Notre Dame Journal of Formal Logic
Volume 57, Number 3, 2016

SCE-Cell Decomposition and OCP in Weakly O-Minimal Structures

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Abstract *Continuous extension (CE) cell decomposition* in o-minimal structures was introduced by Simon Andrews to establish the *open cell property* (OCP) in those structures. Here, we define *strong* CE-*cells* in weakly o-minimal structures, and prove that every weakly o-minimal structure with strong cell decomposition has SCE-cell decomposition if and only if its canonical o-minimal extension has CE-cell decomposition. Then, we show that every weakly o-minimal structure with SCE-cell decomposition satisfies OCP. Our last result implies that every o-minimal structure in which every definable open set is a union of finitely many open CE-cells, has CE-cell decomposition.

1 Introduction and Preliminaries

A first-order expansion $\mathcal{M} = (M, <, \ldots)$ of a dense linear order without endpoints is said to be *o-minimal* (resp., *weakly o-minimal*) if every unary definable (with parameters from M) subset of M is a finite union of open intervals and points (resp., open convex subsets and points). Recall that intervals in the linearly ordered set $(M, <)$ are defined as in $(\mathbb{R}, <)$, and a subset C of M is said to be *convex* if for all $a < b$ in C and any $c \in M$, $a < c < b$ implies that $c \in C$. We use (a, b) to denote an open interval with endpoints a and b, but $\langle a, b\rangle$ denotes an ordered pair with components a and b. When we use $\overline{a} \in A$, we mean $\overline{a} \in A^n$ for some positive integer $n \in \mathbb{N}^+$. Here, we are concerned with interval topology on M for which the set of all open intervals constitutes a basis. Also, for any positive integer n, we equip M^n with the product topology. An open box in M^n is a product of n open intervals $I_1, \ldots, I_n$ with endpoints in M. The set of all open boxes in M^n is a basis for the product topology on M^n. The topological closure of a set $X \subseteq M^n$ will be denoted by $\mathrm{cl}(X)$. Also, $\mathrm{fr}(X)$ denotes the frontier of X; that is, $\mathrm{cl}(X) \setminus X$.

Received March 14, 2013; accepted May 13, 2014
First published online April 20, 2016
2010 Mathematics Subject Classification: Primary 03C64
Keywords: weakly o-minimal structure, open cell property, SCE-cell decomposition
 10.1215/00294527-3507270

Throughout the paper, π will denote a projection map which drops the last coordinate. Let $f : X \to M$ be a definable function, and let $a \in \mathrm{cl}(X)$. Then, we write $\lim_{t \to a} f(t) = b$, where $b \in M \cup \{\pm\infty\}$, if for every open neighborhood V of b there exists an open neighborhood U of a in M^n such that for all $x \in U \cap X$, $f(x) \in V$.

Let $\mathcal{M} = (M, <, \ldots)$ be a weakly o-minimal structure. For $C, D \subseteq M$, let $C < D$ if and only if $c < d$ for all $c \in C$ and $d \in D$. A pair $\langle C, D\rangle$ of nonempty subsets of M is called a (Dedekind) *cut* in $(M, <)$ if $M = C \cup D$, $C < D$, and D has no lowest element. A cut $\langle C, D\rangle$ in $(M, <)$ is said to be *definable* in $\mathcal{M}$ if and only if the sets C, D are definable in $\mathcal{M}$. The set of all definable cuts of $\mathcal{M}$ will be denoted by $\overline{M}$. For $\langle C_1, D_1\rangle, \langle C_2, D_2\rangle$ in $\overline{M}$, let $\langle C_1, D_1\rangle < \langle C_2, D_2\rangle$ if and only if $C_1 \subset C_2$. With this relation, $(\overline{M}, <)$ is a dense linear order without endpoints. By identifying any element $a \in M$ with the definable cut $\langle(-\infty, a], (a, +\infty)\rangle$, $(\overline{M}, <)$ is an extension of $(M, <)$, where M is dense in $\overline{M}$. A function $f : X \to \overline{M}$, where $X \subseteq M^m$ is a definable set in $\mathcal{M}$, is said to be *definable* in $\mathcal{M}$ if and only if the set $A = \{\langle \overline{x}, y\rangle \in X \times M : y < f(\overline{x})\}$ is definable in $\mathcal{M}$. The definable set A is called the *defining set* of f in $\mathcal{M}$.

We will use some basic notions from (weak) o-minimality without giving the definitions, such as *k-cell*, *cell decomposition*, and so on. For further information about o-minimality, we refer the reader to van den Dries [4]; for weak o-minimality, see Macpherson, Marker, and Steinhorn [3] and Wencel [5].

Let $\mathcal{M} = (M, <, \ldots)$ be an o-minimal structure. The structure $\mathcal{M}$ is said to have the *open cell property*, OCP, if every nonempty definable open subset of M^n, for any positive integer n, is a finite union of open cells. Wilkie has shown that if $\mathcal{M} = (M, <, +, \cdot, \ldots)$ is an o-minimal expansion of a real closed field, then every definable bounded open subset $S \subseteq M^n$ is a finite union of open cells (see [7]). It is easy to construct an open definable subset of M^2 that cannot be expressed as a finite union of definable open cells. Edmundo, Elefteriou, and Prelli in [2] extended the result of Wilkie for o-minimal expansions of an ordered group. Andrews in [1] introduced continuous extension cell decomposition (i.e., denoted by CE-cell decomposition) in o-minimal structures. Then, he proved that every o-minimal structure having CE-cell decomposition satisfies OCP.

In this article, we focus on OCP in weakly o-minimal structures. First, we combine CE-cell decomposition of [1] with strong cell decomposition of [5] to introduce *strong continuous extension cell decomposition* (SCE-*cell decomposition*) in weakly o-minimal structures. Wencel [5] showed that every weakly o-minimal structure $\mathcal{M} = (M, <, \ldots)$ with strong cell decomposition has a canonical o-minimal extension $\overline{\mathcal{M}} = (\overline{M}, <, \ldots)$. In Section 2, we show that $\mathcal{M}$ has SCE-cell decomposition if and only if $\overline{\mathcal{M}}$ has CE-cell decomposition. We use this result to show that every weakly o-minimal structure with SCE-cell decomposition has OCP; that is, every nonempty definable open set is a finite union of open strong cells. Let $\mathcal{M} = (M, <, \ldots)$ be a weakly o-minimal structure with strong cell decomposition. In Section 3, we also prove that if every definable open set $X \subseteq M^n$, $n \in \mathbb{N}$, is a union of finitely many SCE-cells, then $\mathcal{M}$ holds SCE-cell decomposition. This implies somehow a partial converse to [1, Theorem 2].

2 SCE-Cell Decomposition in Weakly O-Minimal Structures

Let $\mathcal{M} = (M, <, \ldots)$ be a weakly o-minimal expansion of a dense linear order $(M, <)$. According to [5], strong cells in M^m and their completions in $\overline{M}^m$ are simultaneously defined for any $m \in \mathbb{N}^+$. The completion of a strong cell $C \subseteq M^m$ will be denoted by $\overline{C}$.

(i) Any singleton of M is a 0-strong cell in M and is equal to its completion.

(ii) A nonempty open convex definable subset of M is a 1-strong cell in M. If $C \subseteq M$ is a 1-strong cell, then $\overline{C} := \{x \in \overline{M} \mid (\exists a, b \in C)(a < x < b)\}$.

Let $m \in \mathbb{N}^+$, $k \leq m$, and suppose that we have already defined k-strong cells in M^m and their completions in $\overline{M}^m$.

(iii) If $C \subseteq M^m$ is a k-strong cell in M^m and $f : C \to M$ is a definable continuous function which has a (necessarily unique) continuous extension $\overline{f} : \overline{C} \to \overline{M}$, then the graph of f, which is denoted by $\Gamma(f)$, is a k-strong cell in M^{m+1} and its completion in $\overline{M}^{m+1}$ is defined as $\Gamma(\overline{f})$.

(iv) Assume that $C \subseteq M^m$ is a k-strong cell in M^m and that $f, g : C \to \overline{M} \cup \{\pm\infty\}$ are definable continuous functions with continuous extensions $\overline{f}, \overline{g} : \overline{C} \to \overline{M} \cup \{\pm\infty\}$, respectively, such that $\overline{f}(\overline{a}) < \overline{g}(\overline{a})$ for all $\overline{a} \in \overline{C}$. Then, the set

$$(f, g)_C := \left\{\langle \overline{a}, b\rangle \in C \times M : f(\overline{a}) < b < g(\overline{a})\right\}$$

is a $(k+1)$-strong cell in M^{m+1}. The completion of $(f, g)_C$ in $\overline{M}^{m+1}$ is the set

$$(\overline{f}, \overline{g})_{\overline{C}} := \left\{\langle \overline{a}, b\rangle \in \overline{C} \times \overline{M} : \overline{f}(\overline{a}) < b < \overline{g}(\overline{a})\right\}.$$

For a k-strong cell C, $\dim(C) = k$ is the topological dimension of C. A weakly o-minimal structure $\mathcal{M}$ is said to have *strong cell decomposition* if for any positive integers m, k and any definable sets $X_1, \ldots, X_k \subseteq M^m$, there exists a decomposition of M^m into strong cells that partitions each of the sets $X_1, \ldots X_k$. (For the notions of decomposition and partitioning a set, see [4].)

For $X \subseteq M^m$, let $\mathrm{cl}_{\overline{M}}(X)$ denote the topological closure of X in the space $\overline{M}^m$. Using induction on m, one can see that if C is a strong cell, then $\overline{C} \subseteq \mathrm{cl}_{\overline{M}}(C)$. So, $\mathrm{cl}_{\overline{M}}(C) = \mathrm{cl}(\overline{C})$.

Let $C \subseteq M^m$ be a strong cell, and let $f : C \longrightarrow \overline{M}$ be a definable function. Then, f is said to be *strongly continuous* if and only if it has a (unique) continuous extension $\overline{f} : \overline{C} \longrightarrow \overline{M}$. A function which is identically equal $+\infty$ or $-\infty$, and whose domain is a strong cell will be also strongly continuous.

In [1], continuous extension cells, CE-cells, are defined inductively in an o-minimal structure $(M, <, \ldots)$. In M, CE-cells are exactly the points and open intervals. The cells $\Gamma(f)_C$ and $(g, h)_C$ are CE-cells if C is a CE-cell and there exist continuous extensions of f, g, and h on $\mathrm{cl}(C)$. An o-minimal structure is said to have the CE-*cell decomposition property* if any cell decomposition admits a refinement by CE-cells. Below, we define *strong* CE-*cells* (SCE-*cells*) in a weakly o-minimal structure $\mathcal{M}$ inductively.

(i) Every singleton of M is a 0-SCE-cell, and every definable open convex subset of M is a 1-SCE-cell.

(ii) If $C \subseteq M^n$ is a k-SCE-cell, where $k \leq n$, and $f : C \to M$ is a definable continuous function which has a continuous extension $\overline{f} : \mathrm{cl}_{\overline{M}}(C) \to \overline{M}$, then $\Gamma(f) = \{\langle \overline{x}, y\rangle \in C \times M \mid y = f(\overline{x})\}$ is a k-SCE-cell.

(iii) If $C \subseteq M^n$ is a k-SCE-cell and $f, g : C \to \overline{M}$ are definable continuous functions which have continuous extensions $\overline{f}, \overline{g} : \mathrm{cl}_{\overline{M}}(C) \to \overline{M}$ such that $\overline{f}|_{\overline{C}} < \overline{g}|_{\overline{C}}$, then $(f, g)_C = \{\langle \overline{x}, y\rangle \in C \times M \mid f(\overline{x}) < y < g(\overline{x})\}$ is a $(k + 1)$-SCE-cell.

The dimension of a k-SCE-cell is k. As in the o-minimal case, one can see that for $0 \leq k < m$ and every k-SCE-cell $C \subseteq M^m$ there exists a k-SCE-cell $D \subseteq M^{m-1}$ and a definable homeomorphism π_C from C onto D which has a definable continuous extension $\overline{\pi}_C$ from $\mathrm{cl}_{\overline{M}}(C)$ onto $\mathrm{cl}_{\overline{M}}(D)$. An SCE-*cell decomposition* for $\mathcal{M}$ is a strong cell decomposition of $\mathcal{M}$ in which any strong cell is an SCE-cell. Note that if $\mathcal{C}$ is an SCE-cell decomposition of M^{m+1}, then $\pi(\mathcal{C}) = \{\pi(C) \mid C \in \mathcal{C}\}$ is an SCE-cell decomposition of M^m.

In the following example, we give a weakly o-minimal structure with strong cell decomposition but without SCE-cell decomposition. We first recall the notion of nonvaluational (weakly) o-minimal structure. Let $\mathcal{M} = (M, <, +, \ldots)$ be a (weakly) o-minimal expansion of an ordered abelian group. A cut $\langle C, D\rangle$ of M is said to be *nonvaluational* if for any $\epsilon \in M^{>0}$, there exist $x \in C$ and $y \in D$ such that $|x - y| < \epsilon$. Then, $\mathcal{M}$ is called *nonvaluational* if every definable cut of M is nonvaluational. By Wencel [6, Theorem 2.11], every expansion of a nonvaluational weakly o-minimal structure $\mathcal{M}$ by a family of nonvaluational cuts of M is a nonvaluational weakly o-minimal structure, so it has the strong cell decomposition property by [5, Corollary 2.16].

Example 2.1 Let $\mathcal{M} = (M, <, +, \cdot, 0, 1, \ldots)$ be a weakly o-minimal nonvaluational expansion of a real closed field, for example, the ordered field of real algebraic numbers expanded by a unary predicate for the set of real algebraic numbers less than π. Then, $\mathcal{M}$ has strong cell decomposition but does not have SCE-cell decomposition; as for the definable strongly continuous function $f : (0, +\infty) \to \overline{M}$, where $f(x) = \frac{1}{x}$, the definable strong cell $(-\infty, f)_{(0,+\infty)}$ cannot be a finite union of SCE-cells.

It is worth noting that SCE-cell decomposition for weakly o-minimal structures does not imply the o-minimality. To illustrate that, let $\mathcal{M} = (M, <, +, \{\lambda \mid \lambda \in D\})$ be an ordered vector space over an ordered division ring D, and let $\{C_i \mid i \in I\}$ be a family of nonvaluational convex subsets of M. Then $\mathcal{M}_1 = (M, <, +, \{\lambda \mid \lambda \in D\}, \{C_i \mid i \in I\})$, which is a nonvaluational weakly o-minimal structure, has strong cell decomposition. In the following, we make the method of [4, (1.7.4)] compatible with weak o-minimality to show that $\mathcal{M}_1$ has indeed SCE-cell decomposition.

We say that a function $f : M^n \to \overline{M}$ is *pseudoaffine* if there exist $k \in \mathbb{N}$, $C_1, \ldots, C_k \in \{C_i \mid i \in I\}$, $\lambda_1, \ldots, \lambda_n, \gamma_1, \ldots, \gamma_k \in D$, and $\alpha \in M$ such that

$$f(x_1, \ldots, x_n) = \lambda_1 x_1 + \cdots + \lambda_n x_n + \alpha + \gamma_1 C_1 + \cdots + \gamma_k C_k.$$

Note that by $\gamma_i C_i$ in the above expression, we mean the least upper bound of the (proper) convex set $\gamma_i C_i$ in $\overline{M}$. If $k = 0$, then f is an affine function. Also, a basic pseudo-semilinear set in M^n is a set of the form

$$\left\{x \in M^n \;\middle|\; f_1(x) = \cdots = f_p(x) = 0, g_1(x) > 0, \ldots, g_q(x) > 0\right\},$$

where the f_i's are affine functions and the g_j's are pseudoaffine functions. A pseudo-semilinear set in M^n is a finite Boolean combination of basic pseudo-semilinear sets. As in [4], one can show that $\mathcal{M}_1$ has a cell decomposition property for which all cell-defining functions are pseudoaffine. Thus, every cell here is an SCE-cell and so $\mathcal{M}_1$ has SCE-cell decomposition.

The proof of the following fact is completely routine.

Fact 2.2 Assume that $\mathcal{M} = (M, <, \ldots)$ is a weakly o-minimal structure with SCE-cell decomposition, m is a positive integer, X is a definable subset of M^m, and $f : X \to \overline{M}$ is a definable function. Then, there is an SCE-cell decomposition $\mathcal{C}$ of M^m which partitions X, and for each $C \in \mathcal{C}$ with $C \subseteq X$, $f|_C$ has a definable continuous extension on $\mathrm{cl}_{\overline{M}}(C)$.

Refined strong cells are all those whose boundary functions (cell-defining functions) assume values in one of the sets: $\{-\infty\}$, $\{+\infty\}$, M, $\overline{M} \setminus M$. As mentioned in [5, Fact 2.5], it is easy to see that if $\mathcal{M}$ has strong cell decomposition, then every strong cell can be partitioned into finitely many refined strong cells; so every strong cell decomposition can be refined into a cell decomposition with refined strong cells.

Let $\mathcal{M} = (M, <, \ldots)$ be a weakly o-minimal structure with strong cell decomposition. By [5, Section 3], the structure $\overline{\mathcal{M}} = (\overline{M}, <, \{\overline{C} \mid C$ is a refined strong cell in $\mathcal{M}\})$ is o-minimal. It is called the *canonical o-minimal extension* of $\mathcal{M}$. As noted in [5, comments before Fact 3.4], for every definable set $X \subseteq \overline{M}^n$ in $\overline{\mathcal{M}}$, $X \cap M^n$ is a definable set in $\mathcal{M}$.

In the following fact, which can be proved inductively on m, we compare cell types in $\mathcal{M}$ and $\overline{\mathcal{M}}$.

Fact 2.3 Let $\mathcal{M} = (M, <, \ldots)$ be a weakly o-minimal structure with strong cell decomposition, and let $\overline{\mathcal{M}}$ be its canonical o-minimal extension. Then we have the following.

(i) If $C \subseteq M^m$ is a strong cell (resp., SCE-cell), then $\overline{C}$ is a cell (resp., CE-cell) in $\overline{\mathcal{M}}$. If C is, in addition, open, then $\overline{C}$ is open too.
(ii) If $S \subseteq \overline{M}^m$ is an open cell (resp., open CE-cell) in $\overline{\mathcal{M}}$, then $S \cap M^m$ is a strong open cell (resp., open SCE-cell) in $\mathcal{M}$ whose completion is S.

Note that (ii) does not hold for nonopen cells. To see that, let $\mathcal{M} = (M, <, +, \cdot, 0, 1, P)$ be the ordered field of real algebraic numbers expanded by $P = (-\infty, \pi)$. We remarked in Example 2.1 that $\mathcal{M}$ has strong cell decomposition. Let $S = \Gamma(f)$, where $f : (0, 7) \to \overline{M}$ is a continuous definable function in $\overline{\mathcal{M}}$ such that $f(x) = x$ for $x \in (0, \pi)$, $f(x) = \pi$ for $x \in [\pi, 2\pi]$, and $f(x) = \frac{1}{2}x$ for $x \in (2\pi, 7)$. Then, S is a CE-cell in $\overline{\mathcal{M}}$, but $S \cap M^2$ is a union of two disjoint SCE-cells in $\mathcal{M}$.

In the following, we make a link between the SCE-cell decomposition property of a weakly o-minimal structure $\mathcal{M}$ having strong cell decomposition and the CE-cell decomposition property of the canonical o-minimal extension $\overline{\mathcal{M}}$ of $\mathcal{M}$. First, we provide some requirements in the following lemmas, where we fix an arbitrary weakly o-minimal structure $\mathcal{M} = (M, <, \ldots)$.

Lemma 2.4 *Let $C \subseteq M^m$ be an open* SCE*-cell for some $m \in \mathbb{N}^+$, and let $a \in \mathrm{cl}_{\overline{M}}(C)$. Then, for every open box $B \subseteq M^m$ with $a \in \overline{B}$, there exists an open box $B_1 \subseteq B$ with $a \in \overline{B_1}$ such that $B_1 \cap C$ is an open* SCE*-cell.*

Proof This is done by induction on m. It is clear for $m = 1$. Assume that $C \subseteq M^{m+1}$ is an open SCE-cell, with $a \in \mathrm{cl}_{\overline{M}}(C)$. Then, $C = (f, g)_D$, where D is an open SCE-cell in M^m and f, g are definable continuous functions which have definable continuous extensions $\overline{f}, \overline{g} : \mathrm{cl}_{\overline{M}}(D) \to \overline{M}$ such that $\overline{f}|_{\overline{D}} < \overline{g}|_{\overline{D}}$. Let $B = E \times I$ be an open box with $a = \langle \overline{a}, a_{m+1} \rangle \in \overline{B}$. Since $a \in \mathrm{cl}_{\overline{M}}(C)$, $\lim_{t \to \overline{a}} f(t) \leq a_{m+1} \leq \lim_{t \to \overline{a}} g(t)$ and $\overline{a} \in \mathrm{cl}_{\overline{M}}(D)$. So, we have four cases as follows.

(i) $\lim_{t \to \overline{a}} f(t) < a_{m+1} < \lim_{t \to \overline{a}} g(t)$. Let $J = (c, d)$ be an open subinterval of I where $\lim_{t \to \overline{a}} f(t) < c < a_{m+1} < d < \lim_{t \to \overline{a}} g(t)$. Then, there exists an open box $E' \subseteq E$ with $\overline{a} \in \overline{E'}$ such that $\overline{f}(\overline{E'} \cap \mathrm{cl}_{\overline{M}}(D)) < c$ and $\overline{g}(\overline{E'} \cap \mathrm{cl}_{\overline{M}}(D)) > d$. By the induction hypothesis, there is an open box $E'' \subseteq E'$ containing $\overline{a}$ in $\overline{E''}$ such that $E'' \cap D$ is an open SCE-cell with $\overline{E'' \cap D} \subseteq \overline{D}$. Then, $E'' \times J$ is the desired open box. Note that $(E'' \times J) \cap C = (h_1, h_2)_{E'' \cap D}$, where h_1 and h_2 are the definable constant functions with values c and d, respectively.

(ii) $\lim_{t \to \overline{a}} f(t) < a_{m+1} = \lim_{t \to \overline{a}} g(t)$. In this case, we take an open subinterval $J \subseteq I$ as (c, d) where $\lim_{t \to \overline{a}} f(t) < c < a_{m+1} < d$. Now, similarly to the case (i), there exists an open box E'' with $\overline{a} \in E''$ such that $E'' \cap D$ is an open SCE-cell with $\overline{E'' \cap D} \subseteq \overline{D}$. Then, $(E'' \times J) \cap C = (h, g)_{E'' \cap D}$ is the desired open box in which h is the constant function c.

(iii) $\lim_{t \to \overline{a}} f(t) = a_{m+1} < \lim_{t \to \overline{a}} g(t)$. This case is completely similar to the case before.

(iv) $\lim_{t \to \overline{a}} f(t) = a_{m+1} = \lim_{t \to \overline{a}} g(t)$. In this case, we take $E' \subseteq E$ as an open box with $a \in \overline{E'}$ such that for every $x \in E' \cap \mathrm{cl}_{\overline{M}}(D)$, $\overline{f}(x), \overline{g}(x) \in I$. By the induction hypothesis, there is an open box $E'' \subseteq E'$ containing $\overline{a}$ in $\overline{E''}$ such that $E'' \cap D$ is an open SCE-cell with $\overline{E'' \cap D} \subseteq \overline{D}$. Then, we have $(E'' \times I) \cap C = (f, g)_{E'' \cap D}$. □

Lemma 2.5 *Let $C \subseteq M^m$ be an open* SCE*-cell for some $m \in \mathbb{N}^+$, $a \in \mathrm{cl}_{\overline{M}}(C)$, and let $\mathcal{C}$ be an* SCE*-cell decomposition that partitions C. Assume that $C_1, \ldots, C_n$ are all elements of $\mathcal{C}$ such that $C_i \subseteq C$ and $a \in \mathrm{cl}_{\overline{M}}(C_i)$ for every $1 \leq i \leq n$. Then, there exists an open box B with $a \in \overline{B}$ such that $B \cap C = \bigcup_{1 \leq i \leq n}(B \cap C_i)$.*

Proof This is done by induction on m. It is clear for $m = 1$. Let $C = (f, g)_D \subseteq M^{m+1}$ be an open SCE-cell, let $a = \langle \overline{a}, a_{m+1} \rangle \in \mathrm{cl}_{\overline{M}}(C)$, and let $\mathcal{C}$ be an SCE-cell decomposition of M^{m+1} that partitions C. Let $\mathcal{C}' = \{C_1, \ldots, C_n\}$ be the set of all elements of $\mathcal{C}$ such that $C_i \subseteq C$ and $a \in \mathrm{cl}_{\overline{M}}(C_i)$ for every $1 \leq i \leq n$. Then, for every $E \in \pi(\mathcal{C}')$, we have $(f, g)_E = \bigcup_{0 \leq i \leq k_E}(h_i, h_{i+1})_E \cup \bigcup\{\Gamma(h_i|_E) \mid 1 \leq i \leq k_E \text{ and } \Gamma(h_i|_E) \in \mathcal{C}\}$ for some $k_E \geq 0$, where $h_0 = f, h_{k_E+1} = g$, and any h_i is a definable continuous function which has a definable continuous extension on $\mathrm{cl}_{\overline{M}}(E)$. Let $\mathcal{C}_E = \{(h_i, h_{i+1})_E \mid 0 \leq i \leq k_E\}$. Since $a = \langle \overline{a}, a_{m+1} \rangle \in \mathrm{cl}_{\overline{M}}(C)$, $\overline{a} \in \mathrm{cl}_{\overline{M}}(D)$ and $\lim_{t \to \overline{a}} f(t) \leq a_{m+1} \leq \lim_{t \to \overline{a}} g(t)$. So, $\pi(C_1), \ldots, \pi(C_n)$ are all $\pi(X) \in \pi(\mathcal{C})$ for which $\overline{a} \in \mathrm{cl}_{\overline{M}}(\pi(X))$ and $\pi(X) \subseteq D$. By the induction hypothesis, there exists an open box B_1 with $\overline{a} \in \overline{B_1}$ such that $B_1 \cap D = \bigcup_{1 \leq i \leq n}(B_1 \cap \pi(C_i))$. For every $E \in \pi(\mathcal{C}')$, let

$$l_E = \min\{h_i \mid 0 \leq i \leq k_E \text{ and } (h_i, h_{i+1})_E \in \mathcal{C}_E\},$$
$$h_E = \max\{h_{i+1} \mid 0 \leq i \leq k_E \text{ and } (h_i, h_{i+1})_E \in \mathcal{C}_E\}.$$

Then, it is easy to see that l_E and h_E hold the following conditions:

(i) l_E and h_E are definable continuous functions on E which have definable continuous extensions $\overline{l_E}$ and $\overline{h_E}$ on $\mathrm{cl}_{\overline{M}}(E)$, and $\overline{l_E} < \overline{h_E}$ on $\overline{E}$;

(ii) $\lim_{t\to\overline{a}} l_E(t) \le a_{m+1} \le \lim_{t\to\overline{a}} h_E(t)$;

(iii) if $\lim_{t\to\overline{a}} l_E(t) = a_{m+1}$, then $l_E = f|_E$; also, $h_E = g|_E$ if $\lim_{t\to\overline{a}} h_E(t) = a_{m+1}$.

There exist elements c, d in M such that $c < a_{m+1} < d$ and

(i) $\lim_{t\to\overline{a}} l_E(t) < c$ for every $E \in \pi(\mathcal{C}')$ which $\lim_{t\to\overline{a}} l_E(t) < a_{m+1}$,

(ii) $d < \lim_{t\to\overline{a}} h_E(t)$ for every $E \in \pi(\mathcal{C}')$ which $a_{m+1} < \lim_{t\to\overline{a}} h_E(t)$.

By the continuity of $\overline{l_E}$ and $\overline{h_E}$, there exists an open box $B_2 \subseteq B_1$ with $\overline{a} \in \overline{B_2}$, such that $\overline{l_E}(\overline{B_2} \cap \mathrm{cl}_{\overline{M}}(E)) < c$ or $c < \overline{l_E}(\overline{B_2} \cap \mathrm{cl}_{\overline{M}}(E)) < d$, and $c < \overline{h_E}(B_2 \cap \mathrm{cl}_{\overline{M}}(E)) < d$ or $d < \overline{h_E}(\overline{B_2} \cap \mathrm{cl}_{\overline{M}}(E))$ for every $E \in \pi(\mathcal{C}')$. Then, for $B = B_2 \times (c, d)$, we have $B \cap C = \bigcup_{1\le i\le n}(B \cap C_i)$. □

Lemma 2.6 *Let $C \subseteq M^m$ be an open* SCE-*cell, and let $f : C \to \overline{M}$ be a definable strongly continuous function. Assume that $\mathcal{C}$ is an* SCE-*cell decomposition partitioning C such that for all $D \in \mathcal{C}$, $f|_D$ has a definable continuous extension to* $\mathrm{cl}_{\overline{M}}(D)$. *Then, f has a definable continuous extension to* $\mathrm{cl}_{\overline{M}}(C)$.

Proof It is sufficient to show that for every $D_1, D_2 \in \mathcal{C}$ and $a \in \mathrm{cl}_{\overline{M}}(D_1) \cap \mathrm{cl}_{\overline{M}}(D_2)$, $\lim_{t\to a} f|_{D_1}(t) = \lim_{t\to a} f|_{D_2}(t)$. It is true for every $a \in \overline{C}$, since f is strongly continuous. So, fix an element $a \in \mathrm{fr}(\overline{C})$, and let $D_1, \ldots, D_n$ be all elements of $\mathcal{C}$ such that $D_i \subseteq C$ and $a \in \mathrm{cl}_{\overline{M}}(D_i)$ for every $1 \le i \le n$. If the claim does not hold, then there are elements $b_1 < b_2 < \cdots < b_k$ in $\overline{M}$, where $k > 1$, such that for every $1 \le i \le n$, $\lim_{t\to a} f|_{D_i}(t) = b_j$ for some $1 \le j \le k$. By Lemma 2.5, there exists an open box B_1 with $a \in \overline{B_1}$ such that $B_1 \cap C = \bigcup_{1\le i\le n}(B_1 \cap D_i)$. For every $1 \le j \le k$, let I_j be an open interval $(c_j, d_j) \subset M$ where $c_j < b_j < d_j < c_{j+1}$. We may assume, without loss of generality, that $\forall t \in (B_1 \cap D_i)$, $f|_{D_i}(t) \in I_j$ for all $1 \le i \le n$ and $1 \le j \le k$ such that $\lim_{t\to a} f|_{D_i}(t) = b_j$. By Lemma 2.4, there exists an open box $B_2 \subseteq B_1$ with $a \in \overline{B_2}$ such that $B_2 \cap C$ is an open SCE-cell and $\overline{B_2 \cap C} \subseteq \overline{C}$. Since $(-\infty, f)_C$ is a strong cell with completion $(-\infty, \overline{f})_{\overline{C}}$, $\overline{f} : \overline{C} (\subseteq \overline{M}^n) \to \overline{M}$ is a continuous definable function in the o-minimal structure $\overline{\mathcal{M}}$. Then, $\overline{f}(\overline{B_2 \cap C})$ is a definably connected set in $\overline{M}$, as $\overline{B_2 \cap C}$ is a cell in $\overline{\mathcal{M}}$ and hence definably connected. This is a contradiction, because of $B_2 \cap C = \bigcup_{1\le i\le n}(B_1 \cap D_i)$ and $\forall t \in (B_2 \cap D_i)$, $f|_{D_i}(t) \in I_j$ whenever $\lim_{t\to a} f|_{D_i}(t) = b_j$. □

Theorem 2.7 *Let $\mathcal{M} = (M, <, \ldots)$ be a weakly o-minimal structure with* SCE-*cell decomposition. Then every definable strongly continuous function $f : C \to \overline{M}$, where $C \subseteq M^n$ is an open* SCE-*cell in $\mathcal{M}$, has a definable continuous extension to* $\mathrm{cl}_{\overline{M}}(C)$.

Proof Let $C \subseteq M^n$ be an open SCE-cell in $\mathcal{M}$, and let $f : C \to \overline{M}$ be a definable strongly continuous function. By Fact 2.2, there is an SCE-cell decomposition $\mathcal{C}$ of M^n that partitions C into SCE-cells on each of which f has a continuous extension on its topological closure in $\overline{M}^n$. By Lemma 2.6, f has a definable continuous extension to $\mathrm{cl}_{\overline{M}}(C)$. □

Corollary 2.8 *Let $\mathcal{M} = (M, <, \dots)$ be a weakly o-minimal structure with* SCE*-cell decomposition. Then every open strong cell $C \subseteq M^m$ is a finite union $C_1 \cup \dots \cup C_n$, for some $n \geq 1$, of open* SCE*-cells such that $\overline{C_i} \subseteq \overline{C}$ for $1 \leq i \leq n$.*

Proof This is done by induction on m. It is clear for $m = 1$. Let $C = (f, g)_D \subseteq M^{m+1}$ be an open strong cell. By the induction assumption, D is a finite union $D_1 \cup \dots \cup D_n$ of open SCE-cells such that $\overline{D_i} \subseteq \overline{D}$ for $1 \leq i \leq n$. By Theorem 2.7, for any $1 \leq i \leq n$, the definable strongly continuous functions $f|_{D_i}, g|_{D_i}$ have definable continuous extensions on $\mathrm{cl}_{\overline{M}}(D_i)$. Hence, $(f, g)_{D_1}, \dots, (f, g)_{D_n}$ are all SCE-cells. As $C = (f, g)_{D_1} \cup \dots \cup (f, g)_{D_n}$ and $\overline{(f, g)_{D_i}} = (\overline{f}, \overline{g})_{\overline{D_i}} \subseteq \overline{C}$ for any $1 \leq i \leq n$, we are done. □

Theorem 2.9 *Let $\mathcal{M} = (M, <, \dots)$ be a weakly o-minimal structure with strong cell decomposition. Then, $\mathcal{M}$ has* SCE*-cell decomposition if and only if the canonical o-minimal extension $\overline{\mathcal{M}}$ of $\mathcal{M}$ has* CE*-cell decomposition.*

Proof Assume that $\mathcal{M}$ has SCE-cell decomposition. We show that if $S \subseteq \overline{M}^n$ is a k-CE-cell in $\overline{\mathcal{M}}$ and $f : S \to \overline{M}$ is a definable continuous function, then f has a continuous extension on $\mathrm{cl}(S)$. It is clear for $k = 0$. For $k \geq 1$, we prove it by induction on n. If $S \subseteq \overline{M}$ is a 1-cell in $\overline{\mathcal{M}}$, then $S = \overline{C}$ for some open SCE-cell C in $\mathcal{M}$. Then, $\mathrm{cl}(S) = \mathrm{cl}_{\overline{M}}(C)$ and $f|_C : C \to \overline{M}$ is a definable strongly continuous function in $\mathcal{M}$. Hence by Theorem 2.7, $\lim_{t \to a} f(t)$ exists in $\overline{M}$ for any element $a \in \mathrm{cl}(S)$.

Now, let $S \subseteq \overline{M}^{n+1}$ be a k-CE-cell in $\overline{\mathcal{M}}$, let $f : S \to \overline{M}$ be a definable continuous function, and let $a \in \mathrm{cl}(S)$. If S is open, then by Fact 2.3(ii), $S = \overline{C}$ for some open SCE-cell in $\mathcal{M}$, and so $\mathrm{cl}(S) = \mathrm{cl}_{\overline{M}}(C)$. Thus by Theorem 2.7, $\lim_{t \to a} f(t)$ exists in $\overline{M}$ for any $a \in \mathrm{cl}(S)$. Now, assume that S is not open. In this case, there exists a k-CE-cell $U \subseteq \overline{M}^n$ and a definable homeomorphism π_S from S onto U which has a definable continuous extension $\overline{\pi}_S$ from $\mathrm{cl}(S)$ onto $\mathrm{cl}(U)$. Then, by the induction assumption, $\lim_{y \to \overline{\pi}_S(a)} f \circ \pi_S^{-1}(y)$ exists in $\overline{M}$. Thus, $\lim_{t \to a} f(t)$ exists in $\overline{M}$. Now, using cell decomposition of $\overline{\mathcal{M}}$, we can inductively show that $\overline{\mathcal{M}}$ has CE-cell decomposition.

Conversely, assume that $\overline{\mathcal{M}}$ has CE-cell decomposition. Since $\mathcal{M}$ has strong cell decomposition, the following assertions, which can be easily proved by using simultaneous induction on m, follow the SCE-cell decomposition of $\mathcal{M}$.

(a)$_m$. If $C_1, \dots, C_k$ are strong cells in M^m, then there exists an SCE-cell decomposition $\mathcal{C}$ of M^m that partitions each of the sets $C_1, \dots, C_k$.

(b)$_m$. If $C_1, \dots, C_k$ are strong cells in M^m and $f_1 : C_1 \to \overline{M}, \dots, f_k : C_k \to \overline{M}$ are definable strongly continuous functions, then there exists an SCE-cell decomposition $\mathcal{C}$ of M^m that partitions each of the sets $C_1, \dots, C_k$, and for every $D \in \mathcal{C}$ such that $D \subseteq C_i$, $f_i|_D$ has a definable continuous extension on $\mathrm{cl}_{\overline{M}}(D)$.

(a)$_1$ is clear. For (b)$_1$, there exists a CE-cell decomposition $\mathcal{D}$ of $\overline{M}$ that partitions each of the sets $\overline{C_1}, \dots, \overline{C_k}$, and for every $D \in \mathcal{D}$ such that $D \subseteq \overline{C_i}$, $\overline{f_i}|_D$ has a definable continuous extension on $\mathrm{cl}(D)$. Then, $\mathcal{C} = \{D \cap M \mid D \in \mathcal{D}\}$ is a desired SCE-cell decomposition of M.

For (a)$_{m+1}$, let $C_1, \dots, C_k$ be strong cells in M^{m+1}. There exists a strong cell decomposition $\mathcal{C}$ that partitions each of the sets $C_1, \dots, C_k$. Now, by using (b)$_m$ for all cell-defining functions of $\mathcal{C}$, we get a desired SCE-cell decomposition of M^{m+1}.

For $(b)_{m+1}$, let $C_1, \ldots, C_k$ be strong cells in M^{m+1}, and let $f_1 : C_1 \to \overline{M}, \ldots, f_k : C_k \to \overline{M}$ be definable strongly continuous functions. By Fact 2.3, the functions $\overline{f_1} : \overline{C_1} \to \overline{M}, \ldots, \overline{f_k} : \overline{C_k} \to \overline{M}$ are definable in $\overline{\mathcal{M}}$. There exists a CE-cell decomposition $\mathcal{D}$ of $\overline{M}^{m+1}$ that partitions each of the sets $\overline{C_1}, \ldots, \overline{C_k}$, and for every $D \in \mathcal{D}$ such that $D \subseteq \overline{C_i}$, $\overline{f_i}|_D$ has a definable continuous extension on $\mathrm{cl}(D)$. By the assumption, there exists a strong cell decomposition $\mathcal{C}_1$ that partitions every set of $\{D \cap M^{m+1} \mid D \in \mathcal{D}\}$. Now, using $(a)_{m+1}$ on members of $\mathcal{C}_1$ gives a desired SCE-cell decomposition of M^{m+1}. □

From Theorem 2.9, and [5, Corollary 2.16], we have the following.

Corollary 2.10 *Let $\mathcal{M} = (M, <, +, \ldots)$ be a weakly o-minimal nonvaluational expansion of an ordered abelian group. If $\mathcal{M}$ has* SCE*-cell decomposition, then the canonical o-minimal extension $\overline{\mathcal{M}}$ of $\mathcal{M}$ has* OCP.

It is worth noting that if $\mathcal{D}$ is a CE-cell decomposition of $\overline{M}^m$ in the structure $\overline{\mathcal{M}}$, then $\mathcal{D} \cap M^m = \{D \cap M^m \mid D \in \mathcal{D}\}$ is not necessarily an SCE-cell decomposition for M^m in the structure $\mathcal{M}$. For example, we consider the structure $\mathcal{M} = (\mathbb{Q}, <, +, \{r \mid r \in \mathbb{Q}\}, P)$ as an ordered vector space over $\mathbb{Q}$ expanded by $P = (-\infty, \sqrt{2})$. As we saw above, $\mathcal{M}$ has SCE-cell decomposition. Let $f : (0, 4) \to \overline{M}$ be a continuous definable function in $\overline{\mathcal{M}}$ such that $f(x) = x$ for $x \in (0, \sqrt{2})$, $f(x) = \sqrt{2}$ for $x \in [\sqrt{2}, 2\sqrt{2}]$, and $f(x) = \frac{1}{2}x$ for $x \in (2\sqrt{2}, 4)$. Then,

$$\begin{aligned}\mathcal{D} = \big\{&(-\infty, 0) \times \overline{M}, \{0\} \times \overline{M}, (-\infty, f)_{(0,4)}, \\ &\Gamma(f), (f, +\infty)_{(0,4)}, \{4\} \times \overline{M}, (4, +\infty) \times \overline{M}\big\}\end{aligned}$$

is a CE-cell decomposition of $\overline{M}^2$, while $\mathcal{D} \cap M^2$ is not even a strong cell decomposition.

3 OCP in Weakly O-Minimal Structures

We say that a weakly o-minimal structure $\mathcal{M} = (M, <, \ldots)$ has the *open cell property*, OCP, if every definable open set $X \subseteq M^m$, for each $m \in \mathbb{N}^+$, is a finite union of open strong cells. In this section, we show that weakly o-minimal structures with SCE-cell decomposition have OCP. We also provide a partial converse of that result, which implies a partial converse for Theorem 2 of [1].

Lemma 3.1 *Let $\mathcal{M} = (M, <, \ldots)$ be a weakly o-minimal structure with* SCE*-cell decomposition. Then, for every definable open set $X \subseteq M^m$, $m \in \mathbb{N}^+$, there exists an open set $Y \subseteq \overline{M}^m$ definable in $\overline{\mathcal{M}}$ such that $X = Y \cap M^m$.*

Proof Let $X \subseteq M^m$, where $m \in \mathbb{N}^+$, be a definable open set in $\mathcal{M}$. By [5, Fact 2.5], there exists a strong cell decomposition $\mathcal{C}$ of M^m that partitions X into refined strong cells $C_1, \ldots, C_p$, that is, strong cells with cell-defining functions taking value only in one of the sets M, $\overline{M} \setminus M$, or $\{-\infty, +\infty\}$. Assume that $f_1, \ldots, f_q$ are all cell-defining functions of $C_1, \ldots, C_p$ that take value in $\overline{M} \setminus M$ (if they exist). For $1 \leq j \leq q$, let $\overline{f_j} : \overline{\pi(C_i)} \to \overline{M}$ be the strongly continuous extension of $f_j : \pi(C_i) \to \overline{M} \setminus M$ for some $1 \leq i \leq p$. Now, set $Z = (\bigcup_{1 \leq i \leq p} \overline{C_i}) \cup (\bigcup_{1 \leq j \leq q} \Gamma(\overline{f_j}))$ and $Y = \mathrm{int}(Z)$, the interior of Z in $\overline{M}^m$. It is clear that Y is definable in $\overline{\mathcal{M}}$. We show that $Y \cap M^m = X$. Since $Z \cap M^m = X$, $Y \cap M^m \subseteq X$. For the converse, we show that for every $a \in X$ there exists an open box $B \subseteq \overline{M}^m$ containing a such that $B \subseteq Z$.

By the assumption and Theorem 2.9, there exists a CE-cell decomposition $\mathcal{D}$ of $\overline{M}^m$ that partitions the definable set Z. By refining $\mathcal{D}$, we may assume that for all $E, F \in \pi(\mathcal{D})$, $E \subseteq \mathrm{cl}(F)$ whenever $E \cap \mathrm{cl}(F) \neq \emptyset$. Assume that $Z = \bigcup_{1 \leq l \leq k} D_l$, where $D_1, \ldots, D_k \in \mathcal{D}$. Let $a = (\overline{a}, a_m) \in X$, and let D be a CE-cell from $\{D_1, \ldots, D_k\}$ containing a. Since the assertion of the lemma clearly holds for $m = 1$, we may assume that $m > 1$. If D is open, then we are done. Assume that D is not open. Let $\mathcal{E}$ be the set of all $E \in \pi(\mathcal{D})$ such that $\mathrm{cl}(E) \cap \pi(D) \neq \emptyset$. Also, let $\mathcal{F}$ denote the set of all cell-defining functions of $\mathcal{D}$. Now, for every $E \in \mathcal{E}$, let $l_E = \min\{f \in \mathcal{F} \mid \mathrm{dom}(f) = E \text{ and } a_m < \lim_{t \to \pi(a)} f(t)\}$ and $h_E = \max\{f \in \mathcal{F} \mid \mathrm{dom}(f) = E \text{ and } \lim_{t \to \pi(a)} f(t) < a_m\}$. Then, $(l_E, h_E)_E \subseteq Z$, because X is open in M^m and $a = (\overline{a}, a_m) \in X$. Also, $a \in \mathrm{cl}((l_E, h_E)_E)$. If $\pi(D)$ is open, then for $E = \pi(D)$, the open CE-cell $(l_E, h_E)_E$ contains the desired open box. Otherwise, let $U = \bigcup_{E \in \mathcal{E}} E$. Then, U is a connected open set and $\pi(a) \in U$. For every $E \in \mathcal{E}$, there are $b_E, c_E \in M$ such that $\lim_{t \to \pi(a)} l_E(t) < b_E < a_m < c_E < \lim_{t \to \pi(a)} h_E(t)$. Then, there exists an open box B_E containing $\pi(a)$ such that $h_E(B_E \cap E) > c_E$ and $l_E(B_E \cap E) < b_E$. Then, for $B_1 = \bigcap_{E \in \mathcal{E}} B_E$ and $I = \bigcap_{E \in \mathcal{E}} (b_E, c_E)$, $B = B_1 \times I$ is an open box such that $a \in B \subseteq Z$. □

Theorem 3.2 *Assume that $\mathcal{M} = (M, <, \ldots)$ is a weakly o-minimal structure with* SCE-*cell decomposition. Then, $\mathcal{M}$ has* OCP.

Proof Let $X \subseteq M^m$ be a definable open set. By Lemma 3.1, there is an open set $Y \subseteq \overline{M}^m$ definable in $\overline{\mathcal{M}}$ such that $X = Y \cap M^m$. By Theorem 2.9 and [1, Theorem 2], there are open cells $C_1, \ldots, C_n \subseteq \overline{M}^m$ such that $Y = \bigcup_{1 \leq i \leq n} C_i$. Then, $X = \bigcup_{1 \leq i \leq n} (C_i \cap M^m)$, where each $(C_i \cap M^m)$ is an open strong cell in $\mathcal{M}$ by Fact 2.3(ii). □

From Corollary 2.8 and Theorem 3.2, we have the following result.

Corollary 3.3 *Let $\mathcal{M} = (M, <, \ldots)$ be a weakly o-minimal structure with* SCE-*cell decomposition. Then, every definable open set $X \subseteq M^m$ is a union of finitely many open* SCE-*cells.*

The following theorem is a partial converse to the above result.

Theorem 3.4 *Assume that $\mathcal{M} = (M, <, \ldots)$ is a weakly o-minimal structure with strong cell decomposition and that every open definable set $X \subseteq M^m$, $m \in \mathbb{N}^+$, is a finite union of open* SCE-*cells. Then, $\mathcal{M}$ has* SCE-*cell decomposition.*

Proof By simultaneous induction on m, we prove the assertions $(a)_m$, $(b)_m$, and $(c)_m$ below for any $m > 0$.

$(a)_m$ If $C_1, \ldots, C_k$ are strong cells in M^m, then there exists an SCE-cell decomposition $\mathcal{C}$ of M^m that partitions each of the sets $C_1, \ldots, C_k$.

$(b)_m$ If $C_1, \ldots, C_k$ are strong cells in M^m and $f_1 : C_1 \to \overline{M}, \ldots, f_k : C_k \to \overline{M}$ are definable functions, then there exists an SCE-cell decomposition $\mathcal{C}$ of M^m that partitions each of the sets $C_1, \ldots, C_k$, and for every $1 \leq i \leq k$ and $D \in \mathcal{C}$ such that $D \subseteq C_i$, $f_i|_D$ has a definable continuous extension on $\mathrm{cl}_{\overline{M}}(D)$.

$(c)_m$ If $C \subseteq M^m$ is an SCE-cell and $f : C \to \overline{M}$ is a definable strongly continuous function, then f has a definable continuous extension on $\mathrm{cl}_{\overline{M}}(C)$.

Since $\mathcal{M}$ has strong cell decomposition, the above assertions imply that $\mathcal{M}$ has SCE-cell decomposition.

$(a)_1$ holds because strong cells in M are SCE-cells. For $(b)_1$, let $C_1, \dots, C_k$ be strong cells in M, and let $f_1 : C_1 \to \overline{M}, \dots, f_k : C_k \to \overline{M}$ be definable functions. By [5, Fact 2.5], there exists a strong cell decomposition $\mathcal{C}$ that partitions each of the sets $C_1, \dots, C_k$, and for every $C \in \mathcal{C}$ with $C \subseteq C_i$, $f_i|_C$ is strongly continuous. By [5, Lemma 1.3], $\lim_{t\to a} f_i|_C(t)$ exists in $\overline{M} \cup \{-\infty, +\infty\}$ for every $a \in \mathrm{cl}_{\overline{M}}(C)$. Assume that $\lim_{t\to a} f_i(t) = b \in \{-\infty, +\infty\}$, where $a \in \{\sup C, \inf C\}$. In the case $b = +\infty$ and $a = \sup C$, the definable open set $(-\infty, f_i)_C \cup [a, +\infty) \times M$ cannot be a finite union of open SCE-cells. This contradicts our assumption. Similarly, three other cases conclude a contradiction. So $f_i|_C$ has a definable continuous extension on $\mathrm{cl}_{\overline{M}}(C)$. $(c)_1$ is clear for C a singleton. If $C \subseteq M$ is a definable convex set and $f : C \to \overline{M}$ is a definable strongly continuous function, then as we saw in $(b)_1$, $\lim_{t\to a} f(t)$ exists in $\overline{M}$ for every $a \in \mathrm{cl}_{\overline{M}}(C)$. So, f has a definable continuous extension on $\mathrm{cl}_{\overline{M}}(C)$.

Now suppose that $(a)_i$, $(b)_i$, and $(c)_i$ hold for each $i \leq m$. For $(a)_{m+1}$, let $C_1, \dots, C_k$ be strong cells in M^{m+1}. There exists a strong cell decomposition $\mathcal{C}$ that partitions each of $C_1, \dots, C_k$. Now, by applying $(b)_m$ for all cell-defining functions of $\mathcal{C}$, we get an SCE-cell decomposition of M^{m+1} that partitions each of the sets $C_1, \dots, C_k$.

For $(b)_{m+1}$, let $C_1, \dots, C_k$ be strong cells in M^{m+1}, and let $f_1 : C_1 \to \overline{M}, \dots, f_k : C_k \to \overline{M}$ be definable functions. There exists a strong cell decomposition $\mathcal{C}$ that partitions each of the sets $C_1, \dots, C_k$, and for every $C \in \mathcal{C}$ with $C \subseteq C_i$, $f_i|_C$ is strongly continuous. By $(a)_{m+1}$, we may assume that $\mathcal{C}$ is an SCE-cell decomposition. Let $C \in \mathcal{C}$ be such that $C \subseteq C_i$ for some $1 \leq i \leq k$. We show that $f_i|_C$ has a definable continuous extension on $\mathrm{cl}_{\overline{M}}(C)$. For that, we consider two cases. We first assume that C is nonopen. Then, there exists an SCE-cell $D \subseteq M^m$ and a definable homeomorphism π_C from C onto D which has a definable continuous extension $\overline{\pi}_C$ from $\mathrm{cl}_{\overline{M}}(C)$ onto $\mathrm{cl}_{\overline{M}}(D)$. By $(c)_m$, the definable strongly continuous function $f_i|_C \circ \pi_C^{-1} : D \to \overline{M}$ has a definable continuous extension on $\mathrm{cl}_{\overline{M}}(D)$. Therefore, $f_i|_C$ has a definable continuous extension on $\mathrm{cl}_{\overline{M}}(C)$. For the other case, assume that C is open. Then, $(-\infty, f_i|_C)_C$ is a strong cell and, by the hypothesis of the theorem, is a finite union of open SCE-cells, say, $U_1, \dots, U_n$. Then for any $1 \leq j \leq n$, $f_i|_{\pi(U_j)}$ has a definable continuous extension on $\mathrm{cl}_{\overline{M}}(\pi(U_j))$. By using $(a)_{m+1}$ for SCE-cells $\pi(U_1), \dots, \pi(U_n)$, we get an SCE-cell decomposition $\mathcal{D}$ that partitions each of the SCE-cells $\pi(U_1), \dots, \pi(U_n)$. Hence, for every $D \in \mathcal{D}$ contained in some $\pi(U_i)$, $f_i|_D$ has a definable continuous extension on $\mathrm{cl}_{\overline{M}}(D)$. By Lemma 2.6, $f_i|_C$ has a definable continuous extension on $\mathrm{cl}_{\overline{M}}(C)$.

For $(c)_{m+1}$, let $C \subseteq M^{m+1}$ be an SCE-cell, and let $f : C \to \overline{M}$ be a definable strongly continuous function. If C is nonopen, then, after using a projection map, we use $(c)_m$. If C is open, then by $(b)_{m+1}$ there exists an SCE-cell decomposition $\mathcal{D}$ of M^{m+1} that partitions C, and for every $D \in \mathcal{D}$ such that $D \subseteq C$, $f|_D$ has a definable continuous extension on $\mathrm{cl}_{\overline{M}}(D)$. Then, by Lemma 2.6, f has a definable continuous extension on $\mathrm{cl}_{\overline{M}}(C)$. □

From Theorem 3.4 and Corollary 3.3, we have the following result, which somehow contains a partial converse to [1, Theorem 2].

Corollary 3.5 *Let* $\mathcal{M} = (M, <, \dots)$ *be an o-minimal structure. Then,* $\mathcal{M}$ *has* CE*-cell decomposition if and only if every definable open set* $X \subseteq M^m$, $m \in \mathbb{N}^+$, *is a union of finitely many open* CE*-cells.*

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Acknowledgment

The authors thank the anonymous referee, whose useful comments helped to improve the paper.

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