n is an idempotent: by using the associativity of $\cdot$ and applying n times (11) to $a^n \cdot a^n$, we can obtain $a^n \cdot a^n = a^n$. Moreover, it is not hard to see that, since the operation $\cdot$ is commutative, if a and b are idempotent elements, the element $a \cdot b$ is also idempotent.

Let us recall (see [5], [15], Veroff and Spinks [35]) that for every n-potent hoop $\langle A; \to, \cdot, \mathbf{1} \rangle$ and any idempotent element $e \in A$,

$$e \to (a \to b) = (e \to a) \to (e \to b) \tag{14}$$

and

$$e \to (a \cdot b) = (e \to a) \cdot (e \to b). \tag{15}$$

Now, using (13), (14), and (15), we get

$$\begin{aligned} &td(p,q,r_1 \to r_2) \\ &\quad\approx \big((p \to q)^n \cdot (q \to p)^n\big) \to (r_1 \to r_2) \\ &\quad\approx \big(\big((p \to q)^n \cdot (q \to p)^n\big) \to r_1\big) \to \big(\big((p \to q)^n \cdot (q \to p)^n\big) \to r_2\big) \\ &\quad\approx td(p,q,r_1) \to td(p,q,r_2) \end{aligned}$$

and

$$\begin{aligned} &td(p,q,r_1 \cdot r_2) \\ &\quad\approx \big((p \to q)^n \cdot (q \to p)^n\big) \to (r_1 \cdot r_2) \\ &\quad\approx \big(\big((p \to q)^n \cdot (q \to p)^n\big) \to r_1\big) \cdot \big(\big((p \to q)^n \cdot (q \to p)^n\big) \to r_2\big) \\ &\quad\approx td(p,q,r_1) \cdot td(p,q,r_2). \end{aligned}$$

Thus, we can conclude that operations $\to$ and $\cdot$ are *td*-distributive. □

Corollary 3.8 *A variety of n-potent hoops with additional operations is $\mathcal{C}'$-structurally complete for any set $\mathcal{C}'$ of td-distributive operations. In particular, if all the additional operations are td-distributive, the variety is primitive and the corresponding logic is hereditarily structurally complete.*

3.5 Prucnal's substitutions in the algebraic setting Let us start with an observation that in the proof of Lemma 3.1 we were using a mapping

$$\mu : [c]_\theta \mapsto td(a, b, c).$$

As we saw, for n-potent hoops

$$td(p,q,r) := (p \to q) \xrightarrow{n} \big((q \to p) \xrightarrow{n} r\big)$$

is a TD term. Or, taking into account that in the hoops the identity

$$p \to (q \to r) \approx p \cdot q \to r$$

holds, we can use a different TD term

$$td_0(p,q,r) := (p \to q) \cdot (q \to p) \xrightarrow{n} r.$$

From the fact that td_0 is a TD term and that the following identity holds in the hoops

$$(p \to q) \cdot (q \to p) \approx \big((p \to q) \cdot (q \to p)\big) \cdot \mathbf{1},$$

we can conclude that in any n-potent hoop, $\theta(\mathsf{a}, \mathsf{b}) = \theta((\mathsf{a} \to \mathsf{b}) \cdot (\mathsf{b} \to \mathsf{a}), \mathbf{1})$ for any elements a, b. Thus, every compact congruence is defined by a single element. Recall that $\mathbf{1} \to p \approx p$ and $p \cdot \mathbf{1} \approx p$ hold in every hoop. So, we can consider

$$pr(p, r) = td_0(p, \mathbf{1}, r) = (p \to \mathbf{1}) \cdot (\mathbf{1} \to p) \overset{n}{\to} r = p \overset{n}{\to} r$$

and $pr(\mathsf{a}, \mathsf{b}) = pr(\mathsf{a}, \mathsf{c})$ if and only if $\mathsf{b} \equiv \mathsf{c} \mod(\theta(\mathsf{a}, \mathbf{1}))$.

Consequently, if we take a free algebra and regard its elements as formulas (terms) and if we take a formula A that defined a congruence, we are actually using a mapping

$$\varphi : p \mapsto (A \overset{n}{\to} p),$$

which is exactly a substitution invented by Prucnal (see [28], [29]).

Let us now consider a case of the simplest translation, namely, the case when $\epsilon(A) := (A = \mathbf{1})$. For instance, intermediate and normal modal logics admit such a translation. If we deal with the logics that have n-potent enriched hoops as their algebraic semantic and admit the above translation, we can formulate the following corollary from Theorem 3.2.

Corollary 3.9 (see [29, Theorem 1]) *Let* L *be a logic having n-potent hoops (enriched with operations* $\mathcal{C}$*) as an algebraic semantic, and let the translation be* $\epsilon(A) := (A = \mathbf{1})$. *Then*

- *(a) if* $\mathcal{C}' \subseteq \mathcal{C}$ *is a set of pr-distributive operations, then* L *is hereditarily* $\mathcal{C}'$*-structurally complete;*
- *(b) if all additional operations are pr-distributive, then* L *is hereditarily structurally complete.*

Theorem 2 in [29] immediately follows from the above corollary: all the logics considered in [29, Theorem 2] have n-potent hoops with additional pr-distributive operations as their algebraic semantic.

Remark 3.2 One can repeat the arguments used in this section for enriched BCI-monoids (see Agliano [1]) instead of hoops and obtain the results regarding structural $\mathcal{C}(td)$-completeness for different many-valued logics.

4 Projectivity and Unification

As we mentioned in the Introduction, the study of structural completeness is closely related to the study of projective unifiers. In this section, we show that Prucnal's substitution is a projective unifier for any pair of formulas containing only td-distributive connectives.

4.1 Unification Recall from Mal'cev [24] that given a class of algebras $\mathcal{K}$, a finite set of equalities Γ, and an equality $A \approx B$, an equality $A \approx B$ is said to be a $\mathcal{K}$ *consequence of* Γ (in symbols, $\Gamma \models_{\mathcal{K}} A \approx B$) if a quasi-identity $\Gamma \Rightarrow A \approx B$ holds in $\mathcal{K}$. In other words, if every valuation in algebras from $\mathcal{K}$ makes $A \approx B$ true every time, it makes true all the equalities from Γ. A substitution σ is called a $\mathcal{K}$*-unifier* of $A \approx B$ (and we will omit $\mathcal{K}$ when no confusion arises) if the equality $\sigma(A) \approx \sigma(B)$

is valid in every algebra from $\mathcal{K}$; that is, $\models_{\mathcal{K}} \sigma(A) \approx \sigma(B)$. A $\mathcal{K}$-unifier is called *projective* (see Ghilardi [16]) if

$$A \approx B \models_{\mathcal{K}} p \approx \sigma(p) \tag{16}$$

for every variable occurring in A, B. In other words, a unifier σ is projective if $A \approx B \Rightarrow p \approx \sigma(p)$ holds in $\mathcal{K}$ for all variables p occurring in A, B.

Theorem 4.1 *Suppose that $\mathcal{V}$ is a variety and that $td(p,q,r)$ is its TD term. Suppose that formulas $A(\underline{p})$, $B(\underline{p})$ contain only td-distributive connectives. Then Prucnal's substitution*

$$\sigma : r \to td\big(A(\underline{p}), B(\underline{p}), r\big)$$

is a projective unifier for $A(\underline{p}) \approx B(\underline{p})$.

Proof To prove that σ is a projective unifier, we need to prove that

(a) $\sigma(A(\underline{p})) \approx \sigma(B(\underline{p}))$ is valid in every algebra from $\mathcal{V}$; that is, σ is a unifier;
(b) a quasi-identity $A(\underline{p}) \approx B(\underline{p}) \Rightarrow r \approx \sigma(r)$ is valid in every algebra from $\mathcal{V}$; that is, σ is projective.

(a) Let $\underline{p} := p_1, \dots, p_n$. Since $\sigma(r) = td(A, B, r)$, we have

$$\sigma\big(A(\underline{p})\big) \approx A\big(\sigma(\underline{p})\big) \approx A\big(td(A, B, p_1), \dots, td(A, B, p_n)\big).$$

Recall that all the connectives in A, B are td-distributive. Hence, using a simple induction on the number of connectives in A and (6), we obtain

$$A\big(td(A, B, p_1), \dots, td(A, B, p_n)\big) \approx td\big(A, B, A(p_1, \dots, p_n)\big) = td(A, B, A).$$

Thus,

$$\sigma\big(A(\underline{p})\big) \approx td(A, B, A) \qquad \text{and} \qquad \sigma\big(B(\underline{p})\big) \approx td(A, B, B).$$

Because every variety is closed under homomorphisms, we can use the identity

$$td(p, q, p) \approx td(p, q, q)$$

that holds due to [5, Theorem 2.3(iii)], and we can get

$$td(A, B, A) \approx td(A, B, B).$$

Thus, σ is a $\mathcal{V}$-unifier.

(b) Assume that $\mathcal{A} \in \mathcal{V}$ and $\underline{\mathsf{a}} := \mathsf{a}_1, \dots, \mathsf{a}_n$, where $\mathsf{a}_i \in \mathcal{A}; i = 1, \dots, n$, are elements such that $A(\underline{\mathsf{a}}) = B(\underline{\mathsf{a}})$. We need to demonstrate that for every $\mathsf{a} \in \underline{\mathsf{a}}$,

$$\mathsf{a} = td(A, B, \mathsf{a}).$$

Without losing generality, we can assume that $\mathcal{A}$ is a finitely generated algebra; moreover, we can assume that $\mathcal{A}$ is generated by elements $\underline{\mathsf{a}} = \mathsf{a}_1, \dots, \mathsf{a}_n$. Let us take an algebra $\mathcal{F}_n \in \mathcal{V}$ freely generated by elements $\mathsf{g}_1, \dots, \mathsf{g}_n$ and consider a mapping $\mathsf{g}_i \mapsto \mathsf{a}_i; i = 1, \dots, n$. By properties of free algebras, this mapping can be extended to a homomorphism $\varphi : \mathcal{F}_n \to \mathcal{A}$. Let θ be a kernel congruence of φ, and let $\mathsf{a}' = A(\underline{\mathsf{g}})$ and $\mathsf{b}' = B(\underline{\mathsf{g}})$. Since $A(\underline{\mathsf{a}}) = B(\underline{\mathsf{a}})$, that is, $\varphi(\mathsf{a}') = \varphi(\mathsf{b}')$, we have $[\mathsf{a}']_\theta = [\mathsf{b}']_\theta$. Hence, if $\theta' = \theta(\mathsf{a}', \mathsf{b}')$, then $\theta' \subseteq \theta$, and if we show that $[\mathsf{g}_i]_{\theta'} = [\sigma(\mathsf{g}_i)]_{\theta'}$ for all $i = 1, \dots, n$, we will be able to complete the proof.

By definition of TD term, since θ' is a principal congruence, we have

$$[\mathsf{a}]_{\theta'} = [\mathsf{b}]_{\theta'} \quad \text{iff} \quad td(\mathsf{a}', \mathsf{b}', \mathsf{a}) = td(\mathsf{a}', \mathsf{b}', \mathsf{b}).$$

Therefore,

$$[\mathsf{g}_i]_{\theta'} = \big[td(\mathsf{a}', \mathsf{b}', \mathsf{b})\big]_{\theta'} \quad \text{iff} \quad td(\mathsf{a}', \mathsf{b}', \mathsf{g}_i) = td\big(\mathsf{a}', \mathsf{b}', td(\mathsf{a}', \mathsf{b}', \mathsf{g}_i)\big).$$

Let us recall from [5, Corollary 2.4(2-1)] that for each TD term and each term $C(r_1, \dots, r_m)$, the following identity holds:

$$td\big(p, q, C(r_1, \dots, r_m)\big) \approx td\big(p, q, C\big(td(p, q, r_1), \dots, td(p, q, r_m)\big)\big).$$

So, if we take $C := r$, we have

$$td(p, q, r) \approx td\big(p, q, td(p, q, r)\big),$$

and hence

$$td(\mathsf{a}', \mathsf{b}', \mathsf{g}_i) = td\big(\mathsf{a}', \mathsf{b}', td(\mathsf{a}', \mathsf{b}', \mathsf{g}_i)\big)$$

holds for all $i = 1, \dots, n$. Therefore, for all $i = 1, \dots, n$,

$$[\mathsf{g}_i]_{\theta'} = \big[\sigma(\mathsf{g}_i)\big]_{\theta'},$$

and this completes the proof of the theorem. □

Let us note that Corollary 3.3 immediately follows from the above theorem.

5 Final Remarks

5.1 Compatible connectives Recall from [5] and [1] that a connective (operation) f is called *compatible* (in an algebra $\mathcal{A}$) if an algebra $\mathcal{A}'$ obtained from $\mathcal{A}$ by adding f to its signature has the same set of congruences. Given a variety $\mathcal{V}$, a connective is called *compatible* in $\mathcal{V}$ if it is compatible in each algebra from $\mathcal{V}$. Thus if $\mathcal{V}$ is a variety of algebras in the signature $\mathcal{C}$ with TD term td and td-distributive connectives $td(\mathcal{C})$, adding the compatible connectives to $\mathcal{C}$ preserves the TD term and $td(\mathcal{C})$-hereditary completeness.

Note that if $\mathcal{V}'$ is a variety obtained from $\mathcal{V}$ by adding compatible connectives to the signature, the variety $\mathcal{V}'$ will be hereditarily $\mathcal{C}'$-structurally complete as long as the variety $\mathcal{V}$ is hereditarily $\mathcal{C}'$-structurally complete. That is, any axiomatic extension of a hereditarily $\mathcal{C}'$-structurally complete variety is hereditarily $\mathcal{C}'$-structurally complete.

For example, we can take a variety $\mathcal{V}$ of Brouwerian semilattices (idempotent hoops) in the signature $\wedge, \to, \mathbf{1}$, extend the signature by adding two new connectives $\vee, \neg$ and the new axioms that will define a variety of Heyting algebras $\mathcal{V}'$, and, since $\mathcal{V}$ is hereditarily $\{\wedge, \to, \mathbf{1}\}$-structurally complete, we can conclude that the variety of Heyting algebras is also hereditarily $\{\wedge, \to, \mathbf{1}\}$-structurally complete.

5.2 Derivative connectives From the definition of td-distributivity, it immediately follows that any superposition of any td-distributive connectives is td-distributive. Thus, if $\mathcal{C}$ is a signature, we can extend the signature $\mathcal{C}$ by adding a new connective f that is expressed via connectives from $\mathcal{C}(td)$ and obtain a new variety that will be hereditarily $\mathcal{C}(td) \cup \{f\}$-structurally complete. For example, in the variety $\mathcal{H}$ of Heyting algebras in the signature $\wedge, \vee, \to, \neg$, the connectives $\to, \wedge$ are $td_{\to}$-distributive. If we add new connectives $\mathbf{1} := (p \to p)$ and $p \leftrightarrow q := (p \to q) \wedge (q \to p)$, we can conclude that the obtained variety is hereditarily $\{\wedge, \to, \leftrightarrow, \mathbf{1}\}$-structurally complete. Let us also note that even though $\neg$ is not $td_{\to}$-distributive connective, the connective $\neg\neg$ is $td_{\to}$-distributive. By the

Table 1 Examples of *td*-distributive connectives.

Variety	Signature	*td*-term	*td*-distributive
Hilbert algebras	$\to, \mathbf{1}$	(a)	$\to, \mathbf{1}$
Brouwerian semilattices	$\wedge, \to, \mathbf{1}$	(a)	$\wedge, \to, \mathbf{1}$
Heyting algebras	$\wedge, \vee, \to, \neg$	(a)	$\wedge, \to, \top, \sim, \neg\neg$
Heyting algebras	$\wedge, \vee, \to, \neg$	(b)	$\wedge, \vee, \bot$
KM algebras	$\wedge, \vee, \to, \neg, \Delta$	(a)	$\wedge, \to, \top, \sim, \neg\neg$
KM algebras	$\wedge, \vee, \to, \neg$	(b)	$\wedge, \vee, \bot, \Delta$
n-potent hoops	$\cdot, \to, \mathbf{1}$	(c)	$\cdot, \to, \mathbf{1}$
n-transitive modal algebras	$\wedge, \vee, \to, \neg, \Box$	(d)	$\wedge, \to, \top$
n-transitive modal algebras	$\wedge, \vee, \to, \neg, \Box$	(e)	$\wedge, \vee, \Box, \bot$
Interior algebras	$\wedge, \vee, \to, \neg, \Box$	(f)	$\wedge, \to, \top$
Interior algebras	$\wedge, \vee, \to, \neg, \Box$	(g)	$\wedge, \vee, \Box, \bot$
BCI monoids	$\wedge, \to, \cdot, \mathbf{1}$	(h)	$\wedge, \to, \cdot$
BCI monoids	$\wedge, \to, \cdot, \mathbf{1}$	(i)	$\wedge, \to, \cdot$

Glivenko theorem, in each Heyting algebra the following hold:

$$p \to \neg\neg q \approx \neg(p \wedge \neg q) \approx \neg\neg(\neg p \vee q) \approx \neg\neg(p \to q).$$

Hence, $\mathcal{H}$ is hereditarily $\{\wedge, \to, \mathbf{1}, \leftrightarrow, \neg\neg\}$-structurally complete or, for instance, $\{\leftrightarrow, \neg\neg\}$-structurally complete (see Słomczyńska [34]).

5.3 Examples of *td*-distributive connectives In Table 1, we provide examples of some varieties, their TD terms and corresponding *td*-distributive connectives. We use $p \sim q$ as an abbreviation for $(p \to q) \wedge (q \to p)$ or $(p \to q) \cdot (q \to p)$, and we use $\leftrightarrow$ for a principal connective. We also use $\top$ and $\bot$ as an abbreviation for $(p \to p)$ and $p \wedge \neg p$, while $\mathbf{1}$ and $\mathbf{0}$ are used for principal connectives. We define $p^{\hat{n}}$ as $p^{\hat{1}} = p \wedge \mathbf{1}$ and $p^{\hat{n}} = (p \wedge \mathbf{1}) \wedge p^{\widehat{n-1}}$.

We will consider the following ternary terms:

(a) $td := (p \to q) \to ((q \to p) \to r)$,
(b) $td := (p \to q) \wedge (q \to p) \wedge r$,
(c) $td := (p \to q) \xrightarrow{n-1} ((q \to p) \xrightarrow{n-1} r)$,
(d) $td := (p \sim q) \wedge \Box(p \sim q) \wedge \cdots \wedge \Box^{n-1}(p \sim q) \to r$,
(e) $td := (p \sim q) \wedge \Box(p \sim q) \wedge \cdots \wedge \Box^{n-1}(p \sim q) \wedge r$,
(f) $td := \Box(p \sim q) \to r$,
(g) $td := \Box(p \sim q) \wedge r$,
(h) $td := (p \sim q)^n \cdot r$,
(i) $td := (p \sim q) \xrightarrow{n} r$.

Notes

1. The terms "Prucnal's trick" and "modified Prucnal's trick" were introduced by Wroński in connection with $\to, \wedge$ and $\to, \wedge, \neg$ fragments of IPL.

2. As the anonymous referee pointed out, exactness of formulas suffices.

3. In [5], a discriminator is called an "affine discriminator" to distinguish this notion from the notion of a fixed point discriminator.

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