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End Extensions of Models of Weak Arithmetic Theories

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Abstract We give alternative proofs of results due to Paris and Wilkie concerning the existence of end extensions of countable models of $B\Sigma_1$, that is, the theory of Σ_1 collection.

1 Introduction

We work with subsystems of first-order Peano arithmetic (PA) in the first-order language of arithmetic LA. As usual, for $n \in \mathbb{N}$, $I\Sigma_n$ denotes the induction schema for Σ_n formulas (plus the well-known base theory PA^-), $L\Sigma_n$ denotes the least number axiom schema for Σ_n formulas (plus PA^-), $B\Sigma_n$ denotes $I\Delta_0$ plus the collection schema for Σ_n formulas, and exp denotes the axiom expressing "exponentiation is total" (recall that there is a Δ_0 formula representing the graph of the function 2^x). Finally, $\langle\ ,\ \rangle$ denotes one of the usual pairing functions. (For details, the reader can consult Hájek and Pudlák [10] or Kaye [12].)

Having proved his first incompleteness theorem, Gödel realized that the proof could be formalized and thus he obtained his second incompleteness theorem. The same fundamental insight works for other results, including Gödel's completeness theorem for the predicate calculus. This idea led to the so-called arithmetized completeness theorem (ACT), first formulated by Hilbert and Bernays [11, Section 4.2].

The ACT is undoubtedly an important result, as it can be applied to construct arithmetical models and give alternative proofs of the incompleteness theorems (see, e.g., [12]). Its statement has two forms, a syntactic and a semantic one. Since later in this article we will be considering models of theories in LA, the semantic form seems more appropriate (see, e.g., [12, Section 13.2]). In what follows, T will denote a theory in LA.

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Theorem 1.1 (ACT-semantic form) *Let M be a model of PA, and let T be a theory definable in M. If $M \models \mathrm{Con}(T)$, then there exists a model K of T such that K is "strongly definable" in M.*

Here, *strong definability* means, roughly speaking, that

(a) the universe of K may be taken to be the same as that of M, and
(b) the satisfaction relation for K is parametrically definable in M.

If the theory T contains PA, the relationship between M and K is much nicer; indeed, one can prove the following.

Lemma 1.2 *If M, K are models of PA and K is strongly definable in M, then M is isomorphic to an initial segment of K.*

By condition (b) of strong definability and the (well-known) fixed-point lemma, it follows that M cannot be isomorphic to an elementary substructure of K. However, the ACT can be applied in such a way that M is isomorphic to a Σ_n elementary substructure of K. Indeed, the following result, first stated explicitly by McAloon [14, p. 256], refers to this fact.

Theorem 1.3 *Let M be a model of PA, and let T be a theory definable in M such that $M \models \mathrm{Con}(T + \mathrm{Tr}(\Pi_n))$, where $\mathrm{Tr}(\Pi_n)$ denotes the set of (Gödel numbers of) Π_n sentences true in M. Then there exists a model K of T such that*

1. *K is strongly definable in M (and, therefore),*
2. *M is isomorphic to a proper Σ_n elementary initial segment of K.*

As it was the case with other fundamental theorems that were known to hold for PA, there were attempts to miniaturize the above results, that is, prove their counterparts for fragments of PA. Such a result, described as "a mild refinement of the arithmetized completeness theorem," was proved by Paris [15, p. 252] and is essentially the following.

Theorem 1.4 *Let M be a model of $B\Sigma_n$, let $n \geq 2$, and let $T \supseteq I\Delta_0$ be a theory Δ_{n-1} definable in M such that $M \models \mathrm{Con}(T)$. Then there exists a model K of T which is Δ_n definable in M, and M is isomorphic to a proper initial segment of K.*

By applying this result, the same author showed that (see [15, Theorems 2 and 5])

(i) every model of $B\Sigma_n$, $n \geq 2$, has a proper end extension $J \models B\Sigma_n$, and
(ii) every model of $I\Sigma_n$, $n \geq 2$, has a proper end extension $J \models I\Sigma_n$

(in fact, the author proved stronger results, but we are restricting our attention to versions relevant to our work).

In the previous theorem, in order to obtain K having a nicer relationship to M we need an extra assumption on M, as the following result shows (see Paris and Kirby [17, p. 200]).

Theorem 1.5 *For any countable model M of $I\Delta_0$ and $n \geq 2$,*

(a) $M \models B\Sigma_n \Leftrightarrow$ there exists $K \models I\Delta_0$ such that $M \prec_{n,e} K$, and
(b) if M has a proper Σ_1 elementary end extension, then $M \models B\Sigma_2$.

Concerning part (a) of Theorem 1.5, let us note that the proof of ($\Leftarrow$) does not rely on the countability of M, while the proof of the converse implication relies heavily on this assumption. Despite attempts to show that *any* model M of $B\Sigma_n$, $n \geq 2$, is extendable to a model K of $I\Delta_0$ such that $M \prec_{n,e} K$, this question still remains open (see, e.g., Clote [7]).

In view of Theorem 1.5, a natural question that arises is whether (a) holds for $n = 0, 1$. Concerning the implication ($\Leftarrow$), it holds for both $n = 0$ and $n = 1$, by the fact that $B\Sigma_0 \Leftrightarrow B\Sigma_1$ and the fact that if $M \subset_e K \models I\Delta_0$, then $M \models B\Sigma_1$. Concerning the converse implication, it does not hold for $n = 1$, by part (b) of Theorem 1.5 and the existence of models of the theory $B\Sigma_1 + \neg B\Sigma_2$ (which follows by results in [17]). Therefore, the only remaining question is the following.

Problem 1.6 Is every countable model M of $B\Sigma_1$ extendable to a model K of $I\Delta_0$ such that $M \prec_{0,e} K$?

Since for any structures M, K for LA, $M \subset_e K$ implies $M \prec_0 K$, it follows that Problem 1.6 is equivalent to the following.

Problem 1.7 Is every countable model M of $B\Sigma_1$ extendable to a model K of $I\Delta_0$ such that $M \subset_e K$?

Problem 1.7 is considered one of the main problems concerning fragments of PA (see Clote and Krajíček [8, "Fundamental problem F"]). This problem was examined exhaustively by Wilkie and Paris in [19]. These authors introduced the notion of Γ-*fullness*, Γ being a set of sentences, and showed that this problem has a positive answer, provided that M is $I\Delta_0$-full; that is, they proved the following result.

Theorem 1.8 *For any countable model M of $B\Sigma_1$, if M is $I\Delta_0$-full, then there exists $K \models I\Delta_0$ such that $M \subset_e K$.*

Moreover, Wilkie and Paris proved that certain natural conditions on M imply $I\Delta_0$-fullness. In order to be able to state their result precisely, we need to recall the definition of a notion and some notation.

Definition 1.9 Let M be a structure for LA. We say that M is *short Π_1-recursively saturated* if whenever $\Phi = \{x < a \wedge \varphi_i(x, \vec{b}) : i \in \mathbb{N}\}$ is a recursive set of Π_1 formulas (with parameters from M) finitely satisfiable in M, then Φ is satisfiable in M.

Notation 1.10 $I\Delta_0 \vdash \neg\Delta_0 H$ stands for the hypothesis that the Δ_0 hierarchy provably collapses in $I\Delta_0$; that is, there is a fixed n such that for any formula $\theta \in \Delta_0$, there is a formula $\chi \in \Delta_0$ in prenex normal form with at most n alternations of bounded quantifiers such that $I\Delta_0 \vdash \theta \leftrightarrow \chi$.

Now we can state the result in [19, p. 145] which concerns sufficient conditions for a model of $B\Sigma_1$ to be $I\Delta_0$-full.

Theorem 1.11 *For any countable nonstandard model M of $B\Sigma_1$, each of the following conditions implies that M is $I\Delta_0$-full:*

(I) *M is short Π_1-recursively saturated;*
(II) $M \models \exp$;
(III) $I\Delta_0 \vdash \neg\Delta_0 H$ *and* $\exists \mathbb{N} < \gamma \in M, M \models \forall x \exists y (y = x^\gamma)$;
(IV) $I\Delta_0 \vdash \neg\Delta_0 H$ *and* $\exists a \in M \forall b \in M \exists n \in \mathbb{N}, b \leq a^n$;
(V) $I\Delta_0 \vdash \neg\Delta_0 H$ *and* $\exists M \subset_e K \models B\Sigma_1$.

However interesting the notion of Γ-fullness may be, it is highly technical and, therefore, not very intuitive. For this reason, we found it worthwhile to look for an alternative approach, which would avoid the use of this notion and would be easier to grasp. Actually, the answer lies in a remark in [19, Theorem 5(2), p. 154], made just after the end of the proof, which reads as follows.

Remark A direct proof that any countable model of $I\Delta_0 + B\Sigma_1$ which is closed under exponentiation has a proper end extension to a model of $I\Delta_0$ may be obtained by mimicking the proof of Theorem 4 but with "Semantic Tableau consistency of Γ" in place of "Γ-full" and adding a new constant symbol $\pi > M$ to ensure that the end extension is proper.

Our paper is dedicated to showing how one can apply variants of the ACT to prove in an alternative way that if a countable nonstandard model M of $B\Sigma_1$ satisfies any of the conditions (I)–(IV) of Theorem 1.11, then it is properly end extendable to a model of $I\Delta_0$ (note that working with condition (V) makes no sense in our context, as it presupposes the proper end extendability of M). Although we have obtained no new results, we feel our undertaking is interesting from a methodological point of view, as it connects Problem 1.7 with the approach suggested by the ACT.

It should be noted that work in the same spirit, that is, employing variants of the ACT to study questions of end extendability of models of weak arithmetic theories, was done in Adamowicz [1], [2]. Indeed, in those papers the author provided characterizations of models that are properly end extendable to models of various theories $T \supseteq I\Delta_0$. Although we use consistency statements of the same kind as the ones in [1] and [2], our approach differs in that (most of) the models we consider satisfy properties other than the ones considered therein.

The paper is organized as follows. In the next section we give a more or less detailed proof of [19, Remark]. The third section is used to sketch proofs of variants of the Remark, obtained by replacing the condition that M is closed under exponentiation by another natural condition considered by Wilkie and Paris; that is, one of conditions (I), (III), (IV) in Theorem 1.11. The paper ends with a section containing some remarks and problems.

Before we proceed to the main body of our work, let us note that, when one works with strong theories like PA and $B\Sigma_n$, $n \geq 2$, one can work with simple consistency, while here one has to employ the weaker notion of "semantic tableau consistency" (as is well known, the two notions are equivalent in models closed under the superexponential function).

2 Totality of Exponentiation

To obtain the proof suggested by the Remark mentioned above, we will need some auxiliary definitions and results, which we will give now. First, we need the (formalized) notion of cut-free proof, for which the semantic tableau method seems most appropriate, as discussed in Wilkie and Paris [18].

Definition 2.1 Let T be a set of sentences. We say that a sequence of sets of formulas $\Gamma_0, \Gamma_1, \ldots, \Gamma_s$ is a tableau proof from T of a contradiction if the following hold.

1. For all $X \in \Gamma_s$ there is an atomic formula θ such that $\theta \in X$ and $\neg\theta \in X$.
2. If $X \in \Gamma_0$, then $X \subseteq T \cup \{$the logical equality axioms$\}$.

3. For all $X \in \Gamma_i$ with $i < s$ one of the following holds:
   (a) $X \in \Gamma_{i+1}$,
   (b) $X \cup \{\theta\} \in \Gamma_{i+1}$ for some $\neg\neg\theta \in X$,
   (c) $X \cup \{\neg\theta_1\}, X \cup \{\theta_2\} \in \Gamma_{i+1}$ for some $(\theta_1 \to \theta_2) \in X$,
   (d) $X \cup \{\theta_1, \neg\theta_2\} \in \Gamma_{i+1}$ for some $\neg(\theta_1 \to \theta_2) \in X$,
   (e) $X \cup \{\theta(t)\} \in \Gamma_{i+1}$ for some $\forall x\theta(x) \in X$ and some term t which is freely substitutable for x in $\theta(x)$,
   (f) $X \cup \{\neg\theta(y)\}$ for some $\neg\forall x\theta(x) \in X$ and some variable y which does not occur in any formula in X.
4. For all $Y \in \Gamma_{i+1}$ with $i < s$ there is an $X \in \Gamma_i$ such that Y is obtained from X by one of the rules 3(a)–(f).

Whenever $\Gamma_0, \Gamma_1, \ldots, \Gamma_s$ is a tableau proof of a contradiction from T, we say that s is the *depth* of this proof.

As Wilkie and Paris [18, p. 295] note, for any theory T,

(a) T is inconsistent in the usual sense just if there is a tableau proof from T of a contradiction;
(b) tableau proofs are superior to usual proofs, since they contain only subformulas of the sentences in T, but they suffer from the disadvantage of being in general iteratedly exponentially longer than Hilbert-style proofs.

For any suitable theory T in (a language extending) LA, using standard methods of formalizing syntactic notions, we can obtain a Δ_0 formula $\mathrm{Tabinconpr}(T, x)$ expressing, in any model of $I\Delta_0$, that "x is a tableau proof from T of a contradiction." In what follows, $\mathrm{Tabcon}(T)$ will denote the Π_1 sentence $\neg\exists x\, \mathrm{Tabinconpr}(T, x)$.

It is rather straightforward to formalize the proof of the so-called *elimination lemma* (see, e.g., Bell and Machover [3, Chapter 2, Section 6]) to prove the following result.

Lemma 2.2 *For any model M of $I\Delta_0 + \exp$, any theory T coded in M, and any sentence θ, if $M \models \neg\mathrm{Tabcon}(T + \theta)$ and $M \models \neg\mathrm{Tabcon}(T + \neg\theta)$, then $M \models \neg\mathrm{Tabcon}(T)$.*

Another fact that is necessary for the sequel is a result of Lessan [13, p. 43] (see also Paris and Dimitracopoulos [16, Theorem 2]), concerning the satisfaction of Δ_0 formulas (in models of $I\Delta_0$).

Theorem 2.3 *There exists a Δ_0 formula $\mathrm{Sat}_0(x, y, z)$ such that, for any $M \models I\Delta_0$, $\varphi(\vec{x}) \in \Delta_0$ and $\vec{a}, b \in M$,*

$$M \models b \geq 2^{(\max(\vec{a})+2)^{\ulcorner\varphi\urcorner}} \to \big[\varphi(\vec{a}) \leftrightarrow \mathrm{Sat}_0\big(b, \langle\vec{a}\rangle, \ulcorner\varphi(\vec{x})\urcorner\big)\big].$$

Remark 2.4

(a) Sat_0 acts like a satisfaction relation, for formulas in the sense of M. For example, for any $d, e \in M$, if, in the sense of M, d is the Gödel number of a Δ_0 formula of the form $\exists y \leq x_1 \psi(y, \vec{x})$ and e is the Gödel number of the formula $\psi(y, \vec{x})$, then

$$M \models \forall\vec{z}\forall t{\geq}2^{(\max(\vec{z})+2)^d}\big[\mathrm{Sat}_0\big(t, \langle\vec{z}\rangle, d\big){\leftrightarrow}\exists y{\leq}z_1\, \mathrm{Sat}_0\big(t, \langle y, \vec{z}\rangle, e\big)\big].$$

(b) The particular value of b is insignificant, as long as it exceeds $2^{(\max(\vec{a})+2)^{\ulcorner\varphi\urcorner}}$.

Now we are ready to prove the main result of this section; that is, part (II) of Theorem 1.11.

Theorem 2.5 *If M is a countable model of $B\Sigma_1+\exp$, then there exists $K \models I\Delta_0$ such that $M \subset_e K$.*

Proof With M satisfying the hypothesis, let LA* be the language obtained from LA by adding a new constant symbol c and a set of new constant symbols $\{c_a : a \in M\}$. For T a (suitable) theory in LA*, $T + \Delta + c{>}M$ denotes the theory obtained if we add to T the diagram Δ of M and the set of sentences $\{c{>}c_a : a \in M\}$.

The proof of the theorem relies on a couple of lemmas, the first of which is a variant of [18, Lemma 8.10].

Lemma 2.6 *We have $M \models \mathrm{Tabcon}(I\Delta_0 + \Delta + c{>}M)$.*

Proof of Lemma 2.6. Suppose this fails. Then there exist $a \in M$ and a tableau proof $\Gamma_0, \Gamma_1, \dots, \Gamma_s$ (in M) of a contradiction from hypotheses:

- the first a axioms of $I\Delta_0$,
- certain elements of Δ, with all constants having indices at most a,
- $\{c{>}c_0, \dots, c{>}c_a\}$.

In what follows, we will be using the formula Sat_0 for formulas with Gödel number $< e$, where $e \in M$ is larger than the Gödel number of any formula occurring in any Γ_i and for values of free variables less than $b = 2^{f^e}$, where $f = (a+2)^{2^{s+1}}$.

For all $i < s$ and $X \in \Gamma_i$ we define, by recursion on i in M, a function $F_{i,X}$ with domain the set of variables and constants occurring in formulas in X and range bounded by b, as follows.

- If $i = 0$, $F_{i,X}$ is empty.
- For x a variable in (some formula in) $Y \in \Gamma_{i+1}$ pick $X \in \Gamma_i$ such that Y is derived from X by one of the tableau rules.
  - If x appears in X, set $F_{i+1,Y}(x) = F_{i,X}(x)$.
  - If $Y = X \cup \{\neg\theta(x, x_1, \dots, x_p, c, \vec{c})\}$, where $\vec{c}$ denotes $c_0, \dots, c_a$ and $\neg\forall x\theta(x, x_1, \dots, x_p, c, \vec{c}) \in X$, set

$$F_{i+1,Y}(x) = \begin{cases} \text{the least } d < b \text{ such that} & \text{if such } d \\ M \models \mathrm{Sat}_0(b, \langle d, F_{i,X}(x_1), \dots, F_{i,X}(x_p), a+1, \vec{a}\rangle, \ulcorner\theta\urcorner), & \text{exists,} \\ 0, & \text{otherwise,} \end{cases}$$

    with $\vec{a}$ denoting $0, \dots, a$.
  - In all other cases, set $F_{i+1,Y}(x) = 0$.

Using the fact that $M \models I\Delta_0 + \exp$, one can check that the above definition can be carried out and prove (by induction) that, for each $i \le s$, the following hold.

(a) For all $X \in \Gamma_i$, $\mathrm{Range}(F_{i,X}) \subseteq \{m \in M \mid M \models m < (a+2)^{2^{i+1}}\}$.
(b) There is an $X \in \Gamma_i$ such that for all formulas $\theta(x_1, \dots, x_p, c, \vec{c})$ in X which are either Σ_1 or Π_1,

$$M \models \mathrm{Sat}_0\big(b, \big\langle F_{i,X}(x_1), \dots, F_{i,X}(x_p), a+1, \vec{a}\big\rangle, \ulcorner\theta\urcorner\big).$$

Let $X \in \Gamma_s$ satisfying (b) above. By (1) of Definition 2.1, X must contain θ and $\neg\theta$, for some atomic formula θ. But this clearly leads to a contradiction, which completes the proof of the lemma. □

We now proceed to the second lemma needed to prove the main theorem.

Lemma 2.7 *If $\theta(y, c, \vec{c})$ is a formula of* LA*, $a \in M$ and T is a finite extension of $I\Delta_0$ such that $M \models \mathrm{Tabcon}(T + \Delta + c{>}M)$, then either*

$$M \models \mathrm{Tabcon}\big(T + \Delta + c{>}M + \neg\exists y{\le}c_a\theta(y, c, \vec{c})\big)$$

or there exists $b{\le}^M a$ such that

$$M \models \mathrm{Tabcon}\big(T + \Delta + c{>}M + \theta(c_b, c, \vec{c})\big).$$

Proof of Lemma 2.7. Suppose, toward a contradiction, that T is a finite extension of $I\Delta_0$ such that

$$M \models \mathrm{Tabcon}(T + \Delta + c{>}M), \tag{1}$$

$$M \models \neg\mathrm{Tabcon}\big(T + \Delta + c{>}M + \neg\exists y{\le}c_a\theta(y, c, \vec{c})\big), \tag{2}$$

$$\text{for all } b{\le}^M a, \quad M \models \neg\mathrm{Tabcon}\big(T + \Delta + c{>}M + \theta(c_b, c, \vec{c})\big). \tag{3}$$

Using the fact that M satisfies $B\Sigma_1$, we can deduce from (3) that there exists $q \in M$ such that

$$M \models \forall z{\le}a\exists r{\le}q \quad \text{“}r \text{ is the Gödel number of a tableau proof of a contradiction from } T + \Delta + c{>}M + \theta(c_z, c, \vec{c})\text{.”}$$

Now we use Δ_0 induction and the fact that M is closed under exponentiation, to show that

$$M \models \forall z{\le}a\exists r{\le}q^{2^z} \quad \text{“}r \text{ is the Gödel number of a tableau proof of a contradiction from } T + \Delta + c{>}M + \exists y{\le}c_z\theta(y, c, \vec{c})\text{.”} \tag{4}$$

The case $z = 0$ clearly holds, by (3) above. For the inductive step, suppose that $b{\le}^M a$ and $r_1 \in M$ such that

$$M \models r_1{\le}q^{2^b} \wedge \text{“}r_1 \text{ is the Gödel number of a tableau proof of a contradiction from } T + \Delta + c{>}M + \exists y{\le}c_b\theta(y, c, \vec{c})\text{.”}$$

By (3), there exists $r_2 \in M$ such that

$$M \models r_2{\le}q \wedge \text{“}r_2 \text{ is the Gödel number of a tableau proof of a contradiction from } T + \Delta + c{>}M + \theta(c_{b+1}, c, \vec{c})\text{.”}$$

Recalling that the implication $y{\le}c_{b+1} \rightarrow y{\le}c_b \vee y{=}c_{b+1}$ is provable from $I\Delta_0$, one sees that, by combining proofs, it is possible to obtain $r \in M$ such that

$$M \models \text{“}r \text{ is the Gödel number of a tableau proof of a contradiction from } T + \Delta + c{>}M + \exists y{\le}c_{b+1}\theta(y, c, \vec{c})\text{.”}$$

An easy calculation shows that $M \models r{\le}q^{2^{b+1}}$, and so the proof of the inductive step is complete.

Setting $z = a$ in (2), we deduce that

$$M \models \exists r \leq q^{2^a} \quad \text{"}r \text{ is the Gödel number of a tableau proof of a contradiction from } T + \Delta + c{>}M + \exists y{\leq}c_a\theta(y,c,\vec{c})\text{,"}$$

and so

$$M \models \neg\text{Tabcon}\big(T + \Delta + c{>}M + \exists y{\leq}c_a\theta(y,c,\vec{c})\big).$$

But then, by (2) and Lemma 2.2, it follows that $M \models \neg\text{Tabcon}(T + \Delta + c{>}M)$, which contradicts (1). □

Now we return to the proof of Theorem 2.5. Using Lemma 2.6 and Lemma 2.7, one can construct a sequence $T_0 \subseteq T_1 \subseteq \cdots \subseteq T_n \subseteq \cdots$ of theories in LA* such that

(a) $M \models \text{Tabcon}(T_n + \Delta + c{>}M)$, for any $n \in \mathbb{N}$,
(b) $T_\infty = \bigcup_{n\in\mathbb{N}} T_n$ is a complete theory in LA*, containing the diagram of M and all sentences of the form $c{>}c_a, a \in M$,
(c) whenever $\exists y \leq c_a\theta(y,c,\vec{c}) \in T_\infty$, then there exists $b \leq a$ in M such that $\theta(c_b,c,\vec{c}) \in T_\infty$.

By applying the omitting types theorem now, we obtain a model K^* of T_∞ in which the interpretations of the constant symbols $\{c_a : a \in M\}$ form an initial segment. Let now K be the reduct of K^* to LA. Clearly, K is a model of $I\Delta_0$. Using the fact that T_∞ contains the diagram of M, it follows that M is isomorphic to an initial segment of K (via the identification of each $a \in M$ with the interpretation of c_a in K). Since $c{>}c_a \in T_\infty$, for each $a \in M$, M is actually (isomorphic to) a proper initial segment of K. Therefore, K has all the required properties. □

3 Other Conditions

Our aim in this section is to show that Theorem 2.5 holds, if we replace the assumption that M is closed under exponentiation by each of conditions (I), (III), (IV) of Theorem 1.11. Note that conditions (III)–(IV) contain the assumption that $I\Delta_0 \vdash \neg\Delta_0 H$, which may well be false. However, following [19], we consider it worthwhile to study how it affects Problem 2.

The argument when we adopt one of conditions (I), (III), (IV) is basically similar to that employed when $M \models$ exp. The main difference between the approach in Section 2 and the one taken in this section is that here we have to pay more attention to the behavior of the satisfaction formula Sat_0, so that we can keep on working with (modifications of) it even when M satisfies properties other than being a model of exp. In fact, assuming either one of conditions (III) and (IV), Sat_0 works with b significantly smaller than in Theorem 2.3; this is due to the following result from [16, p. 320].

Theorem 3.1 *Assuming $I\Delta_0 \vdash \neg\Delta_0 H$, the bound $2^{(\max(\vec{a})+2)^{\ulcorner\varphi\urcorner}}$ in Theorem 2.3 can be replaced by $(\max(\vec{a}) + 2)^{\ulcorner\varphi\urcorner}$.*

Remarks 3.2 The assumption $I\Delta_0 \vdash \neg\Delta_0 H$ is necessary only if we need to be able to talk about the satisfiability of all standard formulas. So, if we need to talk about the satisfiability of formulas with Gödel number $< k$, $k \in \mathbb{N}$, it suffices to know that $(\max(\vec{a}) + 2)^k$ exists, which is guaranteed in any model of $I\Delta_0$.

Independently of which condition we will be assuming, we have to work with a restricted form of the formula Tabcon(T). Indeed, we will be using k-Tabcon(T), which denotes the formula expressing "there is no tableau proof of a contradiction from T, using only substitution instances of formulas with Gödel number $\leq k$"; note that this is strongly reminiscent of the formula Con(X, k), which was introduced and used extensively in [18].

It is not difficult to check that Lemma 2.2 holds for the restricted form of the formula expressing the tableau consistency of a theory; that is, we can prove the following result.

Lemma 3.3 *For any model M of $I\Delta_0$ and $i \in M$, any theory T coded in M, and any sentence θ, if $M \models \neg i$-*Tabcon$(T + \theta)$ *and $M \models \neg i$-*Tabcon$(T + \neg\theta)$*, then $M \models \neg i$-*Tabcon(T).

Now we can proceed to proving the following variant of Theorem 2.5.

Theorem 3.4 *If M is a countable model of $B\Sigma_1$ satisfying one of conditions (I), (III), (IV) of Theorem 1.11, then there exists $K \models I\Delta_0$ such that $M \subset_e K$.*

Proof Letting M be as in the hypothesis, we use the same notation as in the proof of Theorem 2.5. Clearly, what we have to prove is modifications of Lemma 2.6 and Lemma 2.7.

Lemma 3.5

(a) If M satisfies condition (I) or (IV), then, for all $k \in \mathbb{N}$,

$$M \models k\text{-Tabcon}(I\Delta_0 + \Delta + c{>}M).$$

(b) If M satisfies condition (III), then there exists $j \in M - \mathbb{N}$ such that

$$M \models j\text{-Tabcon}(I\Delta_0 + \Delta + c{>}M).$$

Proof of Lemma 3.5. (a) We essentially repeat the proof of Lemma 2.6, noting that the formula Sat_0 is still at our disposal, in view of the remark just after Theorem 3.1.

(b) In this case, we can do better than when M satisfies condition (I) or (IV). Indeed, one can mimic the proof of Lemma 2.6, working with j-tableau proofs, for any nonstandard j much smaller than the γ of condition (III). □

Let us now proceed to the counterpart of Lemma 2.7.

Lemma 3.6 *(a) Assume M satisfies condition (I) or (IV). If $\theta(y, c, \vec{c})$ is a formula of* LA**, $a \in M$ and T is a finite extension of $I\Delta_0$ such that*

$$M \models k\text{-Tabcon}(T + \Delta + c{>}M), \quad \textit{for all } k \in \mathbb{N},$$

then either

$$M \models k\text{-Tabcon}\big(T + \Delta + c{>}M + \neg\exists y{\leq}c_a\theta(y, c, \vec{c})\big), \quad \textit{for all } k \in \mathbb{N},$$

or there exists $b{\leq}^M a$ such that

$$M \models k\text{-Tabcon}\big(T + \Delta + c{>}M + \theta(c_b, c, \vec{c})\big), \quad \textit{for all } k \in \mathbb{N}.$$

(b) Assume M satisfies condition (III). If $\theta(y, c, \vec{c})$ is a formula of LA**, $a \in M$ and T is a finite extension of $I\Delta_0$ such that*

$$M \models j\text{-Tabcon}(T + \Delta + c{>}M), \quad \textit{for some } j \in M - \mathbb{N},$$

then either

$$M \models j\text{-Tabcon}\big(T + \Delta + c{>}M + \neg\exists y{\le}c_a\theta(y,c,\vec{c})\big), \quad \textit{for some } j \in M - \mathbb{N},$$

or there exists $b{\le}^M a$ *such that*

$$M \models j\text{-Tabcon}\big(T + \Delta + c{>}M + \theta(c_b,c,\vec{c})\big), \quad \textit{for some } j \in M - \mathbb{N}.$$

Proof of Lemma 3.6. (a) First, we note that, as shown in [19], if M satisfies condition (IV), then it satisfies condition (I). Hence it suffices to work with M satisfying condition (I).

So let us assume M is short Π_1-recursively saturated and T is a finite extension of $I\Delta_0$ such that

$$M \models k\text{-Tabcon}(T + \Delta + c{>}M), \quad \text{for all } k \in \mathbb{N}, \tag{5}$$

$$M \models \neg k_0\text{-Tabcon}\big(T + \Delta + c{>}M + \neg\exists y{\le}c_a\theta(y,c,\vec{c})\big), \quad \text{for some } k_0 \in \mathbb{N}. \tag{6}$$

We will show that there exists $b \le^M a$ such that

$$M \models k\text{-Tabcon}\big(T + \Delta + c{>}M + \theta(c_b,c,\vec{c})\big), \quad \text{for all } k \in \mathbb{N}. \tag{7}$$

Observe that the set

$$Z = \big\{z \le a \wedge k\text{-Tabcon}\big(T + \Delta + c{>}M + \theta(c_b,c,\vec{c})\big) \;\big|\; k \in \mathbb{N}\big\}$$

is a recursive set of Π_1 formulas. We claim that Z is finitely satisfiable in M. Supposing not, there would be some $k_1,\dots,k_m \in \mathbb{N}$ such that

$$M \models \neg\exists z \le a \bigwedge_{1\le i\le m} k_i\text{-Tabcon}\big(T + \Delta + c{>}M + \theta(c_z,c,\vec{c})\big).$$

Letting $K = \max\{k_1,\dots,k_m\}$, we see that

$$M \models \forall z \le a \exists t \neg K\text{-Tabcon}\big(T + \Delta_t + c{>}t + \theta(c_z,c,\vec{c})\big),$$

where Δ_t denotes the restriction of the diagram to sentences involving constants with index less than t.

Since M satisfies $B\Sigma_1$, there exists $b \in M$ such that

$$M \models \forall z \le a \exists t \le b \neg K\text{-Tabcon}\big(T + \Delta_t + c{>}t + \theta(c_z,c,\vec{c})\big).$$

But now note that the size of a K-tableau proof from $T + \Delta_t + c{>}t + \theta(c_z,c,\vec{c})$ of a contradiction cannot exceed $\max(a,b)^L$, for some natural number L depending on K. Therefore, by an inductive argument similar to that used in the proof of Lemma 2.6, we can show that

$$M \models \neg K\text{-Tabcon}\big(T + \Delta + c{>}M + \exists y{\le}c_a\theta(y,c,\vec{c})\big).$$

But then, by (6) and Lemma 3.3, it follows that

$$M \models \neg L\text{-Tabcon}(T + \Delta + c{>}M),$$

with $L = \max(k_0, K)$, which contradicts (5).

It follows that Z is finitely satisfiable in M and so it is satisfied in M, by the saturation hypothesis about M. Therefore, there exists $b \le^M a$ such that (7) holds, as required.

(b) Suppose that M satisfies condition (III) and T is a finite extension of $I\Delta_0$ such that

$$M \models j_0\text{-Tabcon}(T + \Delta + c{>}M), \quad \text{for some } j_0 \in M - \mathbb{N}, \tag{8}$$

$$M \models \neg j\text{-Tabcon}\big(T + \Delta + c{>}M + \neg\exists y{\le}c_a\theta(y, c, \vec{c})\big), \quad \text{for all } j \in M - \mathbb{N}, \tag{9}$$

$$\text{for all } b \le^M a, \quad M \models \neg j\text{-Tabcon}\big(T + \Delta + c{>}M + \theta(c_b, c, \vec{c})\big), \quad \text{for all } j \in M - \mathbb{N}. \tag{10}$$

Clearly, (10) implies that

$$M \models \forall z \le a \exists t \neg j_0\text{-Tabcon}\big(T + \Delta_t + c{>}t + \theta(c_z, c, \vec{c})\big).$$

Since M satisfies $B\Sigma_1$, there exists $b \in M$ such that

$$M \models \forall z \le a \exists t \le b \neg j_0\text{-Tabcon}\big(T + \Delta_t + c{>}t + \theta(c_z, c, \vec{c})\big).$$

As in the first part of the proof, we observe that the size of a j_0-tableau proof of a contradiction from $T + \Delta_t + c{>}t + \theta(c_z, c, \vec{c})$ cannot exceed $\max(a, b)^{j_0}$. Therefore, one can use induction on z to prove that

$$M \models \neg j_0\text{-Tabcon}\big(T + \Delta + c{>}M + \exists y{\le}c_a\theta(y, c, \vec{c})\big). \tag{11}$$

But now, combining (11) with (9) and Lemma 3.3, it follows that

$$M \models \neg j_0\text{-Tabcon}(T + \Delta + c{>}M),$$

which contradicts (8). □

Returning to the proof of Theorem 3.4, we see that Lemma 3.5 and Lemma 3.6 enable us to construct a sequence of theories in LA* satisfying conditions (a)–(c) at the end of the proof of Theorem 2.5, the only difference being that the formula Tabcon$(\cdots)$ has to be replaced by its restricted version. Then we can apply the omitting types theorem as before to obtain a proper end extension $K \models I\Delta_0$ of M. □

4 Concluding Remarks

Whether or not one prefers to use the notion of fullness or the approach suggested by the ACT, several end extendability problems arise naturally, some of which we will discuss in what follows.

Some variants of Problem 2 concern the possibility of proving Theorem 2.5, if M satisfies an axiom weaker than exp. A particularly interesting case is when exp is replaced by Ω_1, that is, the axiom expressing "the function $x^{|x|}$ is total," where $|x|$ denotes the length of x.

Problem 4.1 Is every countable model M of $B\Sigma_1 + \Omega_1$ extendable to a model K of $I\Delta_0$ such that $M \subset_e K$?

It is well known (see, e.g., [10]) that the strength of $B\Sigma_1 + \Omega_1$ lies strictly between that of $B\Sigma_1$ and $B\Sigma_1 + \text{exp}$, so attacking Problem 4.1 seems worthwhile.

Another direction of further work concerns the issue of cardinality of the model M, whose end extendability is studied. Indeed, in view of Theorem 1.4 and the fact that every model of OI (open induction) is properly end extendable to a model of OI, which was proved by Boughattas [4, p. 714], it is natural to ask what happens for arbitrary M; that is, consider the following problems.

Problem 4.2 Is every model M of $B\Sigma_1 + \exp$ extendable to a model K of $I\Delta_0$ such that $M \subset_e K$?

Problem 4.3 Is every model M of $B\Sigma_1 + \Omega_1$ extendable to a model K of $I\Delta_0$ such that $M \subset_e K$?

Finally, we may consider variants of Problem 4.1, which regard the end extendability of models of theories weaker than $B\Sigma_1 + \Omega_1$. For example, one can replace $B\Sigma_1 + \Omega_1$ by $T_2^i + B\Sigma_i^b$, where, as usual, T_2^i denotes the theory of Σ_i^b-induction and $B\Sigma_i^b$ the theory of Σ_i^b-collection, as defined by Buss in [5] and [6].

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