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Phase Transition Results for Three Ramsey-Like Theorems

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Abstract We classify a sharp phase transition threshold for Friedman's finite adjacent Ramsey theorem. We extend the method for showing this result to two previous classifications involving Ramsey theorem variants: the Paris–Harrington theorem and the Kanamori–McAloon theorem. We also provide tools to remove ad hoc arguments from the proofs of phase transition results as much as currently possible.

Phase transitions in logic are a recent development in unprovability. The general program, started by Andreas Weiermann, is to classify parameter functions $f : \mathbb{N} \to \mathbb{N}$ according to the provability of a parameterized theorem φ_f in a theory T. We study these transitions with the goal of gaining a better understanding of unprovability. More details on this program, with an overview of related publications, can be found at Weiermann [16].

In this paper we examine the transition results for three Ramsey theorem variants: Friedman's finite adjacent Ramsey theorem, the Paris–Harrington theorem, and the Kanamori–McAloon theorem. The latter two of these have been studied previously in Weiermann [15] and Carlucci, Lee, and Weiermann [4], but the methods used in the present paper are a natural continuation of the method for the adjacent Ramsey theorem. The emphasis of this method is on connecting the variants φ_k of the theorem for constant functions k with the classification f according to the provability of φ_f. Furthermore, this proof method does not depend on whether the original nonparameterized version of the theorem was shown using proof/recursion theory or model-theoretic constructions.

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We will also provide some general tools to streamline the proofs of phase transition results: the upper bounds Lemmas 3.1 and 3.3, and the lower bounds sharpening Lemmas 2.9 and 2.11. The manner in which these lemmas are stated indicates the most important steps in the proofs of (sharpened) phase transitions. These lemmas remove the need to repeat some ad hoc arguments for each transition result.

This paper is divided into four sections. Section 1 introduces the three Ramsey theorem variants and the transition results. Section 2 is dedicated to independence, and Section 3 is dedicated to provability. We conclude with some observations on phase transitions in Section 4. In Section 2 the three Ramsey theorem variants are each treated in a separate subsection. We advise the reader who wishes to skip one or two of those subsections to first read the intuitive sketch of the proofs at the beginning of Section 2.

1 Three Ramsey-Like Theorems

1.1 Adjacent Ramsey The finite adjacent Ramsey theorem is one of the most recent independence results at the level of PA and was first presented in Friedman [6]. Independence of the variants with fixed dimension is examined extensively in Friedman and Pelupessy [7]. This examination uses proof-theoretic techniques. Showing independence by using model-theoretic constructions is still an open problem. As in the case of the other Ramsey variants, we will call functions $C\colon \{0,\dots,R\}^d \to \mathbb{N}^r$ *colorings*. Notice the distinction between parameter functions, which are provided externally and colorings, which are being quantified over inside the theorems. We denote the ith coordinate of an r-tuple a with $(a)_i$.

Definition 1.1 For r-tuples a, b:

$$a \leq b \Leftrightarrow (a)_1 \leq (b)_1 \wedge \cdots \wedge (a)_r \leq (b)_r.$$

Definition 1.2 A coloring $C\colon \{0,\dots,R\}^d \to \mathbb{N}^r$ is f-limited if

$$\max C(x) \leq f(\max x) + 1 \quad \text{for all } x \in \{0,\dots,R\}^d.$$

Theorem 1.3 (AR_f) *For every d, r there exists R such that for every f-limited coloring $C\colon \{0,\dots,R\}^d \to \mathbb{N}^r$, there exist $x_1 < \cdots < x_{d+1} \leq R$ with*

$$C(x_1,\dots,x_d) \leq C(x_2,\dots,x_{d+1}).$$

Proof We show that, for every $C\colon \mathbb{N}^d \to \mathbb{N}^r$, there exist $x_1 < \cdots < x_{d+1}$ such that $C(x_1,\dots,x_d) \leq C(x_2,\dots,x_{d+1})$ (the proof for this claim is taken from [6]). Given a C as in the claim, define $D\colon [\mathbb{N}]^{d+1} \to 2$ as follows:

$$D(x_1,\dots,x_{d+1}) = \begin{cases} 0 & \text{if } C(x_1,\dots,x_d) \leq C(x_2,\dots,x_{d+1}) \\ 1 & \text{otherwise.} \end{cases}$$

By the infinite Ramsey theorem, there exists an infinite homogeneous set for D. By Dickson's lemma, the value of D on this set must be zero, which finishes the proof of the claim. Apply a compactness argument to obtain AR_f. □

Definition 1.4 We denote the smallest R from AR_f with $\mathrm{AR}_f^d(r)$. The theorem AR_f with fixed d is denoted with AR_f^d.

Theorem 1.5 *We have the following:*

1. $\mathrm{I}\Sigma_{d+1} \nvdash \mathrm{AR}_{\mathrm{id}}^{d+1}$;
2. $\mathrm{PA} \nvdash \mathrm{AR}_{\mathrm{id}}$.

Proof See [7, Theorems 3.7 and 3.8]. □

1.2 Paris–Harrington The Paris–Harrington theorem is one of the earliest examples of natural theorems which are independent of PA. This was first shown using model-theoretic methods in Paris [13]; later this was shown using proof-theoretic methods in Ketonen and Solovay [10], Loebl and Nešetřil [12], and in [7].

Definition 1.6 The set $[X]^d$ is the set of d-element subsets of X, $[m, R]^d = [\{m, \dots, R\}]^d$ and $[R]^d = [0, R]^d$.

Definition 1.7 Given a coloring $C : [m, R]^d \to r$, we call a set H *homogeneous for C* or *C-homogeneous* if C is constant on $[H]^d$.

Theorem 1.8 (PH_f) *For every d, r, m there exists an R such that, for every coloring $C : [m, R]^d \to r$, there exists an $H \subseteq [m, R]$ of size $f(\min H)$ for which C limited to $[H]^d$ is constant.*

Definition 1.9 We denote the smallest R from PH_f with $\mathrm{PH}^d_f(m, r)$. The theorem PH_f with fixed d is denoted with PH^d_f. We call a coloring $C : [m, R]^d \to r$ *bad* if every C-homogeneous set has size strictly less than $f(\min H)$.

Theorem 1.10 *We have the following:*

1. $\mathrm{I}\Sigma_{d+1} \nvdash \mathrm{PH}^{d+2}_{\mathrm{id}}$;
2. $\mathrm{PA} \nvdash \mathrm{PH}_{\mathrm{id}}$.

Proof See [7, Theorems 3.7 and 3.8]. □

1.3 Kanamori–McAloon The Kanamori–McAloon theorem is also known as the *regressive Ramsey theorem*. We will examine the following variant.

Theorem 1.11 (KM_f) *For every d, m, a there exists R such that, for every $C : [a, R]^d \to \mathbb{N}$ with $C(x) \le f(\min x)$, there exists $H \subseteq R$ of size m for which for all $x, y \in [H]^d$ with $\min x = \min y$ we have $C(x) = C(y)$.*

Definition 1.12 We denote the smallest R from KM_f with $\mathrm{KM}^d_f(a, m)$. The theorem KM_f with fixed d is denoted with KM^d_f.

Theorem 1.13 *We have the following:*

1. $\mathrm{I}\Sigma_{d+1} \nvdash \mathrm{KM}^{d+2}_{\mathrm{id}}$;
2. $\mathrm{PA} \nvdash \mathrm{KM}_{\mathrm{id}}$.

Proof See Kanamori and McAloon [9, Theorem A, Corollary 4.5]. □

1.4 Phase transition results All parameter functions are assumed to be nondecreasing. For every $f : \mathbb{N} \to \mathbb{N}$ the inverse is

$$f^{-1}(i) = \min\{j : f(j) \ge i\},$$

$2_0(i) = i$, $2_{n+1}(i) = 2^{2_n(i)}$, log is the inverse of $i \mapsto 2^i$, $\log^n$ is the inverse of $i \mapsto 2_n(i)$, $\log^*$ is the inverse of $i \mapsto 2_i(2)$, $\sqrt[c]{\log^n}$ is the inverse of $i \mapsto 2_n(i^c)$, and $i \mapsto \frac{i}{c}$ is the inverse of $i \mapsto i \cdot c$, where $\frac{x}{0} = 1$. We use φ^d_f to denote the theorem φ_f with fixed dimension d. The function H_α is the αth function from the Hardy hierarchy with canonical fundamental sequences. In items (5) and (6) of the following theorems we assume that the reader is familiar with proof-theoretic results involving this hierarchy and provability in PA and $\mathrm{I}\Sigma_d$. Details on this can be found in, for example, Buss [3], Pohlers [14], Arai [1], or in the lecture notes in Buchholz [2].

Theorem 1.14 *We have the following:*

1. $\mathrm{I}\Sigma_{d+1} \nvdash \mathrm{AR}^{d+1}_{\sqrt[c]{\log^d}}$ *for every* $c > 0$*;*
2. $\mathrm{PA} \nvdash \mathrm{AR}_{\log^n}$ *for every* n*;*
3. $\mathrm{I}\Sigma_1 \vdash \mathrm{AR}^{d+1}_{\log^{d+1}}$*;*
4. $\mathrm{I}\Sigma_1 \vdash \mathrm{AR}_{\log^*}$*;*
5. $\mathrm{PA} \vdash \mathrm{AR}_{f_\alpha} \Leftrightarrow \alpha < \varepsilon_0$*;*
6. $\mathrm{I}\Sigma_{d+1} \vdash \mathrm{AR}^{d+1}_{f^{d+1}_\alpha} \Leftrightarrow \alpha < \omega_{d+2}$*,*

where $f^{d+1}_\alpha(i) = \sqrt[H^{-1}_\alpha(i)]{\log^d(i)}$ *and* $f_\alpha(i) = \log^{H^{-1}_\alpha(i)}(i)$.

Theorem 1.15 *We have the following:*

1. $\mathrm{I}\Sigma_{d+1} \nvdash \mathrm{PH}^{d+2}_{\frac{\log^{d+1}}{c}}$ *for every* $c > 0$*;*
2. $\mathrm{PA} \nvdash \mathrm{PH}_{\log^n}$ *for every* n*;*
3. $\mathrm{I}\Sigma_1 \vdash \mathrm{PH}^{d+2}_{\log^{d+2}}$*;*
4. $\mathrm{I}\Sigma_1 \vdash \mathrm{PH}_{\log^*}$*;*
5. $\mathrm{PA} \vdash \mathrm{PH}_{f_\alpha} \Leftrightarrow \alpha < \varepsilon_0$*;*
6. $\mathrm{I}\Sigma_{d+1} \vdash \mathrm{PH}^{d+2}_{f^{d+1}_\alpha} \Leftrightarrow \alpha < \omega_{d+2}$*;*

where $f^{d+1}_\alpha(i) = \frac{\log^{d+1}(i)}{H^{-1}_\alpha(i)}$ *and* $f_\alpha(i) = \log^{H^{-1}_\alpha(i)}(i)$.

Theorem 1.16 *We have the following:*

1. $\mathrm{I}\Sigma_{d+1} \nvdash \mathrm{KM}^{d+2}_{\sqrt[c]{\log^d}}$ *for every* $c > 0$*;*
2. $\mathrm{PA} \nvdash \mathrm{KM}_{\log^n}$ *for every* n*;*
3. $\mathrm{I}\Sigma_1 \vdash \mathrm{KM}^{d+2}_{\log^{d+1}}$*;*
4. $\mathrm{I}\Sigma_1 \vdash \mathrm{KM}_{\log^*}$*;*
5. $\mathrm{PA} \vdash \mathrm{KM}_{f_\alpha} \Leftrightarrow \alpha < \varepsilon_0$*;*
6. $\mathrm{I}\Sigma_{d+1} \vdash \mathrm{KM}^{d+2}_{f^d_\alpha} \Leftrightarrow \alpha < \omega_{d+2}$*;*

where $f^d_\alpha(i) = \sqrt[H^{-1}_\alpha(i)]{\log^d(i)}$ *and* $f_\alpha(i) = \log^{H^{-1}_\alpha(i)}(i)$.

The first two items and the unprovability parts of the last two items of these theorems will be treated in Section 2. The first item of Theorem 1.14, 1.15, or 1.16 is derived from Theorems 1.5, 1.10, and 1.13 combined with Theorems 2.3, 2.5, and 2.7, respectively. The unprovability parts of items (5) and (6) are shown by combining the first two items with Lemmas 2.9 and 2.11.

Items (3) and (4) and the provability parts of the last two items are shown in Section 3. These are direct consequences of Lemmas 3.1 and 3.3 and upper bound estimates from the literature.

Theorem 1.15 can already be found in [15]; Theorem 1.16 can be found in Lee [11, Corollary 4.1.3] and [4, Theorems 1.1, 1.2].

2 Lower Bounds

In the following three sections we show items (1) and (2) of Theorems 1.14, 1.15, and 1.16. The underlying idea of the three proofs is to show, for the appropriate parameter f, that $\varphi_f \to \varphi_{\mathrm{id}}$ by compressing the colorings C for φ_{id} by using f to obtain a coloring D_1. This causes the problem that if one obtains for such colorings an adjacent/homogeneous/min-homogeneous set H (by φ_f), the set $\{f(x) : x \in H\}$ may not have the right size (to demonstrate φ_{id}) because $f(x) = f(y)$ could be satisfied for some $x < y$. We will solve this by combining D_1 with two colorings D_2 and D_3.

The coloring D_2 will have the property that, for an adjacent/homogeneous/min-homogeneous set H and for $x_1 < \cdots < x_{d+1}$ in H, either $f(x_d) = f(x_{d+1})$ or $f(x_1) < \cdots < f(x_{d+1})$.

The other coloring D_3 will ensure that, in the case $f(x_d) = f(x_{d+1})$, the set H cannot be adjacent/homogeneous/min-homogeneous with the appropriate size. To obtain suitable coloring D_3, we use lower-bound estimates for φ_k with constant function k.

2.1 Adjacent Ramsey For determining the transitions, estimates on $i \mapsto \mathrm{AR}^d_i(r)$ play a central part. A variant of these functions has been examined extensively in [6]. We use the following result.

Lemma 2.1 *For every $d \in \mathbb{N}$ and $i, c > 0$ there exists a coloring*

$$C\colon \big\{0, \ldots, 2_d(i^c)\big\}^{d+1} \to \{0, \ldots, i\}^{32\cdot d + c}$$

such that $C(x) \neq C(y)$ for all $x \neq y$.

Proof Induction on d. If $d = 0$, enumerate $\{0, \ldots, i\}^c = \{x_1, \ldots, x_{(i+1)^c}\}$ and take $C(j) = x_j$. For the induction step apply [6, Lemma 1.9]. □

We modify these colorings slightly.

Lemma 2.2 *For every $d \in \mathbb{N}$ and $c, i > 0$ there exists a coloring*

$$C_{d,c,i}\colon \big\{0, \ldots, 2_d(i^c)\big\}^{d+1} \to \{0, \ldots, i\}^{64\cdot d + 2\cdot c}$$

such that $C(x_1, \ldots, x_{d+1}) \not\leq C(x_2, \ldots, x_{d+2})$ for all $x_1 < \cdots < x_{d+2} \leq 2_d(i^c)$.

Proof Take C' from Lemma 2.1, and define

$$C(x) = \big(C'(x), i - C'(x)\big). \qquad \square$$

With these estimates, we can prove parts (1) and (2) of Theorem 1.14.

Theorem 2.3 *There exists a primitive recursive function h such that*

$$\mathrm{AR}^{d+1}_{\sqrt[c]{\log^d}}\big(h(d, c, r)\big) \geq \mathrm{AR}^{d+1}_{\mathrm{id}}(r).$$

Proof We claim that the inequality holds for $h(d, c, r) = r + 65 \cdot (d + 1) + 2 \cdot c + 3$. Given id-limited coloring $C\colon \{0, \ldots, R\}^{d+1} \to \mathbb{N}^r$, take $f(x) = \sqrt[c]{\log^d}$ and, for $i, j \in \{1, \ldots, d + 1\}$, $(w(i))_j = 1$ if $i = j$, zero otherwise.

Define colorings $D_1\colon\{0,\dots,R\}^{d+1}\to\mathbb{N}^r$, $D_2\colon\{0,\dots,R\}^{d+1}\to\mathbb{N}^{d+1}$, and $D_3\colon\{0,\dots,R\}^{d+1}\to\mathbb{N}^{64\cdot(d+1)+2(c+1)}$ as follows:

$$D_1(x)=C\bigl(f(x_1),\dots,f(x_{d+1})\bigr),$$
$$D_2(x)=w(i),$$

where $i<d+1$ is the biggest such that $f(x_{i-1})=f(x_i)$ if such i exists, one otherwise, and

$$D_3(x)=C_{d,c+1,f(\max x)}(x),$$

where $C_{d,c,i}$ are taken from Lemma 2.2.

Combine these colorings into a single f-limited coloring

$$D\colon\{0,\dots,R\}^{d+1}\to\mathbb{N}^{r+d+1+64\cdot(d+1)+2(c+1)}$$

by taking $D=(D_1,D_2,D_3)$.

Suppose that for $x_1<\dots<x_{d+2}\le R$ we have $D(x_1,\dots,x_{d+1})\le D(x_2,\dots,x_{d+2})$. Observe that if $1<i\ne j<d$, then $w(i)\not\le w(j)$, so $D_2(x_1,\dots,x_{d+1})\le D_2(x_2,\dots,x_{d+2})$ implies either $f(x_1)<\dots<f(x_{d+2})$ or $f(x_1)=\dots=f(x_{d+2})$.

1. If $f(x_1)<\dots<f(x_{d+2})$, then, by definition of D_1, we have
$$C\bigl(f(x_1),\dots,f(x_{d+1})\bigr)\le C\bigl(f(x_2),\dots,f(x_{d+2})\bigr).$$
2. If $f(x_1)=\dots=f(x_{d+2})$, then $C_{d,c+1,f(x_{d+1})}=C_{d,c+1,f(x_{d+2})}$, so by definition $D_3(x_1,\dots,x_{d+1})\not\le D_3(x_2,\dots,x_{d+2})$, which is a contradiction.

□

2.2 Paris–Harrington We will use lower bounds from Ramsey theory from [8, Section 4.7, Theorem 19], which are attributed to Erdős and Hajnal.

Lemma 2.4 *For every $d\ge 2$ there exists constant a_d such that*

$$\mathrm{PH}^d_{i\cdot a_d}(0,r)>2_{d-2}(r^{i-2})$$

for all $r\ge 4$ and $i\ge 3$.

With these estimates, we can prove parts (1) and (2) of Theorem 1.15.

Theorem 2.5 *There exist primitive recursive functions h_1 and h_2 such that*

$$\mathrm{PH}^{d+1}_{\frac{\log^d}{c}}\bigl(h_1(c,d,m),h_2(c,d,m,r)\bigr)\ge \mathrm{PH}^{d+1}_{\mathrm{id}}(m,r)$$

for every $c,r>0$ and m sufficiently large.

Proof We claim this is the case for

$$h_1(c,d,m)=2_d(a_d\cdot c\cdot m),\qquad h_2(c,d,m,r)=r\cdot(d+2)^2\cdot 2^{(c+2)\cdot a_d}.$$

Given $C\colon[m,R]^{d+1}\to r$, take $f(i)=\frac{\log^d(i)}{a_d\cdot c}$ and colorings

$$D_{i\cdot a_d}\colon\bigl[2_{d-1}(2^{(c+1)\cdot a_d\cdot i})\bigr]^{d+1}\to 2^{(c+2)\cdot a_d},$$

where $D_{i\cdot a_d}$ is obtained from Lemma 2.4. Define

$$D\colon\bigl[2_d(a_d\cdot c\cdot m),R\bigr]^{d+1}\to r\times 2^{(c+2)\cdot a_d}\times(d+2)^2$$

as follows.

If $f(x_1) < \cdots < f(x_{d+1})$, then

$$D(x) = \big(C\big(f(x_1), \ldots, f(x_{d+1})\big), 0, 0, d+1\big).$$

If $1 \le i \le d+1$ is the biggest i such that $f(x_1) = \cdots = f(x_i)$ and if $1 \le j < d+1$ is the biggest j such that $f(x_1) < \cdots < f(x_j)$ (if $i > 1$, then $j = 1$, and if $j > 1$, then $i = 1$), then

$$D(x) = (0, 0, i, j).$$

Note that although the values of D are tuples, the number of possible values is bound by $r \cdot 2^{(c+2)\cdot a_d} \cdot (d+2)^2$; hence, D can be converted to a coloring $[2_d(a_d \cdot c \cdot m), R]^{d+1} \to r \cdot 2^{(c+2)\cdot a_d} \cdot (d+2)^2$ by using a suitable encoding. Homogeneous sets for this converted function are also homogeneous for D.

Suppose that H is homogeneous for D and of size greater than $d+2$. In this case, the last two coordinates have value 1 or $d+1$. If not, then there exist $x_1 < \cdots < x_{d+2}$ with $i = (D(x_2, \ldots, x_{d+2}))_3 = (D(x_1, \ldots, x_{d+2}))_3 + 1 = i + 1$, which is a contradiction (same argument for 4th coordinate). If one of those two is 1, then the other must be $d+1$, so either $f(x_1) = \cdots = f(x_{d+1})$ for all $x_1 < \cdots < x_{d+1}$ in H or $f(x_1) < \cdots < f(x_{d+1})$ for all $x_1 < \cdots < x_{d+1}$ in H.

By definition of D, this implies that H is homogeneous for $(D)_1$ or $D_{f(\min H)\cdot a_d}$. In the latter case, H has size strictly less than $f(\min H)$. Hence if H has size larger than $f(\min H)$, then $H' = \{f(h) : h \in H\}$ has size larger than $\min H'$ and is homogeneous for C. $\square$

2.3 Kanamori–McAloon We have the following estimates from [4, Lemma 3.14].

Lemma 2.6 *For every $d \ge 2$ there exists constant a_d such that*

$$\mathrm{KM}^d_{i\cdot a_d\cdot m}\big(0, a_d \cdot (m+1)\big) > 2_{d-2}(i^m).$$

This implies that, for $i > (a_d \cdot m)^m$ and $m > c+2$, we get

$$\mathrm{KM}^d_i\big(0, a_d \cdot (m+1)\big) > 2_{d-2}\big((i+1)^{c+1}\big).$$

According to this estimate, there exist colorings

$$D_i \colon \big[0, 2_{d-2}\big((i+1)^{c+1}\big)\big]^d \to a_d \cdot (m+1)$$

such that for every $H \subseteq [0, 2_{d-2}((i+1)^{c+1})]$ of size i there exist $x, y \in [H]^d$ with $\min x = \min y$ and $D_i(x) \ne D_i(y)$. With these colorings we can prove parts (1) and (2) of Theorem 1.16.

Theorem 2.7 *There exist primitive recursive functions h_1, h_2 such that for $d \ge 2$,*

$$\mathrm{KM}^d_{\sqrt[c]{\log^{d-2}}}\big(h_1(d,m,c), h_2(d,m,c)\big) \ge \mathrm{KM}^d_{\mathrm{id}}(0, m).$$

Proof We claim this inequality holds for $h_1(d,m,c) = 2_{d-2}((a_d \cdot m)^{cm})$, $h_2(d,m,c) = a_d \cdot (m+1) + 1$, and $m > d + c + 3$. Given a coloring $D \colon [R]^d \to \mathbb{N}$ for the identity function, we create an intermediate coloring

$$\tilde{C} \colon [R]^d \to \mathbb{N} \times \mathbb{N} \times (d+2) \times (d+2).$$

Roughly speaking, $\tilde{C}_1$ will be $D(f(x_1), \ldots, f(x_d))$, $\tilde{C}_2$ is $D_{f(x_1)}$, and $\tilde{C}_3$ and $\tilde{C}_4$ will ensure that for min-homogeneous sets either $f(x_1) = \cdots = f(x_d)$ or $f(x_1) < \cdots < f(x_d)$ in the manner similar to what we have seen for adjacent Ramsey and Paris–Harrington. We define f-regressive C to be one of the first two coordinates or zero, where the choice is dependent on and coded by the value of the last coordinate. We emphasize again that the lower-bound estimates for KM_i^d directly influence the functions f for which this construction is useful.

We take $f = \sqrt[c+1]{\log^{d-2}}$ and

$$\tilde{C}(x) = \big(D\big(f(x_1), \dots, f(x_d)\big), D_{f(x_1)}(x), i, j\big),$$

where i is the biggest such that $f(x_1) = \cdots = f(x_i)$ and j is the biggest such that $f(x_1) < \cdots < f(x_j)$ (if $i > 1$, then $j = 1$, and if $j > 1$, then $i = 1$). Note that $\tilde{C}_1$ is not everywhere defined; take it to be zero if it is undefined (same for $\tilde{C}_2$).

If H of size at least $d + 2$ is min-homogeneous for $\tilde{C}_3$, then the value of this coordinate is 1 or $d + 1$. Suppose not. Let $x_1 < \cdots < x_{d+1}$ be the first $d + 1$ elements of H; then

$$i = \tilde{C}_3(x_1, x_3, \dots, x_{d+1}) = \tilde{C}_3(x_1, x_2, \dots, x_d) + 1 = i + 1,$$

which is a contradiction.

If H is min-homogeneous for $\tilde{C}_4$, then it must, by a similar argument, have values 1, 2, or $d + 1$. Let $x_1 < \cdots < x_{d+1}$ be the first $d + 1$ elements of H, and suppose that $\tilde{C}_4(x_1, \dots, x_d) = 2$. Then $f(x_2) = f(x_3)$; hence, $\tilde{C}_4(x_2, x_3, \dots, x_d) = 1$. In other words, in this case $\tilde{C}_4$ has value 1 on $H' = H - \min H$.

Hence either $f(x) < f(y)$ for all $x < y \in H'$ or $f(x) = f(y)$ for all $x < y \in H'$. So H' is min-homogeneous for D in the first case or min-homogeneous for $D_{\min H'}$ in the latter case.

Encode the last two coordinates into a single coloring $E\colon [R]^d \to (d + 1)^2$ such that the first of those two cases is encoded in value 0, the latter in 1. We take

$$C(x) = \begin{cases} (d + 1)^2 + 2 \cdot \tilde{C}_1(x) + 1 & \text{if } E(x) = 0 \\ (d + 1)^2 + 2 \cdot \tilde{C}_2(x) + 2 & \text{if } E(x) = 1 \\ E(x) & \text{otherwise.} \end{cases}$$

Suppose that H of size greater than $d + 2$ is min-homogeneous for C; it then must have value greater than $(d+1)^2+1$. Hence $H' = H - \min H$ is min-homogeneous for either D or $D_{f(\min H')}$. In the latter case, it has size strictly less than $a_d \cdot (m+1)$. Hence if we have a min-homogeneous set for C of size $a_d \cdot (m + 1) + 1$, we obtain a min-homogeneous set for D of size m.

This coloring is $\sqrt[c]{\log^{d-2}}$-regressive because

$$(d + 1)^2 + 2 + 2 \cdot \sqrt[c+1]{\log^{d-2}(x_1)} < \sqrt[c]{\log^{d-2}(x_1)}$$

is ensured by limiting the domain of C to numbers larger than $2_{d-2}((a_d \cdot m)^{cm})$. □

2.4 Sharpening In this section we prove the unprovability parts of (5) and (6) of Theorems 1.14, 1.15, and 1.16. For applying the sharpening lemmas it is of use to note that if we combine Section 2 with lower-bounds estimates from [7, Theorems 4–6] and model constructions from [9, Theorem 4.4], we have the following.

Theorem 2.8 *Fix d. There exist primitive recursive functions h_1, h_2, h_3, h_4, h_5 such that:*

1. $M_{1,l_c^{-1}}(n,c,x) \geq H_{\omega_{d+1}(n)}(x)$ *for*
$$M_{1,f}(n,c,x) = \mathrm{AR}_f^{d+1}\big(h_1(n,c,x)\big) \qquad \text{and} \qquad l_c(i) = 2_d(i^c);$$
2. $M_{2,l_c^{-1}}(n,c,x) \geq H_{\omega_{d+1}(n)}(x)$ *for*
$$M_{2,f}(n,c,x) = \mathrm{PH}_f^{d+2}\big(h_2(d,n,c,x), h_3(d,n,c,x)\big) \qquad \text{and}$$
$$l_c(i) = 2_{d+1}\big(i \cdot (c)\big);$$
3. *in every nonstandard model N of* $\mathrm{I}\Sigma_1$ *with $n \in N$ and nonstandard $c, x \in N$ there exists a model of* $\mathrm{I}\Sigma_d$ *below* $M_{3,l_c^{-1}}(n,c,x)$ *for*
$$M_{3,f}(n,c,x) = \mathrm{KM}_f^{d+2}\big(h_4(d,n,c,x), h_5(d,n,c,x)\big) \qquad \text{and}$$
$$l_c(i) = 2_d(i^c).$$

Lemma 2.9 (Proof-theoretic lower-bounds sharpening) *Suppose that T is a theory that includes* $\mathrm{I}\Sigma_1$, *that M_f is computable for every computable f, and that we have the following.*

1. $(i,c) \to l_c(i)$ *is nondecreasing and provably total in T.*
2. $f(i) \leq g(i)$ *for all* $i \leq M_g(n,c,x)$ *implies* $M_f(n,c,x) \leq M_g(n,c,x)$.
3. *Every provably total function of T can be eventually dominated by H_n for some n and* $H(i) = H_i(i)$.
4. $M_{l_c^{-1}}(n,c,i) \geq H_n(i)$ *for every n;*

then:
$$T \nvdash \forall n,c,x \exists y M_h(n,c,x) = y,$$
where $h(i) = l^{-1}_{H^{-1}(i)}(i)$.

Proof We show that
$$M = M_h(x,x,x) \geq H(x).$$
Suppose, for a contradiction, that
$$M < H(x);$$
then $H^{-1}(i) \leq x$ for all $i \leq M$; hence $h(i) \geq l_x^{-1}(i)$ for all $i \leq M$. Therefore,
$$\begin{aligned} M &\geq M_{l_x^{-1}}(x,x,x) \\ &\geq H_x(x) = H(x), \end{aligned}$$
which contradicts our assumption. □

Corollary 2.10 *We have the following:*

1. $\mathrm{PA} \nvdash \mathrm{AR}_f$, *where* $f(i) = \log^{H_{\varepsilon_0}^{-1}(i)}(i)$;
2. $\mathrm{I}\Sigma_{d+1} \nvdash \mathrm{AR}_f^{d+1}$, *where* $f(i) = \sqrt[H_{\omega_{d+2}}^{-1}(i)]{\log^d(i)}$;
3. $\mathrm{PA} \nvdash \mathrm{PH}_f$, *where* $f(i) = \log^{H_{\varepsilon_0}^{-1}(i)}(i)$;
4. $\mathrm{I}\Sigma_{d+1} \nvdash \mathrm{PH}_f^{d+2}$, *where* $f(i) = \frac{\log^{d+1}(i)}{H_{\omega_{d+2}}^{-1}(i)}$.

Lemma 2.11 (Model-theoretic lower-bounds sharpening) *Suppose that T is a theory that includes* $\mathrm{I}\Sigma_1$, *that M_f is computable for every computable f, and that we have the following.*

1. $(i,c) \to l_c(i)$ *is nondecreasing and provably total in T.*
2. $f(i) \leq g(i)$ *for all* $i \leq M_g(n,c,x)$ *implies* $M_f(n,c,x) \leq M_g(n,c,x)$.

3. *H eventually dominates every provably total function of T.*
4. *In every nonstandard model N of $\mathrm{I}\Sigma_1$ and for every $c \in N$ and nonstandard $n, x, M_{l_c^{-1}}(n,c,x) \in N$ there exists an initial segment $I < M_{l_c^{-1}}(n,c,x)$ which models T;*

then:

$$T \nvdash \forall n,c,x \exists y M_h(n,c,x) = y,$$

where $h(i) = l^{-1}_{H^{-1}(i)}(i)$.

Proof Fix a nonstandard model $N \models \mathrm{I}\Sigma_1 + \forall x \exists y M_h(x,x,x) = y$. If $M_h(x,x,x) \geq H(x)$ for infinitely many standard x, we are finished, so suppose that for all but finitely many standard x we have

$$M_h(x,x,x) < H(x).$$

For these x we know that $h(i) \geq l_x^{-1}(i)$ for all $i \leq M_h(x,x,x)$; by overflow there exists a nonstandard x with these properties, so there exists a nonstandard instance of $M_{l_x^{-1}}(x,x,x)$. Hence there exists an initial segment which models T. □

Corollary 2.12 *We have the following:*

1. PA $\nvdash$ KM$_f$, *where* $f(i) = \log^{H_{\varepsilon_0}^{-1}(i)}(i)$;
2. $\mathrm{I}\Sigma_{d+1} \nvdash \mathrm{KM}_f^{d+2}$, *where* $f(i) = \sqrt[H^{-1}_{\omega_{d+2}}(i)]{\log^d(i)}$.

3 Upper Bounds

In this section we show items (3) and (4) and the provability parts of items (5) and (6) of Theorems 1.14, 1.15, and 1.16.

Lemma 3.1 (Upper-bounds lemma) *Suppose that T is a theory that contains $\mathrm{I}\Sigma_1$, that $M_f : \mathbb{N}^2 \to \mathbb{N}$ is a computable function for all computable f, and that $M_f(d,x) \leq M_g(d,x)$ whenever $f(i) \leq g(i)$ for all $i \leq M_g(d,x)$. Additionally, suppose that there exist nondecreasing, provably total, functions u, h such that, for every d, n and $k \geq h(d,n)$, we have*

$$M_k(d,n) \leq u(k);$$

then

$$T \vdash \forall d, x \exists! y M_{u^{-1}}(d,x) = y.$$

Proof If $i \leq u(h(d,x))$, then $u^{-1}(i) \leq h(d,x)$. Hence

$$M_{u^{-1}}(d,x) \leq M_{h(d,x)}(d,x) \leq u\big(h(d,x)\big). \qquad \square$$

Corollary 3.2 *If φ is one of* AR, PH, KM, *then*

$$\mathrm{I}\Sigma_1 \vdash \varphi_{\log^*}.$$

Proof First note that

$$\mathrm{AR}_k^d(r) \leq R(d, d+1, k^d),$$

$$\mathrm{PH}_k^d(m,r) \leq R(d,k,r) + m,$$

$$\mathrm{KM}_k^d(a,m) \leq R(d,m,k) + a,$$

where $R(d,m,r)$ are the Ramsey numbers for dimension d, size m, and r colors. We know that $R(d,m,k) \leq 2_k(2)$ for $k > 2_d(m)$ by the Erdős–Rado bounds from

[5]; hence, $k \mapsto \mathrm{AR}^d_k(r)$, $k \mapsto \mathrm{PH}^d_k(m,r)$, and $k \mapsto \mathrm{KM}^d_k(a,m)$ are also bounded by the tower function. □

Lemma 3.3 (Upper-bounds sharpening lemma) *Let T, M be as in the upper bounds lemma, and let l_c be unbounded for every c. If $(c,i) \mapsto l_c(i)$ is a nondecreasing provably total function such that there exist provably total functions g_1, g_2 with $g_1(d) \leq g_2(d,x)$ for all x and $M_k(d,x) \leq l_{g_1(d)}(k)$ whenever $k \geq g_2(d,x)$, then*

$$T \vdash \forall d, x \exists! y M_f(d,x) = y,$$

where $f(i) = l^{-1}_{B^{-1}(i)}(i)$, and B is an arbitrary unbounded, nondecreasing, and provably total function.

Proof Assume without loss of generality that $B \geq \mathrm{id}$. If $i \leq l_{g_1(d)}(B(g_2(d,x)))$, then

$$f(i) \leq l^{-1}_{g_2(d,x)}\big(l_{g_1(d)}\big(B\big(g_2(d,x)\big)\big)\big) \leq l^{-1}_{g_2(d,x)}\big(l_{g_2(d,x)}\big(B\big(g_2(d,x)\big)\big)\big).$$

Therefore

$$M_f(d,x) \leq M_{B(g_2(d,x))}(d,x) \leq l_{g_1(d)}\big(B\big(g_2(d,x)\big)\big).$$ □

Corollary 3.4 *We have $\mathrm{I}\Sigma_{d+1} \vdash \mathrm{AR}^{d+1}_{f_\alpha}$ whenever $f_\alpha(i) = \sqrt[H_\alpha^{-1}(i)]{\log^d(i)}$ and $\alpha < \omega_{d+2}$.*

Proof Examine AR^d_f with fixed d and its associated function $\mathrm{AR}^d_f(r)$. The r will have the role of d when applying the upper-bounds sharpening lemma. By the Erdős–Rado bounds on Ramsey numbers from [5],

$$\mathrm{AR}^{d+1}_k(r) \leq 2_d(k^{(r+1)}).$$

Hence, by sharpening, $\mathrm{I}\Sigma_{d+1} \vdash \mathrm{AR}^{d+1}_{f_\alpha}$ whenever $f_\alpha(i) = \sqrt[H_\alpha^{-1}(i)]{\log^d(i)}$ and $\alpha < \omega_{d+2}$. □

Corollary 3.5 *We have $\mathrm{I}\Sigma_d \vdash \mathrm{PH}^{d+1}_{f_\alpha}$ whenever $f_\alpha(i) = \frac{\log^d(i)}{H_\alpha^{-1}(i)}$ and $\alpha < \omega_{d+1}$.*

Proof Examine PH^d_f with fixed d and its associated function $\mathrm{PH}^d_f(m,r)$. The r will have the role of d when applying the upper-bounds sharpening lemma. By the Erdős–Rado bounds on Ramsey numbers from [5], if $k \geq r + m$, then

$$\mathrm{PH}^d_k(m,r) \leq 2_{d-1}(r^{d^2} \cdot k) + m \leq 2_{d-1}\big((r^{d^2}+1)\cdot k\big) = l_{r^{d^2}+1}(k).$$

Hence, by sharpening, $\mathrm{I}\Sigma_d \vdash \mathrm{PH}^{d+1}_{f_\alpha}$ whenever $f_\alpha(i) = \frac{\log^d(i)}{H_\alpha^{-1}(i)}$ and $\alpha < \omega_{d+1}$. □

Corollary 3.6 *We have $\mathrm{I}\Sigma_{d+1} \vdash \mathrm{KM}^{d+2}_{f_\alpha}$ whenever $f_\alpha(i) = \sqrt[H_\alpha^{-1}(i)]{\log^d(i)}$ and $\alpha < \omega_{d+2}$.*

Proof Examine KM^d_f with fixed d and its associated function $\mathrm{KM}^d_f(a,m)$. The m will have the role of d when applying the upper-bounds sharpening lemma. We use bounds from [11, Corollary 4.2.3]:

$$\mathrm{KM}^d_k(a,m) \leq 2_{d-2}(k^{d^2\cdot m}) + a \leq 2_{d-2}(k^{d^2\cdot m+2}) = l_{d^2\cdot n+2}(k),$$

where the second inequality is true for $k \geq (a + d^2 \cdot m + 2)$. Hence, by sharpening, $\mathrm{I}\Sigma_{d+1} \vdash \mathrm{KM}^{d+2}_{f_\alpha}$ whenever $f_\alpha(i) = \sqrt[H_\alpha^{-1}(i)]{\log^d(i)}$ and $\alpha < \omega_{d+2}$. □

Corollary 3.7 *Let φ be one of* AR, PH, KM, *and let $f_\alpha(i) = \log^{H_\alpha^{-1}(i)}(i)$. We have*

$$\mathrm{PA} \vdash \varphi_{f_\alpha}$$

whenever $\alpha < \varepsilon_0$.

4 Some Observations on Transitions

In the phase transitions which have been examined so far, the same heuristics are used to determine the threshold functions: as soon as the upper-bound lemmas cannot be applied, because l is a lower bound, the resulting theorem is not provable for l^{-1}. We conjecture that phase transitions in unprovability always have the following shape.

Conjecture 4.1 (Lower bounds) *Suppose that T is a theory that contains $\mathrm{I}\Sigma_1$, that l is nondecreasing, and that M_f is a nondecreasing computable function for every computable f with the following properties:*

1. *$T \nvdash \forall x \exists y M_{\mathrm{id}}(x) = y$;*
2. *$f(i) \leq g(i)$ for all $i \leq M_g(x)$ implies $M_f(x) \leq M_g(x)$;*
3. *there exists x such that $k \mapsto l(k)$ is eventually strictly dominated by $k \mapsto M_k(x)$;*

then

$$T \nvdash \forall x \exists y M_{l^{-1}}(x) = y.$$

This observed connection between lower- and upper-bound estimates for M_k and the transition threshold leads to a difference between the threshold results for PH_f^d and AR_f^d, KM_f^d. In the first case, PH_k^d is a statement which involves the size of homogeneous sets being of size k, while in AR_k^d and KM_k^d the number of colors is dependent on k. In the estimates for Ramsey numbers there is a difference in the height in the exponential tower at which these two factors occur, leading to the different thresholds.

For the sharpening of the transition results the two lower-bounds sharpening lemmas suffice. These lemmas are dependent on the method of proving independence of φ_{id}. We conjecture that it is possible to generalize the following sharpening.

Conjecture 4.2 (Lower-bounds sharpening) *Suppose that T is a theory that contains $\mathrm{I}\Sigma_1$, that $(c, i) \mapsto l_c(i)$ is nondecreasing, and that M_f is a nondecreasing computable function for every computable f with the following properties:*

1. *$T \nvdash \forall x \exists y M_{l_c^{-1}}(x) = y$ for every c;*
2. *$f(i) \leq g(i)$ for all $i \leq M_g(x)$ implies $M_f(x) \leq M_g(x)$;*
3. *H eventually dominates every provably total function of T;*

then

$$T \nvdash \forall x \exists y M_h(x) = y,$$

where $h(i) = l_{H^{-1}(i)}^{-1}(i)$.

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