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Non-Fregean Propositional Logic with Quantifiers

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Abstract We study the non-Fregean propositional logic with propositional quantifiers, denoted by $\mathsf{SCI_Q}$. We prove that $\mathsf{SCI_Q}$ does not have the finite model property and that it is undecidable. We also present examples of how to interpret in $\mathsf{SCI_Q}$ various mathematical theories, such as the theory of groups, rings, and fields, and we characterize the spectra of $\mathsf{SCI_Q}$-sentences. Finally, we present a translation of $\mathsf{SCI_Q}$ into a classical two-sorted first-order logic, and we use the translation to prove some model-theoretic properties of $\mathsf{SCI_Q}$.

1 Introduction

Non-Fregean logics, introduced by Roman Suszko, can be seen as a realization of Gottlob Frege's semantic program with the exception of a postulate—known in the literature as the *Fregean axiom*—that treats the truth value of a sentence as its denotation. According to Frege, sentences are not only true or false, but are also names denoting the corresponding truth values. The Fregean axiom is a fundamental assumption underlying classical logic. The theory of models based on classical logic does not involve a universe corresponding to the statements in the appropriate language, but only assigns a simple truth value to each statement. Thus, the Fregean axiom is a formal expression of a certain philosophical view concerning the meanings of statements.

Non-Fregean logic was explicitly proposed by Suszko as an alternative to the established standard. Non-Fregean logic rejects the Fregean axiom and introduces a universe of the semantic correlates of statements, known as the universe of *situations*. In order to express claims concerning the universe of situations, a new connective $\equiv$, called the *identity connective*, is added to the language. The identity connective, as the name implies, expresses the identity of two statements; that is, it

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connects two statements and forms a new one, which is true whenever the semantic correlates of the arguments are the same. Suszko presents the central ideas of the non-Fregean framework and the underlying philosophical motivations extensively in his article [11].

The weakest non-Fregean logic is *sentential calculus with identity* or SCI, first introduced by Suszko in [8]. SCI is defined axiomatically: the axioms of classical propositional logic are augmented with new axioms characterizing the identity connective, postulating that identity be an equivalence and obey an extensionality principle. Suszko argues that the identity connective is more basic than other non-truth-functional operators, such as modal operators, pointing out that it cannot be eliminated without trivializing it into another name for the equivalence connective. In general, the identity connective is different from classical equivalence: two sentences that are simultaneously true or simultaneously false do not need to have identical semantic correlates. In other words, the truth value of a sentence is not the same as the situation it describes. Most of the existing literature on non-Fregean logics concentrates on SCI and its propositional extensions. Richer extensions of SCI, such as non-Fregean propositional logic with quantifiers or non-Fregean first- or higher-order logic, have been studied to a limited degree and little is known about their properties. Many questions about them remain unanswered. Arguably the most important result is the definition of a semantics for first-order non-Fregean logic and a corresponding completeness theorem presented by Bloom [1].

The topic of the present article is SCI_Q, the non-Fregean propositional logic with quantifiers. The logic SCI_Q is obtained from SCI by extending the language with the quantifiers $\forall$ and $\exists$, which bind propositional variables, and by adding some fairly natural axioms concerning them.

This seemingly minor change to the language proves to have drastic effects. The logic SCI, despite its many advantages, is inadequate for many purposes because it lacks expressive power. In addition to not having any means of expressing connections between objects and situations, it does not allow one to claim the existence of a given type of situation. The logic SCI_Q offers a much wider repertoire of ways to express interesting properties of the universe of situations.

In this article, we review some previously known facts about SCI_Q and present our new results. Among other things, we show that SCI_Q does not have the finite model property and is undecidable. We also show that in SCI_Q one can formulate many interesting first-order theories, such as the theories of groups, rings, and fields, as well as a weak fragment of Peano arithmetic. Moreover, we explore the computational complexity of SCI_Q, including an analogue of Fagin's theorem for SCI_Q. Finally, we prove the Löwenheim–Skolem theorem for SCI_Q by translating SCI_Q into a two-sorted first-order theory, whose model-theoretic properties translate back into SCI_Q.

The results presented here, as well as other examples in the literature concerning the expressibility of various nonclassical logics in SCI (e.g., modal logics and finite-valued Łukasiewicz logics), show that non-Fregean logic can be seen as a *logical framework* for a majority of the most important logical calculi. This, in turn, means that non-Fregean logic offers good tools for representing and comparing syntactically and/or semantically different logics in a unified logical formalism. It is also worth noting that non-Fregean logic has been an inspiration to several other logical systems. It is known that logics with non-Fregean semantics are not algebraizable in the Rasiowa–Sikorski style. This fact inspired Brown, Suszko, and Bloom [3] to introduce the concept of abstract logics. Currently, many ideas presented in [3] are explored and developed in the framework of abstract algebraic logics.

The paper is organized as follows. In Section 2, we present the basic definitions and properties of the non-Fregean propositional logic SCI. The language, Hilbert-style axiomatization, and non-Fregean semantics for the extension of SCI with propositional quantifiers, that is, for the logic $\mathsf{SCI_Q}$, are presented in Section 3. In Section 4, we show that first-order theories of groups, rings, fields, and a weak fragment of Peano arithmetic are expressible in the logic $\mathsf{SCI_Q}$. One of the main conclusions of this section is the undecidability of $\mathsf{SCI_Q}$. In Section 5, we show an exact correspondence between $\mathsf{SCI_Q}$ and NP (nondeterministic polynomial-time computations). In Section 6, we interpret $\mathsf{SCI_Q}$ as a first-order theory, we prove a translation theorem, and we show how it can be used to get the Löwenheim–Skolem theorem for $\mathsf{SCI_Q}$. Conclusions and open problems are described in Section 7.

2 The Basic Non-Fregean Propositional Logic SCI

The basic minimal non-Fregean propositional logic is the logic SCI, known in the literature as *sentential calculus with identity*. The vocabulary of the language of SCI consists of the symbols from the following pairwise disjoint sets: $\mathbb{V}$, a countably infinite set of propositional variables; $\{\neg, \vee, \wedge, \rightarrow, \leftrightarrow, \equiv\}$, the set of propositional operations of negation $\neg$, disjunction $\vee$, conjunction $\wedge$, implication $\rightarrow$, equivalence $\leftrightarrow$, and identity $\equiv$. The set of SCI-formulas is the smallest set including $\mathbb{V}$ and closed with respect to all the propositional operations.

A Hilbert-style axiomatization of SCI consists of axiom schemas of the classical propositional logic PC, which characterize the operations $\neg, \vee, \wedge, \rightarrow, \leftrightarrow$, and the following axiom schemas for the identity operation $\equiv$:

($\equiv_1$) $\varphi \equiv \varphi$,
($\equiv_2$) $(\varphi \equiv \psi) \rightarrow (\neg\varphi \equiv \neg\psi)$,
($\equiv_3$) $(\varphi \equiv \psi) \rightarrow (\varphi \rightarrow \psi)$,
($\equiv_4$) $[(\varphi \equiv \psi) \wedge (\vartheta \equiv \xi)] \rightarrow [(\varphi \# \vartheta) \equiv (\psi \# \xi)]$, for $\# \in \{\vee, \wedge, \rightarrow, \leftrightarrow, \equiv\}$.

The only rule of inference is modus ponens. The notion of provability of a formula is defined as usual. Thus, an SCI-formula φ is said to be SCI-*provable* whenever there exists a finite sequence $\varphi_1, \ldots, \varphi_n$ of SCI-formulas, $n \geq 1$, such that $\varphi_n = \varphi$ and each φ_i, $i \in \{1, \ldots, n\}$, is an SCI-axiom or follows from earlier formulas in the sequence by the modus ponens rule.

An SCI-*model* is a structure $\mathcal{M} = (M, \sim, \sqcup, \sqcap, \Rightarrow, \Leftrightarrow, \circ, D)$, where M is a nonempty set, D is any nonempty subset of M, $\sim$ is a unary operation on M, and $\sqcup, \sqcap, \Rightarrow, \Leftrightarrow, \circ$ are binary operations on M such that for all $a, b \in M$ the following hold:

(SCI1) $\sim a \in D$ iff $a \notin D$,
(SCI2) $a \sqcup b \in D$ iff $a \in D$ or $b \in D$,
(SCI3) $a \sqcap b \in D$ iff $a \in D$ and $b \in D$,
(SCI4) $a \Rightarrow b \in D$ iff $a \notin D$ or $b \in D$,
(SCI5) $a \Leftrightarrow b \in D$ iff $a \Rightarrow b \in D$ and $b \Rightarrow a \in D$,
(SCI6) $a \circ b \in D$ iff $a = b$.

Let $\mathcal{M}$ be an SCI-model. A *valuation* on $\mathcal{M}$ is any mapping $v \colon \mathbb{V} \rightarrow M$. We will abuse the notation and write v for the extension of v to all SCI-formulas defined inductively as follows:

$$
\begin{aligned}
v(\neg\varphi) &= {\sim}v(\varphi),\\
v(\varphi \vee \psi) &= v(\varphi) \sqcup v(\psi),\\
v(\varphi \wedge \psi) &= v(\varphi) \sqcap v(\psi),\\
v(\varphi \to \psi) &= v(\varphi) \Rightarrow v(\psi),\\
v(\varphi \leftrightarrow \psi) &= v(\varphi) \Leftrightarrow v(\psi),\\
v(\varphi \equiv \psi) &= v(\varphi) \circ v(\psi).
\end{aligned}
$$

Let v be a valuation on an SCI-model $\mathcal{M}$. An SCI-formula φ is *satisfied* by v in $\mathcal{M}$, $\mathcal{M}, v \models \varphi$, whenever $v(\varphi) \in D$. An SCI-formula φ is *true* in $\mathcal{M}$ if it is satisfied by all valuations in $\mathcal{M}$. A formula is SCI-*valid* if it is true in all SCI-models. An SCI-formula φ is said to be *satisfiable in an* SCI-*model* $\mathcal{M}$ whenever there exists a valuation v on $\mathcal{M}$ such that $\mathcal{M}, v \models \varphi$. A model is referred to as *finite* if its universe is finite.

The intuitive interpretation of an SCI-model $\mathcal{M}$ is as follows: the elements of M are *situations* (denotations of sentences), the function symbols correspond to the formation of new formulas with connectives, while D can be thought of as the set of *facts*—that is, it consists of those situations that correspond to true sentences.

Soundness and completeness of SCI with respect to the class of SCI-models was proved in Bloom and Suszko [2, Theorem 1.9].

Theorem 2.1 (Soundness and completeness of SCI) *For every* SCI-*formula φ, the following conditions are equivalent:*

1. *φ is* SCI-*provable;*
2. *φ is* SCI-*valid.*

The logic SCI is *two-valued*, in the sense that for any given SCI-formula φ, SCI-model $\mathcal{M}$, and valuation v in $\mathcal{M}$, it holds that $\mathcal{M}, v \models \varphi$ if and only if $\mathcal{M}, v \not\models \neg\varphi$.

The logic SCI is also *extensional* in the sense that any subformula ψ of an SCI-formula φ can be replaced with another formula ϑ denoting the same as ψ without affecting the denotation of φ. More precisely, the extensionality of SCI can be expressed as follows.

Fact 2.2 Let $\mathcal{M}$ be an SCI-model, let v be a valuation in $\mathcal{M}$, let φ be an SCI-formula containing a subformula ψ, and let φ' be the result of replacing some occurrences of ψ in φ by a formula ϑ. Then, $\mathcal{M}, v \models \psi \equiv \vartheta$ implies $\mathcal{M}, v \models \varphi \equiv \varphi'$.

Note that two-valuedness and extensionality concern different levels. Two-valuedness is a property of the truth values, while extensionality holds for the denotations.

Theorem 2.3 (Finite-model property and decidability of SCI) *The logic* SCI *has the finite model property; that is, every satisfiable* SCI-*formula is satisfiable in a finite* SCI-*model. Furthermore, the logic* SCI *is decidable.*

The proof of the above theorem can be found in [2].

Corollary 2.4 *Let T be a set of* SCI-*formulas such that T is true in all finite* SCI-*models. Then, T is true in all infinite* SCI-*models as well.*

Proof Assume that the claim does not hold. Then, there are an SCI-formula $\varphi \in T$ and an SCI-model $\mathcal{M}$ such that φ is not true in $\mathcal{M}$. So, $\neg\varphi$ is satisfiable in $\mathcal{M}$, and hence $\neg\varphi$ is satisfiable also in some finite SCI-model $\mathcal{M}'$, which means that φ is not true in $\mathcal{M}'$, contradicting the assumption. □

The logic SCI is extremely weak. It does not impose any specific assumptions on the universe of situations except that it must have at least two elements. Any additional assumption on the universe of situations leads to an extension of SCI. For example, if we add to the set of SCI-axioms the *Fregean axiom*, which identifies the denotations of sentences with their truth values, that is,

$$\text{(FA)} \quad (\varphi \leftrightarrow \psi) \to (\varphi \equiv \psi),$$

then we get the classical propositional logic PC. The logic PC is the strongest among all propositional extensions of SCI. However, it is known that there are many other propositional logics which can be formulated as non-Fregean theories that are stronger than SCI and weaker than PC, such as certain modal propositional logics and finite-valued Łukasiewicz logics.[1]

In fact, we know something more: we proved in [6] that the number of different propositional non-Fregean logics between SCI and PC is uncountable (cf. Golińska-Pilarek [5]). However, despite the richness of the class of non-Fregean propositional logics, many of which express interesting properties of the universe of situations, the expressive power of non-Fregean propositional logics is rather limited. In particular, Corollary 2.4 implies that no nontrivial property shared by all finite SCI-models can be expressed in SCI. To widen the scope of applications of the non-Fregean concept, for instance, to formalize an idea on the nature of denotations of statements or other applications to formal philosophy, it seems natural to study extensions of SCI that avoid these limitations. We will show that the logic $\mathsf{SCI_Q}$, presented in the following section, indeed rises to the challenge.

3 The Logic $\mathsf{SCI_Q}$

In this section, we present the logic $\mathsf{SCI_Q}$, which is an extension of SCI with quantifiers that range over propositional variables. The language of $\mathsf{SCI_Q}$ is the SCI-language endowed with quantifiers. More precisely, the $\mathsf{SCI_Q}$-language consists of a countably infinite set of propositional variables $\mathbb{V} = \{x, y, z \ldots\}$, the propositional operations of negation $\neg$, disjunction $\vee$, conjunction $\wedge$, implication $\to$, equivalence $\leftrightarrow$, identity $\equiv$, and the quantifiers—universal $\forall$ and existential $\exists$. The set of $\mathsf{SCI_Q}$-formulas is the smallest set satisfying the following conditions.

- Each propositional variable in $\mathbb{V}$ is an $\mathsf{SCI_Q}$-formula.
- If φ and ψ are $\mathsf{SCI_Q}$-formulas, then so are $\neg\varphi$ and $\varphi\#\psi$, for every $\# \in \{\wedge, \vee, \to, \leftrightarrow, \equiv\}$.
- If φ is an $\mathsf{SCI_Q}$-formula and $x \in \mathbb{V}$, then $\forall x\varphi$ and $\exists x\varphi$ are $\mathsf{SCI_Q}$-formulas.

The notions of free and bound variables are defined in a standard way as in first-order logic. By $\varphi(\bar{x})$ we denote a formula whose free variables are among $\bar{x} = x_1, \ldots, x_n$. An $\mathsf{SCI_Q}$-*sentence* is an $\mathsf{SCI_Q}$-formula with no free variables.

A Hilbert-style axiomatization of $\mathsf{SCI_Q}$ consists of the axiom schemas of SCI adjusted to the $\mathsf{SCI_Q}$-language and the following axiom schemas characterizing the interactions between quantifiers and connectives:

($\mathsf{SCI_Q}$1) $\forall x\varphi \to \varphi(x/\psi)$,

($\mathsf{SCI_Q}$2) $\forall x(\varphi \to \psi) \to (\forall x\varphi \to \forall x\psi)$,
($\mathsf{SCI_Q}$3) $\varphi \to \forall x\varphi$, provided that x is not free in φ,
($\mathsf{SCI_Q}$4) $\exists x\varphi \leftrightarrow \neg\forall x\neg\varphi$,
($\mathsf{SCI_Q}$5) $\forall x(\varphi \equiv \psi) \to (Qx\varphi \equiv Qx\psi)$, for $Q \in \{\forall, \exists\}$,
($\mathsf{SCI_Q}$6) $Qx\varphi \equiv Qy\varphi(x/y)$, where $Q \in \{\forall, \exists\}$ and y does not occur free in φ.

The inference rules of $\mathsf{SCI_Q}$ are modus ponens and generalization. The notion of $\mathsf{SCI_Q}$-provability is defined in a standard way.

Note that the quantifiers obey all the classical laws not involving the identity connective. For instance, $\forall p\varphi(p) \leftrightarrow \neg\exists p\neg\varphi(p)$ is provable in $\mathsf{SCI_Q}$. However, if we replace $\leftrightarrow$ with $\equiv$, the resulting laws do not hold: $\forall p\varphi(p) \equiv \neg\exists p\neg\varphi(p)$ is not provable. For this reason, $\mathsf{SCI_Q}$ can express nontrivial properties of the universe of situations, including complicated structural properties of the whole model and the interaction between the identity operation and the classical ones.

A structure $\mathcal{M} = (M, \sim, \sqcap, \sqcup, \Rightarrow, \Leftrightarrow, \circ, \mathcal{F}, \bigotimes, \bigoplus, D)$ is said to be a *partial* $\mathsf{SCI_Q}$*-model* whenever the following conditions are satisfied:

- $(M, \sim, \sqcap, \sqcup, \Rightarrow, \Leftrightarrow, \circ, D)$ is an SCI-model,
- $\mathcal{F} \subseteq M^M$,
- $\bigotimes, \bigoplus \colon \mathcal{F} \to M$ are operations such that for every $f \in \mathcal{F}$ the following hold:
  - $\bigotimes f \in D$ if and only if for all $t \in M$, $f(t) \in D$,
  - $\bigoplus f \in D$ if and only if for some $t \in M$, $f(t) \in D$.

A *partial valuation* on a partial $\mathsf{SCI_Q}$-model $\mathcal{M}$ is any function $v\colon W \to M$ such that $W \subseteq \mathbb{V}$. A *(total) valuation* on $\mathcal{M}$ is a function $v\colon \mathbb{V} \to M$.

Let $\mathcal{M} = (M, \sim, \sqcap, \sqcup, \Rightarrow, \Leftrightarrow, \circ, \mathcal{F}, \bigotimes, \bigoplus, D)$ be a partial $\mathsf{SCI_Q}$-model, and let v be a partial valuation on $\mathcal{M}$. We define inductively the *value* of an $\mathsf{SCI_Q}$-formula φ, $\|\varphi, v\|$ for short, as follows.

- If $\varphi := x$ for $x \in \mathbb{V}$, then $\|\varphi, v\| \stackrel{\text{df}}{=} v(x)$.
- If $\varphi := \neg\psi$, then $\|\varphi, v\| \stackrel{\text{df}}{=} \sim\|\psi, v\|$.
- If $\varphi := \psi \wedge \theta$, then $\|\varphi, v\| \stackrel{\text{df}}{=} \|\psi, v\| \sqcap \|\theta, v\|$.
- If $\varphi := \psi \vee \theta$, then $\|\varphi, v\| \stackrel{\text{df}}{=} \|\psi, v\| \sqcup \|\theta, v\|$.
- If $\varphi := \psi \to \theta$, then $\|\varphi, v\| \stackrel{\text{df}}{=} \|\psi, v\| \Rightarrow \|\theta, v\|$.
- If $\varphi := \psi \leftrightarrow \theta$, then $\|\varphi, v\| \stackrel{\text{df}}{=} \|\psi, v\| \Leftrightarrow \|\theta, v\|$.
- If $\varphi := \psi \equiv \theta$, then $\|\varphi, v\| \stackrel{\text{df}}{=} \|\psi, v\| \circ \|\theta, v\|$.

If φ is a formula of the form $\forall x\psi$ or $\exists x\psi$, we define first the function $f\colon M \to M$ as $f(t) \stackrel{\text{df}}{=} \|\psi, v_x^t\|$, where

$$v_x^t(y) \stackrel{\text{df}}{=} \begin{cases} t, & \text{if } y = x, \\ v(y), & \text{otherwise.} \end{cases}$$

Then, if $f \in \mathcal{F}$, we set

$$\|\forall x\psi, v\| \stackrel{\text{df}}{=} \bigotimes f \qquad \text{and} \qquad \|\exists x\psi, v\| \stackrel{\text{df}}{=} \bigoplus f.$$

Note that in any of the above clauses the right-hand side may be undefined. In that case, we simply leave the left-hand side undefined as well.

A partial $\mathsf{SCI_Q}$-model $\mathcal{M}$ is called *sufficient* for an $\mathsf{SCI_Q}$-formula φ if and only if for every valuation v in $\mathcal{M}$, the value $\|\varphi, v\|$ is well defined. A partial $\mathsf{SCI_Q}$-model is referred to as an $\mathsf{SCI_Q}$-*model* whenever it is sufficient for every $\mathsf{SCI_Q}$-formula. A partial valuation v is sufficient for φ in a partial $\mathsf{SCI_Q}$-model $\mathcal{M}$ whenever $\|\varphi, v\|$ is well defined. An $\mathsf{SCI_Q}$-formula φ is said to be *satisfied* in an $\mathsf{SCI_Q}$-model $\mathcal{M}$ by a valuation v ($\mathcal{M}, v \models \varphi$ for short) if and only if $\|\varphi, v\| \in D$. A formula is *true* in $\mathcal{M}$ if it is satisfied by all the valuations in $\mathcal{M}$ and it is $\mathsf{SCI_Q}$-*valid* whenever it is true in all $\mathsf{SCI_Q}$-models. A set X of $\mathsf{SCI_Q}$-formulas is said to be *satisfied* in $\mathcal{M}$ by v ($\mathcal{M}, v \models X$) if and only if every formula of X is satisfied in $\mathcal{M}$ by v. A formula φ is *satisfiable* whenever it is satisfied in some $\mathsf{SCI_Q}$-model by some valuation. The *size* of a partial $\mathsf{SCI_Q}$-model $\mathcal{M}$, denoted by $|\mathcal{M}|$, is the cardinality of M. We denote the cardinality of $\mathcal{F}$ by $|\mathcal{M}|_2$. $\mathsf{SCI_Q}$-models with an infinite universe will be referred to as *infinite models*.

Example 3.1 Consider a structure of the form

$$\mathcal{M} = \Big(M, \sim, \sqcap, \sqcup, \Rightarrow, \Leftrightarrow, \circ, \mathcal{F}, \bigotimes, \bigoplus, D\Big)$$

such that $M = \{0, 1, 2\}$, $D = \{1, 2\}$, $\mathcal{F} = M^M$, and for all $a, b \in M$, $f \in \mathcal{F}$ the following conditions are satisfied:

$$\sim 0 \overset{\text{df}}{=} 2, \qquad \sim 1 \overset{\text{df}}{=} \sim 2 = 0, \qquad a \sqcap b \overset{\text{df}}{=} \min\{a, b\}, \qquad a \sqcup b \overset{\text{df}}{=} \max\{a, b\},$$

$$a \Rightarrow b \overset{\text{df}}{=} \sim a \sqcup b, \qquad a \Leftrightarrow b \overset{\text{df}}{=} (a \Rightarrow b) \sqcap (b \Rightarrow a),$$

$$a \circ b \overset{\text{df}}{=} \begin{cases} a, & \text{if } a = b \in D, \\ 1, & \text{if } a = b = 0, \\ 0, & \text{otherwise,} \end{cases}$$

$$\bigotimes f \overset{\text{df}}{=} \min\big\{f(a) \bigm| a \in M\big\}, \qquad \bigoplus f \overset{\text{df}}{=} \max\big\{f(a) \bigm| a \in M\big\}.$$

Clearly, the structure $\mathcal{M}$ is a partial $\mathsf{SCI_Q}$-model. Indeed, by an easy calculation it can be shown that $(M, \sim, \sqcap, \sqcup, \Rightarrow, \Leftrightarrow, \circ, D)$ is an SCI-model. Let $f \in \mathcal{F}$. It is easy to show that $\min\{f(a) \mid a \in M\} \in D$ if and only if $\{f(a) \mid a \in M\} \subseteq D$. Clearly, if $\{f(a) \mid a \in M\} \subseteq D$, then in particular $\min\{f(a) \mid a \in M\} \in D$. On the other hand, if $\{f(a) \mid a \in M\} \nsubseteq D$, then $0 \in \{f(a) \mid a \in M\}$, which means that $\min\{f(a) \mid a \in M\} = 0 \notin D$. This proves that for every $f \in \mathcal{F}$, $\bigotimes f \in D$ if and only if for every $t \in M$, $f(t) \in D$. In a similar way, we may show that $\bigoplus f \in D$ if and only if for some $t \in M$, $f(t) \in D$. Hence, $\mathcal{M}$ is a partial $\mathsf{SCI_Q}$-model. Since $\mathcal{F} = M^M$, $\mathcal{M}$ is an $\mathsf{SCI_Q}$-model. Now, observe that $\mathcal{M}$ is an example of the model in which the formula $\forall x(x \vee \neg x) \equiv \neg\exists x \neg(x \vee \neg x)$ is not true. Indeed, a formula $\forall x(x \vee \neg x) \equiv \neg\exists x \neg(x \vee \neg x)$ is satisfied in $\mathcal{M}$ by a valuation v if and only if $\|\forall x(x \vee \neg x), v\| = \|\neg\exists x \neg(x \vee \neg x), v\|$, which is not the case, since for any valuation v in $\mathcal{M}$ the following hold:

$$\begin{aligned} \big\|\forall x(x \vee \neg x), v\big\| &= \min\{t \sqcup \sim t \mid t \in M\} \\ &= \min\big\{\max\{t, \sim t\} \bigm| t \in M\big\} = \min\{1, 2\} = 1, \\ \big\|\neg\exists x \neg(x \vee \neg x), v\big\| &= \sim \max\big\{\sim(t \sqcup \sim t) \bigm| t \in M\big\} \\ &= \sim \max\big\{\sim \max\{t, \sim t\} \bigm| t \in M\big\} = \sim \max\{0\} = 2. \end{aligned}$$

Thus, we have $1 = \|\forall x(x \vee \neg x), v\| \neq \|\neg\exists x \neg(x \vee \neg x), v\| = 2$.

As in first-order logic, the following fact can be easily proved.

Fact 3.2 For every $\mathsf{SCI_Q}$-sentence φ, for every $\mathsf{SCI_Q}$-model $\mathcal{M}$, and for every valuation v in $\mathcal{M}$, the following conditions are equivalent:

1. $\mathcal{M}, v \models \varphi$,
2. $\mathcal{M} \models \varphi$.

The next theorem follows from the soundness and completeness theorem for the full first-order non-Fregean logic proved in [1] (cf. Suszko [10]).

Theorem 3.3 (Soundness and completeness of $\mathsf{SCI_Q}$) *For every $\mathsf{SCI_Q}$-formula φ, the following conditions are equivalent:*

1. *φ is $\mathsf{SCI_Q}$-provable,*
2. *φ is $\mathsf{SCI_Q}$-valid.*

4 Toward Undecidability of $\mathsf{SCI_Q}$

The logic $\mathsf{SCI_Q}$ is much more expressive than SCI. Furthermore, in contrast to SCI, the logic $\mathsf{SCI_Q}$ is undecidable, which will be shown in this section. We start with a simple example of a property of the logic $\mathsf{SCI_Q}$.

Proposition 4.1 *The logic $\mathsf{SCI_Q}$ does not have the finite model property; that is, there exists an $\mathsf{SCI_Q}$-formula which is satisfiable only in infinite $\mathsf{SCI_Q}$-models.*

Proof Let φ be the following $\mathsf{SCI_Q}$-formula:

$$\exists x\big[x \wedge \forall y \neg\big(x \equiv (y \equiv y)\big) \wedge \forall y \forall z\big((y \equiv y) \equiv (z \equiv z) \to y \equiv z\big)\big].$$

We will show that the formula φ is satisfiable only in infinite $\mathsf{SCI_Q}$-models.

Let $\mathcal{M} = (M, \sim, \sqcap, \sqcup, \Rightarrow, \Leftrightarrow, \circ, \mathcal{F}, \bigotimes, \bigoplus, D)$ be an $\mathsf{SCI_Q}$-model, and let v be a valuation in $\mathcal{M}$ such that $\mathcal{M}, v \models \varphi$. We will show that M is infinite. Let us define

$$\varphi' \overset{\mathrm{df}}{=} x \wedge \forall y \neg\big(x \equiv (y \equiv y)\big) \wedge \forall y \forall z\big((y \equiv y) \equiv (z \equiv z) \to y \equiv z\big).$$

By the definition of the satisfaction relation, φ is satisfied in $\mathcal{M}$ by v if and only if $\|\varphi, v\| \in D$, that is, $\bigoplus \|\varphi', v_x^t\| \in D$. Thus, there must exist $t \in M$ such that $\|\varphi', v_x^t\| \in D$, which means that the following conditions hold:

1. $t \in D$,
2. $\|\forall y \neg(x \equiv (y \equiv y)), v_x^t\| \in D$,
3. $\|\forall y \forall z((y \equiv y) \equiv (z \equiv z) \to y \equiv z), v_x^t\| \in D$.

Condition 2 implies $t \neq (a \circ a)$, for every $a \in M$. By condition 3, we have also that for all $a, b \in M$, $(a \circ a) = (b \circ b)$ implies $a = b$. Let us define the function $f\colon M \to M$ as $f(a) = (a \circ a)$. By condition 3, the function f is an injection, and, by condition 2, it is not a surjection. Hence, M must be infinite. □

The logic $\mathsf{SCI_Q}$ is essentially more expressive than the logic SCI. We may express within $\mathsf{SCI_Q}$ many first-order theories, in particular the theory of groups, rings, fields, and the theory of Peano arithmetic.

The language of the group theory TG is a first-order language with one binary function symbol $\cdot$. For simplicity of the presentation, we identify the set of individual variables of TG with the set of propositional variables of $\mathsf{SCI_Q}$. Furthermore, we assume that the primitive logical symbols of TG are $\neg, \to$, and $\forall$. The other connectives and the existential quantifier are regarded as shorthand notations defined in the usual way. The axioms of TG are

($\mathsf{TG1}$) $\forall x \forall y \forall z(x \cdot (y \cdot z) = (x \cdot y) \cdot z)$, (TG2) $\exists x \forall y \exists z((x \cdot y) = (y \cdot x) = y \wedge x = (y \cdot z) = (z \cdot y))$.

Models of TG are first-order structures of the form $\mathcal{M} = (M, \cdot)$ such that the operation $\cdot$ satisfies axioms (TG1) and (TG2). A valuation in a TG-model is defined as usual in first-order logic. By $\mathcal{M}, v \models \varphi$ we will denote the fact that a TG-formula φ is satisfied in a model $\mathcal{M}$ by a valuation v, where the satisfaction relation $\models$ is defined in a standard Tarskian way. The axiom (TG1) expresses the associativity of the group operation. The axiom (TG2) implies that in every TG-model there exists a neutral element of the operation $\cdot$ and each element of a model has its inverse element. It is known that in each TG-model there exists exactly one neutral element and for each element there exists exactly one inverse element. The neutral element will be denoted by e.

Let $\mathsf{SCI}_{\mathsf{Q}}^{\mathsf{TG}}$ be the class of all $\mathsf{SCI}_{\mathsf{Q}}$-models of the formula $\gamma \stackrel{\text{df}}{=} \gamma_1 \wedge \gamma_2$, where

$$\gamma_1 := \forall x \forall y \forall z\big[(x \wedge y \wedge z) \rightarrow \big(x \wedge (y \wedge z)\big) \equiv \big((x \wedge y) \wedge z\big)\big],$$

$$\gamma_2 := \exists x\big[x \wedge \forall y\big(y \rightarrow \gamma_3(x,y)\big) \wedge \forall y\big(y \rightarrow \exists z\big(z \wedge \gamma_4(x,y,z)\big)\big)\big], \quad \text{for}$$

$$\gamma_3(x,y) := \big(y \equiv (x \wedge y)\big) \wedge \big(y \equiv (y \wedge x)\big),$$

$$\gamma_4(x,y,z) := \big(x \equiv (y \wedge z)\big) \wedge \big(x \equiv (z \wedge y)\big).$$

It is easy to see that the class $\mathsf{SCI}_{\mathsf{Q}}^{\mathsf{TG}}$ is nonempty. Indeed, one $\mathsf{SCI}_{\mathsf{Q}}^{\mathsf{TG}}$-model is a structure $\mathcal{M} = (M, \sim, \sqcap, \sqcup, \Rightarrow, \Leftrightarrow, \circ, \mathcal{F}, \bigotimes, \bigoplus, D)$ such that $M = \{0,1\}$, $D = \{1\}$, $\mathcal{F} = M^M$, $(M, \sim, \sqcap, \sqcup)$ is a Boolean algebra, and for all $a, b \in M$, for every $f \in \mathcal{F}$:

$$a \Rightarrow b \stackrel{\text{df}}{=} {\sim} a \sqcup b, \qquad a \Leftrightarrow b \stackrel{\text{df}}{=} (a \Rightarrow b) \sqcap (b \Rightarrow a),$$

$$a \circ b \stackrel{\text{df}}{=} \begin{cases} 1, & \text{if } a = b, \\ 0, & \text{otherwise,} \end{cases}$$

$$\bigotimes f \stackrel{\text{df}}{=} \min\big\{f(a) \bigm| a \in M\big\}, \qquad \bigoplus f \stackrel{\text{df}}{=} \max\big\{f(a) \bigm| a \in M\big\}.$$

Intuitively, $\mathsf{SCI}_{\mathsf{Q}}^{\mathsf{TG}}$-models are $\mathsf{SCI}_{\mathsf{Q}}$-models such that $(D, \sqcap)$ is a TG-model. This is stated more formally in the following proposition.

Proposition 4.2 *A structure of the form* $\mathcal{M} = (M, \sim, \sqcap, \sqcup, \Rightarrow, \Leftrightarrow, \circ, \mathcal{F}, \bigotimes, \bigoplus, D)$ *is an* $\mathsf{SCI}_{\mathsf{Q}}^{\mathsf{TG}}$*-model if and only if the following conditions hold.*

1. $\mathcal{M}$ *is an* $\mathsf{SCI}_{\mathsf{Q}}$*-model.*
2. *For all* $a, b, c \in D$, $a \sqcap (b \sqcap c) = (a \sqcap b) \sqcap c$.
3. *There exists* $e \in D$ *such that for all* $a \in D$:
   (a) $a = e \sqcap a = a \sqcap e$,
   (b) *there exists* $b \in D$ *such that* $(a \sqcap b) = (b \sqcap a) = e$.

Proof Let $\mathcal{M} = (M, \sim, \sqcap, \sqcup, \Rightarrow, \Leftrightarrow, \circ, \mathcal{F}, \bigotimes, \bigoplus, D)$ be an $\mathsf{SCI}_{\mathsf{Q}}^{\mathsf{TG}}$-model. By the definition it is an $\mathsf{SCI}_{\mathsf{Q}}$-model in which the formula $\gamma = (\gamma_1 \wedge \gamma_2)$ is true. Thus, the formulas γ_1 and γ_2 are true in $\mathcal{M}$, which implies conditions 2 and 3, respectively. The other direction can be proved in a similar way. □

For simplicity of the presentation, in what follows we will use the following notation: $\bar{x}$ denotes a finite sequence of variables, $t(\bar{x})$ denotes a TG-term built with the operation $\cdot$ and variables from a sequence $\bar{x}$, and $\bigwedge \bar{x}$ denotes the conjunction of variables from $\bar{x}$.

Now, we will define the translation function τ from terms and formulas in the language of TG into $\mathsf{SCI_Q}$-formulas.

- $\tau(x) \overset{\text{df}}{=} x$, for any variable x,
- $\tau(t(\bar{x}) \cdot t'(\bar{y})) \overset{\text{df}}{=} \tau(t(\bar{x})) \wedge \tau(t'(\bar{y}))$, for any terms t, t' and finite sequences of variables $\bar{x}$ and $\bar{y}$,
- $\tau(t(\bar{x}) = t'(\bar{y})) \overset{\text{df}}{=} (\bigwedge \bar{x} \wedge \bigwedge \bar{y}) \rightarrow (\tau(t(\bar{x})) \equiv \tau(t'(\bar{y})))$, for any terms t, t' and finite sequences $\bar{x}$ and $\bar{y}$,
- $\tau(\neg(t(\bar{x}) = t'(\bar{y}))) \overset{\text{df}}{=} (\bigwedge \bar{x} \wedge \bigwedge \bar{y}) \rightarrow \neg(\tau(t(\bar{x})) \equiv \tau(t'(\bar{y})))$, for any terms t, t' and finite sequences $\bar{x}$ and $\bar{y}$,
- $\tau(\neg\neg\varphi) \overset{\text{df}}{=} \tau(\varphi)$,
- $\tau(\varphi \rightarrow \psi) \overset{\text{df}}{=} \tau(\neg\varphi) \vee \tau(\psi)$,
- $\tau(\neg(\varphi \rightarrow \psi)) \overset{\text{df}}{=} \tau(\varphi) \wedge \tau(\neg\psi)$,
- $\tau(\forall x\varphi) \overset{\text{df}}{=} \forall x(x \rightarrow \tau(\varphi))$,
- $\tau(\neg\forall x\varphi) \overset{\text{df}}{=} \exists x(x \wedge \tau(\neg\varphi))$.

Lemma 4.3 *For every* TG*-model* $\mathcal{M}$ *and for every valuation* v *in* $\mathcal{M}$ *there exists an* $\mathsf{SCI_Q^{TG}}$*-model* $\mathcal{M}'$ *and a valuation* v' *in* $\mathcal{M}'$ *such that for every formula* φ *in the language of* TG *the following holds:*

$$(\alpha) \quad \mathcal{M}, v \not\models \varphi \textit{ implies } \mathcal{M}', v' \not\models \tau(\varphi).$$

Proof Let $\mathcal{M} = (M, \cdot)$ be a TG-model, let v be a valuation in $\mathcal{M}$, and let e be the neutral element of $\mathcal{M}$. Furthermore, let $\mathcal{M}' = (M', \sim, \sqcap, \sqcup, \Rightarrow, \Leftrightarrow, \circ, \mathcal{F}, \bigotimes, \bigoplus, D)$ be a structure such that $M' = M \cup \{0\}$, where $0 \notin M$, $D = M$, $\mathcal{F} = M'^{M'}$, and for all $a, b \in M'$, for every $f \in \mathcal{F}$ the operations are defined as

$$\sim a \overset{\text{df}}{=} \begin{cases} 0, & \text{if } a \in M, \\ e, & \text{otherwise,} \end{cases} \qquad a \circ b \overset{\text{df}}{=} \begin{cases} e, & \text{if } a = b, \\ 0, & \text{otherwise,} \end{cases}$$

$$a \sqcap b \overset{\text{df}}{=} \begin{cases} a \cdot b, & \text{if } a, b \in M, \\ 0, & \text{otherwise,} \end{cases} \qquad a \sqcup b \overset{\text{df}}{=} \sim(\sim a \sqcap \sim b),$$

$$a \Rightarrow b \overset{\text{df}}{=} \sim a \sqcup b, \qquad a \Leftrightarrow b \overset{\text{df}}{=} (a \Rightarrow b) \sqcap (b \Rightarrow a),$$

$$\bigotimes f \overset{\text{df}}{=} \begin{cases} e, & \text{if } \{f(t) \mid t \in M'\} \subseteq M, \\ 0, & \text{otherwise,} \end{cases}$$

$$\bigoplus f \overset{\text{df}}{=} \begin{cases} e, & \text{if } \{f(t) \mid t \in M'\} \cap M \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to check that $\mathcal{M}'$ is an $\mathsf{SCI_Q}$-model. Moreover, it is also an $\mathsf{SCI_Q^{TG}}$-model. Indeed, $D = M$ and the operation $\sqcap$ is defined on D in the same way as the operation $\cdot$ on M. Thus, conditions 2 and 3 of Proposition 4.2 are satisfied; hence $\mathcal{M}'$ is an $\mathsf{SCI_Q^{TG}}$-model.

Let v' be a valuation in $\mathcal{M}'$ such that $v'(x) = v(x)$, for each variable x. Now, we will prove the claim by induction on the complexity of formulas. First, observe that for every TG-term t, $v(t) = v'(\tau(t))$. Clearly, it is true for variables, as $\tau(x) = x$. Next, let t and t' be TG-terms. Then, we have $v'(\tau(t \cdot t')) = v'(\tau(t) \wedge \tau(t')) = v'(\tau(t)) \sqcap v'(\tau(t'))$. Since $v'(\tau(t))$ and $v'(\tau(t'))$ are elements of D, we obtain

$v'(\tau(t)) \sqcap v'(\tau(t')) = v'(\tau(t)) \cdot v'(\tau(t'))$, which by the induction hypothesis implies $v'(\tau(t)) \cdot v'(\tau(t')) = v(t) \cdot v(t') = v(t \cdot t')$. Therefore, for any TG-terms $t(\bar{x})$ and $t'(\bar{y})$, we have

$$(*) \quad v\big(t(\bar{x})\big) = v\big(t'(\bar{y})\big) \quad \text{iff} \quad v'\big(\tau\big(t(\bar{x})\big)\big) = v'\big(\tau\big(t(\bar{y})\big)\big).$$

Let $\varphi := t(\bar{x}) = t'(\bar{y})$. Assume $\mathcal{M}, v \not\models t(\bar{x}) = t'(\bar{y})$; that is, $v(t(\bar{x})) \neq v(t'(\bar{y}))$. Thus, by $(*)$, $v'(\tau(t(\bar{x}))) \neq v'(\tau(t(\bar{y})))$, which means that $v'(\tau(t(\bar{x})) \equiv \tau(t(\bar{y}))) \notin D$. On the other hand, for any variable z in $\bar{x}$ or $\bar{y}$, $v'(z) \in D$. Therefore, $v'(\bigwedge \bar{x} \wedge \bigwedge \bar{y}) \in D$. Hence, $\tau(\varphi)$, that is, the formula $(\bigwedge \bar{x} \wedge \bigwedge \bar{y}) \to (\tau(t(\bar{x})) \equiv \tau(t'(\bar{y})))$ is not satisfied in $\mathcal{M}'$ by v'.

Let $\varphi := \neg(t(\bar{x}) = t'(\bar{y}))$. Assume $\mathcal{M}, v \not\models \neg(t(\bar{x}) = t'(\bar{y}))$; that is, $v(t(\bar{x})) = v(t'(\bar{y}))$. Thus, by $(*)$, $v'(\tau(t(\bar{x}))) = v'(\tau(t(\bar{y})))$, which means that $v'(\tau(t(\bar{x})) \equiv \tau(t(\bar{y}))) \in D$, so $v'(\neg(\tau(t(\bar{x})) \equiv \tau(t(\bar{y})))) \notin D$. Clearly, we have again $v'(\bigwedge \bar{x} \wedge \bigwedge \bar{y}) \in D$. Hence, the formula $(\bigwedge \bar{x} \wedge \bigwedge \bar{y}) \to \neg(\tau(t(\bar{x})) \equiv \tau(t'(\bar{y})))$ is not satisfied in $\mathcal{M}'$ by v'.

Now, assuming that (α) holds for formulas ψ, ϑ, and their negations, we will show that it holds for formulas of the form $\neg\neg\psi$, $\psi \to \vartheta$, $\neg(\psi \to \vartheta)$, $\forall x \psi$, $\neg\forall x \psi$.

If $\mathcal{M}, v \not\models \neg\neg\psi$, then $\mathcal{M}, v \not\models \psi$. Thus, by the induction hypothesis, $\mathcal{M}', v' \not\models \tau(\psi)$; hence, $\mathcal{M}', v' \not\models \tau(\neg\neg\psi)$.

Assume $\mathcal{M}, v \not\models (\psi \to \vartheta)$, which implies $\mathcal{M}, v \not\models \neg\psi$ and $\mathcal{M}, v \not\models \vartheta$. Thus, by the induction hypothesis, $\mathcal{M}', v' \not\models \tau(\neg\psi)$ and $\mathcal{M}', v' \not\models \tau(\vartheta)$. Therefore, $v'(\tau(\neg\psi)) \notin D$ and $v'(\tau(\vartheta)) \notin D$, which means that $v'(\tau(\neg\psi) \vee \tau(\vartheta)) \notin D$. Hence, $\mathcal{M}', v' \not\models \tau(\psi \to \vartheta)$.

Assume $\mathcal{M}, v \not\models \neg(\psi \to \vartheta)$. Therefore, either $\mathcal{M}, v \not\models \psi$ or $\mathcal{M}, v \not\models \neg\vartheta$. Thus, by the induction hypothesis, either $\mathcal{M}', v' \not\models \tau(\psi)$ or $\mathcal{M}', v' \not\models \tau(\neg\vartheta)$. Therefore, either $v'(\tau(\psi)) \notin D$ or $v'(\tau(\neg\vartheta)) \notin D$, which means that $v'(\tau(\psi) \wedge \tau(\neg\vartheta)) \notin D$. Hence, $\mathcal{M}', v' \not\models \tau(\neg(\psi \to \vartheta))$.

Assume $\mathcal{M}, v \not\models \forall x \psi$. Then, there exists $a \in M$ such that $\mathcal{M}, v_a \not\models \psi$, where $v_a(y) = v(y)$, for all variables $y \neq x$, and $v_a(y) = a$, otherwise. Clearly, $v'^a_x(y) = v_a(y)$, for every variable y. Thus, by the induction hypothesis, $\|\tau(\psi), v'^a_x\| \notin D$. Since $a \in D$, we have $\mathcal{M}', v' \not\models \forall x(x \to \tau(\psi))$; that is, $\mathcal{M}', v' \not\models \tau(\forall x \psi)$.

Assume $\mathcal{M}, v \not\models \neg\forall x \psi$. Then, for every $a \in M$, $\mathcal{M}, v_a \not\models \neg\psi$, where $v_a(y) = v(y)$, for all variables $y \neq x$, and $v_a(y) = a$, otherwise. Clearly, $v'^a_x(y) = v_a(y)$, for every variable y. Thus, by the induction hypothesis, for every $a \in D$, $\|\tau(\neg\psi), v'^a_x\| \notin D$, so for every $a \in D$, $\|x \wedge \tau(\neg\psi), v'^a_x\| \notin D$. Recall that $M' \setminus D = \{0\}$. Since $0 \notin D$, we get $\|x \wedge \tau(\neg\psi), v'^a_x\| \notin D$, for $a = 0$. Thus, for every $a \in M'$, $\|x \wedge \tau(\neg\psi), v'^a_x\| \notin D$. Hence, $\mathcal{M}', v' \not\models \exists x(x \wedge \tau(\neg\psi))$; that is, $\mathcal{M}', v' \not\models \tau(\neg\forall x \psi)$. □

Lemma 4.4 *For every* $\mathsf{SCI}_\mathsf{Q}^\mathsf{TG}$*-model* $\mathcal{M}$ *and for every valuation* v *in* $\mathcal{M}$ *there exists a* TG*-model* $\mathcal{M}'$ *and a valuation* v' *in* $\mathcal{M}'$ *such that for every formula* φ *in the language of* TG *the following holds:*

$$(\beta) \quad \mathcal{M}, v \not\models \tau(\varphi) \text{ implies } \mathcal{M}', v' \not\models \varphi.$$

Proof Let $\mathcal{M} = (M', \sim, \sqcap, \sqcup, \Rightarrow, \Leftrightarrow, \circ, \mathcal{F}, \bigotimes, \bigoplus, D)$ be an $\mathsf{SCI}_\mathsf{Q}^\mathsf{TG}$-model, and let v be a valuation in $\mathcal{M}$. By Proposition 4.2, the structure $\mathcal{M}' = (M', \cdot)$ such that $M' \stackrel{\text{df}}{=} D$ and $a \cdot b \stackrel{\text{df}}{=} a \sqcap b$, for all $a, b \in M'$, is a TG-model. Let $e \in M'$ be the neutral element of $\mathcal{M}'$, and let v' be a valuation in $\mathcal{M}'$ such that for every variable x,

$$v'(x) \stackrel{\text{df}}{=} \begin{cases} v(x), & \text{if } v(x) \in D, \\ e, & \text{otherwise.} \end{cases}$$

As in the case of the previous lemma, we prove (β) by induction on the complexity of formulas. First, observe that the following can be easily proved:

($*$) For all variables $x_1, \ldots, x_n$ and for any TG-term $t(x_1, \ldots, x_n)$, if all the values $v(x_1), \ldots, v(x_n)$ belong to D, then $v(\tau(t(x_1, \ldots, x_n))) = v'(t(x_1, \ldots, x_n))$.

Let $t(\bar{x})$ and $t'(\bar{y})$ be TG-terms. Assume $\mathcal{M}, v \not\models \tau(t(\bar{x}) = t'(\bar{y}))$. Then, it follows that $v(\bigwedge \bar{x} \wedge \bigwedge \bar{y}) \in D$ and $v(\tau(t(\bar{x}))) \neq v(\tau(t'(\bar{y})))$. Therefore, by ($*$), we have $v'(t(\bar{x})) \neq v'(t'(\bar{y}))$, so $\mathcal{M}', v' \not\models t(\bar{x}) = t'(\bar{y})$. In a similar way we can prove (β) for formulas of the form $\neg(t(\bar{x}) = t'(\bar{y}))$. Thus, ($\beta$) holds for atomic TG-formulas and their negations. Now, the proofs of (β) for formulas of the form $\neg\neg\psi$, $\psi \rightarrow \vartheta$, $\neg(\psi \rightarrow \vartheta)$, $\forall x \psi$, $\neg\forall x \psi$ are similar to the corresponding proofs in the proof of Lemma 4.3. By way of example, we will show that (β) holds for formulas with quantifiers.

Assume $\mathcal{M}, v \not\models \tau(\forall x \psi)$. Then, there exists $a \in M$ such that $\|x \rightarrow \tau(\psi), v_x^a\| \notin D$, which means that there is $a \in D$ such that $\|\tau(\psi), v_x^a\| \notin D$. Thus, by the induction hypothesis, $\mathcal{M}', v'_a \not\models \psi$, where $v'_a(y) = v_x^a(y)$, for all variables y. Hence, $\mathcal{M}', v' \not\models \forall x \psi$.

Assume $\mathcal{M}, v \not\models \tau(\neg\forall x \psi)$. Then, for every $a \in M$, $\|x \wedge \tau(\neg\psi), v_x^a\| \notin D$, which means that for all $a \in D$, $\|\tau(\neg\psi), v_x^a\| \notin D$. Thus, by the induction hypothesis, for every $a \in D$, $\mathcal{M}', v'_a \not\models \neg\psi$, where $v'_a(y) = v_x^a(y)$, for all variables y, which implies $\mathcal{M}', v'_a \models \psi$, for every $a \in M'$. Hence, $\mathcal{M}', v' \not\models \neg\forall x \psi$. □

Lemmas 4.3 and 4.4 lead to the following.

Theorem 4.5 *For every* TG*-formula φ, the following conditions are equivalent:*

1. *φ is true in all* TG*-models,*
2. *$\tau(\varphi)$ is true in all* $\mathsf{SCI}_\mathsf{Q}^\mathsf{TG}$*-models,*
3. *$\gamma \rightarrow \tau(\varphi)$ is true in all* SCI_Q*-models.*

Proof Let φ be a TG-formula. Assume it is true in all TG-models, and suppose $\tau(\varphi)$ is not true in all $\mathsf{SCI}_\mathsf{Q}^\mathsf{TG}$-models. Then, there must exist an $\mathsf{SCI}_\mathsf{Q}^\mathsf{TG}$-model $\mathcal{M}$ and a valuation v in $\mathcal{M}$ such that $\mathcal{M}, v \not\models \tau(\varphi)$. Then, by Lemma 4.4, there exist a TG-model $\mathcal{M}'$ and a valuation v' in $\mathcal{M}'$ such that $\mathcal{M}', v' \not\models \varphi$, which contradicts the assumption. Now, assume that $\tau(\varphi)$ is true in all $\mathsf{SCI}_\mathsf{Q}^\mathsf{TG}$-models, and suppose φ is not TG-valid. Then, there are a TG-model $\mathcal{M}$ and a valuation v in $\mathcal{M}$ such that φ is not satisfied in $\mathcal{M}$ by v. Thus, by Lemma 4.3, there must exist an $\mathsf{SCI}_\mathsf{Q}^\mathsf{TG}$-model $\mathcal{M}'$ and a valuation v' in $\mathcal{M}'$ such that $\mathcal{M}', v' \not\models \tau(\varphi)$, a contradiction. Hence, we have proved the equivalence of conditions 1 and 2.

Now, let $\tau(\varphi)$ be such that it is true in all $\mathsf{SCI}_\mathsf{Q}^\mathsf{TG}$-models. Consider an SCI_Q-model $\mathcal{M}$ and valuation v in $\mathcal{M}$ such that $\mathcal{M}, v \models \gamma$. Then, since γ is an SCI_Q-sentence, it is true in $\mathcal{M}$, so $\mathcal{M}$ must be an $\mathsf{SCI}_\mathsf{Q}^\mathsf{TG}$-model, which means that $\mathcal{M}, v \models \tau(\varphi)$. Therefore, 2 implies 3. Now, assume that $\gamma \rightarrow \tau(\varphi)$ is true in all SCI_Q-models, and suppose there is an $\mathsf{SCI}_\mathsf{Q}^\mathsf{TG}$-model $\mathcal{M}$ and a valuation v in $\mathcal{M}$ such that $\mathcal{M}, v \not\models \tau(\varphi)$. Since $\mathcal{M}$ is an $\mathsf{SCI}_\mathsf{Q}^\mathsf{TG}$-model, we have $\mathcal{M}, v \models \gamma$, which implies $\mathcal{M}, v \not\models \gamma \rightarrow \tau(\varphi)$, a contradiction. Hence, 3 implies 2. □

Theorem 4.5 will enable us to prove the undecidability of the logic $\mathsf{SCI_Q}$.

Theorem 4.6 *The logic* $\mathsf{SCI_Q}$ *is undecidable.*

Proof By Theorem 4.5, the problem of TG-validity of a TG-formula φ reduces to the problem of $\mathsf{SCI_Q}$-validity of an $\mathsf{SCI_Q}$-formula $\gamma \rightarrow \tau(\varphi)$. However, since it is known that the former one is undecidable, so is the latter, which means that the logic $\mathsf{SCI_Q}$ is undecidable. □

Using the same idea we can express the theory of rings and fields. The common language of the theory TR of rings and the theory TF of fields is a first-order language with two binary function symbols $\cdot$ and $+$. Below we recall the axioms of theories TR and TF. The axioms of TR are

(TR1) $\forall x \forall y \forall z (x\#(y\#z) = (x\#y)\#z)$, for $\# \in \{+, \cdot\}$,
(TR2) $\forall x \forall y (x + y = y + x)$,
(TR3) $\exists x \forall y \exists z ((x + y) = y \wedge x = (y + z))$,
(TR4) $\forall x \forall y \forall z (x \cdot (y + z)) = ((x \cdot y) + (x \cdot z))$,
(TR5) $\forall x \forall y \forall z\, ((x + y) \cdot z) = ((x \cdot z) + (y \cdot z))$.

The axioms of TF are

(TF1) $\forall x \forall y \forall z (x\#(y\#z) = (x\#y)\#z)$, for $\# \in \{+, \cdot\}$,
(TF2) $\forall x \forall y (x\#y = y\#x)$, for $\# \in \{+, \cdot\}$,
(TF3) $\exists x [\forall y \exists z ((x + y) = y \wedge x = (y + z)) \wedge \exists x'(x \neq x' \wedge \forall y (x' \cdot y = y) \wedge \forall y (y \neq x \rightarrow \exists z (x' = (y + z))))]$,
(TF4) $\forall x \forall y \forall z\, (x \cdot (y + z)) = ((x \cdot y) + (x \cdot z))$.

Models of TR are structures of the form $(M, +, \cdot)$ such that $(M, +)$ is an abelian group (i.e., a TG-model in which the operation $+$ is commutative), and the operation $\cdot$ is associative and distributive over $+$. Models of TF are structures of the form $(M, +, \cdot)$ such that $(M, +)$ and $(M \setminus \{e\}, \cdot)$ are abelian groups, where e is the neutral element of $(M, +)$, and the operation $\cdot$ is distributive over $+$.

The axioms of rings and fields can be expressed in $\mathsf{SCI_Q}$-language. Roughly speaking, we will code the operations $+$ and $\cdot$ as $\mathsf{SCI_Q}$-operations $\vee$ and $\wedge$, respectively. First, let us set

$$\chi_1^\# := \forall x \forall y \forall z \big((x \wedge y \wedge z) \rightarrow \big(\big(x\#(y\#z)\big) \equiv \big((x\#y)\#z\big)\big)\big), \quad \text{for } \# \in \{\vee, \wedge\},$$
$$\chi_2^\# := \forall x \forall y \big((x \wedge y) \rightarrow \big((x\#y) \equiv (y\#x)\big)\big), \quad \text{for } \# \in \{\vee, \wedge\},$$
$$\chi_3 := \forall x \forall y \forall z \big((x \wedge y \wedge z) \rightarrow \big(\big(x \wedge (y \vee z)\big) \equiv \big((x \wedge y) \vee (x \wedge z)\big)\big)\big),$$
$$\chi_4 := \forall x \forall y \forall z \big((x \wedge y \wedge z) \rightarrow \big(\big((x \vee y) \wedge z\big) \equiv \big((x \wedge z) \vee (y \wedge z)\big)\big)\big),$$
$$\chi_5 := \exists x \big[x \wedge \forall y \big(y \rightarrow (\chi_8 \wedge \chi_{10})\big)\big],$$
$$\chi_6 := \exists x \exists x' \big[\chi_7 \wedge \forall y \big(y \rightarrow (\chi_8 \wedge \chi_9 \wedge \chi_{10})\big) \wedge \chi_{11}\big],$$

where

$$\chi_7 := \big(x \wedge x' \wedge \neg(x \equiv x')\big),$$
$$\chi_8 := \big((x \vee y) \equiv y\big) \qquad \text{and} \qquad \chi_9 := \big((x' \wedge y) \equiv y\big),$$
$$\chi_{10} := \exists z \big(z \wedge \big((y \vee z) \equiv x\big)\big),$$
$$\chi_{11} := \forall y \big(\big(y \wedge \neg(y \equiv x)\big) \rightarrow \exists z \big(z \wedge \big((y \wedge z) \equiv x'\big)\big)\big).$$

Let $\mathcal{M} = (M, \sim, \sqcap, \sqcup, \Rightarrow, \Leftrightarrow, \circ, \mathcal{F}, \bigotimes, \bigoplus, D)$ be an $\mathsf{SCI_Q}$-model. It can be easily seen that $\chi_1^\#$ (resp., $\chi_2^\#$) expresses the fact that the interpretation of the operation #

in $\mathcal{M}$ is associative (resp., commutative) on D. The formulas χ_3 and χ_4 reflect left and right distributivity of $\sqcap$ over $\sqcup$ with respect to the set D. The formula χ_5 says that there exists the neutral element of $\sqcup$ in D and each element in D has its inverse element with respect to $\sqcup$. The formula χ_6 says that both operations $\sqcup$ and $\sqcap$ have neutral elements, which are different and belong to D, and, in addition, each element in D has its inverse element with respect to $\sqcup$, and each element in D different from the neutral element of the operation $\sqcup$ has its inverse element with respect to $\sqcap$.

Now, let ρ and φ be the following formulas:

$$\rho := \chi_1^{\vee} \wedge \chi_1^{\wedge} \wedge \chi_2^{\vee} \wedge \chi_3 \wedge \chi_4 \wedge \chi_5,$$
$$\varphi := \chi_1^{\vee} \wedge \chi_1^{\wedge} \wedge \chi_2^{\vee} \wedge \chi_2^{\wedge} \wedge \chi_3 \wedge \chi_6.$$

$\mathsf{SCI_Q}$-models in which the formula ρ (resp., φ) is true will be referred to as $\mathsf{SCI_Q^{TR}}$-*models* (resp., $\mathsf{SCI_Q^{TF}}$-*models*). Then, the following can be proved.

Proposition 4.7 *Let a structure* $\mathcal{M} = (M, \sim, \sqcap, \sqcup, \Rightarrow, \Leftrightarrow, \circ, \mathcal{F}, \bigotimes, \bigoplus, D)$ *be an* $\mathsf{SCI_Q}$*-model. Then*

1. *$\mathcal{M}$ is an $\mathsf{SCI_Q^{TR}}$-model if and only if $(D, +, \cdot)$ is a ring;*
2. *$\mathcal{M}$ is an $\mathsf{SCI_Q^{TF}}$-model if and only if $(D, +, \cdot)$ is a field.*

Now, we define the translation σ from terms and formulas in the first-order language with function symbols $+$ and $\cdot$ into $\mathsf{SCI_Q}$-formulas as follows:

- $\sigma(x) \stackrel{\text{df}}{=} x$, for any variable x,
- $\sigma(t(\bar{x}) + t'(\bar{y})) \stackrel{\text{df}}{=} \sigma(t(\bar{x})) \vee \sigma(t'(\bar{y}))$, for any terms t, t' and finite sequences of variables $\bar{x}$ and $\bar{y}$,
- $\sigma(t(\bar{x}) \cdot t'(\bar{y})) \stackrel{\text{df}}{=} \sigma(t(\bar{x})) \wedge \sigma(t'(\bar{y}))$, for any terms t, t' and finite sequences of variables $\bar{x}$ and $\bar{y}$,
- $\sigma(t(\bar{x}) = t'(\bar{y})) \stackrel{\text{df}}{=} (\bigwedge \bar{x} \wedge \bigwedge \bar{y}) \to (\sigma(t(\bar{x})) \equiv \sigma(t'(\bar{y})))$, for any terms t, t' and finite sequences $\bar{x}$ and $\bar{y}$,
- $\sigma(\neg(t(\bar{x}) = t'(\bar{y}))) \stackrel{\text{df}}{=} (\bigwedge \bar{x} \wedge \bigwedge \bar{y}) \to \neg(\sigma(t(\bar{x})) \equiv \sigma(t'(\bar{y})))$, for any terms t, t' and finite sequences $\bar{x}$ and $\bar{y}$,
- $\sigma(\neg\neg\varphi) \stackrel{\text{df}}{=} \sigma(\varphi)$,
- $\sigma(\varphi \to \psi) \stackrel{\text{df}}{=} \sigma(\neg\varphi) \vee \sigma(\psi)$,
- $\sigma(\neg(\varphi \to \psi)) \stackrel{\text{df}}{=} \sigma(\varphi) \wedge \sigma(\neg\psi)$,
- $\sigma(\forall x\varphi) \stackrel{\text{df}}{=} \forall x(x \to \sigma(\varphi))$,
- $\sigma(\neg\forall x\varphi) \stackrel{\text{df}}{=} \exists x(x \wedge \sigma(\neg\varphi))$.

As in the case of the group theory, it can be proved that σ preserves TR- and TF-validity of formulas with respect to the class of $\mathsf{SCI_Q^{TR}}$- and $\mathsf{SCI_Q^{TF}}$-models, respectively. More precisely, the following hold.

Theorem 4.8 *For every formula φ of first-order language with the operations $+$ and $\cdot$, the following hold:*

1. *φ is true in all TR-models if and only if $\sigma(\varphi)$ is true in all $\mathsf{SCI_Q^{TR}}$-models if and only if $\rho \to \sigma(\varphi)$ is $\mathsf{SCI_Q}$-valid;*
2. *φ is true in all TF-models if and only if $\sigma(\varphi)$ is true in all $\mathsf{SCI_Q^{TF}}$-models if and only if $\varphi \to \sigma(\varphi)$ is $\mathsf{SCI_Q}$-valid.*

Now, we will show how to code a somewhat more complex theory in $\mathsf{SCI_Q}$. To highlight the differences with the previous examples, we restrict ourselves to a very weak fragment of Peano arithmetic, which we call *successor arithmetic*, SA for short. In particular, it is worth noting that negation has to be handled carefully. Moreover, the theory of SA is infinite, and therefore its straightforward translation results in an infinite $\mathsf{SCI_Q}$-theory. Hence, this kind of coding of even full Peano arithmetic would not suffice to show the undecidability of $\mathsf{SCI_Q}$.

The language of SA is a first-order language with a constant symbol 0 and a unary function symbol S.

Along with the usual axioms of equality, SA consists of the following axioms,

(SA1) $\forall x \neg (S(x) = 0)$,
(SA2) $\forall x \forall y (S(x) = S(y)) \to x = y$,

and the induction schema (ind)

$$\forall \bar{y}\big[\big(\varphi(0, \bar{y}) \wedge \forall x\big(\varphi(x, \bar{y}) \to \varphi\big(S(x), \bar{y}\big)\big)\big) \to \forall x \varphi(x, \bar{y})\big],$$

for every formula $\varphi(x, \bar{y})$ (in SA-language) whose free variables are among $x, \bar{y}$. Let us consider the following $\mathsf{SCI_Q}$-formula α:

$$\begin{aligned}
\alpha &:= \exists x(x \wedge \alpha_1 \wedge \alpha_2) \wedge \alpha_3, \quad \text{where}\\
\alpha_1 &:= \forall y\big(y \to \neg\big((y \equiv y) \equiv x\big)\big),\\
\alpha_2 &:= \forall z\big[\big(z \wedge \forall y\big(y \to \neg\big((y \equiv y) \equiv z\big)\big)\big) \to x \equiv z\big],\\
\alpha_3 &:= \forall x \forall y\big[\big(x \wedge y \wedge \big((x \equiv x) \equiv (y \equiv y)\big)\big) \to (x \equiv y)\big].
\end{aligned}$$

Let $\mathsf{SCI_Q^{SA^-}}$ be the class of all $\mathsf{SCI_Q}$-models in which the formula α is true.

Proposition 4.9 *Let a structure $\mathcal{M} = (M, \sim, \sqcap, \sqcup, \Rightarrow, \Leftrightarrow, \circ, \mathcal{F}, \bigotimes, \bigoplus, D)$ be an $\mathsf{SCI_Q}$-model. Then, the following conditions are equivalent.*

1. *$\mathcal{M}$ is an $\mathsf{SCI_Q^{SA^-}}$-model.*
2. *There exists exactly one $\mathbf{1} \in D$ such that the following conditions are satisfied for all $a, b \in D$:*
   - *$(a \circ a) \neq \mathbf{1}$,*
   - *$(a \circ a) = (b \circ b)$ implies $a = b$.*

Proof Let $\mathcal{M} = (M, \sim, \sqcap, \sqcup, \Rightarrow, \Leftrightarrow, \circ, \mathcal{F}, \bigotimes, \bigoplus, D)$ be an $\mathsf{SCI_Q}$-model. Assume that α is true in $\mathcal{M}$. Then, for every valuation v in $\mathcal{M}$ there must exist $\mathbf{1} \in M$ such that $\|x \wedge \alpha_1 \wedge \alpha_2, v_x^{\mathbf{1}}\| \in D$, so $\mathbf{1} \in D$, $\|\alpha_1, v_x^{\mathbf{1}}\| \in D$, and $\|\alpha_2, v_x^{\mathbf{1}}\| \in D$; that is:

(a) $\|\forall y(y \to \neg((y \equiv y) \equiv x)), v_x^{\mathbf{1}}\| \in D$, and
(b) $\|\forall z[(z \wedge \forall y(y \to \neg((y \equiv y) \equiv z))) \to x \equiv z], v_x^{\mathbf{1}}\| \in D$.

By condition (a), we obtain that for all $a \in D$, $(a \circ a) \circ \mathbf{1} \notin D$; that is, $(a \circ a) \neq \mathbf{1}$. From condition (b) it follows that whenever $z \in D$ satisfies $(a \circ a) \neq z$ for all $a \in D$, then $z = \mathbf{1}$. Hence, $\mathbf{1}$ is the unique element in D such that $(a \circ a) \neq \mathbf{1}$, for all $a \in D$. We have also that the formula $\alpha_3 = \forall x \forall y[(x \wedge y \wedge ((x \equiv x) \equiv (y \equiv y))) \to (x \equiv y)]$ is true in $\mathcal{M}$, from which it follows that for all $a, b \in D$, if $(a \circ a) = (b \circ b)$, then $a = b$.

The other direction can be proved in a similar way. □

Hence, in all $\mathsf{SCI}_{\mathsf{Q}}^{\mathsf{SA}^-}$-models the element **1** is uniquely determined. Thus, in what follows, $\mathsf{SCI}_{\mathsf{Q}}^{\mathsf{SA}^-}$-models will be identified with structures of the form

$$\mathcal{M} = \Big(M, \sim, \sqcap, \sqcup, \Rightarrow, \Leftrightarrow, \circ, \mathcal{F}, \bigotimes, \bigoplus, \mathbf{1}, D\Big)$$

that satisfy appropriate conditions listed above. Furthermore, we may extend the language of $\mathsf{SCI}_{\mathsf{Q}}$ with a propositional constant 1 interpreted in $\mathsf{SCI}_{\mathsf{Q}}^{\mathsf{SA}^-}$-models as **1**. Formulas of this new language will be called $\mathsf{SCI}_{\mathsf{Q}}^{\mathsf{SA}}$-*formulas*. Now, we will show that the induction schema (ind) can be expressed by an $\mathsf{SCI}_{\mathsf{Q}}^{\mathsf{SA}}$-formula. Let φ be any $\mathsf{SCI}_{\mathsf{Q}}^{\mathsf{SA}}$-formula, and let β_φ be a formula of the form

$$\begin{aligned}\forall \bar{y}\Big[\Big(\bigwedge \bar{y} \to \big(\varphi(1,\bar{y}) \wedge \forall x\big(x \wedge \varphi(x,\bar{y}) \to \varphi(x/x \equiv x, \bar{y})\big)\big)\Big)\\ \to \forall x\big(x \to \varphi(x,\bar{y})\big)\Big].\end{aligned}$$

$\mathsf{SCI}_{\mathsf{Q}}^{\mathsf{SA}^-}$-models in which formulas β_φ are true for all $\mathsf{SCI}_{\mathsf{Q}}^{\mathsf{SA}}$-formulas φ will be referred to as $\mathsf{SCI}_{\mathsf{Q}}^{\mathsf{SA}}$-*models*. In fact, $\mathsf{SCI}_{\mathsf{Q}}^{\mathsf{SA}}$-models are $\mathsf{SCI}_{\mathsf{Q}}^{\mathsf{SA}^-}$-models that satisfy an infinite number of formulas. We define the translation π from terms and formulas in the language of SA into $\mathsf{SCI}_{\mathsf{Q}}^{\mathsf{SA}}$-formulas as follows:

- $\pi(x) \overset{\text{df}}{=} x$, for any variable x,
- $\pi(0) \overset{\text{df}}{=} 1$,
- $\pi(S(t)) \overset{\text{df}}{=} (\pi(t) \equiv \pi(t))$, for any term t,
- $\pi(t(\bar{x}) = t'(\bar{y})) \overset{\text{df}}{=} (\bigwedge \bar{x} \wedge \bigwedge \bar{y}) \to (\pi(t(\bar{x})) \equiv \pi(t'(\bar{y})))$, for any terms t, t' and finite sequences $\bar{x}$ and $\bar{y}$,
- $\pi(\neg(t(\bar{x}) = t'(\bar{y}))) \overset{\text{df}}{=} (\bigwedge \bar{x} \wedge \bigwedge \bar{y}) \to \neg(\pi(t(\bar{x})) \equiv \pi(t'(\bar{y})))$, for any terms t, t' and finite sequences $\bar{x}$ and $\bar{y}$,
- $\pi(\neg\neg\varphi) \overset{\text{df}}{=} \pi(\varphi)$,
- $\pi(\varphi \to \psi) \overset{\text{df}}{=} \pi(\neg\varphi) \vee \pi(\psi)$,
- $\pi(\neg(\varphi \to \psi)) \overset{\text{df}}{=} \pi(\varphi) \wedge \pi(\neg\psi)$,
- $\pi(\forall x\varphi) \overset{\text{df}}{=} \forall x(x \to \pi(\varphi))$,
- $\pi(\neg\forall x\varphi) \overset{\text{df}}{=} \exists x(x \wedge \pi(\neg\varphi))$.

As in the case of group theory, we can show that the translation π preserves validity of SA-formulas; that is, the following theorem holds.

Theorem 4.10 *For every* SA-*formula* φ*, the following conditions are equivalent:*

1. φ *is* SA-*valid;*
2. $\pi(\varphi)$ *is true in all* $\mathsf{SCI}_{\mathsf{Q}}^{\mathsf{SA}}$-*models.*

Hence, the first-order theory of successor arithmetic is expressible in an $\mathsf{SCI}_{\mathsf{Q}}$-theory. Note that the non-Fregean interpretation of SA presented above is essentially different from the non-Fregean interpretations of group, ring, and field theories. The former involves infinitely many axioms, whereas each of the latter can be expressed as a single conjunction of finitely many $\mathsf{SCI}_{\mathsf{Q}}$-formulas.

Furthermore, we would also like to point out that the method presented in this section can be extended to the full Peano arithmetic with addition and multiplication, and to other first-order mathematical theories.

5 Spectra

In this section, we consider the $\mathsf{SCI_Q}$-spectra, that is, the sets of sizes of finite models satisfying a given $\mathsf{SCI_Q}$-sentence. Our methods and results resemble those of Ronald Fagin in [4]. However, the details differ significantly, as will be seen.

A partial $\mathsf{SCI_Q}$-model $\mathcal{M}'$ is called a *fragment* of another partial $\mathsf{SCI_Q}$-model $\mathcal{M}$ if the universes and the Boolean connectives are the same, $\mathcal{F}' \subseteq \mathcal{F}$, and $\bigoplus' = \bigoplus \restriction \mathcal{F}', \bigotimes' = \bigotimes \restriction \mathcal{F}'$.

Lemma 5.1 *Let $\mathcal{M}$ be a partial $\mathsf{SCI_Q}$-model, and let $\mathcal{M}'$ be a fragment of $\mathcal{M}$ such that $\mathcal{M}'$ is sufficient for an $\mathsf{SCI_Q}$-formula φ. Then, $\mathcal{M}$ is sufficient for φ, and $\|\varphi, v\|_{\mathcal{M}'} = \|\varphi, v\|_{\mathcal{M}}$, for every partial valuation v in $\mathcal{M}$ sufficient for φ.*

Proof We proceed by induction on the complexity of φ. The claim is trivially true for propositional variables, and the connective steps are quite straightforward. So, assume that φ is the formula $\exists x \psi$ and that the claim holds for ψ. Let v be a valuation on $\mathcal{M}$. Now, to find the value $\|\exists x \psi, v\|_{\mathcal{M}}$ according to the definition, we let $f : M \to M$ be the following function:

$$f(t) = \|\psi, v^t_x\|_{\mathcal{M}}.$$

By the induction hypothesis, the value $f(t)$ is well defined for every $t \in M$, and, in fact, $f(t) = \|\psi, v^t_x\|_{\mathcal{M}'}$. On the other hand, since $\mathcal{M}'$ is sufficient for φ, it holds that $f \in \mathcal{F}'$, and hence $f \in \mathcal{F}$. Therefore,

$$\|\varphi, v\|_{\mathcal{M}} = \bigoplus f = \bigoplus{}' f = \|\varphi, v\|_{\mathcal{M}'}.$$

Hence, the claim also holds for φ. The universal quantifier step is similar. □

Lemma 5.2 *For every $\mathsf{SCI_Q}$-formula φ, there is a polynomial p_φ such that for every finite $\mathsf{SCI_Q}$-model $\mathcal{M}$, there is a fragment $\mathcal{M}'$ of $\mathcal{M}$ such that $|\mathcal{M}'|_2 \leq p_\varphi(|\mathcal{M}|)$ and $\mathcal{M}'$ is sufficient for φ.*

Proof Let Φ be the set of all subformulas of φ, and let W be the set of all variables occurring in φ. For $\psi \in \Phi$, $x \in W$, and $v : W \to M$, let $f_{\psi,x,v} : M \to M$ be defined as

$$f_{\psi,x,v}(t) = \|\psi, v^t_x\|_{\mathcal{M}}.$$

Every such function $f_{\psi,x,v}$ is well defined, as $\mathcal{M}$ is an $\mathsf{SCI_Q}$-model. Now, it suffices to let

$$\mathcal{F}' = \{f_{\psi,x,v} \mid \psi \in \Phi, x \in W, v \in {}^W M\}.$$

Clearly, $\mathcal{M}'$ is a fragment of $\mathcal{M}$ sufficient for φ. Furthermore, the sets Φ and W are fixed, and $|\mathcal{F}'| \leq |\Phi| \cdot |W| \cdot |M|^{|W|}$. □

In the rest of this section, we assume that the models and any internal data structures are coded in a reasonable way. In particular, we make the following assumptions concerning the coding $\hat{\mathcal{M}}$ of a model $\mathcal{M}$.

- It can be checked in polynomial time whether a given string is a syntactically valid coding of a partial $\mathsf{SCI_Q}$-model.
- There is a fixed polynomial p such that

$$|\mathcal{M}| + |\mathcal{M}|_2 \leq |\hat{\mathcal{M}}| \leq p\big(|\mathcal{M}| + |\mathcal{M}|_2\big).$$

- Each of the following operations takes at most $p(|\mathcal{M}| + |\mathcal{M}|_2)$ steps:
  - given $a \in M$, find the value of $\sim a$,

- given $a, b \in M$, find the value of $a \sqcap b$, $a \sqcup b$, $a \Rightarrow b$, $a \Leftrightarrow b$, or $a \circ b$,
- given $f\colon M \to M$, find out whether $f \in \mathcal{F}$,
- given $f \in \mathcal{F}$, find the value of $\bigotimes f$ or $\bigoplus f$,
- given $a \in M$, find out whether $a \in D$.

It is easy to see that the above assumptions yield the following.

Fact 5.3 Given a syntactically valid coding, it can be checked in polynomial time whether it meets the semantic requirements of a partial $\mathsf{SCI_Q}$-model.

Lemma 5.4 *For a fixed* $\mathsf{SCI_Q}$*-sentence* φ*, there is a polynomial* q *such that it can be checked in time* $q(|\hat{\mathcal{M}}|)$ *whether* φ *is true in a given finite partial* $\mathsf{SCI_Q}$*-model* $\mathcal{M}$.

Proof Let Φ and W be as in the previous proof. Write Φ as $\{\psi_0, \psi_1, \ldots, \psi_k\}$, where the formulas ψ_i are ordered by increasing length (in particular, $\psi_k = \varphi$), with formulas of equal length ordered arbitrarily. For $i = 0, 1, \ldots, k$, evaluate $v(\psi_i)$ for each partial valuation $v\colon W \to M$. As the immediate subformulas of ψ_i have already been evaluated, this can be done by table lookups. There are $|M|^{|W|}$ partial valuations to consider. When ψ_i is a propositional variable or formed with a connective, its value can be determined with at most three table lookups. When ψ_i is of the form $\exists x\theta$, finding out its value for a given partial valuation v involves the following steps.

1. For each $t \in M$, look up the value $\|\theta, v_x^t\|$ and store it as $f(t)$.
2. For each $g \in \mathcal{F}$ and each $t \in M$, look up $g(t)$ and compare it with $f(t)$. If they match for every t, look up $\bigoplus g$ and return it.
3. If no g matches f, the model $\mathcal{M}$ is not sufficient for ψ_i and hence not for φ either. The input can be rejected.

Hence, the number of storage/lookup operations needed is $O(|M|^{|W|} \cdot |M| \cdot |\mathcal{F}|)$, which is $O(n^{|W|+2})$ for inputs of length n. □

Lemma 5.5 *For a fixed* $\mathsf{SCI_Q}$*-sentence* φ*, the set of inputs* σ *such that there is an* $\mathsf{SCI_Q}$*-model* $\mathcal{M}$ *satisfying* $\mathcal{M} \models \varphi$ *and* $|\mathcal{M}| = |\sigma|$ *is in* NP.

Proof A nondeterministic Turing machine can first guess a polynomial-sized partial $\mathsf{SCI_Q}$-model $\mathcal{M}$ with $|\mathcal{M}| = |\sigma|$ and then check that $\mathcal{M} \models \varphi$. According to Lemma 5.2, it can be assumed without loss of generality that $|\mathcal{M}|_2$ is bounded by a fixed polynomial, and hence $\mathcal{M}$ can be coded as a polynomial-sized string. Checking the truth of φ in $\mathcal{M}$ can again be done in polynomial time. □

To prove the reverse inclusion, we restrict our attention to finite $\mathsf{SCI_Q}$-models $\mathcal{M}$ with a built-in linear ordering and other definable properties that allow us to code a polynomial-length computation with a nondeterministic Turing Machine.

An $\mathsf{SCI_Q}$-model $\mathcal{M}$ is called *linear* if and only if it satisfies the following conditions.

1. The domain of $\mathcal{M}$ is linearly ordered, with the smallest element denoted by 0 and the largest element by N.
2. We have $D = M \setminus \{0\}$.
3. The operations $\sqcup$, $\sqcap$, and $\circ$ are defined as follows:

$$a \sqcup b \stackrel{\text{df}}{=} \max(a, b), \qquad a \sqcap b \stackrel{\text{df}}{=} \min(a, b), \qquad a \circ b \stackrel{\text{df}}{=} \begin{cases} N, & \text{if } a = b, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 5.6 *The class of linear* $\mathsf{SCI_Q}$*-models is* $\mathsf{SCI_Q}$*-definable.*

Proof Let $\varphi_L := \forall x \forall y \forall z (\varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \varphi_4 \wedge \varphi_5 \wedge \varphi_6)$, where

$$\begin{aligned}
\varphi_1 &:= \big(x \equiv (x \wedge y)\big) \vee \big(y \equiv (x \wedge y)\big),\\
\varphi_2 &:= \big(x \wedge (y \wedge z)\big) \equiv \big((x \wedge y) \wedge z\big),\\
\varphi_3 &:= (x \wedge y) \equiv (y \wedge x),\\
\varphi_4 &:= \big(x \equiv (x \wedge y)\big) \equiv \big(y \equiv (x \vee y)\big),\\
\varphi_5 &:= \neg x \to \big(x \equiv (x \wedge y)\big),\\
\varphi_6 &:= \big((x \equiv x) \wedge y\big) \equiv y.
\end{aligned}$$

Now, if φ_1 through φ_3 are true, it follows that $\wedge$ is indeed a minimum operation in a linear ordering. The formula φ_4 ensures that $\vee$ is the corresponding maximum operation. The formula φ_5 implies that the minimal element is the only one not in D. In particular, $a \circ b = 0$ whenever $a \neq b$. Finally, it follows from φ_6 that also $a \circ a = N$ for every a. □

We will code the contents of the tape and the location of the head with the values of the existential quantifier applied to truth functions that encode tuples of elements. The existence of an accepting computation is expressible with an $\mathsf{SCI_Q}$-sentence. Instead of presenting the whole complicated construction at once, we first formulate a suitable sentence in a first-order language and later present a way to translate it into $\mathsf{SCI_Q}$.

As the details of the definition of a Turing machine vary slightly in the literature, we summarize our choices for clarity. A nondeterministic Turing machine T is formally a 6-tuple $\langle \Sigma, a_0, \mathcal{S}, s_1, s_q, \mathcal{R}\rangle$, where $\Sigma = \{a_1, \ldots, a_p\}$ is the alphabet; a_0 is the blank symbol; $\mathcal{S} = \{s_1, s_2, \ldots, s_q\}$ is the set of states, with s_1 being the initial state and s_q the accepting state; and $\mathcal{R}$ is the set of rules. A rule of T is of the form $\langle a, s, a', s', X\rangle$, where $a, a' \in \Sigma \cup \{a_0\}$, $s \in S \setminus \{s_q\}$, $s' \in \mathcal{S}$, and $X \in \{L, R\}$. A computation on input $\sigma \in \Sigma^*$ starts in state s_1, with the elements of σ stored in consecutive cells and the remaining cells blank, and the head on the first element of σ. Whenever T is in state s with the head scanning a, any rule $\langle s, a, s', a', X\rangle$ can be applied, changing the state to s', writing the symbol a' on the tape, and moving the head to the left or right according to X. When there are no applicable rules, the machine halts. In particular, T always halts once it reaches the accepting state s_q. The length of a computation is the number of configurations, that is, one more than the number of times a rule is applied. For instance, the length of a computation that halts right in the starting configuration is 1. We follow the convention that once a computation halts, further steps are defined to retain the configuration unchanged. For technical convenience, we assume a Turing machine always scans its entire input.

Fix a Turing machine $T = \langle \Sigma, a_0, \mathcal{S}, s_1, s_q, \mathcal{R}\rangle$, where $\Sigma = \{a_1, \ldots, a_p\}$ and $\mathcal{S} = \{s_1, \ldots, s_q\}$. Write I and J for the index sets $\{0, 1, \ldots, q\}$ and $\{0, 1, \ldots, p\}$, respectively. Let L_T be the first-order language with the binary relation symbols K_{ij} for $i \in I$, $j \in J$, and another binary relation symbol $<$. A computation of length at most l can be coded as an L_T-model $\mathcal{M} = (M, <, K_{ij})_{i \in I, j \in J}$, where M is a finite universe of size l, whose elements stand for both time steps and tape locations; $<$ is a linear ordering on M; and each K_{ij} is a binary relation such that $K_{0j}(t, x)$ holds if and only if the symbol in cell x at step t is a_j and the head is not on cell x at step t, and $K_{ij}(t, x)$ holds if and only if at step t, the head is on cell x, the machine is in state s_i, and cell x contains a_j. The model codes the contents of l consecutive cells at l consecutive time steps. The minimal element represents the start of the computation, as time, and the leftmost cell scanned during the computation, as tape location. This suffices to code the entire computation, since there are at most l steps to code, by the assumption, and the cells scanned by the head form a contiguous block of length at most l, since the head moves only one step at a time until the computation stops. Moreover, the entire input is coded in the model, as the machine scans it before halting. It follows directly that for all $t, x \in M$, there is exactly one relation K_{ij} such that $K_{ij}(t, x)$ holds. In fact, this is our reason for choosing such a coding instead of a simpler one.

The following lemma is fairly straightforward, but checking all the details gets tedious. We use the notation of the preceding outline.

Lemma 5.7 *There is a first-order sentence φ_T satisfying the following conditions.*

1. *Every finite model of φ_T codes an accepting computation of T on some input $\sigma \in \Sigma^*$.*
2. *Every accepting computation of T of length less than l is coded by a model $\mathcal{M}$ of φ_T with $|M| = l$.*

Proof First, let ψ be the conjunction of the universal closures of the formulas TC1 through TC12 below. We write $h(t, x)$ for the formula $\neg \bigvee_{j \in J} K_{0j}(t, x)$, indicating that the head is scanning cell x at time t, and we write $s(t, t')$ for the formula $t < t' \wedge \neg\exists u(t < u \wedge u < t')$, indicating that t' is the immediate successor of t.

TC1 $(x < y \wedge y < z) \rightarrow x < z,$

TC2 $x < y \leftrightarrow \neg(x = y \vee y < x),$

TC3 $\displaystyle\bigvee_{i \in I, j \in J} K_{ij}(t, x),$

TC4 $\neg\big(K_{ij}(t, x) \wedge K_{i'j'}(t, x)\big),$ for $i \in I$, $j \in J$, $(i, j) \neq (i', j')$,

TC5 $\forall t \exists x h(t, x),$

TC6 $\big(h(t, x) \wedge h(t, y)\big) \rightarrow x = y,$

TC7 $\exists t \exists x \exists y \Big[\neg\exists u(u < t) \wedge h(t, x) \wedge \forall z \Big(\bigvee_{i \in I} K_{i0}(t, z) \leftrightarrow (z < x \vee y < z)\Big)\Big],$

TC8 $\exists t \exists x \big(\neg\exists y(y < x) \wedge h(t, x)\big),$

TC9 $\exists t \exists x \Big(\neg\exists u(u < t) \wedge \bigvee_{j \in J} K_{1j}(t, x)\Big),$

TC10 $\exists t \exists x \Big(\bigvee_{j \in J} K_{qj}(t, x)\Big),$

TC11 $\big(s(t, t') \wedge h(t, x) \wedge h(t', x')\big) \rightarrow \big(s(x, x') \vee (x = x') \vee s(x', x)\big),$

TC12 $\big(s(t, t') \wedge K_{0j}(t, x)\big) \rightarrow \bigvee_{i \in I} K_{ij}(t', x),$ for $j \in J$.

The formulas TC1 and TC2 ensure that $<$ is indeed a linear ordering. The rest of the above formulas depend only on the states and the alphabet of T, but not directly on its rules. The formula TC3 says that at least one of the predicates K_{ij} holds for each pair (t, x). TC4 is actually a finite set of formulas, which together express the requirement that at most one of the K_{ij} holds. The formulas TC5 and TC6 say that the head is at exactly one location x at any given moment t. The formula TC7 says that the nonblank cells form a contiguous string at the start of the computation, with the head scanning the leftmost one. The formula TC8 ensures that the leftmost cell is scanned at some point during the computation. The formulas TC9 and TC10 express the conditions that the computation starts at state s_1 and reaches the accepting state s_q, respectively. The formula TC11 allows the head to move at most one step at a time. Finally, the formulas TC12 say that the contents of a cell not being scanned remain the same between successive steps.

Now, let us consider how to take the rules of T into account. We want to find a formula $\theta(t, t'x, x')$ that is satisfied by consecutive time steps t and t' and the corresponding head locations x and x' exactly when the rules of T are followed. As T is a nondeterministic machine, it is most natural to look for a disjunction of all the possible configuration changes allowed by the rules.

We are only interested in accepting computations. Therefore, we let the nonaccepting halting states be dead ends, with no satisfiable clauses attached to them. On the other hand, we allow for an accepting state to be simply copied to the next step. So, let $\theta_1(t, t', x)$ be the formula

$$\bigvee_{j \in J} \big(K_{qj}(t, x) \wedge K_{qj}(t', x)\big).$$

Further, for each rule $r = \langle s_i, a_j, s_{i'}, a_{j'}, L\rangle$, let $\xi_r(t, t', x, x')$ be the formula

$$s(x', x) \wedge K_{ij}(t, x) \wedge K_{0j'}(t', x) \wedge \bigvee_{u \in J} K_{i'u}(t', x'),$$

and for a rule $r = \langle s_i, a_j, s_{i'}, a_{j'}, R\rangle$, let $\xi_r(t, t', x, x')$ be

$$s(x, x') \wedge K_{ij}(t, x) \wedge K_{0j'}(t', x) \wedge \bigvee_{u \in J} K_{i'u}(t', x').$$

Let θ_2 be the disjunction of all the ξ_r corresponding to the rules of T, and let θ be the sentence

$$\forall t \forall t' \forall x \forall x' \big[\big(s(t, t') \wedge h(t, x) \wedge h(t', x')\big) \rightarrow \big(\theta_1(t, t', x) \vee \theta_2(t, t', x, x')\big)\big].$$

Suppose that the model $\mathcal{M}$ codes an accepting computation of T. Let t and t' be two consecutive time steps, and let x and x' be the respective head locations. If the machine is in the accepting state at time t, then some disjunct of $\theta_1(t, t', x)$ is satisfied. If not, a rule r has been applied at t, and it can be checked that $\xi_r(t, t', x, x')$ is satisfied. So, θ is true in every model that codes an accepting computation of T.

Now, let φ_T be the sentence $\psi \wedge \theta$. It is a matter of simple but tedious checking that φ_T is indeed as required. □

Assume now that T runs in polynomial time. Then, there is an exponent r such that T runs for fewer than n^r steps before stopping on input of length n, for $n \geq 2$. Consider a linear $\mathsf{SCI_Q}$-model $\mathcal{M} = (M, \sim, \sqcap, \sqcup, \Rightarrow, \Leftrightarrow, \circ, \mathcal{F}, \bigotimes, \bigoplus, D)$ such that $|D| \geq \max(2r, (p+1)(q+1))$. We will define an L_T-model $\mathcal{M}^*$, whose elements are r-tuples of elements of M, ordered lexicographically, and the relations K_{ij} are defined in terms of the values of $\bigoplus$ on certain functions defined from the arguments.

Let $a_0, a_1, \ldots, a_{r-1}, b_0, b_1, \ldots, b_{r-1}$ be the smallest $2r$ elements of D, listed in ascending order. Let $\alpha_i(x)$ and $\beta_i(x)$ be the defining formulas for a_i and b_i, respectively. That is,

$$\begin{aligned}
\alpha_0(x) &\overset{\text{df}}{\Leftrightarrow} \neg\exists y(y < x),\\
\alpha_{i+1}(x) &\overset{\text{df}}{\Leftrightarrow} \exists y\big(\alpha_i(y) \wedge y < x \wedge \neg\exists z(y < z \wedge z < x)\big),\\
\beta_0(x) &\overset{\text{df}}{\Leftrightarrow} \exists y\big(\alpha_{r-1}(y) \wedge y < x \wedge \neg\exists z(y < z \wedge z < x)\big),\\
\beta_{i+1}(x) &\overset{\text{df}}{\Leftrightarrow} \exists y\big(\beta_i(y) \wedge y < x \wedge \neg\exists z(y < z \wedge z < x)\big).
\end{aligned}$$

For $\bar{s}, \bar{t} \in M^r$ and $u \in M$, define

$$f_{\bar{s},\bar{t}}(u) = \begin{cases} N, & \text{if } u = 0,\\ s_i, & \text{if } u = a_i \text{ for some } i < r,\\ t_i, & \text{if } u = b_i \text{ for some } i < r,\\ 0, & \text{otherwise.} \end{cases}$$

Define $F(\bar{s}, \bar{t}) = \bigoplus f_{\bar{s},\bar{t}}$.

Now, let c_{ij}, for $i \in I$ and $j \in J$, be the smallest $(p+1)(q+1)$ elements of D, listed according to the lexicographic ordering of the pairs (i, j), and define $K^*_{ij}(\bar{s}, \bar{t})$ if $F(\bar{s}, \bar{t}) = c_{ij}$. Let $\mathcal{M}^* = (M^r, <^*, K^*_{ij})_{i\in I, j\in J}$, where $<^*$ is the lexicographic ordering on M^r.

Lemma 5.8 *For every L_T-sentence φ, there is an* $\mathsf{SCI_Q}$*-sentence φ' such that for every linear* $\mathsf{SCI_Q}$*-model $\mathcal{M}$, it holds that $\mathcal{M} \models \varphi'$ if and only if $\mathcal{M}^* \models \varphi$.*

Proof We define a translation inductively for all L_T-formulas, with each free variable x replaced with an r-tuple $(x_0, x_1, \ldots, x_{r-1})$. As the universe of $\mathcal{M}^*$ is M^r, we can translate a valuation v in $\mathcal{M}$ into a valuation v^* in $\mathcal{M}^*$ simply by

$$v^*(x) = \big(v(x_0), v(x_1), \ldots, v(x_{r-1})\big).$$

Now the claim is that for every valuation v in $\mathcal{M}$, it holds that $\mathcal{M}, v \models \varphi'$ if and only if $\mathcal{M}^*, v^* \models \varphi$. The case of $=$ and $<$ is straightforward. When φ is $x = y$, let φ' be

$$(x_0 \equiv y_0) \wedge (x_1 \equiv y_1) \wedge \cdots \wedge (x_{r-1} \equiv y_{r-1}).$$

When φ is $x < y$, let φ' be

$$\begin{aligned}
&(x_0 < y_0) \vee \big((x_0 \equiv y_0) \wedge (x_1 < y_1)\big) \vee \cdots\\
&\qquad \vee \big((x_0 \equiv y_0) \wedge (x_1 \equiv y_1) \wedge \cdots \wedge (x_{r-2} \equiv y_{r-2}) \wedge (x_{r-1} < y_{r-1})\big),
\end{aligned}$$

where $x_i < y_i$ is short for $(x_i \equiv (x_i \wedge y_i)) \wedge \neg(x_i \equiv y_i)$. It is easy to see that these translations preserve truth.

Now, assume that φ is $K_{ij}(x, y)$. We define φ' in two stages. First, let φ^+ be the formula $(\exists z \eta(\bar{x}, \bar{y}, z, \bar{s}, \bar{t}, u)) \equiv w$, where η is the following formula:

$$(z \equiv u) \vee \big((z \equiv s_0) \wedge x_0\big) \vee \big((z \equiv s_1) \wedge x_1\big) \vee \cdots \vee \big((z \equiv t_{r-1}) \wedge y_{r-1}\big).$$

Then, to get rid of the free variables u, s_k, t_k, and w, which stand for 0, a_k, b_k, and c_{ij}, respectively, we use the fact that their intended values are uniformly definable in linear models. Let φ' be the formula

$$\begin{aligned}
&\exists u \exists s_0 \exists s_1 \cdots \exists s_{r-1} \exists t_0 \exists t_1 \cdots \exists t_{r-1} \exists w\\
&\quad \big[\neg u \wedge \alpha_0(s_0) \wedge \alpha_1(s_1) \wedge \cdots \wedge \beta_0(t_0) \wedge \cdots \wedge \beta_{r-1}(t_{r-1}) \wedge \gamma_{ij}(w) \wedge \varphi^+\big].
\end{aligned}$$

It follows from the definition of an $\mathsf{SCI_Q}$-valuation for the quantifiers that, outside the scope of $\equiv$, the quantifiers work exactly as in first-order logic. So, $\mathcal{M}, v \models \varphi'$ if and only if $\mathcal{M}, v^+ \models \varphi^+$, where $v^+(u) = 0$; $v^+(w) = c_{ij}$; $v^+(s_k) = a_k$ and $v^+(t_k) = b_k$, for $k < r$; and $v^+(x) = v(x)$ for all other variables x. To determine the value $\|\varphi^+, v^+\|$, we first examine the values $\|\eta, (v^+)^m_z\|$ for each $m \in M$. For $m = 0$, we have $\|z \equiv u, (v^+)^0_z\| = N$, and therefore $\|\eta, (v^+)^0_z\| = N$, as disjunction corresponds to the maximum. As the values $a_0, a_1, \ldots, b_{r-1}$ are all distinct and nonzero, at most one of the clauses in η can be true for any given valuation. In particular,

$$\begin{aligned}\big\|\eta, (v^+)^{a_k}_z\big\| &= \big\|(z \equiv a_k) \wedge x_k, (v^+)^{a_k}_z\big\| = \min\big(N, \big\|x_k, (v^+)^{a_k}_z\big\|\big)\\ &= v^+(x_k) = v(x_k),\end{aligned}$$

for any $k < r$. Likewise, $\|\eta, (v^+)^{b_k}_z\| = v(y_k)$ for $k < r$. For the remaining values $m \in M$, none of the clauses of η are satisfied, so $\|\eta, (v^+)^m_z\| = 0$ for them. Hence, it holds for every $m \in M$ that $\|\eta, (v^+)^m_z\| = f_{v^*(x),v^*(y)}(m)$, and consequently

$$\|\exists z\eta, v^+\| = \bigoplus f_{v^*(x),v^*(y)} = F\big(v^*(x), v^*(y)\big).$$

It follows that

$$\begin{aligned}\mathcal{M}, v \models \varphi' &\Leftrightarrow \|\varphi', v\| \in D\\ &\Leftrightarrow \|\varphi^+, v^+\| \in D\\ &\Leftrightarrow \|\exists z\eta, v^+\| = \|w, v^+\|\\ &\Leftrightarrow F\big(v^*(x), v^*(y)\big) = c_{ij}\\ &\Leftrightarrow \mathcal{M}^*, v^* \models \varphi.\end{aligned}$$

Thus, the formula φ' preserves truth.

The rest is again simple, due to the fact that we do not need to pay attention to the actual values of the formulas, only to whether the values belong to D. The connectives are translated trivially:

$$\begin{aligned}(\neg\varphi)' &= \neg(\varphi'),\\ (\varphi \wedge \psi)' &= (\varphi') \wedge (\psi'),\\ (\varphi \vee \psi)' &= (\varphi') \vee (\psi'),\\ (\varphi \to \psi)' &= (\varphi') \to (\psi'),\\ (\varphi \leftrightarrow \psi)' &= (\varphi') \leftrightarrow (\psi').\end{aligned}$$

The quantifiers only need adjustment for the number of variables:

$$\begin{aligned}(\exists x\varphi)' &= \exists x_0 \exists x_1 \cdots \exists x_{r-1}\varphi',\\ (\forall x\varphi)' &= \forall x_0 \forall x_1 \cdots \forall x_{r-1}\varphi'.\end{aligned}$$ □

Now that we have a truth-preserving translation, we need to make sure the models we need actually exist.

Lemma 5.9 *For every L_T-model $\mathcal{N}$ such that $\mathcal{N} \models \psi$, where ψ is as in the proof of Lemma 5.7 above, and $|\mathcal{N}| = n^r$ for some $n \geq \max(2r, (p+1)(q+1)) + 1$, there is a linear $\mathsf{SCI_Q}$-model $\mathcal{M}$ such that $\mathcal{M}^* \cong \mathcal{N}$.*

Proof Let $M = \{0, 1, \dots, n-1\}$, and let $<$ be the natural ordering on M. Then $<^*$ is a well-defined linear ordering on M^r. Moreover, the elements a_i, b_i, and c_{ij} can be chosen as above. Let D, $\sqcup$, $\sqcap$, and $\circ$ be defined according to the definition of a linear model, and let $\sim 0 = n-1$. Furthermore, let $\mathcal{F} = {}^M M$. As $\mathcal{N} \models \mathsf{TC1} \wedge \mathsf{TC2}$, the universe of $\mathcal{N}$ is linearly ordered by $<^{\mathcal{N}}$. Moreover, $|\mathcal{N}| = n^r = |M^r|$. Hence, there is a unique order-preserving bijection $g\colon M^r \to \mathrm{dom}(\mathcal{N})$. Now, for any tuples $\bar{x}, \bar{y} \in M^r$, we define $\bigoplus f_{\bar{x},\bar{y}} = c_{ij}$ for the unique i, j such that $\mathcal{N} \models K_{ij}(g(\bar{x}), g(\bar{y}))$. For other functions $f\colon M \to M$, let

$$\bigoplus f = \begin{cases} 0, & \text{if } f(x) = 0 \text{ for all } x, \\ N, & \text{otherwise.} \end{cases}$$

Finally, for $f\colon M \to M$, let

$$\bigotimes f = \begin{cases} 0, & \text{if } f(x) = 0 \text{ for some } x, \\ N, & \text{otherwise.} \end{cases}$$

Now, $\mathcal{M} = (M, \sim, \sqcup, \sqcap, \circ, \mathcal{F}, \bigotimes, \bigoplus)$ is a linear $\mathsf{SCI_Q}$-model by construction, and it is straightforward to check that $\mathcal{M}^*$ is indeed isomorphic to $\mathcal{N}$. □

We still need another $\mathsf{SCI_Q}$-sentence, which cannot be expressed in L_T, to make sure the size of the input equals the size of the model.

Lemma 5.10 *There is an* $\mathsf{SCI_Q}$*-sentence* φ_s *that is true in a linear* $\mathsf{SCI_Q}$*-model* $\mathcal{M}$ *satisfying* φ_T *if and only if the size of the input coded by* $\mathcal{M}^*$ *equals the size of* $\mathcal{M}$.

Proof Let φ_s say that for each $m \in M$, there is exactly one r-tuple $\bar{y} = (y_0, y_1, \dots, y_{r-1})$ such that $y_{r-1} = m$ and $F(\bar{x}, \bar{y}) \neq c_{00}$ holds for the minimal possible $\bar{x}$. So, if $\mathcal{M} \models \varphi_s \wedge \varphi_T$, then at the beginning of the computation coded by $\mathcal{M}^*$, there are exactly $|\mathcal{M}|$ many cells c such that c is being scanned or contains a nonblank symbol. As $|\mathcal{M}| \geq 2$ for any $\mathsf{SCI_Q}$-model $\mathcal{M}$, the input cannot be empty, so the head is scanning a nonblank symbol and the length of the input equals $|\mathcal{M}|$. On the other hand, any $|\mathcal{M}|$ many consecutive r-tuples satisfy the given condition, so the converse implication holds as well.

To define φ_s explicitly, let first $\psi(y, z)$ be the L_T-formula

$$\exists x\big(\neg\exists w(w < x) \wedge \neg K_{00}(x, y) \wedge \big(\neg K_{00}(x, z) \to z = y\big)\big).$$

Now, let φ_s be the following $\mathsf{SCI_Q}$-sentence:

$$\forall y \exists y_0 \exists y_1 \cdots \exists y_{r-1} \forall z_0 \forall z_1 \cdots \forall z_{r-1}\big(y_{r-1} = y \wedge \big(z_{r-1} = y \to \psi'(\bar{y}, \bar{z})\big)\big),$$

where ψ' is as in Lemma 5.8 above. It is straightforward to check that φ_s is as required. □

The coding works only for large enough models. A finite number of exceptional cases can be handled directly.

Lemma 5.11 *Let* $B \subseteq \mathbb{N} \setminus \{0, 1\}$ *be finite. Then, there is an* $\mathsf{SCI_Q}$*-sentence* φ_B *such that for any* $\mathsf{SCI_Q}$*-model* $\mathcal{M}$*, it holds that* $\mathcal{M} \models \varphi_B$ *if and only if* $|\mathcal{M}| \in B$.

Proof For each $n \in B$, let φ_n be the sentence

$$\exists x_0 \exists x_1 \cdots \exists x_{n-1}\Big[\Big(\bigwedge_{i<j<n} \neg(x_i \equiv x_j)\Big) \wedge \forall y \bigvee_{i<n} (y \equiv x_i)\Big].$$

It is clear that $\mathcal{M} \models \varphi_n$ if and only if $|\mathcal{M}| = n$. Let now φ_B be the disjunction $\bigvee_{n \in B} \varphi_n$. □

Finally, we are able to prove the main theorem of this section.

Theorem 5.12 *A set $A \subseteq \mathbb{N} \setminus \{0, 1\}$ is the spectrum of an $\mathsf{SCI_Q}$-sentence if and only if A is the set of lengths of inputs accepted by a polynomial-time nondeterministic Turing machine.*

Proof The implication from left to right is Lemma 5.5. Assume then that A is accepted by a polynomial-time nondeterministic Turing machine T. Let $n \in A$ be large enough for the coding described above. Then, there is an L_T-model $\mathcal{N}$ such that $|\mathcal{N}| = n^r$ and $\mathcal{N}$ codes a computation of T with input of length n. By Lemma 5.9, there is a linear $\mathsf{SCI_Q}$-model $\mathcal{M}$ such that $\mathcal{M}^* \cong \mathcal{N}$. Hence, $\mathcal{M}^* \models \varphi_T$, and therefore $\mathcal{M} \models \varphi'_T$. Moreover, $|\mathcal{M}| = n$, so $\mathcal{M} \models \varphi_s$. Of course, $\mathcal{M} \models \varphi_L$.

On the other hand, let $\mathcal{M}$ be an $\mathsf{SCI_Q}$-model such that $\mathcal{M} \models \varphi'_T \wedge \varphi_s \wedge \varphi_L$. Then, $\mathcal{M}$ is linear, and hence $\mathcal{M}^*$ exists. Now, $\mathcal{M}^* \models \varphi_T$, which means that $\mathcal{M}^*$ codes an accepting computation of T. Furthermore, the size of the input equals $|\mathcal{M}|$. Thus, $|\mathcal{M}| \in A$.

Finally, let $B = \{n \in A \mid n \leq \max(2r, (p+1)(q+1))\}$. Then, B is finite, and hence, by Lemma 5.11, there is an $\mathsf{SCI_Q}$-sentence φ_B that is true in a model $\mathcal{M}$ whenever $\mathcal{M}$ is too small for the coding and $|\mathcal{M}| \in A$. Hence, the spectrum of $(\varphi'_T \wedge \varphi_s) \vee \varphi_B$ is exactly A. □

6 Translation Theorem for $\mathsf{SCI_Q}$ and Its Applications

In this section we show how to interpret $\mathsf{SCI_Q}$ as a theory of classical two-sorted first-order language. Many-sorted first-order logic is suitable for describing models that divide naturally into several different parts, such as linear spaces (vectors and scalars).

In a many-sorted language, every syntactically valid term has a unique sort, which determines how the term may be used. The sort of a variable symbol is inherent in the symbol. A predicate signature indicates the sorts of the arguments. A function signature indicates the sorts of the arguments and the sort of the value. For instance, a vector space has two different multiplications, of signatures $(S, V) \to V$ and $(S, S) \to S$ (scalar-by-vector and scalar-by-scalar, respectively). The first one may be applied to a scalar and a vector, in this order, resulting in a vector term. Trying to apply the scalar-by-vector multiplication operator to two scalar terms does not yield a syntactically valid term. Equality is defined for two arguments of the same sort. (A purist might prefer to have a separate equality symbol for each sort.) A quantifier can bind a variable of any sort.

A many-sorted model has one nonempty universe of each sort. The function symbols are interpreted as functions defined on the Cartesian product of the universes corresponding to the signature, and relation symbols as subsets of the appropriate Cartesian products. The syntactic restrictions ensure that every valid term has a value under a given interpretation.

In suitable contexts, many-sorted logic simplifies definitions by syntactically disallowing semantically meaningless expressions, such as the sum of a vector and a scalar, thereby avoiding the need to introduce a dummy element to represent an undefined value or to find another method to resolve the clash between the usual first-order formalism and the constraints of the application.

Note that the sort symbols do not need to appear explicitly in the many-sorted formulas; rather, they can be thought of as meta-level constructs affecting the syntax.

Now, we define a two-sorted first-order language appropriate to express the $\mathsf{SCI_Q}$-language and the translation between the languages. Next, we construct the class of first-order models that correspond to $\mathsf{SCI_Q}$-models, and then we prove a correspondence theorem about the translation.

The language $\mathsf{L}_{\mathsf{SCI_Q}}$ contains two sorts, I (individuals) and F (functions). The vocabulary of $\mathsf{L}_{\mathsf{SCI_Q}}$ consists of the symbols from the following pairwise disjoint sets: $\mathbb{IV} = \{x, y, z, \ldots\}$, a countably infinite set of individual variables; $\mathbb{FV} = \{f, g, h, \ldots\}$, a countably infinite set of function variables; $\{\neg^*, \bigotimes^*, \bigoplus^*\}$, the set of unary function symbols; $\{\vee^*, \wedge^*, \rightarrow^*, \leftrightarrow^*, \equiv^*, \mathcal{V}^*\}$, the set of binary function symbols; $\{=, D^*\}$, the set of predicate symbols, where $=$ is the equality predicate and D^* is a unary predicate; $\{\neg, \vee, \wedge, \rightarrow, \leftrightarrow\}$, the set of the classical propositional operations of negation, disjunction, conjunction, implication, and equivalence, respectively; and $\{\forall, \exists\}$, the set of quantifiers.

All arguments and values of the function and relation symbols are individuals, with the exceptions of $\bigotimes^*$, $\bigoplus^*$, and $\mathcal{V}^*$, with signatures $F \rightarrow I$, $F \rightarrow I$, and $(F, I) \rightarrow I$, respectively, and $=$, which is the standard exception, as explained above. The set of $\mathsf{L}_{\mathsf{SCI_Q}}$-formulas is defined in a standard way, subject to the constraints imposed by the sorts.

In what follows, we will identify the set $\mathbb{IV}$ of individual variables in $\mathsf{L}_{\mathsf{SCI_Q}}$ with the set $\mathbb{V}$ of propositional variables in $\mathsf{SCI_Q}$. $\mathsf{L}_{\mathsf{SCI_Q}}$-*models* are structures of the form

$$\mathcal{M} = \left(M, \sim, \sqcap, \sqcup, \Rightarrow, \Leftrightarrow, \circ, \mathcal{F}, \bigotimes, \bigoplus, \mathcal{V}, D\right),$$

where M and $\mathcal{F}$ are the universes corresponding to the sorts I and F, respectively; $\sim$, $\sqcap$, $\sqcup$, $\Rightarrow$, $\Leftrightarrow$, $\circ$, $\bigotimes$, $\bigoplus$, and $\mathcal{V}$ are the functions that interpret the function symbols $\neg^*$, $\wedge^*$, $\vee^*$, $\rightarrow^*$, $\leftrightarrow^*$, $\equiv^*$, $\bigotimes^*$, $\bigoplus^*$, and $\mathcal{V}^*$, respectively; and, moreover, D is a relation that interprets the predicate symbol D^*. The satisfaction and the truth of an $\mathsf{L}_{\mathsf{SCI_Q}}$-formula in an $\mathsf{L}_{\mathsf{SCI_Q}}$-model are defined in the standard way, as usual in first-order logics.

Most of the symbols we use to denote the functions and relations in an $\mathsf{L}_{\mathsf{SCI_Q}}$-model are the same that we use in the context of $\mathsf{SCI_Q}$-models, and their intended meanings are indeed the same. The lone exception is $\mathcal{V}$, which we use to denote the evaluation function. That is, intuitively $\mathcal{V}(f, x) = f(x)$ for a unary function f and an element x. (Technically, of course, we cannot express any such condition in $\mathsf{L}_{\mathsf{SCI_Q}}$.)

The fact that a quantification in $\mathsf{SCI_Q}$ does not correspond to any simple algebraic operation presents a slight technical difficulty, which we resolve by defining our translation from $\mathsf{SCI_Q}$ to $\mathsf{L}_{\mathsf{SCI_Q}}$ in two steps: first, we define inductively for each $\mathsf{SCI_Q}$-formula $\varphi(\bar{x})$ a quasitranslation $\varphi'(\bar{x}, y)$, which is really a truth-preserving translation of $\varphi(\bar{x}) = y$; then, a truth-preserving translation of φ itself is easy to define.

Let $\varphi(\bar{x})$ be an $\mathsf{SCI_Q}$-formula. An $\mathsf{L_{SCI_Q}}$-formula $\varphi'(\bar{x}, y)$ is said to be a *quasitranslation* of $\varphi(\bar{x})$ for $\varphi'(\bar{x}, y)$, defined inductively as

$$
\begin{aligned}
(x)' &\stackrel{\text{df}}{=} (y = x), \quad \text{for any propositional variable } x \in \mathbb{V},\\
\big(\neg\varphi(\bar{x})\big)' &\stackrel{\text{df}}{=} \exists z\big[\varphi'(\bar{x}, z) \wedge y = \neg^* z\big],\\
\big(\varphi(\bar{x}) \vee \psi(\bar{z})\big)' &\stackrel{\text{df}}{=} \exists w \exists u\big[\varphi'(\bar{x}, w) \wedge \psi'(\bar{z}, u) \wedge y = (w \vee^* u)\big],\\
\big(\varphi(\bar{x}) \wedge \psi(\bar{z})\big)' &\stackrel{\text{df}}{=} \exists w \exists u\big[\varphi'(\bar{x}, w) \wedge \psi'(\bar{z}, u) \wedge y = (w \wedge^* u)\big],\\
\big(\varphi(\bar{x}) \to \psi(\bar{z})\big)' &\stackrel{\text{df}}{=} \exists w \exists u\big[\varphi'(\bar{x}, w) \wedge \psi'(\bar{z}, u) \wedge y = (w \to^* u)\big],\\
\big(\varphi(\bar{x}) \leftrightarrow \psi(\bar{z})\big)' &\stackrel{\text{df}}{=} \exists w \exists u\big[\varphi'(\bar{x}, w) \wedge \psi'(\bar{z}, u) \wedge y = (w \leftrightarrow^* u)\big],\\
\big(\varphi(\bar{x}) \equiv \psi(\bar{z})\big)' &\stackrel{\text{df}}{=} \exists w \exists u\big[\varphi'(\bar{x}, w) \wedge \psi'(\bar{z}, u) \wedge y = (w \equiv^* u)\big],\\
\big(\forall z \varphi(\bar{x}, z)\big)' &\stackrel{\text{df}}{=} \exists f\Big[\forall w \varphi'\big(\bar{x}, w, \mathcal{V}^*(f, w)\big) \wedge y = \bigotimes\nolimits^* f\Big],\\
\big(\exists z \varphi(\bar{x}, z)\big)' &\stackrel{\text{df}}{=} \exists f\Big[\forall w \varphi'\big(\bar{x}, w, \mathcal{V}^*(f, w)\big) \wedge y = \bigoplus\nolimits^* f\Big].
\end{aligned}
$$

An $\mathsf{L_{SCI_Q}}$-formula φ^* is said to be a *translation of* $\varphi(\bar{x})$ whenever

$$\varphi^*(\bar{x}) \stackrel{\text{df}}{=} \exists y\big[\varphi'(\bar{x}, y) \wedge D^*(y)\big].$$

The *non-Fregean theory* in $\mathsf{L_{SCI_Q}}$-language, NF-theory for short, is the $\mathsf{L_{SCI_Q}}$-theory with the following axioms:

(NF1) $\forall x[D^*(\neg^* x) \leftrightarrow \neg D^*(x)]$,
(NF2) $\forall x \forall y[D^*(x \vee^* y) \leftrightarrow (D^*(x) \vee D^*(y))]$,
(NF3) $\forall x \forall y[D^*(x \wedge^* y) \leftrightarrow (D^*(x) \wedge D^*(y))]$,
(NF4) $\forall x \forall y[D^*(x \to^* y) \leftrightarrow (D^*(x) \to D^*(y))]$,
(NF5) $\forall x \forall y[D^*(x \leftrightarrow^* y) \leftrightarrow (D^*(x) \leftrightarrow D^*(y))]$,
(NF6) $\forall x \forall y[D^*(x \equiv^* y) \leftrightarrow x = y]$,
(NF7) $\forall f[D^*(\bigotimes^* f) \leftrightarrow (\forall x D^*(\mathcal{V}^*(f, x)))]$,
(NF8) $\forall f[D^*(\bigoplus^* f) \leftrightarrow (\exists x D^*(\mathcal{V}^*(f, x)))]$,
(NF9) $\forall f, g[\forall x(\mathcal{V}^*(f, x) = \mathcal{V}^*(g, x)) \to f = g]$,
(NF10) $\forall \bar{x} \exists f \forall z \psi'(\bar{x}, z, \mathcal{V}^*(f, z))$, for every $\mathsf{SCI_Q}$-formula $\psi(\bar{x}, z)$.

Let $\mathcal{N} = (N, \sim, \sqcap, \sqcup, \Rightarrow, \Leftrightarrow, \circ, \mathcal{F}, \bigotimes, \bigoplus, \mathcal{V}, D)$ be an $\mathsf{L_{SCI_Q}}$-model of the NF-theory, and let $f \in \mathcal{F}$. We define

$$f^-(t) \stackrel{\text{df}}{=} \mathcal{V}(f, t), \text{for all } t \in N, \bigotimes\nolimits^- f^- \stackrel{\text{df}}{=} \bigotimes f, \bigoplus\nolimits^- f^- \stackrel{\text{df}}{=} \bigoplus f.$$

Claim 6.1 *For an* $\mathsf{L_{SCI_Q}}$*-model* $\mathcal{N} = (N, \sim, \sqcap, \sqcup, \Rightarrow, \Leftrightarrow, \circ, \mathcal{F}, \bigotimes, \bigoplus, \mathcal{V}, D)$ *of the* NF*-theory, the structure* $\mathcal{N}^- = (N, \sim, \sqcap, \sqcup, \Rightarrow, \Leftrightarrow, \circ, \mathcal{F}^-, \bigotimes^-, \bigoplus^-, D)$ *is a partial* $\mathsf{SCI_Q}$*-model, where* $\mathcal{F}^- \stackrel{\text{df}}{=} \{f^- \mid f \in \mathcal{F}\}$.

Indeed, by axioms (NF1)–(NF6), the operations $\sim$, $\sqcap$, $\sqcup$, $\Rightarrow$, $\Leftrightarrow$, $\circ$, and the relation D have to be defined so that $(N, \sim, \sqcap, \sqcup, \Rightarrow, \Leftrightarrow, \circ, D)$ is an SCI-model. From the definitions of the functions $\bigotimes^-$ and $\bigoplus^-$, and axioms (NF7)–(NF8), it follows that these functions are well defined and satisfy the conditions required of partial $\mathsf{SCI_Q}$-models.

For a valuation v on an $\mathsf{L_{SCI_Q}}$-model $\mathcal{N}$, we define a valuation v^- on $\mathcal{N}^-$ as $v^-(x) = v(x)$, for all variables $x \in \mathbb{V}$.

Lemma 6.2 *Let φ be an* $\mathsf{SCI_Q}$*-formula, let $\mathcal{N}$ be an* $\mathsf{L_{SCI_Q}}$*-model of the* NF*-theory, and let v be a valuation on $\mathcal{N}$. Then*

(i) $\|\varphi, v^-\|$ *is defined in* $\mathcal{N}^-$*, and*
(ii) $\mathcal{N}, v \models \varphi'(\bar{x}, y)$ *if and only if* $\mathcal{N}^-, v^- \models (y \equiv \varphi(\bar{x}))$.

Proof Let $\bar{a} = v(\bar{x}), b = v(y)$. We prove (i) and (ii) simultaneously by induction on the complexity of the formula φ. For an atomic formula $\varphi(x) := x$, we have

$$\begin{aligned} \mathcal{N}^-, v^- \models y \equiv \varphi(x) \quad & \text{iff} \quad \big\|(y \equiv x), v^-\big\| \in D \\ & \text{iff} \quad b = a \\ & \text{iff} \quad \mathcal{N}, v \models (y = x) \\ & \text{iff} \quad \mathcal{N}, v \models \varphi'(x, y). \end{aligned}$$

Assume (i) and (ii) hold for $\psi(\bar{x})$ and $\theta(\bar{x})$. By way of example, we will show that they hold for formulas $\neg\psi(\bar{x})$, $\psi(\bar{x}) \wedge \theta(\bar{x})$, and $\forall z \psi(\bar{x}, z)$.

$$\begin{aligned} \mathcal{N}^-, v^- \models y \equiv \neg\psi(\bar{x}) \quad & \text{iff} \quad \|(y \equiv \neg\psi(\bar{x})), v^-\| \in D \\ & \text{iff} \quad b = {\sim}\|\psi(\bar{x}), v^-\| \\ & \text{iff} \quad \text{there is } c \in N \text{ such that } c = \|\psi(\bar{x}), v^-\| \text{ and} \\ & \qquad b = {\sim}c \\ & \text{iff} \quad \mathcal{N}, v \models \exists z[\psi'(\bar{x}, z) \wedge y = \neg^* z] \\ & \text{iff} \quad \mathcal{N}, v \models \varphi'(\bar{x}, y). \end{aligned}$$

$$\begin{aligned} \mathcal{N}^-, v^- \models y \equiv \psi(\bar{x}) \wedge \theta(\bar{x}) \quad & \text{iff} \quad \|(y \equiv \psi(\bar{x}) \wedge \theta(\bar{x})), v^-\| \in D \\ & \text{iff} \quad b = (\|\psi(\bar{x}), v^-\| \sqcap \|\theta(\bar{x}), v^-\|) \\ & \text{iff} \quad \text{there are } c, d \in N \text{ such that } c = \|\psi(\bar{x}), v^-\|, \\ & \qquad d = \|\theta(\bar{x}), v^-\|, \text{ and } b = c \sqcap d \\ & \text{iff} \quad \mathcal{N}, v \models \exists z \exists w[\psi'(\bar{x}, z) \wedge \theta'(\bar{x}, w) \wedge y = (z \\ & \qquad \wedge^* w)] \\ & \text{iff} \quad \mathcal{N}, v \models \varphi'(\bar{x}, y). \end{aligned}$$

Now, let f be the function determined by the formula $\psi(\bar{x}, z)$ and valuation v^-. Then

$$\mathcal{N}^-, v^- \models y \equiv \forall z \psi(\bar{x}, z) \quad \text{iff} \quad \big\|\big(y \equiv \forall z \psi(\bar{x}, z)\big), v^-\big\| \in D \quad \text{iff} \quad b = \bigotimes f.$$

Assume $b = \bigotimes f$. By (NF10), there is $g \in \mathcal{F}$ such that for all $c \in N$, the formula $\psi'(\bar{x}, c, \mathcal{V}^*(g, c))$ is true in the model $\mathcal{N}$ and $b = \bigotimes^* g$. Thus, the formula $\exists f[\forall z \psi'(\bar{x}, z, \mathcal{V}^*(f, z)) \wedge y = \bigotimes^* f]$ is satisfied by v in $\mathcal{N}$; hence $\mathcal{N}, v \models \varphi'(\bar{x}, y)$. So there must be $f \in \mathcal{F}$ such that for all $c \in N$, the formula $\psi'(\bar{x}, c, \mathcal{V}^*(f, c)) \wedge y = \bigotimes^* f$ is satisfied in a model $\mathcal{N}$ by v.

Let $g(c) = \mathcal{V}^*(f, c)$, for all $c \in N$. Then, for every $c \in N$, the formula $\psi'(\bar{x}, c, g(c))$ is satisfied in $\mathcal{N}$ by v. Thus, for every $c \in N$, the formula $g(c) \equiv \psi(\bar{x}, c)$ is satisfied in $\mathcal{N}^-$ by v^-; that is, $g = f$. Since the formula $y = \bigotimes^* f$ is satisfied in $\mathcal{N}$, $b = \bigotimes f$ holds in $\mathcal{N}$. Therefore, the formula $y \equiv \forall z \psi(\bar{x}, z)$ is satisfied in $\mathcal{N}^-$ by v^-. □

Theorem 6.3 (Translation theorem) *For every* $\mathsf{SCI_Q}$*-formula* φ*, for every model* $\mathcal{N}$ *of the* NF*-theory, and for every valuation* v *on* $\mathcal{N}$*, the following holds:*

$$\mathcal{N}, v \models \varphi^* \quad \textit{if and only if} \quad \mathcal{N}^-, v^- \models \varphi.$$

Proof Let φ be an $\mathsf{SCI_Q}$-formula, let $\mathcal{N}$ be a model of NF-theory, and let v be a valuation on $\mathcal{N}$. Then, we have

$$\mathcal{N}, v \models \varphi^*(\bar{x}) \quad \text{if and only if} \quad \mathcal{N}, v \models \exists y\big[\varphi'(\bar{x}, y) \wedge D^*(y)\big].$$

Thus, $\mathcal{N}, v \models \varphi^*(\bar{x})$ holds if and only if there exists $c \in N$ such that the formula $\varphi'(\bar{x}, c) \wedge D^*(c)$ is satisfied in a model $\mathcal{N}$ by v. By Lemma 6.2, the latter holds if and only if there exists $c \in N$ such that $\mathcal{N}^-$ satisfies $c \circ \|\varphi(\bar{x})\| \in D$ and $c \in D$, which is equivalent to $\mathcal{N}^-, v^- \models \varphi(\bar{x})$. □

Given an $\mathsf{SCI_Q}$-model $\mathcal{M} = (M, \sim, \sqcap, \sqcup, \Rightarrow, \Leftrightarrow, \circ, \mathcal{F}, \bigotimes, \bigoplus, D)$, an $\mathsf{L}_{\mathsf{SCI_Q}}$-*structure determined by* $\mathcal{M}$ is of the form

$$\mathcal{M}^* = \Big(M, \sim, \sqcap, \sqcup, \Rightarrow, \Leftrightarrow, \circ, \mathcal{F}, \bigotimes, \bigoplus, \mathcal{V}, D\Big)$$

such that $\mathcal{V}\colon \mathcal{F} \times M \to M$ is a function that satisfies $\mathcal{V}(f, t) = f(t)$, for all $f \in \mathcal{F}$, $t \in M$. Clearly, for every $\mathsf{SCI_Q}$-model $\mathcal{M}$, the structure $\mathcal{M}^*$ is an $\mathsf{L}_{\mathsf{SCI_Q}}$-model, as the following fact states.

Fact 6.4 For every $\mathsf{SCI_Q}$-model $\mathcal{M}$, the structure $\mathcal{M}^*$ is an $\mathsf{L}_{\mathsf{SCI_Q}}$-model of NF-theory and $(\mathcal{M}^*)^- = \mathcal{M}$.

Thus, by Theorem 6.3 we get the following.

Corollary 6.5 *For any* $\mathsf{SCI_Q}$*-formula* φ*,* $\mathsf{SCI_Q}$*-model* $\mathcal{M}$*, and valuation* v *on* $\mathcal{M}^*$*, the following holds:*

$$\mathcal{M}^*, v \models \varphi^* \quad \textit{if and only if} \quad \mathcal{M}, v^- \models \varphi.$$

We will apply the translation theorem to prove the Löwenheim–Skolem theorem for the logic $\mathsf{SCI_Q}$.

Theorem 6.6 (Löwenheim–Skolem theorem) *Let* T *be an* $\mathsf{SCI_Q}$*-theory. If there is an infinite* $\mathsf{SCI_Q}$*-model for* T*, then* T *has an* $\mathsf{SCI_Q}$*-model of every cardinality* $\geq \omega$*.*

Proof Let T be an $\mathsf{SCI_Q}$-theory, and let $\mathcal{M}$ be an infinite $\mathsf{SCI_Q}$-model for T. By Fact 6.4, the model $\mathcal{M}^*$ is a model for T^*, where T^* is an $\mathsf{L}_{\mathsf{SCI_Q}}$-theory over translations of theorems from T. Clearly, $\mathcal{M}^*$ is infinite. Let $\kappa \geq \omega$. By the classical results from model theory for first-order logic, there is a model $\mathcal{N}$ of T^* such that N is of cardinality κ. Moreover, $\mathcal{N}$ is a model of NF-theory. By the translation theorem (Theorem 6.3), $\mathcal{N}^-$ is a model for T. The model $\mathcal{N}^-$ has the same size as $\mathcal{N}$, so T has models of every infinite cardinality. □

Furthermore, the translation theorem implies the following.

Proposition 6.7 *If two models of* NF*-theory* $\mathcal{N}_1$ *and* $\mathcal{N}_2$ *are elementarily equivalent, then* $\mathsf{SCI_Q}$*-models* $\mathcal{N}_1^-$ *and* $\mathcal{N}_2^-$ *are also elementarily equivalent.*

Therefore, due to the above proposition, properties of elementarily equivalent NF-models can be transferred on corresponding $\mathsf{SCI_Q}$-models. In particular, if a property cannot be expressed in $\mathsf{L}_{\mathsf{SCI_Q}}$-language, then the same holds for $\mathsf{SCI_Q}$-language. Hence, the translation theorem can be seen as a tool for showing unexpressibility results for $\mathsf{SCI_Q}$.

7 Conclusions and Open Problems

In the paper we discussed some properties of the non-Fregean propositional logic with quantifiers. We showed that it is much more expressive than the basic non-Fregean propositional logic SCI, as it can express infiniteness and many mathematical first-order theories. We proved the undecidability of SCI_Q and an exact correspondence between SCI_Q and NP. We characterized a two-sorted first-order theory NF appropriate for expressing SCI_Q, and we proved the translation theorem which states the relationship between NF and SCI_Q. Finally, we showed how the translation theorem can be used to prove the Löwenheim–Skolem theorem for SCI_Q. In view of these results, some natural questions arise, in particular:

1. Are there other results of classical model theory that can be transferred to SCI_Q?
2. Is it decidable whether a given finite partial SCI_Q-model is an SCI_Q-model? If yes, what is the complexity of the decision problem?
3. Can generalized spectra be characterized computationally?

   Our method of coding a computation relies heavily on the concept of a linear model, whose SCI-structure is almost completely determined by its size. Can an arbitrary SCI-structure and a subsequent computation be coded entirely in terms of the quantifiers?
4. Does SCI_Q have weaker fragments which are stronger than SCI?

Potentially interesting fragments of SCI_Q could be obtained by allowing only one of the two quantifiers. In this way, we get two logics, $\mathsf{SCI}_\mathsf{Q}^\forall$ with the universal quantifiers only, and $\mathsf{SCI}_\mathsf{Q}^\exists$ with the existential quantifier only. Of course, as in the case of SCI_Q, both of these logics obey those classical quantifier laws that do not involve the identity connective. However, these logics may differ with respect to some other laws and/or definable properties. We conjecture that the logics $\mathsf{SCI}_\mathsf{Q}^\forall$ and $\mathsf{SCI}_\mathsf{Q}^\exists$ are *different* and that they are both proper fragments of SCI_Q stronger than SCI.

Note

1. For instance, $\Box\varphi$ can be treated as a shorthand notation for $\varphi \equiv (p \vee \neg p)$ in a non-Fregean formulation of a modal logic. Then, in order to obtain an extension of SCI equivalent with a modal logic S5 we expand SCI-axiomatization with axioms expressing Boolean laws and a rule corresponding to the necessitation modal rule (see Suszko [9] for details). On the other hand, an interpretation of situations of an SCI-model as logical values will lead to a non-Fregean reinterpretation of finite many-valued logics (see, e.g., Malinowski [7]).

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