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Algebraicity and Implicit Definability in Set Theory

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Abstract We analyze the effect of replacing several natural uses of definability in set theory by the weaker model-theoretic notion of algebraicity. We find, for example, that the class of hereditarily ordinal algebraic sets is the same as the class of hereditarily ordinal definable sets; that is, HOA $=$ HOD. Moreover, we show that every (pointwise) algebraic model of ZF is actually pointwise definable. Finally, we consider the implicitly constructible universe Imp—an algebraic analogue of the constructible universe—which is obtained by iteratively adding not only the sets that are definable over what has been built so far, but also those that are algebraic (or, equivalently, implicitly definable) over the existing structure. While we know that Imp can differ from L, the subtler properties of this new inner model are just now coming to light. Many questions remain open.

1 Introduction

We aim here to analyze the effect of replacing several natural uses of definability in set theory by the weaker model-theoretic notion of algebraicity and its companion concept of implicit definability. In place of the class HOD of hereditarily ordinal definable sets, for example, we consider the class HOA of hereditarily ordinal-algebraic sets. In place of the pointwise definable models of set theory, we examine its (pointwise) algebraic models. And in place of Gödel's constructible universe L, obtained by iterating the definable power set operation, we introduce the implicitly constructible universe Imp, obtained by iterating the algebraic or implicitly definable power set operation. In each case we investigate how the change from definability to algebraicity affects the nature of the resulting concept. We are especially intrigued by Imp, a new inner model of ZF whose subtler properties are just now coming to light. Open questions about Imp abound.

Before proceeding further, let us review the basic definability definitions. In the model theory of first-order logic, an element a is *definable* (without parameters) in

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a structure M if it is the unique object in M satisfying some first-order property φ there, that is, if $M \models \varphi[b]$ just in case $b = a$. More generally, an element a is *algebraic* in M if it has a property φ exhibited by only finitely many objects in M, so that $\{ b \in M \mid M \models \varphi[b] \}$ is a finite set containing a. For each class $P \subseteq M$ we can similarly define what it means for an element to be P-definable or P-algebraic by allowing the formula φ to have parameters from P. Since each element of a structure M for a language with equality is trivially M-definable, the notion of a P-definable element is interesting only when the inclusion $P \subseteq M$ is proper.

In the second-order context, a subset or class $A \subseteq M^n$ is said to be *definable* in M (again without parameters, unless otherwise specified) if $A = \{ \vec{a} \in M \mid M \models \varphi[\vec{a}] \}$ for some first-order formula φ. In other words, A is definable in M if it is the unique class in M^n such that $\langle M, A \rangle \models \forall \vec{x}\, [\varphi(\vec{x}) \leftrightarrow A(\vec{x})]$, where $\forall \vec{x}\, [\varphi(\vec{x}) \leftrightarrow A(\vec{x})]$ is a first-order formula in the expanded language with a predicate symbol for A. Generalizing this condition, we say that a class $A \subseteq M^n$ is *implicitly definable* in M if there is a first-order formula $\psi(A)$ in the expanded language, not necessarily of the form $\forall \vec{x}\, [\varphi(\vec{x}) \leftrightarrow A(\vec{x})]$, with the property that A is the unique class for which $\langle M, A \rangle \models \psi(A)$. While every (explicitly) definable class is thus also implicitly definable, we will see below that the converse can fail. Generalizing even more, we say that a class $A \subseteq M^n$ is *algebraic* in M if there is a first-order formula $\psi(A)$ in the expanded language such that $\langle M, A \rangle \models \psi(A)$ and there are only finitely many $B \subseteq M^n$ for which $\langle M, B \rangle \models \psi(B)$. Allowing parameters from a fixed class $P \subseteq M$ to appear in ψ yields the notions of P-definability, implicit P-definability, and P-algebraicity in M. Simplifying terminology, we say that A is definable, implicitly definable, or algebraic *over* (rather than *in*) M if it is M-definable, implicitly M-definable, or M-algebraic in M, respectively. (Note that the notion of M-definable class is not the same as the trivial notion of M-definable element.) A natural generalization of these concepts arises by allowing second-order quantifiers to appear in ψ. Thus we may speak of a class A as second-order definable, implicitly second-order definable, or second-order algebraic. Further generalizations are of course possible by allowing ψ to use resources from other strong logics.

2 Ordinal Algebraicity

To begin our project, let us consider the class HOA of hereditarily ordinal algebraic sets. In a strong second-order theory such as KM, which proves the existence of satisfaction classes for first-order set-theoretic truth, we may stipulate explicitly that a set a is *ordinal algebraic* if it is Ord-algebraic in V, that is, if for some first-order formula φ and ordinal parameters $\vec{\alpha}$ we have that $\varphi(a, \vec{\alpha})$ holds and there are only finitely many b for which $\varphi(b, \vec{\alpha})$ holds. But because this definition invokes a truth predicate for first-order formulas, we cannot formalize it in the theory GBC, which does not prove the existence of such a predicate, or in ZFC, which by Tarski's nondefinability theorem cannot consistently have such a predicate. Of course, the same problem arises already with the notion of ordinal definability, which one would like to characterize by saying that a set a is ordinal definable if for some first-order φ and ordinal parameters $\vec{\alpha}$ we have that $\varphi(b, \vec{\alpha})$ holds just in case $b = a$. In that case, the familiar solution is to stipulate instead that a set a is ordinal definable if it is θ-definable in some V_θ, that is, using ordinal parameters below θ and referring only to truth in the set V_θ. This condition can be formalized in ZFC, and, moreover (by the Lévy–Montague reflection theorem for first-order formulas), it is provably equivalent in KM to the condition that a is Ord-definable in V.[1] Similarly, in the present context we will stipulate that a set a is *ordinal algebraic* if it is θ-algebraic in some V_θ. Again, this can be formalized in ZFC and is provably equivalent in KM to the condition that a is Ord-algebraic in V. The point is that if a is one of finitely many satisfying instances of $\varphi(\cdot, \vec{\alpha})$ in V, then this fact will reflect to some V_θ with θ exceeding each of the parameters in $\vec{\alpha}$. Conversely, if a is θ-algebraic in V_θ via $\varphi(\cdot, \vec{\alpha})$, then because V_θ is definable from parameter θ, the object a will be algebraic in V via $\varphi(\cdot, \vec{\alpha})^{V_\theta}$. Thus the collection OA of ordinal algebraic sets is definable as a class in V. Likewise, then, for the collection HOA of *hereditarily* ordinal algebraic sets, that is, ordinal algebraic sets whose transitive closures contain only ordinal algebraic sets.[2]

Theorem 1 *The class of hereditarily ordinal algebraic sets is the same as the class of hereditarily ordinal definable sets:*

$$\mathrm{HOA} = \mathrm{HOD}.$$

Proof Clearly HOD $\subseteq$ HOA. Conversely, we show by $\in$-induction that every element of HOA is in HOD. Suppose that $a \in$ HOA and assume inductively that every element of a is in HOD, so that $a \subseteq$ HOD. Since a is ordinal algebraic, there is an ordinal definable set $A = \{a_0, \ldots, a_n\}$ containing it. We may assume that each a_i is a subset of HOD, by adding this condition to the definition of A. The definable well-ordering of HOD in V gives rise to a definable bijection $h : \mathrm{Ord} \to \mathrm{HOD}$, where $h(\beta)$ is the βth element of HOD in that ordering. Thus subsets of HOD are definably identified via h with sets of ordinals, and these in turn are definably linearly ordered, giving rise to a definable linear order $\prec$ on subsets of HOD. Namely, $b \prec c$ if and only if the HOD-least element of the symmetric difference of b and c is in c, that is, if $h(\min(h^{-1}b \mathbin{\triangle} h^{-1}c)) \in c$, where $h^{-1}x = \{\beta \mid h(\beta) \in x\}$. Thus A is a finite ordinal definable set with a definable linear order $\prec$. So each a_i is ordinal definable as the kth element of A with respect to $\prec$, for some k, using the same parameters as the definition of A itself. In particular, a is ordinal definable as desired. □

Since the foregoing proof uses the hereditary nature of HOA, it does not seem to show OA = OD, and for now this question remains open. Perhaps it is consistent with ZFC that some ordinal algebraic set is not ordinal definable. Or perhaps algebraicity simply coincides with definability in models of set theory. We are unsure.[3]

3 Pointwise Algebraic Models

A second application of definability in set theory concerns *pointwise definable* models, that is, structures in which each element is definable without parameters. For example, it is well known that the minimal transitive model of ZFC is pointwise definable, if it exists. Moreover, every countable model of ZFC has a pointwise definable extension by class forcing, and in fact every countable model of GBC has an extension by means of class forcing in which each set and each class is definable. Proofs of these facts and more appear in [4] by Hamkins, Linetsky, and Reitz, which also gives references to earlier literature on the topic.

Here we define a structure to be pointwise algebraic, or simply *algebraic*, if each of its elements is algebraic in it. In some mathematical theories, this is not equivalent to pointwise definability. For instance, in the language of rings $\{+,\cdot,0,1\}$, every algebraic field extension of the rational field is algebraic in the sense just defined, since each of its elements is one of finitely many solutions to a particular polynomial equation over the integers, a property expressible in this language. But since such fields can have nontrivial automorphisms—a major focus of Galois theory—they can fail to be pointwise definable. Set theory is different.

Theorem 2 *Every algebraic model of* ZF *is a pointwise definable model of* $\mathrm{ZFC} + V = \mathrm{HOD}$.

Proof If $M \models \mathrm{ZF}$ is algebraic, then of course each of its elements is ordinal algebraic in the sense of M. By Theorem 1, it follows that each element of M is ordinal definable in the sense of M, and so $M \models \mathrm{ZFC} + V = \mathrm{HOD}$. But each ordinal of M, being algebraic, belongs to a finite definable set of ordinals of M, a set that is definably linearly ordered by the membership relation of M. So each ordinal of M is in fact definable in M. Since M satisfies $V = \mathrm{HOD}$, this implies that every object in M is definable, and so M is pointwise definable. □

Corollary 3 *An extension of* ZF *has an algebraic model if and only if it is consistent with* $V = \mathrm{HOD}$.

Proof In light of Theorem 2, we need only prove the right-to-left direction. So suppose T extends ZF and is consistent with $V = \mathrm{HOD}$. Then some $M \models T$ satisfies $V = \mathrm{HOD}$. Since M thinks the universe is definably well ordered, it has definable Skolem functions. Hence the $N \subseteq M$ consisting of precisely the definable elements of M is closed under these Skolem functions and is therefore an elementary substructure of M. So every element of N is definable in N by the same formula that defines it in M. Therefore $N \models T$ is pointwise definable and hence algebraic. □

Corollary 4 *A complete extension of* ZF *has an algebraic model if and only if it has, up to isomorphism, a unique model in which each ordinal is definable.*

Proof By a result of Paris, a complete extension T of ZF proves $V = \mathrm{HOD}$ if and only if it has, up to isomorphism, a unique model in which each ordinal is definable.[4] In light of this, Corollary 3 yields the desired conclusion. □

Although Theorem 2 shows that algebraicity and definability coincide in algebraic models of set theory, let us reiterate that we do not know whether there can be a model of ZF with an algebraic nondefinable element.

4 The Implicitly Constructible Universe

Finally, we consider the algebraic analogue of Gödel's constructible universe. Gödel builds his universe L by iterating the definable power set operation P_{def}, where for any structure M the definable power set $P_{\mathrm{def}}(M)$ consists of all classes $A \subseteq M$ that are definable over M. Similarly, we define the *implicitly definable* power set of M to be the collection $P_{\mathrm{imp}}(M)$ of all classes $A \subseteq M$ that are implicitly definable over M, and we define the *algebraic* power set of M to be the collection $P_{\mathrm{alg}}(M)$ of all $A \subseteq M$ that are algebraic over M.

Observation 5 The algebraic power set of a structure M is identical to its implicitly definable power set:

$$P_{\mathrm{alg}}(M) = P_{\mathrm{imp}}(M).$$

Proof Obviously a class $A \subseteq M$ is algebraic over M if it is implicitly definable over M. For the converse, suppose $A \subseteq M$ is algebraic over M via φ. Note that each of the finitely many $B \neq A$ satisfying φ in M is distinguished from A by some parameter $a \in M$ that is in A but not B or vice versa. Conjoining to φ assertions about these parameters (either that $a \in A$ or $a \notin A$, as appropriate) produces a formula ψ witnessing that A is implicitly definable over M. □

Though algebraic classes are thus always implicitly definable, implicitly definable classes need not be (explicitly) definable. For example, if M is an ω-standard model of set theory, then its full satisfaction class $\mathrm{Sat}^M = \{\langle \ulcorner\varphi\urcorner, \vec{a}\rangle \mid M \models \varphi[\vec{a}]\}$ is implicitly definable in M as the unique class satisfying the familiar Tarskian recursive truth conditions. But Tarski's theorem on the nondefinability of truth shows precisely that Sat^M cannot be defined in M, even with parameters. It is interesting to note that this argument can fail when M is ω-nonstandard, for satisfaction classes need not be unique in such models, as shown in Krajewski [7]; see Hamkins and Yang [5] for further results.

Let us make a somewhat more attractive example, where the structure M can be ω-nonstandard and the relevant implicitly definable class A must be *amenable* to M, meaning that $A \cap a \in M$ for each $a \in M$. This is an improvement over the above example, for Sat^M can fail to be amenable to M. (When Sat^M is amenable to M, it follows that $\mathrm{Th}(M)$ is in M. But this fails, for example, in pointwise definable models of set theory, as mentioned in [4].) For the modified argument, let N be any model of ZFC, and let α_n be the least Σ_n-reflecting ordinal in N, that is, the least ordinal such that $(V_{\alpha_n})^N \prec_n N$. Such an α_n exists by the reflection theorem and is definable in N using the definability of Σ_n satisfaction. Let $M_n = (V_{\alpha_n})^N$, and let $M = \bigcup_n M_n$ be the union of the progressively elementary chain $M_0 \prec_0 M_1 \prec_1 \cdots$ of models. Since the union of a Σ_n-elementary chain is a Σ_n-elementary extension of each component of the chain, we have $M_n \prec_n M$ and so $M \prec N$. Note that $A = \{\langle \alpha_n, \ulcorner\varphi\urcorner, \vec{a}\rangle \mid \vec{a} \in M_n, \varphi \in \Sigma_n, n \in \omega, M_n \models \varphi[\vec{a}]\}$ is amenable to M. For if $a \in M$, then $a \in M_n$ for some n, whence $A \cap a$ does not contain any triples $\langle \alpha_m, \ulcorner\varphi\urcorner, \vec{a}\rangle$ for $m \geq n$, and consequently $A \cap a$ is constructible from a and the Σ_n satisfaction class of M_n together with the finitely many parameters α_k for $k < n$, all of which are in M. Furthermore, A is implicitly definable in M using a version of the Tarskian satisfaction conditions. For any two truth predicates must agree that each α_n is least such that $(V_{\alpha_n})^M \prec_n M$, and the nonstandard formulas never get a chance to appear in A, as we have defined M_n only for standard n even when N (and hence M itself) is ω-nonstandard. But A cannot be definable in M, even with parameters, since from A we can define a truth predicate for M.

Having appreciated these facts, we introduce the algebraic analogue of L, the *implicitly constructible universe*, hereby dubbed Imp and built as follows:

$$\mathrm{Imp}_0 = \emptyset, \qquad \mathrm{Imp}_{\alpha+1} = P_{\mathrm{imp}}(\mathrm{Imp}_\alpha),$$
$$\mathrm{Imp}_\lambda = \bigcup_{\alpha<\lambda} \mathrm{Imp}_\alpha, \quad \text{for limit } \lambda, \qquad \mathrm{Imp} = \bigcup_{\alpha} \mathrm{Imp}_\alpha.$$

Theorem 6 Imp *is an inner model of* ZF *with* $L \subseteq \mathrm{Imp} \subseteq \mathrm{HOD}$.

Proof Clearly Imp is a transitive class containing all ordinals and closed under the Gödel operations. Imp is almost universal as well, for each of its subsets is included in some Imp_α, each of which belongs to Imp. Any such class is an inner model of

ZF.[5] Since L is the least such inner model, we therefore have $L \subseteq \text{Imp}$. To see that $\text{Imp} \subseteq \text{HOD}$, recall from Myhill and Scott [9] that HOD is identical to the class obtained by transfinite iteration of the second-order definable power set operation. (In other words, HOD is just the second-order constructible universe.) Since the second-order definable power set of a structure M includes the implicitly definable power set $P_{\text{imp}}(M)$, the inclusion $\text{Imp} \subseteq \text{HOD}$ is immediate. One can also obtain this inclusion by transfinite induction: if $\text{Imp}_\alpha \subseteq \text{HOD}$ and $A \in \text{Imp}_{\alpha+1}$, then A is implicitly definable over Imp_α by a specific formula with parameters from Imp_α, and since these parameters and Imp_α are ordinal definable, it follows that A is ordinal definable as the unique subset of Imp_α that is implicitly defined by that formula from those parameters. That is, although A is only implicitly definable as a subset of Imp_α, this fact serves as a first-order definition of A as an element of V. □

We are unsure whether Imp must satisfy the axiom of choice. This is related to the subtle issue of whether $\text{Imp}^{\text{Imp}} = \text{Imp}$, that is, whether Imp can see that it is Imp. If so, then Imp would know that $\text{Imp}^{\text{Imp}} \subseteq \text{HOD}$ and so would satisfy the statement $V = \text{HOD}$, which implies the axiom of choice. In essence, if Imp can see that it is Imp, then it can define a well-ordering of the universe: one set precedes another when it appears earlier in the Imp hierarchy or at the same time but with a smaller formula, or with the same formula but with earlier parameters.

Unfortunately, we do not know whether $\text{Imp}^{\text{Imp}} = \text{Imp}$ must hold. To highlight a difficulty, note that one might aim to prove $\text{Imp}^{\text{Imp}} = \text{Imp}$ by showing inductively that it is true in a level-by-level manner, that is, by proving $\text{Imp}_\alpha^{\text{Imp}} = \text{Imp}_\alpha$ for each α. Of course, if this identity holds at α, then $\text{Imp}_{\alpha+1} \subseteq \text{Imp}_{\alpha+1}^{\text{Imp}}$, because any $A \in \text{Imp}_{\alpha+1}$ will be implicitly definable over $\text{Imp}_\alpha^{\text{Imp}}$ and contained in Imp, and so it will be in $\text{Imp}_{\alpha+1}^{\text{Imp}}$. But the problem for the converse is that perhaps some $B \subseteq \text{Imp}_\alpha^{\text{Imp}}$, belonging not to Imp_α but still belonging to some later stage Imp_β, will be implicitly definable over Imp_α in the sense of Imp but not in the sense of V. This will happen, for example, if $V \neq \text{Imp}$ and every formula witnessing in Imp that B is implicitly definable over Imp_α is satisfied in V also by some set other than B. So we seem not to be able to argue that $\text{Imp}_{\alpha+1}^{\text{Imp}} \subseteq \text{Imp}_{\alpha+1}$, and the inductive method of showing $\text{Imp}^{\text{Imp}} = \text{Imp}$ therefore appears to break down.

Putting that aside for the moment, let us examine the relationship between Imp and L. Observe first that for any countable structure M, the statement that $A \subseteq M$ is implicitly definable over M is a Π^1_1 assertion in the codes for A and M. Shoenfield absoluteness therefore ensures that Imp_α is absolute from V to L for $\alpha < \omega_1^L$, and so $\text{Imp}_{\omega_1^L} = (\text{Imp}_{\omega_1})^L = L_{\omega_1^L}$. Meanwhile, the Imp_α hierarchy grows faster than the L_α hierarchy. For as noted above, the satisfaction class $\{\, \langle \ulcorner\varphi\urcorner, \vec{a} \rangle \mid \text{Imp}_\alpha \models \varphi[\vec{a}] \,\}$ is implicitly, but not explicitly, definable over Imp_α. Furthermore, the satisfaction class for hyperarithmetic truth is in $P_{\text{imp}}(\text{Imp}_\omega) = \text{Imp}_{\omega+1}$ but does not appear in L until stage ω_1^{CK}. The satisfaction relation for $L_{\kappa,\lambda}$ logic over Imp_α, using formulas in Imp_α, is likewise implicitly definable in Imp_α. One naturally wonders whether the L_α hierarchy ever catches up to the Imp_α hierarchy, in the sense that each Imp_α is contained in some L_β. The following theorem shows that this is not necessary.

Theorem 7 *It is relatively consistent with ZFC that* $\text{Imp} \neq L$.

Proof Let T be a Souslin tree in L with the unique branch property, a strong notion of rigidity described in Fuchs and Hamkins [2]. Forcing over L with T yields a model $L[b]$ containing exactly one cofinal branch b through T. Since $T \in L$, there is an α with $T \in \mathrm{Imp}_\alpha^{L[b]}$. And in $\mathrm{Imp}_\alpha^{L[b]}$ the formula "X is a cofinal branch through T" is satisfied uniquely by b. So $b \in \mathrm{Imp}_{\alpha+1}^{L[b]}$. But $b \notin L$. So $L[b]$ thinks $\mathrm{Imp} \neq L$. □

Corollary 8 Imp *is not absolute to forcing extensions.*

A careful inspection of the proof of Theorem 7 shows that a copy of T becomes definable in $\mathrm{Imp}_{\omega_1}^{L[b]}$, and so in fact $b \in \mathrm{Imp}_{\omega_1+1}^{L[b]}$. Moreover, the branch b is implicitly definable over L inside any universe in which b is isolated in T. In general, we would like to know what else is contained in $P_{\mathrm{imp}}(L)$. For example, is $0^\sharp$, considered as a set of natural numbers, implicitly definable over L? Can any nonconstructible real be implicitly definable over L?

Refining the notion of implicit constructibility captured by Imp, we define the class gImp, the *generic* implicitly constructible universe, to consist of those sets a that belong to $\mathrm{Imp}^{V[G]}$ for every set-forcing extension $V[G]$ of the universe. This class is first-order definable in V. One can compare gImp with the generic HOD class gHOD, an inner model of ZFC defined in Fuchs, Hamkins, and Reitz [3]. Since the proof of Theorem 6 works in each forcing extension, we note that $\mathrm{gImp} \subseteq \mathrm{gHOD}$.

Theorem 9 *In any set-forcing extension $L[G]$ of L, there is a further extension $L[G][H]$ with* $\mathrm{gImp}^{L[G][H]} = \mathrm{Imp}^{L[G][H]} = L$.

Proof Let $L[G][H]$ be the forcing extension obtained by absorbing the G forcing into a large collapse $\mathrm{Coll}(\omega, \theta)$ forcing that is almost homogeneous. Since L is the HOD of $L[G][H]$, it follows that $\mathrm{gImp}^{L[G][H]} = \mathrm{Imp}^{L[G][H]} = L$ as well. □

Open questions about Imp abound. Can $\mathrm{Imp}^{\mathrm{Imp}}$ differ from Imp? Does Imp satisfy the axiom of choice? Can Imp have measurable cardinals? Must $0^\sharp$ be in Imp when it exists?[6] Which large cardinals are absolute to Imp? Does Imp have fine structure? Should we hope for any condensation-like principle? Can CH or GCH fail in Imp? Can reals be added at uncountable construction stages of Imp? Can we separate Imp from HOD? How much can we control Imp by forcing? Can we put arbitrary sets into the Imp of a suitable forcing extension? What can be said about the universe $\mathrm{Imp}(\mathbb{R})$ of sets implicitly constructible relative to $\mathbb{R}$ and, more generally, about $\mathrm{Imp}(X)$ for other sets X? Here we hope at least to have aroused interest in these questions.

Notes

1. See Theorem 12.14 and equation 13.26 of Jech [6] on reflection and ordinal definability, respectively.

2. Nevertheless, there are some metamathematical subtleties to this approach. In the case of definability, suppose a model M believes that a is defined in V_θ^M by a formula $\varphi(\cdot, \vec{\alpha})$. If φ has standard length, then, since $\theta \mapsto V_\theta^M$ is definable in M, we can see externally that a is Ord^M-definable in M as the unique object satisfying the formula $\varphi(\cdot, \vec{\alpha})^{V_\theta^M}$ with parameters $\vec{\alpha}$ and θ. In fact, a remains Ord^M-definable in M even when φ is a nonstandard formula of M, for a is the unique object thought by M to satisfy, in V_θ^M,

the formula coded by the ordinal $\ulcorner\varphi\urcorner$ with parameters coded by the ordinal $\ulcorner\vec{\alpha}\urcorner$. Thus OD, as defined in M by our official definition, coincides with the class of objects in M that really are Ord^M-definable in M.

For algebraicity, however, there is a wrench in the works of the analogous absoluteness argument: although each object that is externally Ord^M-algebraic in M is also internally ordinal algebraic in M, it is conceivable that an ω-nonstandard model M may regard an object a as ordinal algebraic even when it is not really Ord^M-algebraic in M. This could happen if n were a nonstandard integer and M regarded a as ordinal algebraic due to its membership in a set $\{\, b \mid V_\theta^M \models \varphi(b, \vec{\alpha}) \,\}$ of "finite" size n. In short, the more expansive concept of finiteness inside an ω-nonstandard model M may lead it to a more generous concept of algebraicity. Of course, since we do not yet know whether algebraicity differs from definability in any model of set theory, we cannot now confirm this possibility by presenting a particular ω-nonstandard model M for which OD^M contains sets that are not really Ord^M-algebraic in M.

3. See the previous note for a related discussion.

4. See Enayat [1, Theorem 3.6] for a proof.

5. See [6, Theorem 13.9] for a definition of *almost universality* and a proof that any almost universal transitive class containing all ordinals and closed under the Gödel operations is a model of ZF.

6. An affirmative answer arose in conversation with Menachem Magidor and Gunter Fuchs, and we hope that Imp will subsume further large cardinal features. We anticipate a future article on the implicitly constructible universe.

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