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The Distributivity on Bi-Approximation Semantics

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Abstract In this paper, we give a possible characterization of the distributivity on bi-approximation semantics. To this end, we introduce new notions of special elements on polarities and show that the distributivity is first-order definable on bi-approximation semantics. In addition, we investigate the dual representation of those structures and compare them with bi-approximation semantics for intuitionistic logic. We also discuss that two different methods to validate the distributivity—by the splitters and by the adjointness—can be explicated with the help of the axiom of choice as well.

1 Introduction

Bi-approximation semantics is a universal relational-type semantics for substructural and lattice-based logics, not necessarily including distributive substructural logics such as orthonormal logic or lattice-based modal logics (see Suzuki [11]). Unlike other relational semantics for nondistributive lattice-based logics (see, e.g., Goldblatt [7], Hartonas [8], Hartonas and Dunn [9], Gehrke [5]; see also Restall [10]), the novelty of bi-approximation semantics is to reason not only about formulas but also sequents, that is, logical consequences, based on polarities. As bi-approximation semantics was introduced to explicate Ghilardi and Meloni's [6] canonicity methodology via relational-type structures, we can enjoy the canonicity results of lattice-based logics in Suzuki [12] (cf. residuated frames in Galatos and Jipsen [3]). In other words, we may say that bi-approximation semantics is a canonicity-friendly relational semantics for lattice-based logics. In addition, a Sahlqvist-type first-order definability for substructural logic was already shown in Suzuki [14]. Therefore, this completes the so-called Sahlqvist theorem for substructural and lattice-based logics.

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However, while the notion of *consistent variable occurrence* in [12] successfully characterizes many canonical and first-order definable sequents as a natural extension of the original Sahlqvist theorem for modal logic, there are still interesting sequents which do *not* have consistent variable occurrence in our setting. One of the remarkable examples is the main topic of the current paper, that is, the *distributivity*: $\phi \wedge (\psi \vee \chi) \Rightarrow (\phi \wedge \psi) \vee (\phi \wedge \chi)$. In fact, it is known that the distributivity is canonical but does not have consistent variable occurrence. Then, it is natural to ask how the distributivity can be characterized on bi-approximation semantics, and whether it is first-order definable. If the distributivity is not first-order definable, our Sahlqvist-type theorem would not be very attractive anymore, because it would mean that we can apply our theorem only on nondistributive lattice-based logics. The main aim of this paper is to give positive answers to these questions.

To study the distributivity on bi-approximation semantics, we will introduce *distributive polarity frames* by means of special elements in polarity, named *splitters*. Then we will investigate how to validate the distributivity on distributive polarity frames and how splitters essentially work on distributive polarity frames. We will also consider the *prime skeletons* of distributive polarity frames as well. To this end, we find a connection to generalized Kripke frames over distributive lattices (see [5]).

Furthermore, we think about bi-approximation semantics for intuitionistic logic. There are two reasons that we focus on intuitionistic logic. One is that intuitionistic logic is a well-known logic that can derive distributivity. Another reason is that intuitionistic logic can be seen as a substructural logic extended only with sequents that have consistent variable occurrence: *weakening* and *contraction*. Hence, by applying the Sahlqvist theorem in [14], we can describe bi-approximation semantics for intuitionistic logic by first-order sentences. So, in this paper, we also study how to validate the distributivity on bi-approximation semantics and compare the underlying polarity frames with distributive polarity frames. Interestingly, the polarity frames for intuitionistic logic do not, at least explicitly, mention anything about splitters. Also different is how the distributivity on bi-approximation semantics for intuitionistic logic is validated compared with distributive polarity frames. More precisely, bi-approximation semantics for intuitionistic logic validates by means of the adjointness of *fusion* and *residuals*, instead of *splitters*. However, with the help of the *axiom of choice*, we can claim that bi-approximation semantics for intuitionistic logic are also based on distributive polarity frames as well.

We outline the structure of the this paper as follows. In Section 2, we briefly recall the idea of bi-approximation semantics, that is, how to reason about sequents on polarity frames, and we review fundamental properties of bi-approximation semantics. We also see morphisms and invariance of sequents here. In Section 3, we introduce the pivotal notions for our purpose, that is, splitters, splitting counterparts, and splitting pairs. By means of the splitting pairs, we give a possible characterization of distributive polarity frames and show how it works on distributive polarity frames. In addition, we prove a dual representation of distributive lattices and distributive polarity frames. Unlike what happens in the setting of the dual representation of lattices and polarity frames, the axiom of choice is necessary here. To investigate how splitters essentially work on distributive polarity frames, in Section 4, we consider the prime skeletons and prove the characteristic properties. In Section 5, we compare bi-approximation semantics for intuitionistic logic, which is obtained by the Sahlqvist theorem for substructural logic with distributive polarity frames. Finally, we give concluding remarks in Section 6.

2 Bi-Approximation Semantics: Reasoning with Logical Consequences on Polarities

Bi-approximation semantics is a universal relational-style semantics for lattice-based logics. A novelty of our semantics is to evaluate not only formulas but also logical consequences, that is, sequents, based on polarities. Here we briefly recall fundamental results for polarities (see, e.g., Davey and Priestley [2] for a polarity, and [11] for bi-approximation semantics).

A *polarity* is a triple $\mathbb{F} = \langle X, Y, B \rangle$ of two nonempty sets X and Y, which are not necessarily disjoint,[1] and a binary relation B between them, that is, $B \subseteq X \times Y$. Given a polarity $\mathbb{F} = \langle X, Y, B \rangle$, the binary relation B can be naturally extended to a preorder $\leq_B$ on $X \cup Y$ as follows: for all $x, x_1, x_2 \in X$ and $y, y_1, y_2 \in Y$, we let

1. $x_1 \leq_B x_2 \iff$ for each $y' \in Y$, if $x_2 B y'$, then $x_1 B y'$;
2. $y_1 \leq_B y_2 \iff$ for each $x' \in X$, if $x' B y_1$, then $x' B y_2$;
3. $x \leq_B y \iff xBy$;
4. $y \leq_B x \iff$ for all $x' \in X$ and $y' \in Y$, if $x' B y$ and $x B y'$, then $x' B y'$.

Hence we may sometimes refer to the triple $\mathbb{F} = \langle X, Y, \leq_B \rangle$, instead of $\langle X, Y, B \rangle$, as a polarity. Also, we may sometimes omit the subscript $_B$ for $\leq_B$, that is, $\leq$.

For the time being, until Section 5, we consider formulas constructed simply by propositional variables $p, q, \ldots$ and two logical connectives: *conjunction* $\wedge$ and *disjunction* $\vee$ only. We denote by Φ the set of propositional variables and by Λ the set of formulas. A *sequent* (logical consequence) is a pair of formulas ϕ and ψ, denoted by $\phi \mapsto \psi$.

To reason about sequents on polarities, it is necessary to introduce appropriate valuations. Let $\mathbb{F} = \langle X, Y, \leq \rangle$ be a polarity, let $\wp(X)$ be the poset of the powerset of X and the set-theoretical inclusion $\subseteq$, and let $\wp(Y)^\partial$ be the poset of the powerset of Y and the *reverse* set-theoretical inclusion $\supseteq$. A pair $V = (V^\downarrow, V_\uparrow)$ of two functions $V^\downarrow \colon \Phi \to \wp(X)$ and $V_\uparrow \colon \Phi \to \wp(Y)^\partial$ is a *doppelgänger valuation on* $\mathbb{F}$, if the pair of two functions satisfy

1. $V^\downarrow(p) = \{x \in X \mid \forall y \in V_\uparrow(p).\, x \leq y\}$,
2. $V_\uparrow(p) = \{y \in Y \mid \forall x \in V^\downarrow(p).\, x \leq y\}$,

for each propositional variable $p \in \Phi$. We denote by $\mathcal{D}_V$ the set of all doppelgänger valuations on $\mathbb{F}$. Intuitively speaking, doppelgänger valuations make, for each propositional variable p, the sequent $p \mapsto p$ valid on polarities. Also, we mention a connection to the Dedekind–MacNeille completion as well. We can define a Galois connection $\lambda \dashv \upsilon$ between $\wp(X)$ and $\wp(Y)^\partial$ as follows:

1. $\lambda \colon \wp(X) \to \wp(Y)^\partial$ with $\lambda(\mathfrak{X}) := \{y \in Y \mid \forall x \in \mathfrak{X}.\, x \leq y\}$ for $\mathfrak{X} \in \wp(X)$;
2. $\upsilon \colon \wp(Y)^\partial \to \wp(X)$ with $\upsilon(\mathfrak{Y}) := \{x \in X \mid \forall y \in \mathfrak{Y}.\, x \leq y\}$ for $\mathfrak{Y} \in \wp(Y)^\partial$.

Since λ and υ form a Galois connection, the images $\lambda[\wp(X)]$ and $\upsilon[\wp(Y)^\partial]$ are isomorphic. In fact, these images are the (generalized) Dedekind–MacNeille completions of $\mathbb{F}$, and the elements in $\lambda[\wp(X)]$ and $\upsilon[\wp(Y)^\partial]$ are the so-called *Dedekind cuts*. By doppelgänger valuations, we assign each propositional variable to the corresponding points on these images $\lambda[\wp(X)]$ and $\upsilon[\wp(Y)^\partial]$. Note that the order in the (generalized) Dedekind–MacNeille completion reflects the extended preorder $\leq_B$.

Remark 2.1 Each function $f\colon \Phi \to \wp(X \cup Y)$ can be extended to a doppelgänger valuation V_f by first-order sentences. Conversely, all doppelgänger valuations are first-order definable from some functions (see [14]; cf. also Ciabattoni, Galatos, and Terui [1], Galatos and Jipsen [3]).

Hereinafter, we call a polarity $\mathbb{F}$ a *polarity frame*, and we call a pair $\mathbb{M} = \langle \mathbb{F}, V \rangle$ of a polarity frame $\mathbb{F}$ and a doppelgänger valuation V on $\mathbb{F}$ a *bi-approximation model*. On a bi-approximation model $\mathbb{M} = \langle \mathbb{F}, V \rangle$, we inductively define two satisfaction relations for formulas, that is, one for *premises* on X and the other for *conclusions* on Y, as follows. For all formulas ϕ and ψ, each $x \in X$ and each $y \in Y$, we let

X-1: $\mathbb{M} \Vdash^{x} p \iff x \in V^{\downarrow}(p)$ for each propositional variable $p \in \Phi$;

X-2: $\mathbb{M} \Vdash^{x} \phi \vee \psi \iff$ for each $y' \in Y$, if $\mathbb{M} \Vdash_{y'} \phi \vee \psi$, then $x \leq y'$;

X-3: $\mathbb{M} \Vdash^{x} \phi \wedge \psi \iff \mathbb{M} \Vdash^{x} \phi$ and $\mathbb{M} \Vdash^{x} \psi$;

Y-1: $\mathbb{M} \Vdash_{y} p \iff y \in V_{\uparrow}(p)$ for each propositional variable $p \in \Phi$;

Y-2: $\mathbb{M} \Vdash_{y} \phi \vee \psi \iff \mathbb{M} \Vdash_{y} \phi$ and $\mathbb{M} \Vdash_{y} \psi$;

Y-3: $\mathbb{M} \Vdash_{y} \phi \wedge \psi \iff$ for each $x' \in X$, if $\mathbb{M} \Vdash^{x'} \phi \wedge \psi$, then $x' \leq y$.

On these satisfaction relations, we also introduce a satisfaction relation for logical consequences as follows:

S-1: $\mathbb{M} \Vdash^{x}_{y} \phi \mapsto \psi \iff$ if $\mathbb{M} \Vdash^{x} \phi$ and $\mathbb{M} \Vdash_{y} \psi$, then $x \leq y$;

S-2: $\mathbb{M} \Vdash \phi \mapsto \psi \iff$ for all $x \in X$ and $y \in Y$, $\mathbb{M} \Vdash^{x}_{y} \phi \mapsto \psi$;

S-3: $\mathbb{F} \Vdash \phi \mapsto \psi \iff$ for each $V \in \mathfrak{D}_V$, $\langle \mathbb{F}, V \rangle \Vdash \phi \mapsto \psi$.

On bi-approximation semantics, these satisfaction relations are interpreted as follows:

1. $\mathbb{M} \Vdash^{x} \phi$: a formula ϕ is *assumed* at x in $\mathbb{M}$;
2. $\mathbb{M} \Vdash_{y} \psi$: a formula ψ is *concluded* at y in $\mathbb{M}$;
3. $\mathbb{M} \Vdash^{x}_{y} \phi \mapsto \psi$: a sequent $\phi \mapsto \psi$ is *true* at (x, y) in $\mathbb{M}$;
4. $\mathbb{M} \Vdash \phi \mapsto \psi$: a sequent $\phi \mapsto \psi$ is *universally true* in $\mathbb{M}$;
5. $\mathbb{F} \Vdash \phi \mapsto \psi$: a sequent $\phi \mapsto \psi$ is *valid* on $\mathbb{F}$.

Proposition 2.2 (Hereditary) *For all $x, x' \in X$ and $y, y' \in Y$,*

1. *if $x' \leq x$ and $\mathbb{M} \Vdash^{x} \phi$, then $\mathbb{M} \Vdash^{x'} \phi$;*
2. *if $y \leq y'$ and $\mathbb{M} \Vdash_{y} \phi$, then $\mathbb{M} \Vdash_{y'} \phi$.*

Proposition 2.3 (Extension of doppelgänger valuations) *For all $x \in X$ and $y \in Y$,*

1. $\mathbb{M} \Vdash^{x} \phi \iff$ *for each $y' \in Y$, if $\mathbb{M} \Vdash_{y'} \phi$, then $x \leq y'$;*
2. $\mathbb{M} \Vdash_{y} \phi \iff$ *for each $x' \in X$, if $\mathbb{M} \Vdash^{x'} \phi$, then $x' \leq y$.*

Remark 2.4 Note that each doppelgänger valuation is naturally extended from Φ to Λ. It means that, for each formula $\phi \in \Lambda$, the sequent $\phi \mapsto \phi$ is valid on polarity frames. Therefore, on the dual algebras, doppelgänger valuations become homomorphisms from Λ.

Initial sequents: $\phi \mapsto \phi$ **Cut rule:** $\dfrac{\phi \mapsto \chi \qquad \chi \mapsto \psi}{\phi \mapsto \psi}\ (\mathrm{cut})$

Rules for logical connectives:

$$\frac{\phi \mapsto \chi \qquad \psi \mapsto \chi}{\phi \vee \psi \mapsto \chi}\ (\vee \mapsto) \qquad \frac{\chi \mapsto \phi}{\chi \mapsto \phi \vee \psi}\ (\mapsto \vee_1) \qquad \frac{\chi \mapsto \psi}{\chi \mapsto \phi \vee \psi}\ (\mapsto \vee_2)$$

$$\frac{\phi \mapsto \chi}{\phi \wedge \psi \mapsto \chi}\ (\wedge_1 \mapsto) \qquad \frac{\psi \mapsto \chi}{\phi \wedge \psi \mapsto \chi}\ (\wedge_2 \mapsto) \qquad \frac{\chi \mapsto \phi \qquad \chi \mapsto \psi}{\chi \mapsto \phi \wedge \psi}\ (\mapsto \wedge)$$

Figure 1 The sequent calculus of the lattice logic.

Theorem 2.5 (Soundness and completeness) *The lattice logic, the collection of provable sequents in Figure 1, is sound and complete with respect to the class of all polarity frames.*

Remark 2.6 In Figure 1, the distributivity $\phi \wedge (\psi \vee \chi) \mapsto (\phi \wedge \psi) \vee (\phi \wedge \chi)$ is not provable. Also, on polarity frames, since $\mathbb{M} \overset{x}{\Vdash} \phi \vee \psi$ is not the same as $\mathbb{M} \overset{x}{\Vdash} \phi$ or $\mathbb{M} \overset{x}{\Vdash} \psi$ (dually, $\mathbb{M} \underset{y}{\Vdash} \phi \wedge \psi$ is not the same as $\mathbb{M} \underset{y}{\Vdash} \phi$ or $\mathbb{M} \underset{y}{\Vdash} \psi$), the distributivity is not valid in general.

Dedekind-cut-preserving morphisms Next we briefly summarize morphisms for bi-approximation semantics (see [13]).

Definition 2.7 (Dedekind-cut-preserving morphism) Given two polarity frames $\mathbb{F} = \langle X_1, Y_1, \leq_1 \rangle$ and $\mathbb{G} = \langle X_2, Y_2, \leq_2 \rangle$, a pair $\langle \sigma | \tau \rangle$ of two functions $\sigma: X_1 \to X_2$ and $\tau: Y_1 \to Y_2$ forms a *Dedekind-cut-preserving morphism from* $\mathbb{F}$ *to* ?, a *d-morphism* for short and denoted by $\langle \sigma | \tau \rangle: \mathbb{F} \to \mathbb{G}$, if

1. for all $x \in X_1$ and $y \in Y_1$, if $\sigma(x) \leq_2 \tau(y)$, then $x \leq_1 y$;
2. for all $x \in X_1$ and $y' \in Y_2$, if, for each $y \in Y_1$, $y' \leq_2 \tau(y)$ implies $x \leq_1 y$, then $\sigma(x) \leq_2 y'$;
3. for all $x' \in X_2$ and $y \in Y_1$, if, for each $x \in X_1$, $\sigma(x) \leq_2 x'$ implies $x \leq_1 y$, then $x' \leq_2 \tau(y)$.

In addition, a d-morphism $\langle \sigma | \tau \rangle: \mathbb{F} \to \mathbb{G}$ is called *d-embedding*, *d-separating*, and *d-reflecting*, if it also satisfies the following item 4, item 5, and both items 4 and 5, respectively:

4. for all $x \in X_1$ and $y \in Y_1$, if $x \leq_1 y$, then $\sigma(x) \leq_2 \tau(y)$;
5. for all $x' \in X_2$ and $y' \in Y_2$, if, for all $x \in X_1$ and $y \in Y_1$, $\sigma(x) \leq_2 y'$ and $x' \leq_2 \tau(y)$ imply $x \leq_1 y$, then $x' \leq_2 y'$.

For every doppelgänger valuation U on $\mathbb{F}$ and every doppelgänger valuation V on $\mathbb{G}$, a d-morphism $\langle \sigma | \tau \rangle: \mathbb{F} \to \mathbb{G}$ is a *Dedekind-cut-preserving morphism from* $\langle \mathbb{F}, U \rangle$ *to* $\langle \mathbb{G}, V \rangle$, a *d-morphism* for short, and denoted by $\langle \sigma | \tau \rangle: \langle \mathbb{F}, U \rangle \to \langle \mathbb{G}, V \rangle$, if $\langle \sigma | \tau \rangle$ also satisfies

6. $x \in U^{\downarrow}(p) \iff \sigma(x) \in V^{\downarrow}(p)$;
7. $y \in U_{\uparrow}(p) \iff \tau(y) \in V_{\uparrow}(p)$,

for each propositional variable $p \in \Phi$ as well.

Based on d-morphisms, we can obtain the so-called p-morphism lemma as in the case of modal logic and invariance of validity of sequents for bi-approximation semantics.

Lemma 2.8 (For formulas) *Let $\mathbb{M}_1$ and $\mathbb{M}_2$ be bi-approximation models, and let $\langle\sigma|\tau\rangle\colon \mathbb{M}_1 \to \mathbb{M}_2$. For all $\phi, \psi \in \Lambda$, $x \in X_1$, and $y \in Y_1$,*

1. $\mathbb{M}_1 \overset{x}{\Vdash} \phi \iff \mathbb{M}_2 \overset{\sigma(x)}{\Vdash} \phi$;
2. $\mathbb{M}_1 \underset{y}{\Vdash} \psi \iff \mathbb{M}_2 \underset{\tau(y)}{\Vdash} \psi$.

Lemma 2.9 (For sequents) *Let $\mathbb{M}_1$ and $\mathbb{M}_2$ be bi-approximation models, and let $\langle\sigma|\tau\rangle\colon \mathbb{M}_1 \to \mathbb{M}_2$. For every sequent $\phi \mapsto \psi$,*

$$\mathbb{M}_1 \Vdash \phi \mapsto \psi \iff \textit{for all } x \in X_1, y \in Y_1, \mathbb{M}_2 \overset{\sigma(x)}{\underset{\tau(y)}{\Vdash}} \phi \mapsto \psi.$$

Theorem 2.10 *Let $\mathbb{F}$ and $\mathbb{G}$ be polarity frames, and let $\langle\sigma|\tau\rangle\colon \mathbb{F} \to \mathbb{G}$. For every sequent $\phi \mapsto \psi$,*

1. *if $\langle\sigma|\tau\rangle$ is d-embedding, then $\mathbb{G} \Vdash \phi \mapsto \psi$ implies $\mathbb{F} \Vdash \phi \mapsto \psi$;*
2. *if $\langle\sigma|\tau\rangle$ is d-separating, then $\mathbb{F} \Vdash \phi \mapsto \psi$ implies $\mathbb{G} \Vdash \phi \mapsto \psi$;*
3. *if $\langle\sigma|\tau\rangle$ is d-reflecting, then $\mathbb{F} \Vdash \phi \mapsto \psi$ if and only if $\mathbb{G} \Vdash \phi \mapsto \psi$.*

3 Distributive Polarity Frames

On lattice-based logics, we have already shown that sequents which have *consistent variable occurrence* are canonical (see [12]) and first-order definable on bi-approximation semantics (see [14]). While the property "consistent variable occurrence" successfully accounts for many canonical and first-order definable sequents on bi-approximation semantics, there are also some interesting sequents which cannot be explicated by it. The distributivity $\phi \wedge (\psi \vee \chi) \mapsto (\phi \wedge \psi) \vee (\phi \wedge \chi)$ is, in fact, a remarkable example of a canonical sequent which does not have consistent variable occurrence. In this section, we will propose a possible characterization of the distributivity on polarity frames.

First, we consider special elements in X and in Y to handle the distributivity on a polarity frame $\mathbb{F} = \langle X, Y, \leq\rangle$. An element $x \in X$ is a *splitter* if there exists a *splitting counterpart* $y_x \in Y$ *of* x such that $x \not\leq y_x$ and, for each $y \in Y$, if $x \not\leq y$, then $y \leq y_x$. Analogously, an element $y \in Y$ is a *splitter* if there exists a *splitting counterpart* $x_y \in X$ *of* y such that $x_y \not\leq y$ and, for each $x \in X$, if $x \not\leq y$, then $x_y \leq x$.

Proposition 3.1 *Let $\mathbb{F} = \langle X, Y, \leq\rangle$ be a polarity frame. For all $x \in X$ and $y \in Y$,*

1. *if x is a splitter, then every splitting counterpart y_x of x is also a splitter which has x as a splitting counterpart;*
2. *if y is a splitter, then every splitting counterpart x_y of y is also a splitter which has y as a splitting counterpart.*

Furthermore, for each splitter, the splitting counterparts are unique up to $\leq$-equivalence; that is, if y_x and y'_x (x_y and x'_y) are splitting counterparts of x (y), then $y_x \leq y'_x$ and $y'_x \leq y_x$ ($x_y \leq x'_y$ and $x'_y \leq x_y$) hold.

Proof It suffices to show that, for each $x' \in X$, if $x' \not\leq y_x$, then $x \leq x'$, for each $y' \in Y$, if $y' \not\leq x_y$, then $y' \leq y$ and the uniqueness up to $\leq$-equivalence.

Recall the definition of $\leq$; that is, $x \leq x' \iff$ for each $y'' \in Y$, if $x' \leq y''$, then $x \leq y''$. For arbitrary $x' \in X$ and $y'' \in Y$, if $x' \not\leq y_x$ and $x' \leq y''$, by the definition of $\leq$ on Y, we obtain that $y'' \not\leq y_x$. Since y_x is a splitting counterpart of

x, we have that, for each $y' \in Y$, if $x \not\leq y'$, then $y' \leq y_x$. By contraposition, we get $x \leq y''$, which concludes $x \leq x'$.

Analogously, for arbitrary $x'' \in X$ and $y' \in Y$, if $x_y \not\leq y'$ and $x'' \leq y'$, by the definition of $\leq$ on X, we have $x_y \not\leq x''$. As x_y is a splitting counterpart of y, we obtain $x'' \leq y$. Hence $y' \leq y$.

Let y_x and y'_x be splitting counterparts of x. By definition, we have $x \not\leq y_x$ and $x \not\leq y'_x$. Since y_x and y'_x are splitting counterparts of x, for every $y' \in Y$, if $x \not\leq y'$, then we have $y' \leq y_x$ and $y' \leq y'_x$. Therefore, by $x \not\leq y_x$ and $x \not\leq y'_x$, we conclude that $y_x \leq y'_x$ and $y'_x \leq y_x$. The other case is analogous. □

Thanks to Proposition 3.1, we can claim that a pair (x, y) of $x \in X$ and $y \in Y$ is a *splitting pair*, if $x \not\leq y$, for each $y' \in Y$, if $x \not\leq y'$, then $y' \leq y$ and, for each $x' \in X$, if $x' \not\leq y$, then $x \leq x'$.

Theorem 3.2 *Let $\mathbb{M}$ be a bi-approximation model, and let (x, y) be a splitting pair. For every formula $\phi \in \Lambda$, $\mathbb{M} \Vdash^{x} \phi \iff \mathbb{M} \nVdash_{y} \phi$, and equivalently $\mathbb{M} \Vdash_{y} \phi \iff \mathbb{M} \nVdash^{x} \phi$.*

Proof $(\Rightarrow)$. Assume $\mathbb{M} \Vdash^{x} \phi$. Since (x, y) is a splitting pair, we have $x \not\leq y$. Then, this x is a witness to show that there exists $x' \in X$ such that $\mathbb{M} \Vdash^{x'} \phi$ but $x' \not\leq y$. Hence, by Proposition 2.3, we conclude that $\mathbb{M} \nVdash_{y} \phi$.

$(\Leftarrow)$. Suppose that $\mathbb{M} \nVdash_{y} \phi$. By Proposition 2.3, there exists $x' \in X$ such that $\mathbb{M} \Vdash^{x'} \phi$ but $x' \not\leq y$. Now, as (x, y) is a splitting pair, for each $x'' \in X$, if $x'' \not\leq y$, then $x \leq x''$. So we have $x \leq x'$. Since $\mathbb{M} \Vdash^{x'} \phi$, together with Proposition 2.2, we complete this direction; that is, $\mathbb{M} \Vdash^{x} \phi$. □

Also, with the help of Theorem 3.2, we can prove the following lemma.

Lemma 3.3 *Let $\mathbb{M}$ be a bi-approximation model, and let ϕ and ψ be formulas. For all splitters $x \in X$ and $y \in Y$,*

1. $\mathbb{M} \Vdash^{x} \phi \vee \psi \iff \mathbb{M} \Vdash^{x} \phi$ *or* $\mathbb{M} \Vdash^{x} \psi$;
2. $\mathbb{M} \Vdash_{y} \phi \wedge \psi \iff \mathbb{M} \Vdash_{y} \phi$ *or* $\mathbb{M} \Vdash_{y} \psi$.

Proof In both cases, the $\Leftarrow$-directions are trivial.

For the $\Rightarrow$-direction of item 1, we consider the contraposition. Suppose that $\mathbb{M} \nVdash^{x} \phi$ and $\mathbb{M} \nVdash^{x} \psi$. Since x is a splitter, there exists a splitting counterpart $y_x \in Y$; hence (x, y_x) forms a splitting pair. By Theorem 3.2, we obtain that $\mathbb{M} \Vdash_{y_x} \phi$ and $\mathbb{M} \Vdash_{y_x} \psi$, which means $\mathbb{M} \Vdash_{y_x} \phi \vee \psi$. However, as $x \not\leq y_x$, by Proposition 2.3, we conclude that $\mathbb{M} \nVdash^{x} \phi \vee \psi$.

For the $\Rightarrow$-direction of item 2, we also consider the contraposition. Suppose that $\mathbb{M} \nVdash_{y} \phi$ and $\mathbb{M} \nVdash_{y} \psi$. Since y is a splitter, there exists a splitting counterpart x_y of y in X, so (x_y, y) forms a splitting pair. By Theorem 3.2, we obtain that $\mathbb{M} \Vdash^{x_y} \phi \wedge \psi$. On the other hand, as $x_y \not\leq y$, by Proposition 2.3, we conclude that $\mathbb{M} \nVdash_{y} \phi \wedge \psi$. □

Next, by means of splitters, we characterize the distributivity on polarity frames.

Definition 3.4 (Distributive polarity frame) A polarity frame $\mathbb{F} = \langle X, Y, \leq\rangle$ is *distributive* if it satisfies

> **Splitting:** for all $x \in X$ and $y \in Y$, if $x \not\leq y$, there exists a splitting pair (x_s, y_s) such that $x_s \leq x$ and $y \leq y_s$.

Remark 3.5 The condition **Splitting** is a first-order sentence.

Theorem 3.6 *The distributivity $\phi \wedge (\psi \vee \chi) \mapsto (\phi \wedge \psi) \vee (\phi \wedge \chi)$ is valid on any distributive polarity frame.*

Proof Let $\mathbb{F} = \langle X, Y, \leq\rangle$ be a distributive polarity frame, and let V be an arbitrary doppelgänger valuation on $\mathbb{F}$. Assume $\langle \mathbb{F}, V\rangle \Vdash^{x} \phi \wedge (\psi \vee \chi)$ and $\langle \mathbb{F}, V\rangle \Vdash_{y} (\phi \wedge \psi) \vee (\phi \wedge \chi)$. So it suffices to show that $x \leq y$. We will prove it by contradiction.

Suppose that $x \not\leq y$. Since $\mathbb{F}$ is distributive, by the splitting condition, there exists a splitting pair (x_s, y_s) such that $x_s \leq x$ and $y \leq y_s$. By Proposition 2.2, we have that $\langle \mathbb{F}, V\rangle \Vdash^{x_s} \phi \wedge (\psi \vee \chi)$; that is, $\langle \mathbb{F}, V\rangle \Vdash^{x_s} \phi$ and $\langle \mathbb{F}, V\rangle \Vdash^{x_s} \psi \vee \chi$. In addition, as x_s is a splitter, by Lemma 3.3, we obtain that $\langle \mathbb{F}, V\rangle \Vdash^{x_s} \psi$ or $\langle \mathbb{F}, V\rangle \Vdash^{x_s} \chi$. Hence, we conclude that $\langle \mathbb{F}, V\rangle \Vdash^{x_s} \phi \wedge \psi$ or $\langle \mathbb{F}, V\rangle \Vdash^{x_s} \phi \wedge \chi$.

On the other hand, by Proposition 2.2, we also have $\langle \mathbb{F}, V\rangle \Vdash_{y_s} (\phi \wedge \psi) \vee (\phi \wedge \chi)$, which means $\langle \mathbb{F}, V\rangle \Vdash_{y_s} \phi \wedge \psi$ and $\langle \mathbb{F}, V\rangle \Vdash_{y_s} \phi \wedge \chi$. So either way, that is, either $\langle \mathbb{F}, V\rangle \Vdash^{x_s} \phi \wedge \psi$ or $\langle \mathbb{F}, V\rangle \Vdash^{x_s} \phi \wedge \chi$ holds, by Proposition 2.3, $x_s \leq y_s$ must hold, which contradicts the fact that (x_s, y_s) is a splitting pair. Therefore, we conclude that $x \leq y$. □

The dual representation of distributive lattices. We will also check the dual representation of distributive lattices and distributive polarity frames.

Given a lattice $\mathbb{L} = \langle L, \vee, \wedge\rangle$, a subset F of L is a *filter* if it is nonempty, upward-closed, and down-directed. Ordered dually, a subset I of L is an *ideal* if it is nonempty, downward-closed, and up-directed. We denote by $\mathcal{F}$ the set of filters and by $\mathcal{I}$ the set of ideals. Moreover, a filter F is *prime* if $F \neq L$ and, for all $a, b \in L$, $a \vee b \in F$ implies $a \in F$ or $b \in F$. An ideal I is *prime* if $I \neq L$ and, for all $a, b \in L$, $a \wedge b \in I$ implies $a \in I$ or $b \in I$.

Definition 3.7 (Dual polarity frame) Let $\mathbb{L}$ be a lattice. A triple $\mathbb{L}_+ = \langle \mathcal{F}, \mathcal{I}, \sqsubseteq\rangle$ is the *dual polarity frame of* $\mathbb{L}$, where the binary relation $\sqsubseteq$ on $\mathcal{F} \times \mathcal{I}$ is defined by $F \sqsubseteq I \iff F \cap I \neq \emptyset$.

Now, we will show that, for every distributive lattice, the dual polarity frame is distributive (see Theorem 3.9). To prove it, we first recall the following lemma.

Lemma 3.8 *Let $\mathbb{L} = \langle L, \vee, \wedge\rangle$ be a distributive lattice. For every prime filter F, the set-theoretical complement $L \setminus F$ is a prime ideal. Also, for every prime ideal I, the set-theoretical complement $L \setminus I$ is a prime filter.*

Theorem 3.9 *Let $\mathbb{L} = \langle L, \vee, \wedge\rangle$ be a distributive lattice. The dual polarity frame $\mathbb{L}_+ = \langle \mathcal{F}, \mathcal{I}, \sqsubseteq\rangle$ is distributive.*

Proof It is straightforward to show that $\mathbb{L}_+$ is a polarity frame. So the only nontrivial part is the splitting condition.

For arbitrary $F \in \mathcal{F}$ and $I \in \mathcal{I}$, assume that $F \not\sqsubseteq I$; that is, $F \cap I = \emptyset$. Then, by the prime filter (ideal) theorem, there exists a prime filter P_F (or a prime ideal P_I) such that $F \subseteq P_F$ but $P_F \cap I = \emptyset$ ($I \subseteq P_I$ but $F \cap P_I = \emptyset$). Thanks to Lemma 3.8, the set-theoretical complement $L \setminus P_F$ ($L \setminus P_I$) of the prime filter P_F (the prime ideal P_I) is a prime ideal (a prime filter). Now, we claim that the pair $(P_F, L \setminus P_F)$ ($(L \setminus P_I, P_I)$) is an appropriate splitting pair for F and I. It is obvious that $P_F \not\sqsubseteq L \setminus P_F$ ($L \setminus P_I \not\sqsubseteq P_I$). Since $F \subseteq P_F$ ($I \subseteq P_I$), we have $P_F \sqsubseteq F$ ($I \sqsubseteq P_I$). Furthermore, since $P_F \cap I = \emptyset$ ($F \cap P_I = \emptyset$), we also have $I \subseteq L \setminus P_F$ ($F \subseteq L \setminus P_I$) as well. Hence $I \sqsubseteq L \setminus P_F$ ($L \setminus P_I \sqsubseteq F$). Finally, for every ideal $J \in \mathcal{I}$, if $P_F \not\sqsubseteq J$, then $P_F \cap J = \emptyset$. So we obtain $J \subseteq L \setminus P_F$; that is, $J \sqsubseteq L \setminus P_F$. Therefore, the splitting condition holds. Note that we can also analogously prove that the pair $(L \setminus P_I, P_I)$ is an appropriate splitting pair as well. □

Remark 3.10 The axiom of choice is essential to prove Theorem 3.9 for the prime filter (ideal) theorem.

From distributive polarity frames, we can construct the dual algebras as the (generalized) Dedekind–MacNeille completion (see Section 2). Namely, for a distributive polarity frame $\mathbb{F}$, the *dual algebra of* $\mathbb{F}$, denoted by $\mathbb{F}^+$, is the (generalized) Dedekind–MacNeille completion of $\mathbb{F}$.

Theorem 3.11 *For every distributive polarity frame* $\mathbb{F}$*, the dual algebra* $\mathbb{F}^+$ *is a distributive lattice.*

Proof The nontrivial part follows from Theorem 3.6 (see also [11, Theorem 5.3]). □

4 Prime Skeletons

In the previous section, we saw that, on distributive polarity frames, splitters or splitting pairs play central roles. In other words, splitters and splitting pairs essentially work to validate the distributivity on polarity frames. In this section, we will carefully investigate splitters on distributive polarity frames. To this end, hereinafter, we take care only of nontrivial polarity frames, that is, polarity frames $\langle X, Y, B \rangle$ without $B = X \times Y$. This allows us to build a consistent theory. Otherwise, we would encounter a special situation: every trivial polarity frame is distributive by definition, but there is no splitting pair (splitter) on it.

Let $\mathbb{F} = \langle X, Y, \leq \rangle$ be a nontrivial distributive polarity frame. We let X_s be the set of all splitters in X, Y_s be the set of all splitters in Y, and $\leq_s$ be the restriction of $\leq$ on $X_s \cup Y_s$. Then, we call the triple $\mathbb{F}_s = \langle X_s, Y_s, \leq_s \rangle$ the *prime skeleton of* $\mathbb{F}$. Note that, when we consider a trivial polarity frame, the prime skeleton is not a polarity since X_s and Y_s are empty.

Proposition 4.1 *For every nontrivial distributive polarity frame* $\mathbb{F}$*, the prime skeleton* $\mathbb{F}_s$ *of* $\mathbb{F}$ *is a nontrivial distributive polarity frame as well.*

Next, we compare distributive polarity frames with their prime skeletons via d-morphisms. Let $\mathbb{F} = \langle X, Y, \leq \rangle$ be a distributive polarity frame, and let $\mathbb{F}_s$ be the prime skeleton of $\mathbb{F}$. By the construction of the prime skeleton $\mathbb{F}_s$, we easily notice that there are two natural embedding functions $\varepsilon_X : X_s \to X$ and $\varepsilon_Y : Y_s \to Y$, because X_s and Y_s are subsets of X and Y, respectively. For these embedding functions, we prove the following proposition.

Proposition 4.2 *Let $\mathbb{F} = \langle X, Y, \leq \rangle$ be a distributive polarity frame. Then the pair $\langle \varepsilon_X | \varepsilon_Y \rangle$ of the natural embeddings $\varepsilon_X : X_s \to X$ and $\varepsilon_Y : Y_s \to Y$ forms a d-reflecting morphism from $\mathbb{F}_s$ to $\mathbb{F}$; that is, $\langle \varepsilon_X | \varepsilon_Y \rangle : \mathbb{F}_s \to \mathbb{F}$ is d-reflecting.*

Proof We will show that the pair $\langle \varepsilon_X | \varepsilon_Y \rangle$ satisfies the conditions in Definition 2.7.

(Item 1). For arbitrary $x_s \in X_s$ and $y_s \in Y_s$, if $\varepsilon_X(x_s) \leq_s \varepsilon_Y(y_s)$, then $x_s \leq y_s$, because ε_X and ε_Y are the natural embeddings and $\leq_s$ is the restriction of $\leq$ on $X_s \cup Y_s$.

(Item 2). We prove this by contraposition. For arbitrary $x_s \in X_s$ and $y \in Y$, suppose that $\varepsilon_X(x_s) \not\leq y$; that is, $x_s \not\leq y$. Since x_s is a splitter, there exists a splitting counterpart $y_x \in Y$ of x_s such that $x_s \not\leq y_x$ and, for each $y' \in Y$, if $x_s \not\leq y'$, then $y' \leq y_x$. So we obtain $y \leq y_x$. Moreover, since y_x is also a splitter (see Proposition 3.1), $y_x \in Y_s$, which means $y \leq \varepsilon_Y(y_x)$. However, by definition, we have $x_s \not\leq_s y_x$ as well. Therefore, there exists $y_x \in Y_s$ such that $y \leq \varepsilon_Y(y_x)$ but $x_s \not\leq_s y_x$.

(Item 3). We prove this by contraposition. For arbitrary $x \in X$ and $y_s \in Y_s$, suppose that $x \not\leq \varepsilon_Y(y_s)$; that is, $x \not\leq y_s$. Since y_s is a splitter, there exists a splitting counterpart $x_y \in X$ of y_s such that $x_y \not\leq y_s$ and, for each $x' \in X$, if $x \not\leq y_s$, then $x_y \leq x'$. Hence we have $x_y \leq x$. By Proposition 3.1, x_y is also a splitter, so $x_y \in X_s$. Therefore $\varepsilon_X(x_y) \leq x$. However, as $x_y \not\leq_s y_s$ by definition, there exists $x_y \in X_s$ such that $\varepsilon_X(x_y) \leq x$ but $x_y \not\leq_s y_s$.

(Item 4). This is trivial, because ε_X and ε_Y are the natural embeddings and $\leq_s$ is the restriction of $\leq$ on $X_s \cup Y_s$.

(Item 5). We prove this by contraposition. Suppose that $x \not\leq y$ for arbitrary $x \in X$ and $y \in Y$. Because $\mathbb{F}$ is distributive, there exists a splitting pair (x_s, y_s) such that $x_s \leq x$ and $y \leq y_s$. Thanks to Proposition 3.1, we know that $x_s \in X_s$ and $y_s \in Y_s$; hence $\varepsilon_X(x_s) \leq x$ and $y \leq \varepsilon_Y(y_s)$. However, since $x_s \not\leq y_s$ by definition, we conclude that $x_s \not\leq_s y_s$, which completes this clause. □

Thanks to Proposition 4.2, with help of Theorem 2.10, we can show invariance of validity between distributive polarity frames and the prime skeletons.

Theorem 4.3 *Let $\mathbb{F}$ be a distributive polarity frame, and let $\mathbb{F}_s$ be the prime skeleton of $\mathbb{F}$. For every sequent $\phi \Rightarrow \psi$,*

$$\mathbb{F} \models \phi \Rightarrow \psi \iff \mathbb{F}_s \models \phi \Rightarrow \psi.$$

Finally, we also show a property of prime skeletons which relates somewhat to results on RS-frames of distributive lattices (see [5]).

Theorem 4.4 *Let $\mathbb{F} = \langle X, Y, \leq \rangle$ be a nontrivial distributive polarity frame, and let $\mathbb{F}_s = \langle X_s, Y_s, \leq_s \rangle$ be the prime skeleton of $\mathbb{F}$. Then the two preordered subsets $\langle X_s, \leq_s \rangle$ and $\langle Y_s, \leq_s \rangle$ of $\langle X_s \cup Y_s, \leq_s \rangle$ are essentially isomorphic, namely, isomorphic up to $\leq_s$-equivalence. Hence, the quotients of $\langle X_s, \leq_s \rangle$ and $\langle Y_s, \leq_s \rangle$ with respect to $\leq_s$-equivalence are isomorphic.*

Proof For each splitter $x_s \in X_s$, we let a splitting counterpart $y_s \in Y_s$ of x_s be the corresponding element. Then, thanks to Proposition 3.1, the sets X_s and Y_s are essentially bijective. This is because every splitter is a splitting counterpart of the splitting counterpart of the splitter. Also, for each splitter, the splitting counterpart is unique up to $\leq$-equivalence. Therefore, it suffices to show that $x_s \leq_s x_t$ if and only if $y_s \leq y_t$ for all splitting pairs (x_s, y_s) and (x_t, y_t).

($\Rightarrow$). We prove this by contradiction. Suppose that $x_s \leq_s x_t$ but $y_s \not\leq_s y_t$. By the definition of $\leq$ on Y, there exists $x \in X$ such that $x \leq y_s$ but $x \not\leq y_t$. Further, as (x_t, y_t) forms a splitting pair, we have that $x_t \leq x$. By our assumption $x_s \leq_s x_t$, we also obtain $x_s \leq x$. By the transitivity of $\leq$ on $X \cup Y$, we conclude that $x_s \leq x \leq y_s$. However, this contradicts the fact that (x_s, y_s) forms a splitting pair.

($\Leftarrow$). We prove this by contradiction. Suppose that $y_s \leq_s y_t$ but $x_s \not\leq_s x_t$. By the definition of $\leq$ on X, there exists $y \in Y$ such that $x_t \leq y$ but $x_s \not\leq y$. Since (x_s, y_s) forms a splitting pair, we obtain $y \leq y_t$. Now, by the transitivity of $\leq$ on $X \cup Y$, we also obtain $x_t \leq y \leq y_t$, which contradicts the fact that (x_t, y_t) forms a splitting pair.

Therefore, $\langle X_s, \leq_s \rangle$ and $\langle Y_s, \leq_s \rangle$ are essentially isomorphic. □

5 The Distributivity on Bi-Approximation Semantics for Intuitionistic Logic

Intuitionistic logic is one of the well-studied distributive substructural logics, which can derive (or possess) the distributivity $\phi \wedge (\psi \vee \chi) \mapsto (\phi \wedge \psi) \vee (\phi \wedge \chi)$ (see, e.g., Galatos, Jipsen, and Kowalski [4]). In this section, we will introduce bi-approximation semantics for intuitionistic logic. On the framework, we study how to validate the distributivity and discuss a connection to distributive polarity frames.

For formulas of substructural logic, we extend our formulas discussed so far with three binary logical connectives *fusion* $\circ$, *residuals* $\rightarrow$, and $\leftarrow$, and two logical constants *truth* $\mathbf{t}$ and *false* $\mathbf{f}$. In substructural logic, a sequent is usually introduced as a pair of a finite list of formulas $\phi_1, \dots, \phi_n$ and a possibly empty formula ψ, denoted by $\phi_1, \dots, \phi_n \mapsto \psi$. However, since it is equivalent to a pair of two formulas $\phi_1 \circ \cdots \circ \phi_n$ and ψ, that is, $\phi_1 \circ \cdots \circ \phi_n \mapsto \psi$, we look at every sequent as a pair of two formulas ϕ and ψ, denoted by $\phi \mapsto \psi$, for consistency for the other sections. To evaluate these additional logical connectives and constants, we expand polarity frames to p-frames for substructural logic (see [11]).

Definition 5.1 (Polarity frame for substructural logic) A *polarity frame for substructural logic*, a *p-frame* for short, is an octuple $\mathbb{F} = \langle X, Y, \leq, R, O_X, O_Y, N_X, N_Y \rangle$ if $\langle X, Y, \leq \rangle$ is a polarity frame, R is a ternary relation on $X \times X \times Y$; that is, $R \subseteq X \times X \times Y$, O_X is a nonempty subset of X, N_X is a subset of X, O_Y and N_Y are subsets of Y, and $\mathbb{F}$ satisfies

- **R-order:** for all $x, x' \in X$, $x' \leq x \iff R^\circ(x, o, x')$ or $R^\circ(o, x, x')$ for some $o \in O_X$;
- **R-identity:** for each $x \in X$, $R^\circ(x, o_2, x)$ for some $o_2 \in O_X$ and $R^\circ(o_1, x, x)$ for some $o_1 \in O_X$;
- **R-transitivity:** for all $x_1, x_2, x_1', x_2' \in X$ and $y, y' \in Y$, if $x_1' \leq x_1$, $x_2' \leq x_2$, $y \leq y'$ and $R^\circ(x_1, x_2, y)$, then $R(x_1', x_2', y')$;
- **R-associativity:** for all $x_1, x_2, x_3, x \in X$, $R^\circ(x_1, x', x)$ and $R^\circ(x_2, x_3, x')$ for some $x' \in X$, if and only if $R^\circ(x_1, x_2, x'')$ and $R^\circ(x'', x_3, x)$ for some $x'' \in X$;
- **O-isom:** $O_X = \{x \in X \mid \forall y \in O_Y .\, x \leq y\}$ and $O_Y = \{y \in Y \mid \forall x \in O_X .\, x \leq y\}$;

N-isom: $N_X = \{x \in X \mid \forall y \in N_Y.\, x \leq y\}$ and $N_Y = \{y \in Y \mid \forall x \in N_X.\, x \leq y\}$;

∘-tightness: for all $x_1, x_2 \in X$ and $y \in Y$, if, for every $x \in X$, $R^\circ(x_1, x_2, x)$ implies $x \leq y$, then $R(x_1, x_2, y)$;

→-tightness: for all $x_1, x_2 \in X$ and $y \in Y$, if, for every $y_2 \in Y$, $R^\rightarrow(x_1, y_2, y)$ implies $x_2 \leq y_2$, then $R(x_1, x_2, y)$;

←-tightness: for all $x_1, x_2 \in X$ and $y \in Y$, if, for all $y_1 \in Y$, $R^\leftarrow(y_1, x_2, y)$ implies $x_1 \leq y_1$, then $R(x_1, x_2, y)$;

where $R^\circ(x_1, x_2, x)$, $R^\rightarrow(x_1, y_2, y)$, and $R^\leftarrow(y_1, x_2, y)$ are abbreviations of, for every $y \in Y$, $R(x_1, x_2, y)$ implies $x \leq y$, for every $x_2 \in X$, $R(x_1, x_2, y)$ implies $x_2 \leq y_2$, and for every $x_1 \in X$, $R(x_1, x_2, y)$ implies $x_1 \leq y_1$, respectively.

Note that doppelgänger valuations are the same as those on polarity frames. Given a bi-approximation model $\mathbb{M} = \langle \mathbb{F}, V \rangle$, we evaluate the additional logical connectives and constants as follows: for each $x \in X$, each $y \in Y$, and all formulas ϕ and ψ,

X-4: $\mathbb{M} \stackrel{x}{\models} \phi \circ \psi \iff$ for each $y' \in Y$, if $\mathbb{M} \underset{y'}{\models} \phi \circ \psi$, then $x \leq y'$;

X-5: $\mathbb{M} \stackrel{x}{\models} \phi \to \psi \iff$ for all $x_1 \in X$ and $y' \in Y$, if $\mathbb{M} \stackrel{x_1}{\models} \phi$ and $\mathbb{M} \underset{y'}{\models} \psi$, then $R(x_1, x, y')$;

X-6: $\mathbb{M} \stackrel{x}{\models} \psi \leftarrow \phi \iff$ for all $x_2 \in X$ and $y' \in Y$, if $\mathbb{M} \stackrel{x_2}{\models} \phi$ and $\mathbb{M} \underset{y'}{\models} \psi$, then $R(x, x_2, y')$;

X-7: $\mathbb{M} \stackrel{x}{\models} \mathbf{t} \iff x \in O_X$;

X-8: $\mathbb{M} \stackrel{x}{\models} \mathbf{f} \iff x \in N_X$;

Y-4: $\mathbb{M} \underset{y}{\models} \phi \circ \psi \iff$ for all $x_1, x_2 \in X$, if $\mathbb{M} \stackrel{x_1}{\models} \phi$ and $\mathbb{M} \stackrel{x_2}{\models} \psi$, then $R(x_1, x_2, y)$;

Y-5: $\mathbb{M} \underset{y}{\models} \phi \to \psi \iff$ for each $x' \in X$, if $\mathbb{M} \stackrel{x'}{\models} \phi \to \psi$, then $x' \leq y$;

Y-6: $\mathbb{M} \underset{y}{\models} \psi \leftarrow \phi \iff$ for each $x' \in X$, if $\mathbb{M} \stackrel{x'}{\models} \psi \leftarrow \phi$, then $x' \leq y$;

Y-7: $\mathbb{M} \underset{y}{\models} \mathbf{t} \iff y \in O_Y$;

Y-8: $\mathbb{M} \underset{y}{\models} \mathbf{f} \iff y \in N_Y$.

The fundamental properties for polarity frames in Section 2 are naturally extended to those on p-frames as well. In particular, the basic substructural logic **FL**, named after the *full Lambek calculus (FL)*, is sound and complete with respect to the class of p-frames. In the light of substructural logic, intuitionistic logic can be explicated as a collection of derivable sequents on the sequent calculus FL extended by two types of axioms: *weakening* $p \mapsto \mathbf{t}$ and $\mathbf{f} \mapsto p$, and *contraction* $p \mapsto p \circ p$. Hence we may sometimes denote intuitionistic logic by $\mathbf{FL}_{wc}$, which means the set of derivable sequents in FL extended by weakening and contraction.

In [14], we have shown that every substructural logic extended by sequents (axioms) which have consistent variable occurrence is complete with respect to a class of first-order definable p-frames. In addition, the first-order sentences are algorithmically computable. In fact, since the weakening axioms and the contraction axiom have consistent variable occurrence, we can calculate the first-order correspondents as follows:

Weakening: $\forall x \in X.\ x \in O_X$ and $\forall y \in Y.\ y \in N_Y$,
Contraction: $\forall x \in X.\ R^\circ(x, x, x)$.

Therefore, intuitionistic logic is sound and complete with respect to the class of p-frames satisfying the above first-order sentences. Hereinafter, we call p-frames satisfying the above first-order sentences *p-frames for intuitionistic logic*, or *intp-frames* for short. Because the distributivity $\phi \wedge (\psi \vee \chi) \mapsto (\phi \wedge \psi) \vee (\phi \wedge \chi)$ is derivable in intuitionistic logic, each intp-frame must validate the distributivity. Now let us check this fact. Note that, in the following argument, we fully investigate that fusion $\circ$ on $\mathbf{FL}_{wc}$ corresponds to conjunction $\wedge$ on the level of intp-frames.

Let $\mathbb{F} = \langle X, Y, \leq, R, O_X, O_Y, N_X, N_Y \rangle$ be an intp-frame, and let V be a doppelgänger valuation on $\mathbb{F}$. For arbitrary $x \in X$ and $y \in Y$, suppose that $\langle \mathbb{F}, V \rangle \overset{x}{\Vdash} \phi \wedge (\psi \vee \chi)$ and $\langle \mathbb{F}, V \rangle \overset{x}{\Vdash} (\phi \wedge \psi) \vee (\phi \wedge \chi)$. By definition, we have $\langle \mathbb{F}, V \rangle \overset{x}{\Vdash} \phi$, $\langle \mathbb{F}, V \rangle \overset{x}{\Vdash} \psi \vee \chi$, $\langle \mathbb{F}, V \rangle \underset{y}{\Vdash} \phi \wedge \psi$, and $\langle \mathbb{F}, V \rangle \underset{y}{\Vdash} \phi \wedge \chi$. Now we claim the following:

$$\langle \mathbb{F}, V \rangle \underset{y}{\Vdash} \phi \wedge \psi \iff \langle \mathbb{F}, V \rangle \underset{y}{\Vdash} \phi \circ \psi.$$

($\Rightarrow$). For arbitrary $x_1, x_2 \in X$, assume that $\langle \mathbb{F}, V \rangle \overset{x_1}{\Vdash} \phi$ and $\langle \mathbb{F}, V \rangle \overset{x_2}{\Vdash} \psi$. Let $x \in X$ satisfying $R^\circ(x_1, x_2, x)$. Because of the weakening conditions, we have x_1 in O_X and $x_2 \in O_X$, which derives $x \leq x_1$ and $x \leq x_2$ by the R-order of Definition 5.1. Thanks to the hereditary Proposition 2.2, we also obtain that $\langle \mathbb{F}, V \rangle \overset{x}{\Vdash} \phi$ and $\langle \mathbb{F}, V \rangle \overset{x}{\Vdash} \psi$; hence $\langle \mathbb{F}, V \rangle \overset{x}{\Vdash} \phi \wedge \psi$. By our assumption $\langle \mathbb{F}, V \rangle \underset{y}{\Vdash} \phi \wedge \psi$ and Proposition 2.3, we derive $x \leq y$. Finally, by the $\circ$-tightness condition of Definition 5.1, we conclude $R(x_1, x_2, y)$, which means $\langle \mathbb{F}, V \rangle \underset{y}{\Vdash} \phi \circ \psi$.

($\Leftarrow$). For any $x \in X$, if $\langle \mathbb{F}, V \rangle \overset{x}{\Vdash} \phi \wedge \psi$, by definition we have $\langle \mathbb{F}, V \rangle \overset{x}{\Vdash} \phi$ and $\langle \mathbb{F}, V \rangle \overset{x}{\Vdash} \psi$. So by our assumption $\langle \mathbb{F}, V \rangle \underset{y}{\Vdash} \phi \circ \psi$, we obtain that $R(x, x, y)$. Moreover, as $\mathbb{F}$ satisfies the contraction condition, we also have $R^\circ(x, x, x)$. Hence, by the definition of R°, we get $x \leq y$, which concludes $\langle \mathbb{F}, V \rangle \underset{y}{\Vdash} \phi \wedge \psi$.

Analogously, we also have that $\langle \mathbb{F}, V \rangle \underset{y}{\Vdash} \phi \wedge \chi \iff \langle \mathbb{F}, V \rangle \underset{y}{\Vdash} \phi \circ \chi$. Furthermore, by [11, Theorem 4.4], we also obtain

1. $\langle \mathbb{F}, V \rangle \underset{y}{\Vdash} \phi \wedge \psi \iff$ for all $x_1 \in X$ and $y_2 \in Y$, if $\langle \mathbb{F}, V \rangle \overset{x_1}{\Vdash} \phi$ and $R^{\rightarrow}(x_1, y_2, y)$, then $\langle \mathbb{F}, V \rangle \underset{y_2}{\Vdash} \psi$;
2. $\langle \mathbb{F}, V \rangle \underset{y}{\Vdash} \phi \wedge \chi \iff$ for all $x_1 \in X$ and $y_2 \in Y$, if $\langle \mathbb{F}, V \rangle \overset{x_1}{\Vdash} \phi$ and $R^{\rightarrow}(x_1, y_2, y)$, then $\langle \mathbb{F}, V \rangle \underset{y_2}{\Vdash} \chi$.

Now let us come back to the original argument. We currently have $\langle \mathbb{F}, V \rangle \overset{x}{\Vdash} \phi$, $\langle \mathbb{F}, V \rangle \overset{x}{\Vdash} \psi \vee \chi$, $\langle \mathbb{F}, V \rangle \underset{y}{\Vdash} \phi \wedge \psi$, and $\langle \mathbb{F}, V \rangle \underset{y}{\Vdash} \phi \wedge \chi$. To conclude the validity, it suffices to show that $R(x, x, y)$. This is because, on $\mathbb{F}$, we have $R^\circ(x, x, x)$, so if we have $R(x, x, y)$ we can conclude $x \leq y$ because of the definition of R°.

To prove $R(x, x, y)$, we use the $\rightarrow$-tightness condition of Definition 5.1. For any $y' \in Y$, if $R^{\rightarrow}(x, y', y)$ holds, as $\langle \mathbb{F}, V \rangle \overset{x}{\Vdash} \phi$, we obtain that $\langle \mathbb{F}, V \rangle \underset{y'}{\Vdash} \psi$ and $\langle \mathbb{F}, V \rangle \underset{y'}{\Vdash} \chi$ by the above item 1 and item 2. Therefore, we have $\langle \mathbb{F}, V \rangle \underset{y'}{\Vdash} \psi \vee \chi$.

Now, since $\langle \mathbb{F}, V\rangle \underset{x}{\Vdash} \psi \vee \chi$, we have $x \leq y'$ by Proposition 2.3. Furthermore, by the $\rightarrow$-tightness condition of Definition 5.1, we conclude $R(x, x, y)$. Therefore, the distributivity is valid on any intp-frame.

Remark 5.2 The weakening conditions and the contraction condition are used to prove that $\langle \mathbb{F}, V\rangle \underset{y}{\Vdash} \phi \wedge \psi \iff \langle \mathbb{F}, V\rangle \underset{y}{\Vdash} \phi \circ \psi$. Note that the tightness conditions are essentially working to show the adjointness (residuation).

Comparison with distributive polarity frames As we saw above, to validate the distributivity on intp-frames, we have not, at least explicitly, used any condition such as splitters or splitting pairs at all. Instead, we have manipulated the tightness conditions and the adjointness to show the distributivity. So it is natural to ask how intp-frames relate to distributive polarity frames and the splitting condition. Before tackling this question, we first consider the ternary relation R on intp-frames.

Let $\mathbb{F}$ be an intp-frame. Here we claim that for all $x_1, x_2 \in X$ and $y \in Y$,

$$R(x_1, x_2, y) \iff \forall x \in X.[x \leq x_1 \text{ and } x \leq x_2 \Longrightarrow x \leq y]. \tag{1}$$

($\Rightarrow$). Assume $R(x_1, x_2, y)$. For any $x \in X$, if $x \leq x_1$ and $x \leq x_2$, we obtain that $R(x, x, y)$ by the R-transitivity condition of Definition 5.1. Since $\mathbb{F}$ satisfies the contraction condition, we have that $R^\circ(x, x, x)$; that is, for each $y' \in Y$, if $R(x, x, y')$, then $x \leq y'$. Hence, $x \leq y$.

($\Leftarrow$). By [11, Lemma 4.2], on p-frames, $R(x_1, x_2, y)$ is equivalent to the following condition: for each $x \in X$, if $R^\circ(x_1, x_2, x)$, then $x \leq y$. So we will show this condition. Assume $R^\circ(x_1, x_2, x)$ for an arbitrary $x \in X$. As $\mathbb{F}$ satisfies the weakening conditions, x_1 and x_2 are in O_X; hence we have $x \leq x_1$ and $x \leq x_2$ by the R-order condition of Definition 5.1. By our assumption, we conclude $x \leq y$, which shows the equivalence in equation (1).

By the above equivalence, we can say that the auxiliary condition $R^\circ(x_1, x_2, x)$ is the same as $x \leq x_1$ and $x \leq x_2$. Furthermore, the above equivalence tells us that, on intp-frames, the ternary relation R is reconstructed from the binary relation B (or $\leq$).

Remark 5.3 It does not mean that every polarity frame is transferable to an intp-frame. However, from some specific distributive polarity frames, we can reconstruct intp-frames.

Let us look at an example of how to construct an intp-frame from a distributive polarity frame. Let $\langle X, Y, B\rangle$ be a polarity frame with $X = \{x_1, x_2, x_3, x_4, x_5\}$, $Y = \{y_1, y_2, y_3\}$, and the binary relation B is given by Table 1. In Table 1, we let 1 and 0 represent "related" and "not related," respectively. For example, we can learn $x_1 B y_1$ but $x_1 \not B y_2$ from the table.

Based on this binary relation B, we can obtain the posets $\langle X, \leq\rangle$ and $\langle Y, \leq\rangle$ represented by the following Hasse diagrams in Figure 2. Now we induce a ternary relation R by the above equivalence in equation (1). Table 2 shows the results. On the induced ternary relation R, if we let $O_X = \{x_1, x_2, x_3, x_4, x_5\}$, $O_Y = \emptyset$, $N_X = \{x_5\}$, and $N_Y = \{y_1, y_2, y_3\}$, they form a p-frame. Since this is a routine check, we can safely leave the proof of this part for readers. Furthermore, we claim that the p-frame $\langle X, Y, \leq, R, O_X, O_Y, N_X, N_Y\rangle$ satisfies the splitting condition; hence the base polarity $\langle X, Y, \leq\rangle$ is distributive. That is, for each pair (x_i, y_j) of disconnected elements, namely, $x_i \not\leq y_j$, we can find a splitting pair (x_s, y_s) satisfying $x_s \leq x_i$ and $y_j \leq y_s$.

Table 1 The binary relation B

B	y_1	y_2	y_3
x_1	1	0	0
x_2	1	1	0
x_3	0	0	1
x_4	1	0	1
x_5	1	1	1

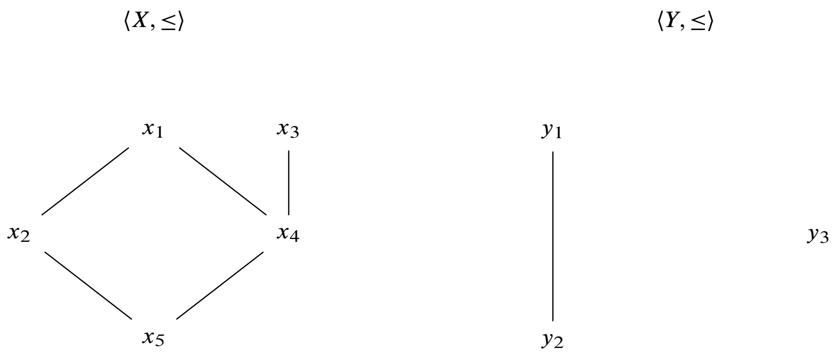

Figure 2 The posets $\langle X, \leq \rangle$ and $\langle Y, \leq \rangle$.

Table 2 The induced ternary relation R

R	y_1	y_2	y_3
x_1, x_1	1	0	0
x_1, x_2	1	1	0
x_1, x_3	1	0	1
x_1, x_4	1	0	1
x_1, x_5	1	1	1
x_2, x_1	1	1	0
x_2, x_2	1	1	0
x_2, x_3	1	1	1
x_2, x_4	1	1	1

R	y_1	y_2	y_3
x_2, x_5	1	1	1
x_3, x_1	1	0	1
x_3, x_2	1	1	1
x_3, x_3	0	0	1
x_3, x_4	1	0	1
x_3, x_5	1	1	1
x_4, x_1	1	0	1
x_4, x_2	1	1	1

R	y_1	y_2	y_3
x_4, x_3	1	0	1
x_4, x_4	1	0	1
x_4, x_5	1	1	1
x_5, x_1	1	1	1
x_5, x_2	1	1	1
x_5, x_3	1	1	1
x_5, x_4	1	1	1
x_5, x_5	1	1	1

On this p-frame, we note that there are three splitting pairs, that is, (x_2, y_3), (x_4, y_2), and (x_3, y_1). Then, for each disconnected pair, we have an appropriate splitting pair as follows:

1. for $x_1 \not\leq y_2$, we have (x_4, y_2) as a splitting pair (note that, in this case, we can also take (x_2, y_3) as a splitting pair);
2. for $x_1 \not\leq y_3$, we have (x_4, y_2) as a splitting pair (note that, in this case, we can also take (x_2, y_3) as a splitting pair);
3. for $x_2 \not\leq y_3$, we have (x_2, y_3) itself as a splitting pair;
4. for $x_3 \not\leq y_1$, we have (x_3, y_1) itself as a splitting pair;
5. for $x_3 \not\leq y_2$, we have (x_3, y_1) as a splitting pair;
6. for $x_4 \not\leq y_2$, we have (x_4, y_2) itself as a splitting pair.

Therefore, this p-frame is distributive. Here we mention that, although one may feel that it is too complicated to find an appropriate splitting pair for each disconnected pair, there is a simple algorithm for finding an appropriate splitting pair, which is in fact the main technique used to prove Theorem 5.4. Before proving Theorem 5.4, let us look at the algorithm. For any disconnected pair $x_i \not\leq y_j$,

1. take the set of all elements in Y which are disconnected to x_i and are greater than or equal to y_j;
2. choose a maximal element, say, y_m, in the set;
3. collect all elements in X which are disconnected to y_m and are less than or equal to x_i;
4. take the minimum element, say, x_m, in the set.

Then, we have obtained an appropriate splitting pair (x_m, y_m) for the disconnected pair $x_i \not\leq y_j$.

Theorem 5.4 *Every p-frame for intuitionistic logic is distributive.*

Proof It suffices to show that every intp-frame satisfies the splitting condition. Let x_i and y_j be a disconnected pair; that is, $x_i \not\leq y_j$. Then, as in item 1 of the algorithm, we take the following set $\mathfrak{Y}_{(x_i,y_j)}$:

$$\mathfrak{Y}_{(x_i,y_j)} := \{y_k \in Y \mid x_i \not\leq y_k \text{ and } y_j \leq y_k\}.$$

To use Zorn's lemma, we will show that $\mathfrak{Y} = (x_i, y_j)$ is an inductive set. By our assumption, that is, $x_i \not\leq y_j$, we have that $y_j \in \mathfrak{Y}_{(x_i,y_j)}$; hence $\mathfrak{Y}_{(x_i,y_j)}$ is nonempty. Note that each element y_k in Y is represented by the subset of all elements of X which are connected to y_k. Also note that this representation is not always unique. For example, in Table 1, y_1, y_2, and y_3 are represented by $\{x_1, x_2, x_4, x_5\}$, $\{x_2, x_5\}$, and $\{x_3, x_4, x_5\}$, respectively. Actually, the order $\leq$ on Y is nothing but the inclusion relation of these subsets. Therefore, for each chain in $\mathfrak{Y}_{(x_i,y_j)}$, the supremum is calculated as the union of these representing subsets of X. Moreover, for each chain, the representing subsets do not contain x_i by definition. As the union of every chain in $\mathfrak{Y}_{(x_i,y_j)}$ is in $\mathfrak{Y}_{(x_i,y_j)}$, the set $\mathfrak{Y}_{(x_i,y_j)}$ is inductive. Thanks to the axiom of choice, we can take a maximal element y_m in $\mathfrak{Y}_{(x_i,y_j)}$. We also mention that, by the definition of $\mathfrak{Y}_{(x_i,y_j)}$, we have that $y_j \leq y_m$.

Next, as in item 3 of the algorithm, we take the following set $\mathfrak{X}_{(x_i,y_m)}$:

$$\mathfrak{X}_{(x_i,y_m)} := \{x_k \in X \mid x_k \not\leq y_m \text{ and } x_k \leq x_i\}.$$

Again, we claim that $\mathfrak{X}_{(x_i,y_m)}$ is an inductive set, to use Zorn's lemma. We take the same strategy. As $y_m \in \mathfrak{Y}_{(x_i,y_j)}$, we have $x_i \in \mathfrak{X}_{(x_i,y_m)}$. Hence, $\mathfrak{X}_{(x_i,y_m)}$

is nonempty. As above, each element x_k in X is representable by the subset of all elements in Y which are connected to x_k. Then, the order $\leq$ on X is the *reverse* inclusion $\supseteq$ of these representing subsets of Y. For each chain in $\mathfrak{X}_{(x_i, y_m)}$, the infimum (the supremum with respect to the inclusion relation) is computable as the union of the representing subsets. Because each element in $\mathfrak{X}_{(x_i, y_m)}$ is disconnected to y_m, no representing subset contains y_m. So $\mathfrak{X}_{(x_i, y_m)}$ is inductive. Therefore, by the axiom of choice, we can take an infimum x_m (a supremum with respect to the inclusion order $\subseteq$).

Furthermore, we also want to state that this infimum x_m is actually the minimum element of $\mathfrak{X}_{(x_i, y_m)}$. For this purpose, we show that the set $\mathfrak{X}_{(x_i, y_m)}$ is down-directed; namely, for arbitrary $x_k, x_l \in \mathfrak{X}_{(x_i, y_m)}$, there exists $x \in X$ such that $x \leq x_k$, $x \leq x_l$, and $x \in \mathfrak{X}_{(x_i, y_m)}$. We prove it by contradiction. That is, we will derive a contradiction on the assumption that there exist $x_k, x_l \in \mathfrak{X}_{(x_i, y_m)}$ such that, for any x, if $x \leq x_k$ and $x \leq x_l$, then $x \notin \mathfrak{X}_{(x_i, y_m)}$; that is, $x \leq y_m$.

Suppose that, for $x_k, x_l \in \mathfrak{X}_{(x_i, y_m)}$ and any $x \in X$, if $x \leq x_k$ and $x \leq x_l$, then $x \leq y_m$. Since $\mathbb{F}$ is an intp-frame, we obtain that $R(x_k, x_l, y_m)$ by the equivalence in equation (1).

Next, to obtain $R(x_k, x_i, y_m)$, we consider the $\rightarrow$-tightness condition of Definition 5.1 for x_k, x_i, and y_m:

$$\forall y_2 \in Y.\big[R^{\rightarrow}(x_k, y_2, y_m) \Longrightarrow x_i \leq y_2\big] \Longrightarrow R(x_k, x_i, y_m). \tag{2}$$

For any $y_2 \in Y$, if $R^{\rightarrow}(x_k, y_2, y_m)$, then $x_l \leq y_2$, because we have already obtained $R(x_k, x_l, y_m)$. By the definition of $\leq$ on Y and the fact that $x_l \in \mathfrak{X}_{(x_i, y_m)}$, we have that $x_l \not\leq y_m$; hence $y_2 \not\leq y_m$. Here, we want to show that $y_m \leq y_2$, namely, y_2 is strictly greater than y_m, to state that $x_i \leq y_2$. For any $x' \in X$, suppose that $x' \leq y_m$. Now, we look at the $\circ$-tightness condition of Definition 5.1 for x_k, x', and y_m:

$$\forall x'' \in X.\big[R^{\circ}(x_k, x', x'') \Longrightarrow x'' \leq y_m\big] \Longrightarrow R(x_k, x', y_m). \tag{3}$$

For any $x'' \in X$, if $R^{\circ}(x_k, x', x'')$, since $\mathbb{F}$ satisfies the weakening conditions, $x'' \leq x'$; hence $x'' \leq y_m$ by our assumption $x' \leq y_m$. Therefore, by the condition (3), we obtain $R(x_k, x', y_m)$. Furthermore, by our assumption $R^{\rightarrow}(x_k, y_2, y_m)$, we also obtain that $x' \leq y_2$; hence $y_m \leq y_2$. So y_2 is strictly greater than y_m. Since y_m is a maximal element in $\mathfrak{Y}_{(x_i, y_j)}$, $x_i \leq y_2$ holds. Hence, by the condition (2), $R(x_k, x_i, y_m)$ as we claimed above.

Next, to obtain $R(x_i, x_i, y_m)$, we think about the $\rightarrow$-tightness condition of Definition 5.1 for x_i, x_i, and y_m:

$$\forall y_1 \in Y.\big[R^{\leftarrow}(y_1, x_i, y_m) \Longrightarrow x_i \leq y_1\big] \Longrightarrow R(x_i, x_i, y_m). \tag{4}$$

For any $y_l \in Y$, if $R^{\leftarrow}(y_l, x_i, y_m)$, then $x_k \leq y_l$, since we already have $R(x_k, x_i, y_m)$. By the definition of $\leq$ on Y and the fact that $x_k \in \mathfrak{X}_{(x_i, y_m)}$, we have that $x_k \not\leq y_m$; hence $y_l \not\leq y_m$. Again, here we would like to show that $y_m \leq y_l$, that is, that y_l is strictly greater than y_m. For any $x' \in X$, suppose that $x' \leq y_m$. Now, apply the $\circ$-tightness condition of Definition 5.1 for x', x_i, and y_m:

$$\forall x'' \in X.\big[R^{\circ}(x', x_i, x'') \Longrightarrow x'' \leq y_m\big] \Longrightarrow R(x', x_i, y_m). \tag{5}$$

For any $x'' \in X$, if $R^{\circ}(x', x_i, x'')$, since $\mathbb{F}$ satisfies the weakening conditions, we have $x'' \leq x'$; hence $x'' \leq y_m$ by our assumption $x' \leq y_m$. By the condition (5), we obtain that $R(x', x_i, y_m)$. Moreover, by our assumption $R^{\leftarrow}(y_l, x_i, y_m)$, we also have that $x' \leq y_l$. Then, we get $y_m \leq y_l$, which means that y_l is strictly greater than

y_m. Since y_m is a maximal element in $\mathfrak{Y}_{(x_i,y_j)}$, hence $x_i \leq y_l$ holds. Therefore, by the condition (4), we obtain $R(x_i, x_i, y_m)$.

Here we have that $R^\circ(x_i, x_i, x_i)$, because $\mathbb{F}$ satisfies the contraction condition. We also have that $R(x_i, x_i, y_m)$. Hence, we conclude that $x_i \leq y_m$. However, it contradicts the fact that y_m is a maximal element in $\mathfrak{Y}_{(x_i,y_j)}$. Therefore, $\mathfrak{X}_{(x_i,y_m)}$ is down-directed. Note that the fact derives that the infimum x_m in $\mathfrak{X}_{(x_i,y_m)}$ is actually the least element in $\mathfrak{X}_{(x_i,y_m)}$. We also mention that $x_m \leq x_i$ by definition.

Finally, we show that the pair (x_m, y_m) is a splitting pair. Namely, $x_m \not\leq y_m$ and $\forall x' \in X.[x' \not\leq y_m \Longrightarrow x_m \leq x']$. However, these follow straightforwardly from the fact that x_m is the minimum element in $\mathfrak{X}_{(x_i,y_m)}$. □

Remark 5.5 There are three comments for the proof of Theorem 5.4.

1. The axiom of choice is essential to find a maximal element y_m in $\mathfrak{Y}_{(x_i,y_j)}$ and the minimum element x_m of $\mathfrak{X}_{(x_i,y_m)}$.
2. We manipulate all three tightness conditions (residuation) to prove that $\mathfrak{X}_{(x_i,y_m)}$ is down-directed in general. However, as in the above example, for p-frame for intuitionistic logic satisfying
$$R(x_1, x_2, y) \iff x_1 \leq y \text{ or } x_2 \leq y,$$
we have a simpler way to show the down-directedness of $\mathfrak{X}_{(x_i,y_m)}$.
3. The algorithm to find splitting pairs is symmetric. That is, we can choose a minimal element x_m, first. After that, we can take the maximum element y_m.

6 Conclusion

In the current paper, we have introduced special elements in polarity frames, named *splitters*, to characterize the distributivity on bi-approximation semantics. As we saw in Section 3, splitters play central roles to validate the distributivity. Plus, we have also studied that, to obtain the dual representation between distributive lattices and distributive polarity frames, the axiom of choice essentially works. By introducing the prime skeletons, we have also learned how splitters work on distributive polarity frames, and we have shown invariance of validity of sequents between distributive polarity frames and the prime skeletons. On the other hand, by applying the Sahlqvist theorem for substructural logic on bi-approximation semantics, we have also considered bi-approximation semantics for intuitionistic logic, which obviously validate the distributive law, but which does not, at least explicitly, guarantee the existence of splitting pairs. The interesting things are:

1. To prove that bi-approximation semantics for intuitionistic logic is based on distributive polarity frames, the axiom of choice is needed.
2. Nevertheless, we can validate the distributive law on bi-approximation semantics for intuitionistic logic without the axiom of choice but by manipulating the adjointness $\wedge \vdash \rightarrow$.

As a natural consequence, we obtain interesting questions: is there any constructive characterization of the distributivity depending neither on the axiom of choice nor on the adjointness? And: is there any interesting connection between the axiom of choice and the adjointness?

One may also question how the first-order definability for Kripke semantics of distributive substructural logics relates to bi-approximation semantics of distributive substructural logics. But, we already have a possible answer for this question, and it will appear in this author's subsequent work.

Note

1. In case $x = y$ for $x \in X$ and $y \in Y$, xBy has to hold.

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