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Semantic Completeness of First-Order Theories in Constructive Reverse Mathematics

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Abstract We introduce a general notion of semantic structure for first-order theories, covering a variety of constructions such as Tarski and Kripke semantics, and prove that, over Zermelo–Fraenkel set theory (ZF), the completeness of such semantics is equivalent to the Boolean prime ideal theorem (BPI). Using a result of McCarty (2008), we conclude that the completeness of Kripke semantics is equivalent, over intuitionistic Zermelo–Fraenkel set theory (IZF), to the Law of Excluded Middle plus BPI. Along the way, we also prove the equivalence, over ZF, between BPI and the completeness theorem for Kripke semantics for both first-order and propositional theories.

In Henkin [8], the equivalence between the compactness theorem for classical theories and the Boolean prime ideal theorem (BPI) is discussed (see, also Jech [9]). As a consequence, the completeness for such theories is also equivalent to BPI, while, on the other hand, the soundness theorem can be easily proved in Zermelo–Fraenkel set theory ZF by induction on the complexity of the formulas. As we shall see in this note, the fact that the completeness theorem supposes an unavoidable use of a choice principle is not only a feature of classical semantics but also of intuitionistic semantics, and in fact of any semantic structure that is sound and satisfies some reasonable property that we will consider below. This condition puts reasonable restrictions on the interpretations of the connectives, but as we shall see, also conveys to such semantics an intrinsic choice appeal.

That completeness theorems for intuitionistic theories imply nonconstructive principles has been known since 1957, when Gödel derived Markov's principle from the assumption of completeness (see this and related ideas in Kreisel [11], McCarty [13], [14], Carter [5], [6]). Our result will focus on the completeness for the general

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case, in which there is a priori no restriction on the cardinality of the signatures of the theories in question and will provide the exact strength of the completeness statement in the sense of constructive reverse mathematics. Indeed, in [15], McCarty proves, among other things, that a form of completeness of Kripke semantics (even for theories restricted to the negative fragment) entails the Law of Excluded Middle (LEM). It will follow from this that over intuitionistic Zermelo–Fraenkel set theory IZF (an account of this theory can be found, e.g., in Myhill [17]), such form of completeness is precisely equivalent to LEM plus BPI.

It is not unusual to combine intuitionistically invalid logical principles such as LEM with choice principles such as BPI. In fact, such logical principles can be seen as restricted choice principles in constructive set theories, as explained, for instance, in Bell [2]. For example, as proven is Bell [3], BPI implies (constructively) the Law of Testability, which is the schema $\neg\varphi \vee \neg\neg\varphi$.

In what follows, we will assume that our theories are based on intuitionistic logic. Given a theory Γ over a language $\mathcal{L}$ (containing only closed formulas), we consider the set $\mathcal{F}$ of closed formulas over $\mathcal{L}$ and associate a semantics for Γ as follows.

Definition 1 A semantic structure for Γ is a pair $(\mathcal{C}, \vDash)$, where $\mathcal{C}$ is a nonempty class of sets, called models, and $\vDash$ is a relation in $\mathcal{C} \times \mathcal{F}$ such that, for every $\mathcal{M} \in \mathcal{C}$ and $\psi \in \Gamma$ we have $\mathcal{M} \vDash \psi$, and furthermore, the following two properties are satisfied.

1. Soundness: If $\Delta \vdash \varphi$ and $\mathcal{M} \vDash \psi$ for every $\psi \in \Delta$, then $\mathcal{M} \vDash \varphi$.
2. Consistency property: If $\mathcal{M} \vDash \varphi \vee \neg\varphi$, then either $\mathcal{M} \vDash \varphi$ or $\mathcal{M} \vDash \neg\varphi$.

The first example arises, of course, when Γ contains all instances of the principle of excluded middle (hence resulting in a classical theory), and $\vDash$ is interpreted as the usual set-valued semantics for first-order theories. In that case the class $\mathcal{C}$ can be taken to be the class of ordinary set-valued models.

A second example is given by taking $\mathcal{C}$ to be the class of pairs (M_Γ, p), where M_Γ is a nonempty Kripke model of Γ and p is a fixed node. Here the satisfaction relation is $\Vdash_p$, the forcing at node p (for an account of Kripke models, see, e.g., Troelstra and van Dalen [19]).

There are several notions of completeness for the semantic structures just defined; these are classically equivalent but are constructively different. Some of these completeness notions are described in sources such as Dummett [7] and Troelstra [18], to which the reader is referred for further details. The notion we will work with is contained in the following.

Definition 2 A semantic structure $(\mathcal{C}, \vDash)$ for Γ is said to be complete if whenever $\mathcal{C} \vDash \varphi$ (i.e., whenever $\mathcal{M} \vDash \varphi$ for each $\mathcal{M} \in \mathcal{C}$), then $\Gamma \vdash \varphi$.

Note that the previous definition of completeness classically implies the Model Existence Lemma, asserting that every consistent theory Γ has a model $\mathcal{M} \in \mathcal{C}$; that is, there exists $\mathcal{M} \in \mathcal{C}$ such that $\mathcal{M} \vDash \psi$ for every $\psi \in \Gamma$. Therefore, whenever we have LEM available, we will make free use of the second form, since it follows from the first. Moreover, when working in ZF, we will implicitly assume that $\mathcal{C}$ is a set.

We state now our main result.

Theorem 3 *Over Zermelo–Fraenkel set theory, the following are equivalent:*

1. *the Boolean prime ideal theorem;*
2. *every consistent first-order theory has a complete semantic structure.*

Proof $(1 \implies 2)$ We may assume that open formulas of the theory are replaced by their universal closure. Consider the (intuitionistic) Lindenbaum–Tarski algebra of the theory, that is, the Heyting algebra $(\mathcal{H}, \leq)$ of equivalence classes of sentences ordered by entailment. We will define a semantic structure that is complete with respect to the theory. For this purpose, define $\mathcal{C}$ as the set of prime filters of $\mathcal{H}$ (which is nonempty, by BPI), and for each $\mathcal{M} \in \mathcal{C}$ define $\mathcal{M} \vDash \varphi \leftrightarrow \varphi \in \mathcal{M}$. It follows that $(\mathcal{C}, \vDash)$ is a semantic structure. To see that it is complete, note that any valid sentence belongs to every prime filter, and hence, by the prime separation theorem for distributive lattices (which is provable from BPI; see Banaschewski [1]), it has to be provably true.

$(2 \implies 1)$ Let $\mathcal{L}$ be a nontrivial distributive lattice, and consider the theory Γ over a language which has a constant for every element of $\mathcal{L}$ (we shall identify such elements with the constants themselves), a unary relation F ($F(a)$ is thought of as the assertion "a is in the filter"), and whose axioms are the following:

1. $\top \to F(1)$;
2. $F(a) \to F(b)$ for each pair $a \leq b$ in $\mathcal{L}$;
3. $F(a) \wedge F(b) \to F(a \wedge b)$ for every pair a, b in $\mathcal{L}$;
4. $F(a \vee b) \to F(a) \vee F(b)$ for each pair a, b in $\mathcal{L}$;
5. $F(0) \to \bot$.

Since this theory is finitely satisfiable (because every finite subset of $\mathcal{L}$ generates a finite sublattice where one can construct a prime filter), it is consistent, and hence by hypothesis there is a model $\mathcal{M}$ with a satisfaction relation $\vDash$. Consider the (intuitionistic) Lindenbaum–Tarski algebra $\mathcal{H}$ of Γ. Let $\mathcal{B}$ be the Boolean algebra of all complemented objects of $\mathcal{H}$; that is, its objects are (equivalence classes of) sentences φ such that $\Gamma \vdash \varphi \vee \neg\varphi$. Define now $\mathcal{U} = \{\varphi \in \mathcal{B} : \mathcal{M} \vDash \varphi\}$. It is easy to prove that $\mathcal{U}$ is an ultrafilter of $\mathcal{B}$. Indeed, $\top \in \mathcal{U}$ since $\mathcal{M} \vDash \top$ by soundness, and $\bot \notin \mathcal{U}$ since $\mathcal{M} \nvDash \bot$ by the consistency property. If φ and ψ are in $\mathcal{U}$, then $\mathcal{M} \vDash \varphi$ and $\mathcal{M} \vDash \psi$, so by soundness we have $\mathcal{M} \vDash \varphi \wedge \psi$ and hence $\varphi \wedge \psi$ belongs to $\mathcal{U}$. If $\varphi \vdash_\Gamma \psi$ and φ is in $\mathcal{U}$, then $\Gamma, \varphi \vdash \psi$, and since $\mathcal{M} \vDash \varphi$, by soundness we get $\mathcal{M} \vDash \psi$ and so ψ is in $\mathcal{U}$. Finally, since for every φ in $\mathcal{B}$ we have $\Gamma \vdash \varphi \vee \neg\varphi$, then by soundness and the consistency property we have that either $\mathcal{M} \vDash \varphi$ or $\mathcal{M} \vDash \neg\varphi$, that is, either φ or $\neg\varphi$ is in $\mathcal{U}$.

We define now a two-valued model of Γ, that is, a 0–1 valuation V compatible in the usual sense with $\vee$, $\wedge$, $\to$, $\top$ and $\bot$. This will give a prime filter in $\mathcal{L}$ as the set $\{a \in \mathcal{L} : V(F(a)) = 1\}$, and so we will prove that every distributive lattice has an prime filter, which is a well-known equivalent of the Boolean prime ideal theorem. Define $V(\varphi) = 1$ if and only if there is some ψ in $\mathcal{U}$ such that $\psi \vdash_\Gamma \varphi$. To prove this is an adequate valuation, it is enough to show the following four conditions:

1. $V(\bot) = 0$ and $V(\top) = 1$;
2. $\Gamma \vdash \varphi \to \psi$ and $V(\varphi) = 1$ implies $V(\psi) = 1$;
3. $V(\varphi \wedge \psi) = 1$ if and only if $V(\varphi) = 1$ and $V(\psi) = 1$;
4. $V(\varphi \vee \psi) = 1$ if and only if $V(\varphi) = 1$ or $V(\psi) = 1$.

Condition 1 follows immediately from the fact that $\bot \notin \mathcal{U}$ and $\top \in \mathcal{U}$.

Condition 2 is proved as follows: that $V(\varphi) = 1$ implies that $\eta \vdash_\Gamma \varphi$ for some η in $\mathcal{U}$. Since $\varphi \vdash_\Gamma \psi$, then $\eta \vdash_\Gamma \psi$ and hence $V(\psi) = 1$.

For condition 3, assume that $V(\varphi \wedge \psi) = 1$. Then for some η in $\mathcal{U}$ we have $\eta \vdash_\Gamma \varphi \wedge \psi$, and hence both $\eta \vdash_\Gamma \varphi$ and $\eta \vdash_\Gamma \psi$; that is, φ and ψ are in $\mathcal{U}$. Conversely, if $V(\varphi) = 1$ and $V(\psi) = 1$, there are η, δ in $\mathcal{U}$ such that $\eta \vdash_\Gamma \varphi$ and $\delta \vdash_\Gamma \psi$. But then $\eta \wedge \delta$ is in $\mathcal{U}$ and $\eta \wedge \delta \vdash_\Gamma \varphi \wedge \psi$, so $V(\varphi \wedge \psi) = 1$.

Let us look at condition 4. If $V(\varphi \vee \psi) = 1$, then for some η in $\mathcal{U}$ we have $\eta \vdash_\Gamma \varphi \vee \psi$. Now $\eta \wedge \varphi$ is complemented (by $\eta \wedge \psi \vee \neg\eta$), and similarly $\eta \wedge \psi$ is complemented (by $\eta \wedge \varphi \vee \neg\eta$). If $\eta \wedge \varphi$ is in $\mathcal{U}$, then $V(\varphi) = 1$, since $\eta \wedge \varphi \vdash_\Gamma \varphi$. Otherwise, $\neg(\eta \wedge \varphi) = \eta \wedge \psi \vee \neg\eta$ is in $\mathcal{U}$, and hence so is $\eta \wedge (\eta \wedge \psi \vee \neg\eta) = \eta \wedge \psi$. Then $V(\psi) = 1$, since $\eta \wedge \psi \vdash_\Gamma \psi$. Conversely, if $V(\varphi) = 1$ or $V(\psi) = 1$, then for some η, η' in $\mathcal{U}$ we have either $\eta \vdash_\Gamma \varphi$ or $\eta' \vdash_\Gamma \psi$, and hence either $\eta \vdash_\Gamma \varphi \vee \psi$ or $\eta' \vdash_\Gamma \varphi \vee \psi$. In either case, $V(\varphi \vee \psi) = 1$. This concludes the proof. □

Remark 4 Henkin's result can now be seen as a particular instance of Theorem 3 applied to the semantic structure considered in our first example, and taking into consideration that completeness is equivalent to the model existence formulation of the theorem. The proof above, however, actually shows a stronger result, namely, that it is enough to consider completeness of nonclassical theories to derive BPI.

Remark 5 The case of propositional theories can be thought of as included in the first-order case by considering 0-ary relations as propositions. The theory of prime filters can then be reformulated by replacing $F(a)$ with a proposition F_a.

Theorem 3 proves that whenever there is a semantic structure that is complete, we can derive BPI. This can be applied to Kripke semantics of our second example.

Corollary 6 *Over Zermelo–Fraenkel set theory, the following are equivalent:*

1. *the Boolean prime ideal theorem;*
2. *every consistent first-order theory has a nonempty Kripke model.*

Proof For the implication $1 \implies 2$ we refer to [12] for the proof of Joyal's completeness theorem from the Boolean prime ideal theorem. Consider a consistent first-order theory Γ, and let $\mathcal{P}$ be the set of finite composable sequences of embeddings $M_1 \to \cdots \to M_n$ between coherent models of Γ of a certain bounded cardinality. (This can be decided depending on the cardinality of the signature, and is done to avoid a proper class of models.) This set $\mathcal{P}$ has a partial order where one sequence is below another if the first is an initial section of the latter. Then one can define a Kripke model over $\mathcal{P}$ by assigning to each $M_1 \to \cdots \to M_n \in \mathcal{P}$ the underlying set S_n of M_n, and setting for each $a \in S_n$, $(M_1 \to \cdots \to M_n) \Vdash \varphi(a)$ if and only if M_n forces $\varphi(a)$ in the presheaf semantics (an account of such semantics can be found, e.g., in [16]). It follows that the defined Kripke model forces the axioms of the theory under consideration and is nonempty due to the classical completeness theorem for coherent theories, which follows from the Boolean prime ideal theorem (see Makkai and Reyes [12] or Johnstone [10] for details).

Conversely, to prove that $2 \implies 1$ one can use Theorem 3 and the completeness of Kripke semantics. There is, however, a shorter proof in this case; starting with the axioms of the theory of prime filters as exposed in the proof of Theorem 3 and given a Kripke model of that theory with a node p, one can simply define a prime filter in $\mathcal{L}$ as $\{a \in \mathcal{L} : p \Vdash F(a)\}$. □

Remark 7 Again, the case of propositional theories can be thought as included in the first-order case. The first part of the proof above can be adapted by defining $\mathcal{P}$ as the poset of prime filters of the Heyting algebra of propositional formulas, while a node F in the resulting Kripke model forces a formula φ if and only if $\varphi \in F$. The second part of the proof above goes through by just replacing $F(a)$ with a proposition F_a in the theory of prime filters.

Remark 8 It follows, in particular, that there can be no proof in IZF of the completeness of semantical interpretations of first-order theories as long as such semantics satisfy Definition 1. Indeed, since BPI is known to be independent of Zermelo–Fraenkel set theory (see, e.g., [9]), Theorem 3 shows that the existence of a complete semantic structure (or the completeness of Kripke semantics) for a given first-order theory cannot be derived in IZF, for it is known that ZF is equivalent to IZF plus LEM, and so, if the statement in question S was derivable in IZF, we would have ZF=IZF+LEM+S=ZF+BPI, which is absurd.

Corollary 9 *Over IZF, the completeness of Kripke semantics (even for theories restricted to the negative fragment) is equivalent to LEM+BPI.*

Proof One implication (from right to left) follows from Joyal's completeness theorem using that IZF+LEM=ZF. The other direction follows immediately from Theorem 6 and the result of McCarty [15] by which the completeness of Kripke semantics entails LEM. □

Remark 10 Note that over IZF, none of the statements LEM and BPI imply each other; that LEM does not imply BPI follows from the fact that the latter is independent from ZF, while the fact that BPI does not entail LEM is proven, for example, in Banaschewski and Bhutani [4].

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