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Finiteness Classes and Small Violations of Choice

Horst Herrlich, Paul Howard, and Eleftherios Tachtsis

To Ulrich Felgner on the occasion of his 70th birthday[†]

Abstract We study properties of certain subclasses of the Dedekind finite sets (addressed as finiteness classes) in set theory without the axiom of choice (AC) with respect to the comparability of their elements and to the boundedness of such classes, and we answer related open problems from Herrlich's "The Finite and the Infinite." The main results are as follows:

1. It is relatively consistent with ZF that the class of all finite sets is not the only finiteness class such that any two of its elements are comparable.
2. The principle "Small Violations of Choice" (SVC)—introduced by A. Blass—implies that the class of all Dedekind finite sets is bounded above.
3. "The class of all Dedekind finite sets is bounded above" is true in every permutation model of ZFA in which the class of atoms is a set, and in every symmetric model of ZF.
4. There exists a model of ZFA set theory in which the class of all atoms is a proper class and in which the class of all infinite Dedekind finite sets is not bounded above.
5. There exists a model of ZF in which the class of all infinite Dedekind finite sets is not bounded above.

1 Background, Terminology, and Known Results

The classical definition of a *finite set* is that a set X is finite if there exists a bijection $f : X \to n$, where n is a natural number ($n = \{m \in \omega : m < n\}$, where as usual ω denotes the set of all natural numbers). Otherwise, X is said to be *infinite*. To put it in other words, X is finite if there exists an injection $f : X \to \omega$ and there is no

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injection $g : \omega \to X$. In this paper we shall use the word *finite* in this classical sense and, as usual, *infinite* will mean "not finite."

Dedekind provided an alternative definition of finite (and consequently of infinite) which did not use the notion of natural numbers. In particular, a set X is called *Dedekind finite* if there is no bijection between X and a proper subset of X. Otherwise, X is called *Dedekind infinite*. Equivalently, X is Dedekind finite if and only if there is no injection $f : \omega \to X$. Following the notation in Herrlich [5], we shall address such sets as *D-finite* (*D-infinite*) sets. Furthermore, an infinite Dedekind-finite set shall be addressed here as a *Dedekind set*.

Now, in ZFC set theory, that is, Zermelo–Fraenkel set theory plus axiom of choice (AC), "finite" coincides with "D-finite" and "infinite" coincides with "D-infinite," so it follows that no new information is added upon the notion of finite when one lives in the ZFC world. However, if AC is discarded from the list of axioms of set theory (and it is consistent to do so, as there are models of set theory in which AC is false; for an extensive survey on such models, the reader is referred to Howard and Rubin [8]), then the above notions of finite split and Dedekind indeed provided an alternative way to talk about finite and infinite. In particular, there are models of set theory without choice in which Dedekind sets exist. So a D-finite set may not be finite; equivalently, an infinite set may not be D-infinite.

We shall adopt the standard notation for the comparability of sets as follows.

Definition 1.1 Let X and Y be two sets.

1. $X \leq Y$ if there exists an injection $f : X \to Y$.
2. $X \approx Y$ if there exists a bijection $f : X \to Y$.
3. $X < Y$ if $X \leq Y$ and $X \not\approx Y$.

So according to the terminology of Definition 1.1, a set X is finite if and only if $X < \omega$ and it is D-finite if and only if $\omega \not\leq X$.

Since the appearance of Dedekind's notion of finite, several other possible definitions of finite have been considered and examined extensively in the literature (see, e.g., De la Cruz [2], Herrlich [5], Howard and Yorke [9], Lévy [13], Tarski [14], Truss [15]). In the absence of AC, most of the various definitions of finite are *not* equivalent, and therefore without AC one cannot be certain of what may be the right definition of a finite set (though, it is agreed that the classical one occupies the most prominent place).

Besides the finite and the D-finite sets, we consider here three other notions of finite which were considered by Herrlich in [5].

Definition 1.2 Let X be a set.

1. X is called *A-finite* if X cannot be expressed as the disjoint union of two infinite sets. Otherwise, X is called *A-infinite*. If X is an infinite, A-finite set, then X is called *amorphous*.
2. X is called *B-finite* if X has no infinite linearly orderable subsets. Otherwise, X is called *B-infinite*.
3. X is called *C-finite* if there is no surjection $f : X \to \omega$. Equivalently, X is C-finite if there is no injection from $\mathcal{P}(X)$ (the power set of X) into a proper subset of $\mathcal{P}(X)$. Otherwise, X is called *C-infinite*.

We note that A-finite was called Ia-*finite* by Lévy in [13], and in the paper by Truss [15] the class of A-finite sets is called Δ_1.[1] Also, C-finite is III-*finite* in [13],[2] and the class of C-finite sets is called Δ_4 in [15]. Finally, in [15], the class of B-finite sets is called Δ_3 and the class of D-finite sets is called Δ.

We also make the following remark.

Remark 1.3 A-finite implies B- and C-finite and each of the latter two notions of finite implies D-finite; none of the former implications is reversible in ZF (Zermelo–Fraenkel set theory minus AC), and it is consistent with ZF that there exists a B-finite set which is C-infinite (see [5, Theorem 9]). Problem 1 in [5] asks whether it is provable in set theory without choice that every C-finite set is a B-finite set. We note here that the answer to this question is already known to be in the negative. In particular, Truss [15] shows that in Mostowski's linearly ordered model (for its description, see [8, p. 182, model $\mathcal{N}3$]) the B-infinite set A of atoms is a C-finite set. Although Lévy has not considered B-finiteness in his paper [13], nevertheless the cited result also follows from [13, Lemma 2, Theorems 3 and 4].

Truss and Herrlich worked from the same point of view—looking at subclasses of the class of D-finite sets rather than at definitions of finite. For example, Truss [15, Corollary, p. 197] proved the following.

Theorem 1.4 *If any two members of the class of D-finite sets are comparable with respect to $\leq$, then the class of A-finite sets is identical to the class of finite sets. (The conclusion of this theorem is equivalent to the assertion that there are no amorphous sets.)*

In the same spirit, Herrlich [5] considered certain subclasses of the D-finite sets (addressed as "finiteness classes" in [5]) and investigated closure properties (called "stability properties" in [5]) under basic set-theoretical operations (e.g., it was examined whether such classes are closed under unions, products, the power set operation, etc.), where "finite" is one of the above prescribed four notions of finite (A- or B- or C- or D-finite) plus the classical concept of finite.

Herrlich [5, Definition 6] gave the following definition.

Definition 1.5 A class $\mathfrak{U}$ of sets is called a *finiteness class* if it satisfies the following conditions:

1. $\mathfrak{U}$ contains every finite set;
2. $\mathfrak{U}$ contains with any set A every set X with $X \leq A$;
3. $\omega \notin \mathfrak{U}$.

We note that parts 2 and 3 of the definition imply that all finiteness classes are comprised of D-finite sets.

Definition 1.6

1. **Fin** is the class of all finite sets.
2. D-**Fin** is the class of all D-finite sets.
3. For any Dedekind set X, $\mathbf{Fin}(X)$ is the class of all sets Y such that $Y \leq X$.

Proposition 1.7 ([5, Theorem 10]) *The following statements hold.*

1. **Fin** *is the smallest (with respect to inclusion) finiteness class.*
2. *For any Dedekind set X, the class $\mathbf{Fin}(X)$ is a finiteness class which is properly larger than* **Fin**.
3. D-**Fin** *is the largest (with respect to inclusion) finiteness class.*

Proposition 1.7 follows immediately from the definitions.

Regarding finiteness classes, the following two questions were posed in Problems 2 and 3, respectively, in [5].

Question 1.8

(1) *Are all finiteness classes bounded with respect to* $\leq$? (Here, "bounded with respect to $\leq$" naturally means "bounded *above* with respect to $\leq$," where a finiteness class $\mathfrak{U}$ is *bounded above with respect to* $\leq$ if there exists a set U such that, for each $X \in \mathfrak{U}$, $X \leq U$.)

(2) *Can* **Fin** *be characterized as the only finiteness class in which any two elements are comparable with respect to* $\leq$?

The research in this paper is motivated by the above problems and our aim is to provide negative answers to both questions in the setting of ZF set theory.

In particular, regarding the boundedness of finiteness classes, we shall prove that it is consistent with ZF + (there exists a Dedekind set) both to have all finiteness classes bounded as well as to have an unbounded finiteness class. For the first assertion, we shall consider Blass's principle of small violations of choice (SVC) introduced in [1] (and listed as *Form 191* in [8]), and which is the following statement:

(SVC) *There is a set* S *such that, for every set* a*, there exist an ordinal* α *and a function from* $S \times \alpha$ *onto* a.

We will prove that SVC implies that the class D-**Fin** is bounded with respect to $\leq$. Since SVC is true in every permutation model with the class of atoms being a set and true in every symmetric model of ZF (see [1, Theorems 4.2 and 4.3]), we shall obtain that D-**Fin** (hence, by Proposition 1.7(1), every finiteness class) is bounded in all such models.

On the other hand, we shall establish that there is a model of ZFA with a proper class of atoms and a model of ZF (using Easton's proper class forcing) in each of which there is a finiteness class which is not bounded above.

Regarding Question 1.8(2), besides providing a negative answer, we shall scrutinize conditions that yield comparability of elements of finiteness classes (see Section 2.1 below), and we shall also show that it is consistent to have finiteness classes in which any two of their elements are comparable with respect to $\leq$, but may fail to share some naturally expected condition of comparability.

For constructions of ZFA-models with a proper class of atoms, the reader is referred to Blass [1], Felgner [3], Felgner and Jech [4], Howard, Rubin, and Rubin [7], and Jech [10, Section 11.2]. For a treatment of Easton forcing, the reader is referred to either Jech [11] or Kunen [12].

2 Main Results

2.1 Comparability of members of finiteness classes Here, we work toward a negative answer to Question 1.8(2), namely, "Can **Fin** be characterized as the only finiteness class in which any two elements are comparable with respect to $\leq$?"

We first observe that two sets X and Y are comparable with respect to $\leq$ if one of the following four conditions is satisfied:

(a) $X \subseteq Y$ or $Y \subseteq X$;

(b) at least one of the sets X and Y is finite;

(c) $X \approx W_1$ and $Y \approx W_2$, where the symmetric difference $W_1 \triangle W_2$ of W_1 and W_2 is finite (then X and Y are comparable with respect to $\leq$, since

$W_1 - W_2$ and $W_2 - W_1$ are both finite sets, hence $W_1 - W_2 \leq W_2 - W_1$ or $W_2 - W_1 \leq W_1 - W_2$, and consequently $W_1 \leq W_2$ or $W_2 \leq W_1$);

(d) $X \approx W_1$ and $Y \approx W_2$, where $W_1 - W_2$ or $W_2 - W_1$ is finite.

Note that (a) and (c) strictly imply (d) and that two sets X and Y are comparable if and only if they satisfy (d) (clearly, (d) implies that X and Y are comparable; if X and Y are comparable, say, $X \leq Y$, and we let $f : X \to Y$ be an injection, then $X \approx f[X]$, $Y \approx Y$, and $f[X] - Y = \emptyset$, and hence it is a finite set).

So there are two extremes concerning the comparability (with respect to $\leq$) of the members of some class $\mathfrak{C}$ of sets.

(A) Any two members of $\mathfrak{C}$ are comparable.

(B) Members X and Y of $\mathfrak{C}$ are comparable only if (a) or (b) or (c) above is satisfied.

We shall prove (see Theorem 2.2(3) and Theorem 2.4) that, for a finiteness class $\mathfrak{C}$, any of the four possibilities for (A) and (B) can happen (i.e., (A) and (B) both hold, exactly one of (A) and (B) holds, none of (A) and (B) holds).

Our first result below shows that, for a finiteness class $\mathfrak{C}$, condition (B) above is equivalent to "members X and Y of $\mathfrak{C}$ are comparable only if (b) or (c) above is satisfied." Note that this will not be true for arbitrary classes $\mathfrak{C}$.

Theorem 2.1 *Let $\mathfrak{C}$ be a finiteness class. Then $\mathfrak{C}$ satisfies* (B) *if and only if it satisfies the condition "members X and Y of $\mathfrak{C}$ are comparable only if (b) or (c) above is satisfied."*

Proof ($\leftarrow$) This is clear.

($\rightarrow$) Assume that $\mathfrak{C}$ satisfies condition (B), and let X and Y be two comparable members of $\mathfrak{C}$. Then, by our assumption, (a) or (b) or (c) holds. If (b) or (c) holds, then the proof is complete. If (a) holds, say, with $X \subseteq Y$, then $X \leq Y \times \{Y\}$. Since $\mathfrak{C}$ is a finiteness class and $Y \approx Y \times \{Y\}$, $Y \times \{Y\}$ is in $\mathfrak{C}$. By our assumption, one of (a), (b), or (c) must hold with X and Y replaced by X and $Y \times \{Y\}$, respectively. Since $X \subseteq Y$ and $Y \cap (Y \times \{Y\}) = \emptyset$, (a) cannot hold (unless $X = \emptyset$, in which case (b) holds). Hence (b) or (c) must hold for X and $Y \times \{Y\}$. But if either (b) or (c) holds for X and $Y \times \{Y\}$, then the one that holds must also hold for X and Y. This completes the proof of the implication and of the theorem. $\square$

Theorem 2.2 *The following statements hold.*

1. *Let $\mathfrak{C}$ be a finiteness class. Then $\mathfrak{C}$ satisfies* (B) *if and only if every member of $\mathfrak{C}$ is A-finite.*
2. *The class consisting of all A-finite sets is the largest finiteness class satisfying* (B).
3. *For any amorphous set X, the finiteness class* $\mathbf{Fin}(X)$ *satisfies* (A) *and* (B).

Proof 1. ($\rightarrow$) Assume that the finiteness class $\mathfrak{C}$ satisfies (B), and let $X \in \mathfrak{C}$. If X is finite, then there is nothing to prove. So we may assume that X is infinite. Toward a proof by contradiction, assume that $X = X_1 \cup X_2$, where $X_1 \cap X_2 = \emptyset$ and X_1 and X_2 are infinite. Then by letting $Y = X_2$, X and Y are comparable. Since (b) does not hold for X and Y, it follows by Theorem 2.1 that (c) must. Therefore, there are two sets W_1 and W_2 and injective functions f from X onto W_1 and g from Y onto W_2 such that $W_1 \triangle W_2$ is finite. Since $W_1 - W_2$ is finite and X_1 is infinite, we may assume without loss of generality that, for all $t \in X_2$, $f(t)$ is in W_2. (This may require a finite number of alterations of f of the following type. If $t \in X_2$ and $f(t) \in (W_1 - W_2)$, then choose $t' \in X_1$ such that $f(t') \in W_2$ and replace f with the function f' which agrees with f except that $f'(t) = f(t')$ and $f'(t') = f(t)$.) Then the set $Z = X - f^{-1}(W_1 - W_2) \supsetneqq X_2$. Further, for all $t \in Z$, $f(t) \in W_2 = \text{Range}(g)$. Therefore $g^{-1} \circ f$ restricted to Z is an injective function whose domain is Z and whose range is a subset of X_2. Since X_2 is a proper subset of Z, it follows that Z is Dedekind infinite. Since $Z \in \mathfrak{C}$, $\mathfrak{C}$ is not a finiteness class (recall that ω is not a member of a finiteness class). This is a contradiction, finishing the proof of the implication.

1. ($\leftarrow$) Assume that every element of $\mathfrak{C}$ is either finite or amorphous. Let X, Y be two comparable elements of $\mathfrak{C}$. If X or Y is finite, then (b) holds and we have nothing to prove. So assume that both X and Y are infinite sets; hence they are amorphous due to our hypothesis. Assume that $X \leq Y$, and let $f : X \to Y$ be an injection. Then $X \approx f[X]$ and $Y \approx Y$, and since Y is amorphous, it follows that $f[X] \triangle Y$ is finite. Hence, (c) is satisfied for X and Y, and consequently $\mathfrak{C}$ satisfies (B) as required.

2. This follows immediately from (1).

3. Let X be an amorphous set. The result follows immediately from (1) and the fact that, since X is A-finite, the power set of X is the finite-cofinite algebra. □

In the following theorem, we provide a negative answer to Question 1.8(2).

Theorem 2.3 *It is relatively consistent with set theory without choice that there exists an amorphous set. In particular, it is relatively consistent with* ZF *that* **Fin** *is not the only finiteness class such that any two of its elements are comparable with respect to* $\leq$.

Proof The first assertion of the theorem is well known (see the basic Fraenkel model for ZFA in [8, p. 176, model $\mathcal{N}1$]).

The second assertion of the theorem follows, on one hand, from the fact that the latter result about the model $\mathcal{N}1$ can be transferred to ZF using [10, Theorem 6.1] (first embedding theorem by Jech and Sochor) and [10, Section 6.3, Problems 1 and 5], and on the other hand, from Theorem 2.2(3). This completes the proof of the theorem. □

Next we investigate the relationship between conditions (A) and (B) for certain finiteness classes. It turns out that, in general, (A) and (B) are mutually independent.

Theorem 2.4 *The following statements hold:*

1. *For any amorphous set X, the finiteness class* $\mathbf{Fin}(X \times 2)$ *satisfies* (A), *but not* (B).
2. *There exists a* ZF*-model with a Dedekind set X such that the finiteness class* $\mathbf{Fin}(X)$ *satisfies neither* (A) *nor* (B).
3. *There exists a* ZFA*-model with a finiteness class which satisfies* (B), *but not* (A).

Proof 1. Fix an amorphous set X, and let $Y, Z \in \mathbf{Fin}(X \times 2)$. By the definition of the class $\mathbf{Fin}(U)$ (for a Dedekind set U), we may assume without loss of generality that Y, Z are subsets of $X \times 2$. We consider the following cases:

(i) Y or Z is finite. Then Y and Z are comparable in view of condition (b).

(ii) Y and Z meet $X \times \{0\}$ and $X \times \{1\}$ each in infinite sets. Then Y and Z are comparable in view of condition (c) (for $X \times \{0\}$ and $X \times \{1\}$ are amorphous sets, hence $Y \triangle Z$ is finite).

(iii) Y and Z meet $X \times \{i\}$ in an infinite set and $X \times \{j\}$ in a finite set, where i, j are distinct elements of 2. Then Y and Z are comparable in view of condition (c) since their symmetric difference is finite.

(iv) Y meets exactly one of the sets $X \times \{0\}$ and $X \times \{1\}$ in an infinite set, and Z meets exactly the other one in an infinite set. Let $Y_0 = Y \cap (X \times \{0\})$, $Y_1 = Y \cap (X \times \{1\})$, $Z_0 = Z \cap (X \times \{0\})$, and $Z_1 = Z \cap (X \times \{1\})$, and assume that Y_0, Z_1 are infinite and that Y_1, Z_0 are finite. Since $X \times \{0\} \approx X \times \{1\}$ ($\approx X$), we obtain that $Y_0 \approx W$ for some $W \subseteq X \times \{1\}$. Since $X \times \{1\}$ is amorphous, it follows that W and Z_1 are comparable. By case (iii), it follows that $W \cup Y_1$ and Z are comparable. Since $Y \approx (W \cup Y_1)$, we obtain that Y and Z are comparable.

That $\mathbf{Fin}(X \times 2)$ does not satisfy (B) follows from the fact that $X \times 2$ is not amorphous and from Theorem 2.2(1).

2. This follows from Herrlich and Tachtsis [6, Proposition 18]. In the second Cohen model (see [8, model $\mathcal{M}7$]) there is a Dedekind set $X = \bigcup_{i \in \omega} X_i$, where $\{X_i : i \in \omega\}$ is a pairwise disjoint family of two-element sets. In [6], it was established that, for any two subsets M and K of ω such that $M - K$ and $K - M$ are infinite, the sets $\bigcup_{m \in M} X_m$ and $\bigcup_{k \in K} X_k$ are incomparable in this model. Thus, $\mathbf{Fin}(X)$ does not satisfy (A).

Consider now the elements X and $Y = \bigcup_{n \in \omega} X_{2n}$ of $\mathbf{Fin}(X)$. Clearly, X and Y are comparable, but neither (b) nor (c) is satisfied for X and Y. Hence, by Theorem 2.1, $\mathbf{Fin}(X)$ does not satisfy condition (B).

3. We start with a ground model $\mathcal{M}$ of ZFA+AC with a set of atoms $A = A_0 \cup A_1$, where $A_0 \cap A_1 = \emptyset$ and each A_i, $i = 0, 1$, is a countably infinite set of atoms. The group G of permutations of A is the group of all permutations ψ such that $\psi(A_0) = A_0$ and $\psi(A_1) = A_1$. The normal ideal I of supports is the set of all finite subsets of A. Let $\mathcal{N}$ be the permutation model which is determined by G and I.

Using standard techniques in Fraenkel–Mostowski models one verifies that both A_0 and A_1 are amorphous sets in $\mathcal{N}$, so we leave this as an easy exercise for the reader.

Consider the finiteness class $\mathfrak{M} = \mathbf{Fin}(A_0) \cup \mathbf{Fin}(A_1)$. Clearly, $\mathfrak{M} \in \mathcal{N}$ since every permutation of A in G fixes $\mathfrak{M}$. In particular, any $\psi \in G$ fixes $\mathbf{Fin}(A_0)$ and $\mathbf{Fin}(A_1)$. We complete the proof by establishing the following two claims.

Claim 2.5 *The class $\mathfrak{M}$ does not satisfy* (A).

Proof Assume the contrary. Then since $A_0, A_1 \in \mathfrak{M}$, $A_0 \leq A_1$ or $A_1 \leq A_0$. Suppose that $A_0 \leq A_1$, and let $f : A_0 \to A_1$ be an injection in $\mathcal{N}$, say, with support E. Since E is finite and A_0 is infinite, let a, b be two distinct elements of $A_0 - E$, and assume that $f(a) = c$ for some $c \in A_1$. Consider the permutation ψ of A in G which swaps a with b and fixes $A - \{a, b\}$ pointwise. Then $\psi \in \mathrm{fix}(E)$, hence $\psi(f) = f$. However,

$$(a, c) \in f \to \big(\psi(a), \psi(c)\big) \in \psi(f) \to (b, c) \in f,$$

meaning that f is not injective, which is a contradiction. Hence, $A_0 \not\leq A_1$ and in a similar manner one verifies that $A_1 \not\leq A_0$. Thus, $\mathfrak{M}$ does not satisfy (A) as claimed. $\square$

Claim 2.6 *The class* $\mathfrak{M}$ *satisfies* (B).

Proof In view of part 1 of Theorem 2.2, it suffices to show that every element of $\mathfrak{M}$ is A-finite. Let $X \in \mathfrak{M}$. By the definition of $\mathfrak{M}$, $X \in \mathbf{Fin}(A_i)$ for some $i \in 2$. Since $X \leq A_i$ and A_i is amorphous in the model, it follows that X is A-finite. Hence, we have the result of the claim. $\square$

The proof of part 3, as well as of the theorem, is complete. $\square$

2.2 All of the finiteness classes can be bounded with respect to $\leq$ In this section as well as the next one, we attack Question 1.8(1) of Section 1—namely, "Are all finiteness classes bounded with respect to $\leq$?" We will prove that the statement "Every finiteness class is bounded above" is *undecidable* in ZF + (there exist Dedekind sets), which of course means that there exist two models of ZF + (there exist Dedekind sets), one of which satisfies the statement and the other satisfies its negation.

Lemma 2.7 *Assume S witnesses* SVC. *Then for every set a there exists an ordinal α such that $a \leq \alpha \times \mathcal{P}(S)$, where $\mathcal{P}(S)$ is the power set of S.*

Proof Let a be any set, and let, by SVC (as witnessed by S), α be an ordinal and f be a function from $S \times \alpha$ onto a. For every $x \in a$, let

$$\kappa_x = \min\big\{\lambda \in \alpha : f^{-1}\big(\{x\}\big) \cap \big(S \times \{\lambda\}\big) \neq \emptyset\big\}.$$

Clearly, the mapping $g : a \to \alpha \times \mathcal{P}(S)$ defined by

$$g(x) = \big(\kappa_x, \big\{s \in S : f\big((s, \kappa_x)\big) = x\big\}\big), \quad x \in a,$$

is an injection. $\square$

Theorem 2.8 SVC *implies there exists a set B such that, for every Dedekind finite set X, $X \leq B$. Hence,* SVC *implies that every finiteness class is bounded above with respect to $\leq$.*

Proof Assume S witnesses SVC, and let X be any Dedekind finite set. We will prove that $X \leq \omega \times \mathcal{P}(S)$. By Lemma 2.7, consider an ordinal α and an injective function $f : X \to \alpha \times \mathcal{P}(S)$. Without loss of generality assume that, $\forall i < \alpha$, $\mathrm{Ran}(f) \cap (\{i\} \times \mathcal{P}(S)) \neq \emptyset$.

Definition 2.9 $\forall A \in \mathcal{P}(S)$, $U_A := \{\beta < \alpha : (\beta, A) \in \mathrm{Ran}(f)\}$ and $T_A := \{x \in X : \exists \beta < \alpha, f(x) = (\beta, A)\}$.

Lemma 2.10 $\forall A \in \mathcal{P}(S)$, *the sets U_A and T_A are finite, and $(T_A)_{A \in \mathcal{P}(S)}$ is a partition of X.*

Proof Assume by way of contradiction that, for some $A \subseteq S$, U_A is infinite. Since f is injective, it follows that, for each $\gamma \in U_A$, there exists *exactly one* $x_\gamma \in X$ such that $f(x_\gamma) = (\gamma, A)$. Since U_A is infinite, $V = \{x_\gamma : \gamma \in U_A\}$ is an infinite well-ordered subset of X, contradicting the fact that X is Dedekind finite. Hence, U_A is finite as required.

It is also easy to verify that $T_A \approx U_A$, $T_A \cap T_B = \emptyset$ for distinct subsets A and B of S, and $\bigcup\{T_A : A \in \mathcal{P}(S)\} = X$. This completes the proof of the lemma. $\square$

Since, for every $A \in \mathcal{P}(S)$, $U_A \in [\alpha]^{<\omega}$ and α is an ordinal, it follows that each U_A has a fixed well-ordering and so it is feasible without using any form of choice to define for every $A \in \mathcal{P}(S)$ an injection $g_A : U_A \to |U_A|$.

Letting $B = \omega \times \mathcal{P}(S)$, we define a function $h : X \to B$ as follows. Let $x \in X$. Then there exist a unique $A_x \subseteq S$ such that $x \in T_{A_x}$ and a unique ordinal $\beta_x \in U_{A_x}$ such that $f(x) = (\beta_x, A_x)$. Define

$$h(x) = \big(g_{A_x}(\beta_x), A_x\big).$$

Then h is an injection. Indeed, let x and y be two distinct elements of X. There are two cases:

(a) $A_x = A_y$. Then $\beta_x \neq \beta_y$, so $g_{A_x}(\beta_x) \neq g_{A_x}(\beta_y) = g_{A_y}(\beta_y)$ for g_{A_x} is injective on U_{A_x}. Thus, $h(x) \neq h(y)$ since they have different first coordinates.

(b) $A_x \neq A_y$. Since A_x and A_y are the second coordinates of $h(x)$ and $h(y)$, respectively, it follows that $h(x) \neq h(y)$.

The second assertion of the theorem follows from Proposition 1.7(3), finishing the proof of the theorem. □

Corollary 2.11 *The statement "Every finiteness class is bounded with respect to $\leq$" is true in every permutation model of* ZFA *(in which the class of atoms is a set) and in every symmetric model of* ZF.

Proof SVC is true in all of these models (see [1, Theorems 4.2 and 4.3]). Hence, we have the result. □

Corollary 2.12 *The statement "Every finiteness class is bounded with respect to $\leq$" is consistent with* ZFA + *(there exist Dedekind sets) and with* ZF + *(there exist Dedekind sets).*

Proof The basic Fraenkel permutation model and the basic Cohen symmetric model are, respectively, models of ZFA + (there exist Dedekind sets) and of ZF + (there exist Dedekind sets) (see [8]). By Corollary 2.11, "every finiteness class is bounded above" is true in each of these models. □

2.3 Two models of ZFA and ZF set theory in which D-Fin is not bounded We commence this section by constructing a suitable model for ZFA set theory which has a *proper class of atoms* and which will witness a finiteness class which is not bounded with respect to $\leq$.

We start with a transitive model $\mathcal{M}$ of ZFA+AC with a set of atoms $A = \bigcup_{i \in \aleph_1} A_i$, where $\mathcal{A} = \{A_i : i \in \aleph_1\}$ is pairwise disjoint and, for each $i \in \aleph_1$, $A_i = \{a_i, b_i\}$. Let G be the group of all permutations of A which fix $\mathcal{A}$ pointwise. Let I be the normal ideal of all finite subsets of A, and let $\mathcal{M}'$ be the permutation model which is determined by G and I. Then $\mathcal{M}'$ is a model of ZFA which contains Dedekind sets (e.g., any infinite subset of A).

For every set x of $\mathcal{M}'$, we denote by $\ker(x)$ the kernel of x, that is, the set $\mathrm{TC}(x) \cap A$, where $\mathrm{TC}(x)$ is the transitive closure of x, that is, the set $x \cup (\bigcup x) \cup (\bigcup\bigcup x) \cup \cdots$.

Let $\mathcal{M}''$ be the subclass of $\mathcal{M}'$ consisting of all atoms and of all sets $x \in \mathcal{M}'$ such that $\ker(x) \subseteq \bigcup_{i \in K} A_i$ for some $K \in [\aleph_1]^{\leq\omega}$ (= the set of all countable subsets of $\aleph_1$). Note that, by the definition of $\mathcal{M}''$, A is a proper class in $\mathcal{M}''$ and $\mathcal{M}''$ is a transitive class.

Theorem 2.13 *$(\mathcal{M}'', \in)$ is a model of* ZFA *set theory in which there is a finiteness class which is not bounded with respect to $\leq$. Hence, D-**Fin** is not bounded with respect to $\leq$ in $\mathcal{M}''$.*

Proof $\mathcal{M}''$ is similar to the model given in [1, Theorem 3.1, p. 36], so we shall give a sketchy proof of $(\mathcal{M}'', \in)$ being a model of ZFA. For the reader's convenience, we discuss the validity of the axiom scheme of replacement in $\mathcal{M}''$.

Lemma 2.14 *If π is a permutation of A (not necessarily in the group G) such that, when π is extended to $\mathcal{M}$ (by recursion on the ranks; $\pi(\emptyset) = \emptyset$, $\pi(x) = \pi[x] = \{\pi(y) : y \in x\}$), $\pi \restriction \mathcal{M}''$ is an $\in$-automorphism of $\mathcal{M}''$, then the sentence $(\forall x_1)\ldots(\forall x_n)\ (F(x_1, x_2, \ldots, x_n) \leftrightarrow F(\pi x_1, \pi x_2, \ldots, \pi x_n))$, where F is a formula in the language of set theory with n free variables $x_1, x_2, \ldots, x_n$, is true when relativized to $\mathcal{M}''$.*

Proof The proof is by straightforward induction on the complexity of F. □

Let S be a set in $\mathcal{M}''$, and let $F(x, y, \vec{p})$, where $\vec{p} = (p_1, p_2, \ldots, p_n) \in \mathcal{M}''$ and the p_i's are parameters, be a formula in the language of set theory such that $f = \{(x, y) : F(x, y, \vec{p})$ is true when relativized to $\mathcal{M}''\}$ is a function on S in $\mathcal{M}''$, that is, S is included in the domain of f, $f \restriction S$ is a function, and $f \restriction S$ is in $\mathcal{M}''$. We will prove that $f[S] = \{y \in \mathcal{M}'' : \exists x \in S$ such that $F(x, y, \vec{p})\} \in \mathcal{M}''$.

To this end, let E be a support of S and of $\vec{p}$, and let $K \in [\aleph_1]^{\leq\omega}$ be chosen so that $E \cup \ker(S) \cup \ker(\vec{p}) \subseteq \bigcup_{j\in K} A_j$. It is not hard to verify, using Lemma 2.14, that $f[S]$ has support E, and consequently $f[S] \in \mathcal{M}'$. In order to complete the proof, we show that $\ker(f[S]) \subseteq \bigcup_{j\in K} A_j$. It suffices to verify that, $\forall y \in f[S]$, $\ker(y) \subseteq \bigcup_{j\in K} A_j$. Assume the contrary, and let $y \in f[S]$ and $x \in S$ be such that $F(x, y, \vec{p})$ is true in $\mathcal{M}''$, but there exists $a \in \ker(y) - \bigcup_{j\in K} A_j$. Let $i \in \aleph_1$ be such that $a \in A_i$, and let $k \in \aleph_1$ be such that $A_k \cap (\ker(y) \cup \bigcup_{j\in K} A_j) = \emptyset$. (Note here that the cardinal $\aleph_1$ is considered in $\mathcal{M}'$ and that it is the same as the $\aleph_1$ of $\mathcal{M}$, since the latter is a *pure set*, i.e., its kernel is empty, and $\mathcal{M}'$ contains the pure part (that is, the pure sets) of the model $\mathcal{M}$. Since $\mathcal{M}$ satisfies AC, $\aleph_1$ is regular in $\mathcal{M}$, thus also regular in $\mathcal{M}'$. This justifies the existence of a $k \in \aleph_1$ with the cited property.) Let $\varphi = (a_i, b_k) \circ (b_i, a_k)$; that is, φ swaps A_i with A_k and fixes all the other atoms pointwise. Then $\varphi \notin G$, and φ satisfies the following.

(1) φ, when extended to $\mathcal{M}'$, is an $\in$-automorphism of $\mathcal{M}'$ (note that φ is well defined, for $\varphi(x) \in \mathcal{M}'$ for all $x \in \mathcal{M}'$. In particular, if $x \in \mathcal{M}'$ with support E, then $\varphi(E)$ is a support for $\varphi(x)$). Furthermore, since $\forall x \in \mathcal{M}'$, $\ker(\varphi(x)) = \varphi(\ker(x))$, we obtain that $\varphi \restriction \mathcal{M}''$ becomes an $\in$-automorphism of $\mathcal{M}''$. Thus, by Lemma 2.14, $F(\varphi(x), \varphi(y), \varphi(\vec{p}))$ is true in $\mathcal{M}''$.

(2) Since φ fixes $\bigcup_{j\in K} A_j$ pointwise and $\ker(S) \cup \ker(\vec{p}) \subseteq \bigcup_{j\in K} A_j$, it follows that $\varphi(S) = S$ and $\varphi(\vec{p}) = \vec{p}$. Furthermore, as $x \in S$, it follows that $\ker(x) \subseteq \ker(S)$, hence $\varphi(x) = x$ too.

(3) $\varphi(y) \neq y$; otherwise, $\varphi(\ker(y)) = \ker(\varphi(y)) = \ker(y)$ and so $\varphi(a) \in \ker(y)$, which is a contradiction.

From the above observations, we conclude that $F(x, y, \vec{p})$ and $F(x, \varphi(y), \vec{p})$ are true in $\mathcal{M}''$, contradicting the functionality of F on S. Thus, the axiom scheme of replacement is valid in $\mathcal{M}''$ as required.

We now prove the second assertion of the theorem. For each $K \in [\aleph_1]^{\leq\omega}$, let $S_K = \bigcup_{k\in K} A_k$, and let $\mathfrak{M}_K = (\mathbf{Fin}(S_K))^{\mathcal{M}''}$ $(= \{Y \in \mathcal{M}'' : \mathcal{M}'' \models Y \leq S_K\})$.

Clearly, $\mathfrak{M} = \bigcup\{\mathfrak{M}_K : K \in [\aleph_1]^{\leq\omega}\}$ is a (proper) finiteness subclass of $\mathcal{M}''$. Toward a proof by contradiction, assume that there exists a set $U \in \mathcal{M}''$ such that, for every $X \in \mathfrak{M}$, $X \leq U$ in $\mathcal{M}''$.

Fix $K \in [\aleph_1]^{\leq\omega}$ such that $\ker(U) \subseteq S_K$, and let $L \in [\aleph_1]^{\omega}$ be such that $K \cap L = \emptyset$. Then $S_K \cap S_L = \emptyset$, $S_L \in \mathfrak{M}$, and, by our hypothesis, $S_L \leq U$. Let $f \in \mathcal{M}''$ be an injection of S_L into U, and let E be a support of f. Pick an $l \in L$ such that $A_l \cap E = \emptyset$ (recall that E is a finite subset of A). Assume that $f(a_l) = x$ and $f(b_l) = y$ for some $x, y \in U$. Since $a_l \neq b_l$ and f is injective, $x \neq y$. Consider now the permutation $\pi = (a_l, b_l)$ (i.e., π transposes the atoms a_l and b_l and fixes all the other atoms pointwise). Then $\pi \in \mathrm{fix}_G(E)$, hence $\pi(f) = f$. Furthermore, since $\ker(x)$, $\ker(y)$ are both subsets of $\ker(U)$, hence of S_K, and π fixes S_K pointwise, it follows that $\pi(x) = x$ and $\pi(y) = y$. Now, we have the following:

$$(a_l, x) \in f \rightarrow \big(\pi(a_l), \pi(x)\big) \in \pi(f) \rightarrow (b_l, x) \in f.$$

But this contradicts the fact that f is an injection. Therefore, the class $\mathfrak{M}$ is not bounded with respect to $\leq$, finishing the proof of the theorem. $\square$

Next we construct a ZF-model M (which can be roughly thought of as the ZF-analogue of the model $\mathcal{M}''$ of Theorem 2.13) in which Dedekind sets exist and the class of all Dedekind finite sets is not bounded above. Since our model resembles the ZF-model given in [1, Theorem 3.2], most of the parts of our proof below will be sketchy.

We start with Gödel's constructible universe L (which satisfies ZFC and the axiom of constructibility $V = L$). Let F be an Easton index function in L on the proper class of all regular cardinals of L (see [11, Theorem 15.18, pp. 232–34] for the latter notion). We define a proper class P of forcing conditions as follows: P consists of all functions p with values 0 and 1, whose domain consists of triples (κ, i, λ), where $\kappa \in \mathrm{Dom}(F)$, $i < 2 < F(\kappa)$, and $\lambda < \kappa$, and such that, for every regular cardinal μ of L, $|\{(\kappa, i, \lambda) \in \mathrm{Dom}(p) : \kappa \leq \mu\}| < \mu$. Note that this implies that, for $\mu = \kappa$, the function whose domain consists of all triples with κ as the first coordinate is a partial function from $2 \times \kappa$ into 2 whose domain has cardinality less than κ. (P is the Easton product of P_κ, κ regular, where P_κ consists of all partial functions p from $\{\kappa\} \times 2 \times \kappa$ into 2 of cardinality less than κ; see [11, p. 233].) A condition p is stronger than q if and only if $p \supseteq q$. Let $G \subseteq P$ be a P-generic filter over L, and let $V = L[G]$ be the resulting extension model. Exactly as in the proof of [11, Theorem 15.18] one verifies that cardinals and cofinalities are preserved by the extension.

So the forcing notion P adjoins for every regular cardinal $\kappa \in L$ a pair of generic subsets of κ, namely, $b^0_\kappa = \{\gamma < \kappa : \exists p \in G, p(\kappa, 0, \gamma) = 1\}$ and $b^1_\kappa = \{\gamma < \kappa : \exists p \in G, p(\kappa, 1, \gamma) = 1\}$. For every regular cardinal κ of L and for $i \in 2$, let $a^i_\kappa = \{x \subseteq \kappa : x \triangle b^i_\kappa \in L \text{ and } |x \triangle b^i_\kappa| < \kappa\}$, where $\triangle$ denotes the operation of symmetric difference between sets. Extend the language of ZF by adding a unary function symbol S, which in V is interpreted by defining $S(a^i_\kappa) = a^{1-i}_\kappa$ and $S(x)$ is undefined for $x \in V$ such that $x \neq a^i_\kappa$ for all regular cardinals κ and $i \in 2$. (Note that (the interpreted) S behaves as the permutations of A in the group G of the above ZFA-model).

Let $B = \{b^i_\kappa : \kappa \in \mathrm{Dom}(F), i \in 2\}$, and let $M = \mathrm{HOD}(B)$; that is, M is the submodel of V which consists of all sets that are hereditarily ordinal-definable (HOD), in the extended language, from finitely many elements of B (for ordinal-definable and HOD sets, see [11, pp. 194–96]).

For every regular cardinal $\kappa \in L$ and for every $x \in \mathcal{P}(\kappa)$ consisting of regular cardinals, let $D_{x,\kappa} = \bigcup\{\{a^0_\lambda, a^1_\lambda\} : \lambda \in x\}$. Using the fact that $M = \mathrm{HOD}(B)$, one may verify that each $D_{x,\kappa}$ is a Dedekind finite set. Indeed, fix a regular cardinal $\kappa \in L$ and an infinite subset $x \subseteq \kappa$ consisting of regular cardinals. By way of contradiction, assume that $D_{x,\kappa}$ is Dedekind infinite, hence there exists an infinite subset $y \subseteq x$ and a bijection $f : \omega \to D_{y,\kappa}$ in M. Pick a cardinal $\xi \in y$ such that ξ does not appear as a subscript of any of the (finitely many) generic b's used in the definition of the function f, and let $n, m \in \omega$ be such that $f(n) = a^0_\xi$ and $f(m) = a^1_\xi$. Let $p \in G$ be such that

$$p \Vdash (\dot{f} \text{ is injective}) \wedge \big(\dot{f}(\check{n}) = \dot{a}^0_\xi\big) \wedge \big(\dot{f}(\check{m}) = \dot{a}^1_\xi\big), \tag{1}$$

where $\check{n}$ and $\check{m}$ are the canonical names of n and m, respectively (see [12, Definition 2.10]). As in part of the proof of [1, Theorem 3.2, p. 40], one may define two automorphisms g and h of P such that

(a) g fixes the name $\dot{f}$, but sends $\dot{a}^0_\xi$ to $\dot{a}^1_\xi$,

(b) h fixes the names $\dot{f}$, $\dot{a}^0_\xi$, and $\dot{a}^1_\xi$,

(c) $hg(p)$ is compatible with p,

(d) $hg(p) \Vdash \dot{f}(\check{n}) = \dot{a}^1_\xi$.

In view of (a)–(d) and equation (1), we conclude that there is a forcing condition $s \in G$ such that

$$s \Vdash \big(\dot{f}(\check{n}) = \dot{a}^1_\xi\big) \wedge \big(\dot{f}(\check{m}) = \dot{a}^1_\xi\big),$$

which yields that f is not injective, which is a contradiction. Therefore, $D_{x,\kappa}$ is Dedekind finite as asserted.

For every regular cardinal $\kappa \in L$ and for every infinite $x \in \mathcal{P}(\kappa)$ consisting of regular cardinals, let $\mathfrak{M}_{x,\kappa} = \mathbf{Fin}(D_{x,\kappa})$. Let also $\mathfrak{M}$ be the union of the $\mathfrak{M}_{x,\kappa}$'s. Clearly, $\mathfrak{M}$ is a finiteness class of M.

Theorem 2.15 *The finiteness class $\mathfrak{M}$ is not bounded above in the* ZF*-model M.*

Proof By way of contradiction, assume that $\mathfrak{M}$ has a $\leq$-upper bound in M, say, U. Since U is a set, there exists a cardinal number $\kappa \in \mathrm{Dom}(F)$ such that every element of U is in $\mathrm{HOD}(\{b^i_\lambda : \lambda < \kappa, i < 2\})$ and such that *infinitely* many regular cardinals less than κ do not appear as subscripts of the generic b's used in the previous definitions. Let Z_κ be the infinite set of all *those* regular cardinals less than κ, and consider the set $D_{Z_\kappa,\kappa} = \bigcup\{\{a^0_\xi, a^1_\xi\} : \xi \in Z_\kappa\}$.

By our assumption, there exists an injection $f : D_{Z_\kappa,\kappa} \to U$ in M. By the properties of κ and Z_κ, we may conclude that there is a $\xi \in Z_\kappa$ such that f and all the elements of U are ordinal-definable in V using b's with subscripts different than ξ. Assume that $f(a^0_\xi) = q$ and $f(a^1_\xi) = r$ for some distinct elements q and r of U. Let $p \in G$ be such that

$$p \Vdash (\dot{f} \text{ is injective}) \wedge \big(\dot{f}(\dot{a}^0_\xi) = \dot{q}\big) \wedge \big(\dot{f}(\dot{a}^1_\xi) = \dot{r}\big). \tag{2}$$

Similarly to the above proof (prior to the current theorem), one defines two automorphisms g and h of P such that

(i) g fixes the names $\dot{f}$, $\dot{q}$, $\dot{r}$, but sends $\dot{a}^0_\xi$ to $\dot{a}^1_\xi$,

(ii) h fixes the names $\dot{f}$, $\dot{q}$, $\dot{r}$, $\dot{a}^0_\xi$, and $\dot{a}^1_\xi$,
(iii) $hg(p)$ is compatible with p,
(iv) $hg(p) \Vdash \dot{f}(\dot{a}^1_\xi) = \dot{q}$.

Due to (i)–(iv) and equation (2), there is a forcing condition $s \in G$ such that

$$s \Vdash \big(\dot{f}(\dot{a}^0_\xi) = \dot{q}\big) \wedge \big(\dot{f}(\dot{a}^1_\xi) = \dot{q}\big),$$

which yields that f is not injective, which is a contradiction. Thus, $\mathfrak{M}$ is not bounded in the model M, finishing the proof of the theorem. □

Notes

1. Actually, Truss [15] considers the above classes of D-finite sets as classes of cardinalities of such sets where, without AC, the cardinality $|X|$ of a set X is defined as the set of all sets Y of minimum rank such that $X \approx Y$.

2. Lévy's formulation of III-finiteness is, in particular, the second clause of (3) in Definition 1.2, that is, X is III-finite if and only if $\mathscr{P}(X)$ is D-finite.

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Herrlich
Feldhäuser Str. 69
28865 Lilienthal
Germany

Howard
Department of Mathematics
Eastern Michigan University
Ypsilanti, Michigan 48197
USA
phoward@emich.edu

Tachtsis
Department of Mathematics
University of the Aegean
Karlovassi, Samos 83200
Greece
ltah@aegean.gr
http://www.actuar.aegean.gr/eng/Dep/Taxtsis_E.html