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Guessing, Mind-Changing, and the Second Ambiguous Class

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Abstract In his dissertation, Wadge defined a notion of guessability on subsets of the Baire space and gave two characterizations of guessable sets. A set is guessable if and only if it is in the second ambiguous class ($\mathbf{\Delta}^0_2$), if and only if it is eventually annihilated by a certain remainder. We simplify this remainder and give a new proof of the latter equivalence. We then introduce a notion of guessing with an ordinal limit on how often one can change one's mind. We show that for every ordinal α, a guessable set is annihilated by α applications of the simplified remainder if and only if it is guessable with fewer than α mind changes. We use guessability with fewer than α mind changes to give a semi-characterization of the Hausdorff difference hierarchy, and indicate how Wadge's notion of guessability can be generalized to higher-order guessability, providing characterizations of $\mathbf{\Delta}^0_\alpha$ for all successor ordinals $\alpha > 1$.

1 Introduction

Let $\mathbb{N}^\mathbb{N}$ be the set of sequences $s : \mathbb{N} \to \mathbb{N}$, and let $\mathbb{N}^{<\mathbb{N}}$ be the set $\bigcup_n \mathbb{N}^n$ of finite sequences. If $s \in \mathbb{N}^{<\mathbb{N}}$, we will write $[s]$ for $\{f \in \mathbb{N}^\mathbb{N} : f \text{ extends } s\}$. We equip $\mathbb{N}^\mathbb{N}$ with a second-countable topology by declaring $[s]$ to be a basic open set whenever $s \in \mathbb{N}^{<\mathbb{N}}$.

Throughout the paper, S will denote a subset of $\mathbb{N}^\mathbb{N}$. We say that $S \in \mathbf{\Delta}^0_2$ if S is simultaneously a countable intersection of open sets and a countable union of closed sets in the above topology. In classic terminology, $S \in \mathbf{\Delta}^0_2$ just in case S is both G_δ and F_σ.

The following notion was discovered by Wadge [9, pp. 141–42] and independently by this author [1, Section 2, Definition 1, p. 2].[1]

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Definition 1.1 We say that S is *guessable* if there is a function $G : \mathbb{N}^{<\mathbb{N}} \to \{0, 1\}$ such that for every $f \in \mathbb{N}^{\mathbb{N}}$,

$$\lim_{n\to\infty} G(f \restriction n) = \chi_S(f) = \begin{cases} 1, & \text{if } f \in S, \\ 0, & \text{if } f \notin S. \end{cases}$$

If so, we say G *guesses* S, or that G is an S*-guesser*.

The intuition behind the above notion is captured eloquently by Wadge [9, p. 142] (notation changed):

> Guessing sets allow us to form an opinion as to whether an element f of $\mathbb{N}^{\mathbb{N}}$ is in S or S^c, given only a finite initial segment $f \restriction n$ of f.

Game theoretically, one envisions an asymmetric game where *II* (the guesser) has perfect information, *I* (the sequence chooser) has zero information, and *II*'s winning set consists of all sequences $(a_0, b_0, a_1, b_1, \ldots)$ such that $b_i \to 1$ if $(a_0, a_1, \ldots) \in S$ and $b_i \to 0$ otherwise.

The following result was proved in [9, pp. 144–45] by infinite game-theoretical methods. The present author found a second proof [1, Section 5, Theorem 25, p. 11] by using mathematical logical methods.

Theorem 1.2 (Wadge [9, pp. 144–45]) *The set* S *is guessable if and only if* $S \in \mathbf{\Delta}^0_2$.

Wadge defined the following remainder operation.

Definition 1.3 ([9, pp. 113–14]) For $A, B \subseteq \mathbb{N}^{\mathbb{N}}$, define $\mathrm{Rm}_0(A, B) = \mathbb{N}^{\mathbb{N}}$. For $\mu > 0$ an ordinal, define

$$\mathrm{Rm}_\mu(A, B) = \bigcap_{\nu<\mu} \left(\overline{\mathrm{Rm}_\nu(A, B) \cap A} \cap \overline{\mathrm{Rm}_\nu(A, B) \cap B} \right).$$

(Here $\overline{\bullet}$ denotes topological closure.) Write $\mathrm{Rm}_\mu(S)$ for $\mathrm{Rm}_\mu(S, S^c)$.

By countability considerations, there is some (in fact countable) ordinal μ, depending on S, such that $\mathrm{Rm}_\mu(S) = \mathrm{Rm}_{\mu'}(S)$ for all $\mu' \geq \mu$; Wadge writes $\mathrm{Rm}_\Omega(S)$ for $\mathrm{Rm}_\mu(S)$ for such a μ. He then proves the following theorem.

Theorem 1.4 (Wadge [9, Theorem B6, Section B, Chapter II, p. 46] attributed to Hausdorff) *We have* $S \in \mathbf{\Delta}^0_2$ *if and only if* $\mathrm{Rm}_\Omega(S) = \emptyset$.

In Section 2, we introduce a simpler remainder $(S, \alpha) \mapsto S_\alpha$ and use it to give a new proof of Theorem 1.4.

In Section 3, we introduce the notion of S being guessable while changing one's mind fewer than α many times ($\alpha \in \mathrm{Ord}$) and show that this is equivalent to $S_\alpha = \emptyset$.

In Section 4, we show that for $\alpha > 0$, S is guessable while changing one's mind fewer than $\alpha + 1$ many times if and only if at least one of S or S^c is in the αth level of the difference hierarchy.

In Section 5, we generalize guessability, introducing the notion of μth-order guessability ($1 \leq \mu < \omega_1$). We show that S is μth-order guessable if and only if $S \in \mathbf{\Delta}^0_{\mu+1}$.

2 Guessable Sets and Remainders

In this section we give a new proof of Theorem 1.4. We find it easier to work with the following remainder[2] which is closely related to the remainder defined by Wadge. For $X \subseteq \mathbb{N}^{<\mathbb{N}}$, we will write $[X]$ to denote the set of infinite sequences all of whose finite initial segments lie in X.

Definition 2.1 Let $S \subseteq \mathbb{N}^{\mathbb{N}}$. We define $S_\alpha \subseteq \mathbb{N}^{<\mathbb{N}}$ ($\alpha \in \mathrm{Ord}$) by transfinite recursion as follows. We define $S_0 = \mathbb{N}^{<\mathbb{N}}$, and $S_\lambda = \bigcap_{\beta<\lambda} S_\beta$ for every limit ordinal λ. Finally, for every ordinal β, we define

$$S_{\beta+1} = \left\{x \in S_\beta \,:\, \exists x', x'' \in [S_\beta] \text{ such that } x \subseteq x', x \subseteq x'', x' \in S, x'' \notin S\right\}.$$

We write $\alpha(S)$ for the minimal ordinal α such that $S_\alpha = S_{\alpha+1}$, and we write S_∞ for $S_{\alpha(S)}$.

Clearly $S_\alpha \subseteq S_\beta$ whenever $\beta < \alpha$. This remainder notion is related to Wadge's as follows.

Lemma 2.2 *For each ordinal α,* $\mathrm{Rm}_\alpha(S) = [S_\alpha]$.

Proof Since $S_\alpha \subseteq S_\beta$ whenever $\beta < \alpha$, for all α, we have $S_\alpha = \bigcap_{\beta<\alpha} S_{\beta+1}$ (with the convention that $\bigcap \emptyset = \mathbb{N}^{<\mathbb{N}}$). We will show by induction on α that $\mathrm{Rm}_\alpha(S) = [S_\alpha] = [\bigcap_{\beta<\alpha} S_{\beta+1}]$.

Suppose that $f \in [\bigcap_{\beta<\alpha} S_{\beta+1}]$. Let $\beta < \alpha$. Let $\mathcal{U}$ be an open set around f; we can assume that $\mathcal{U}$ is basic open, so $\mathcal{U} = [f_0]$, f_0 a finite initial segment of f. Since $f \in [\bigcap_{\beta<\alpha} S_{\beta+1}]$, $f_0 \in S_{\beta+1}$. Thus there are $x', x'' \in [S_\beta]$ extending f_0 (hence in $\mathcal{U}$), $x' \in S$, $x'' \notin S$. In other words, $x' \in [\bigcap_{\gamma<\beta} S_{\gamma+1}] \cap S$ and $x'' \in [\bigcap_{\gamma<\beta} S_{\gamma+1}] \cap S^c$. By induction, $x' \in \mathrm{Rm}_\beta(S) \cap S$ and $x'' \in \mathrm{Rm}_\beta(S) \cap S^c$. By arbitrariness of $\mathcal{U}$, $f \in \overline{\mathrm{Rm}_\beta(S) \cap S} \cap \overline{\mathrm{Rm}_\beta(S) \cap S^c}$. By arbitrariness of β, $f \in \mathrm{Rm}_\alpha(S)$.

The reverse inclusion is similar. □

Note that Lemma 2.2 does not say that $\mathrm{Rm}_\alpha(S) = \emptyset$ if and only if $S_\alpha = \emptyset$. It is (at least a priori) possible that $S_\alpha \neq \emptyset$ while $[S_\alpha] = \emptyset$. Lemma 2.2 does, however, imply that $\mathrm{Rm}_\Omega(S) = \emptyset$ if and only if $S_\infty = \emptyset$, since it is easy to see that if $[S_\alpha] = \emptyset$, then $S_{\alpha+1} = \emptyset$. Thus to prove Theorem 1.4, it suffices to show that S is guessable if and only if $S_\infty = \emptyset$. The $\Rightarrow$ direction requires no additional machinery.

Proposition 2.3 *If S is guessable, then $S_\infty = \emptyset$.*

Proof Let $G : \mathbb{N}^{<\mathbb{N}} \to \{0, 1\}$ be an S-guesser. Assume (for contradiction) that $S_\infty \neq \emptyset$, and let $\sigma_0 \in S_\infty$. We will build a sequence on whose initial segments G diverges, contrary to Definition 1.1. Inductively, suppose we have finite sequences $\sigma_0 \subset_{\neq} \cdots \subset_{\neq} \sigma_k$ in S_∞ such that $\forall 0 < i \leq k$, $G(\sigma_i) \equiv i \bmod 2$. Since $\sigma_k \in S_\infty = S_{\alpha(S)} = S_{\alpha(S)+1}$, there are $\sigma', \sigma'' \in [S_\infty]$, extending σ_k, with $\sigma' \in S$, $\sigma'' \notin S$. Choose $\sigma \in \{\sigma', \sigma''\}$ with $\sigma \in S$ if and only if k is even. Then $\lim_{n\to\infty} G(\sigma \restriction n) \equiv k + 1 \bmod 2$. Let $\sigma_{k+1} \subset \sigma$ properly extend σ_k such that $G(\sigma_{k+1}) \equiv k + 1 \bmod 2$. Note that $\sigma_{k+1} \in S_\infty$ since $\sigma \in [S_\infty]$.

By induction, there are $\sigma_0 \subset_{\neq} \sigma_1 \subset_{\neq} \cdots$ such that for $i > 0$, $G(\sigma_i) \equiv i \bmod 2$. This contradicts Definition 1.1 since $\lim_{n\to\infty} G((\bigcup_i \sigma_i) \restriction n)$ ought to converge. □

The $\Leftarrow$ direction requires a little machinery.

Definition 2.4 If $\sigma \in \mathbb{N}^{<\mathbb{N}}$, $\sigma \notin S_\infty$, let $\beta(\sigma)$ be the least ordinal such that $\sigma \notin S_{\beta(\sigma)}$.

Note that whenever $\sigma \notin S_\infty$, $\beta(\sigma)$ is a successor ordinal.

Lemma 2.5 *Suppose $\sigma \subseteq \tau$ are finite sequences. If $\tau \in S_\infty$, then $\sigma \in S_\infty$. And if $\sigma \notin S_\infty$, then $\beta(\tau) \leq \beta(\sigma)$.*

Proof It is enough to show that $\forall \beta \in \mathrm{Ord}$, if $\tau \in S_\beta$, then $\sigma \in S_\beta$. This is by induction on β, the limit and zero cases being trivial. Assume β is successor. If $\tau \in S_\beta$, this means $\tau \in S_{\beta-1}$ and there are $\tau', \tau'' \in [S_{\beta-1}]$ extending τ with $\tau' \in S$, $\tau'' \notin S$. Since τ' and τ'' extend τ, and τ extends σ, τ' and τ'' extend σ; and since $\sigma \in S_{\beta-1}$ (by induction), this shows $\sigma \in S_\beta$. □

Lemma 2.6 *Suppose $f : \mathbb{N} \to \mathbb{N}$, $f \notin [S_\infty]$. There is some i such that for all $j \geq i$, $f \upharpoonright j \notin S_\infty$ and $\beta(f \upharpoonright j) = \beta(f \upharpoonright i)$. Furthermore, $f \in [S_{\beta(f \upharpoonright i)-1}]$.*

Proof The first part follows from Lemma 2.5 and the well-foundedness of Ord. For the second part we must show that $f \upharpoonright k \in S_{\beta(f \upharpoonright i)-1}$ for every k. If $k \leq i$, then $f \upharpoonright k \in S_{\beta(f \upharpoonright i)-1}$ by Lemma 2.5. If $k \geq i$, then $\beta(f \upharpoonright k) = \beta(f \upharpoonright i)$, and so $f \upharpoonright k \in S_{\beta(f \upharpoonright i)-1}$ since it is in $S_{\beta(f \upharpoonright k)-1}$ by definition of β. □

Definition 2.7 If $S_\infty = \emptyset$, then we define $G_S : \mathbb{N}^{<\mathbb{N}} \to \{0, 1\}$ as follows. Let $\sigma \in \mathbb{N}^{<\mathbb{N}}$. Since $S_\infty = \emptyset$, $\sigma \notin S_\infty$, so $\sigma \in S_{\beta(\sigma)-1} \backslash S_{\beta(\sigma)}$. Since $\sigma \notin S_{\beta(\sigma)}$, this means for every two extensions x', x'' of σ in $[S_{\beta(\sigma)-1}]$, either $x', x'' \in S$ or $x', x'' \in S^c$. So either all extensions of σ in $[S_{\beta(\sigma)-1}]$ are in S, or all such extensions are in S^c.

- (i) If there are no extensions of σ in $[S_{\beta(\sigma)-1}]$, and $\mathrm{length}(\sigma) > 0$, then let $G_S(\sigma) = G_S(\sigma^-)$, where σ^- is obtained from σ by removing the last term.
- (ii) If there are no extensions of σ in $[S_{\beta(\sigma)-1}]$, and $\mathrm{length}(\sigma) = 0$, let $G_S(\sigma) = 0$.
- (iii) If there are extensions of σ in $[S_{\beta(\sigma)-1}]$ and they are all in S, define $G_S(\sigma) = 1$.
- (iv) If there are extensions of σ in $[S_{\beta(\sigma)-1}]$ and they are all in S^c, define $G_S(\sigma) = 0$.

Proposition 2.8 *If $S_\infty = \emptyset$, then G_S guesses S.*

Proof Assume $S_\infty = \emptyset$. Let $f \in S$. I will show that $G_S(f \upharpoonright n) \to 1$ as $n \to \infty$. Since $f \notin [S_\infty]$, let i be as in Lemma 2.6. I claim that $G_S(f \upharpoonright j) = 1$ whenever $j \geq i$. Fix $j \geq i$. We have $\beta(f \upharpoonright j) = \beta(f \upharpoonright i)$ by choice of i, and $f \in [S_{\beta(f \upharpoonright i)-1}] = [S_{\beta(f \upharpoonright j)-1}]$. Since $f \upharpoonright j$ has one extension (namely, f itself) in both $[S_{\beta(f \upharpoonright j)-1}]$ and S, $G_S(f \upharpoonright j) = 1$.

Identical reasoning shows that if $f \notin S$, then $\lim_{n\to\infty} G_S(f \upharpoonright n) = 0$. □

Theorem 2.9 *We have $S \in \boldsymbol{\Delta}^0_2$ if and only if $S_\infty = \emptyset$. That is, Theorem 1.4 is true.*

Proof The theorem is proved by combining Propositions 2.3 and 2.8 and Theorem 1.2. □

3 Guessing Without Changing One's Mind Too Often

In this section our goal is to tease out additional information about $\mathbf{\Delta}^0_2$ from the operation defined in Definition 2.1.

Definition 3.1 For each function G with domain $\mathbb{N}^{<\mathbb{N}}$, if $G(f\restriction(n+1))\neq G(f\restriction n)$ ($f\in\mathbb{N}^{\mathbb{N}}$, $n\in\mathbb{N}$), we say that G *changes its mind on* $f\restriction(n+1)$. Now let $\alpha\in\mathrm{Ord}$. We say that S is *guessable with fewer than* α *mind changes* if there is an S-guesser G along with a function $H:\mathbb{N}^{<\mathbb{N}}\to\alpha$ such that the following hold, where $f\in\mathbb{N}^{\mathbb{N}}$ and $n\in\mathbb{N}$:

(i) $H(f\restriction(n+1))\leq H(f\restriction n)$;
(ii) if G changes its mind on $f\restriction(n+1)$, then $H(f\restriction(n+1))<H(f\restriction n)$.

This notion bears some resemblance to the notion of a set $Z\subseteq\mathbb{N}$ being f-computably enumerable in Figueira et al. [4], or g-computably approximable in Nies [7].

Theorem 3.2 *For* $\alpha\in\mathrm{Ord}$*,* S *is guessable with fewer than* α *mind changes if and only if* $S_\alpha=\emptyset$*.*

Proof ($\Rightarrow$) Assume S is guessable with fewer than α mind changes. Let G,H be as in Definition 3.1. We claim that for all $\beta\in\mathrm{Ord}$, if $\sigma\in S_\beta$, then $H(\sigma)\geq\beta$. This will prove ($\Rightarrow$) because it implies that if $S_\alpha\neq\emptyset$, then there is some σ with $H(\sigma)\geq\alpha$, which is absurd since $\mathrm{codomain}(H)=\alpha$.

We attack the claim by induction on β. The zero and limit cases are trivial. Assume $\beta=\gamma+1$. Suppose $\sigma\in S_{\gamma+1}$. There are $x',x''\in[S_\gamma]$ extending σ, $x'\in S$, $x''\notin S$. Pick $x\in\{x',x''\}$ so that $\chi_S(x)\neq G(\sigma)$, and pick $\sigma^+\in\mathbb{N}^{<\mathbb{N}}$ with $\sigma\subseteq\sigma^+\subseteq x$ such that $G(\sigma^+)=\chi_S(x)$ (some such σ^+ exists since G guesses S). Since $x\in[S_\gamma]$, $\sigma^+\in S_\gamma$. By induction, $H(\sigma^+)\geq\gamma$. The fact $G(\sigma^+)\neq G(\sigma)$ implies $H(\sigma^+)<H(\sigma)$, forcing $H(\sigma)\geq\gamma+1$.

($\Leftarrow$) Assume $S_\alpha=\emptyset$. For all $\sigma\in\mathbb{N}^{<\mathbb{N}}$, define $H(\sigma)=\beta(\sigma)-1$ (by definition of $\beta(\sigma)$, since $S_\alpha=\emptyset$, clearly $H(\sigma)\in\alpha$). I claim that G_S,H witness that S is guessable with fewer than α mind changes.

By Proposition 2.8, G_S guesses S. Let $f\in\mathbb{N}^{\mathbb{N}}$, $n\in\mathbb{N}$. By Lemma 2.5, $H(f\restriction(n+1))\leq H(f\restriction n)$. Now suppose G_S changes its mind on $f\restriction(n+1)$; we must show $H(f\restriction(n+1))<H(f\restriction n)$. Assume for sake of contradiction that $H(f\restriction(n+1))=H(f\restriction n)$. Assume $G_S(f\restriction n)=0$; the other case is similar. By definition of G_S, ($*$) for every infinite extension f' of $f\restriction n$, if $f'\in[S_{\beta(f\restriction n)-1}]$, then $f'\in S^c$. Since G_S changes its mind on $f\restriction(n+1)$, $G_S(f\restriction(n+1))=1$. Thus ($**$) for every infinite extension f'' of $f\restriction(n+1)$, if $f''\in[S_{\beta(f\restriction(n+1))-1}]$, then $f''\in S$. And $f\restriction(n+1)$ does actually have some such infinite extension f'', because if it had none, that would make $G_S(f\restriction(n+1))=G_S(f\restriction n)$ by case 1 of the definition of G_S (see Definition 2.7). Being an extension of $f\restriction(n+1)$, f'' also extends $f\restriction n$; and by the assumption that $H(f\restriction(n+1))=H(f\restriction n)$, $f''\in[S_{\beta(f\restriction n)-1}]$. By ($*$), $f''\in S^c$, and by ($**$), $f''\in S$, which is absurd. □

It is not hard to show S is a Boolean combination of open sets if and only if S is guessable with fewer than ω mind changes, so Theorem 3.2 and Lemma 2.2 give a new proof of a special case of the main theorem of Dougherty and Miller [3, p. 1348] (see also Allouche [2]).

4 Mind Changing and the Difference Hierarchy

We recall the following definition from Kechris [5, p. 175] (stated in greater generality—we specialize it to the Baire space). In this definition, $\mathbf{\Sigma}^0_1(\mathbb{N}^\mathbb{N})$ is the set of open subsets of $\mathbb{N}^\mathbb{N}$, and the *parity* of an ordinal η is the equivalence class modulo 2 of n, where $\eta = \lambda + n$, λ a limit ordinal (or $\lambda = 0$), $n \in \mathbb{N}$.

Definition 4.1 Let $(A_\eta)_{\eta<\theta}$ be an increasing sequence of subsets of $\mathbb{N}^\mathbb{N}$ with $\theta \geq 1$. Define the set $D_\theta((A_\eta)_{\eta<\theta}) \subseteq \mathbb{N}^\mathbb{N}$ by

$x \in D_\theta((A_\eta)_{\eta<\theta}) \Leftrightarrow x \in \bigcup_{\eta<\theta} A_\eta$ and the least $\eta < \theta$ with $x \in A_\eta$ has parity opposite to that of θ.

Let

$$D_\theta(\mathbf{\Sigma}^0_1)(\mathbb{N}^\mathbb{N}) = \left\{D_\theta\big((A_\eta)_{\eta<\theta}\big) \,:\, A_\eta \in \mathbf{\Sigma}^0_1(\mathbb{N}^\mathbb{N}),\ \eta < \theta\right\}.$$

This hierarchy offers a constructive characterization of $\mathbf{\Delta}^0_2$: it turns out that

$$\mathbf{\Delta}^0_2 = \bigcup_{1\leq\theta<\omega_1} D_\theta(\mathbf{\Sigma}^0_1)(\mathbb{N}^\mathbb{N})$$

(see [5, Theorem 22.27, p. 176], attributed to Hausdorff and Kuratowski).

For brevity, we will write D_α for $D_\alpha(\mathbf{\Sigma}^0_1)(\mathbb{N}^\mathbb{N})$.

Theorem 4.2 (Semicharacterization of the difference hierarchy) *Let $\alpha > 0$. The following are equivalent:*

(i) S is guessable with fewer than $\alpha + 1$ mind changes; and
(ii) $S \in D_\alpha$ or $S^c \in D_\alpha$.

We will prove Theorem 4.2 by a sequence of smaller results.

Definition 4.3 For $\alpha, \beta \in \mathrm{Ord}$, write $\alpha \equiv \beta$ to indicate that α and β have the same parity (i.e., $2 \mid n - m$, where $\alpha = \lambda + n$ and $\beta = \kappa + m$, $n, m \in \mathbb{N}$, λ a limit ordinal or 0, κ a limit ordinal or 0).

Proposition 4.4 *Let $\alpha > 0$. If $S \in D_\alpha$, say, $S = D_\alpha((A_\eta)_{\eta<\alpha})$ ($A_\eta \subseteq \mathbb{N}^\mathbb{N}$ open), then S is guessable with fewer than $\alpha + 1$ mind changes.*

Proof Define $G : \mathbb{N}^{<\mathbb{N}} \to \{0, 1\}$ and $H : \mathbb{N}^{<\mathbb{N}} \to \alpha + 1$ as follows. Suppose $\sigma \in \mathbb{N}^{<\mathbb{N}}$. If there is no $\eta < \alpha$ such that $[\sigma] \subseteq A_\eta$, let $G(\sigma) = 0$, and let $H(\sigma) = \alpha$. If there is an $\eta < \alpha$ (we may take η minimal) such that $[\sigma] \subseteq A_\eta$, then let

$$G(\sigma) = \begin{cases} 0, & \text{if } \eta \equiv \alpha; \\ 1, & \text{if } \eta \not\equiv \alpha, \end{cases} \qquad\qquad H(\sigma) = \eta.$$

Let $f : \mathbb{N} \to \mathbb{N}$.

Claim 1 *We have* $\lim_{n\to\infty} G(f \upharpoonright n) = \chi_S(f)$.

If $f \notin \bigcup_{\eta<\alpha} A_\eta$, then $f \notin D_\alpha((A_\eta)_{\eta<\alpha}) = S$, and $G(f \upharpoonright n)$ will always be 0, so $\lim_{n\to\infty} G(f \upharpoonright n) = 0 = \chi_S(f)$. Assume $f \in \bigcup_{\eta<\alpha} A_\eta$, and let $\eta < \alpha$ be minimum such that $f \in A_\eta$. Since A_η is open, there is some n_0 so large that $\forall n \geq n_0$, $[f \upharpoonright n] \subseteq A_\eta$. For all $n \geq n_0$, by minimality of η, $[f \upharpoonright n] \not\subseteq A_{\eta'}$ for any $\eta' < \eta$, so $G(f \upharpoonright n) = 0$ if and only if $\eta \equiv \alpha$. The following are equivalent:

$$f \in S \text{ iff } f \in D_\alpha\big((A_\eta)_{\eta<\alpha}\big)$$
$$\text{iff } \eta \not\equiv \alpha$$

$$\text{iff } G(f \upharpoonright n) \neq 0$$
$$\text{iff } G(f \upharpoonright n) = 1.$$

This shows that $\lim_{n\to\infty} G(f \upharpoonright n) = \chi_S(f)$.

Claim 2 *We have* $\forall n \in \mathbb{N}$, $H(f \upharpoonright (n+1)) \leq H(f \upharpoonright n)$.

If $H(f \upharpoonright n) = \alpha$, there is nothing to prove. If $H(f \upharpoonright n) < \alpha$, then $H(f \upharpoonright n) = \eta$, where η is minimal such that $[f \upharpoonright n] \subseteq A_\eta$. Since $[f \upharpoonright (n+1)] \subseteq [f \upharpoonright n]$, we have $[f \upharpoonright (n+1)] \subseteq A_\eta$, implying $H(f \upharpoonright (n+1)) \leq \eta$.

Claim 3 *For all* $n \in \mathbb{N}$, *if* $G(f \upharpoonright (n+1)) \neq G(f \upharpoonright n)$, *then* $H(f \upharpoonright (n+1)) < H(f \upharpoonright n)$.

Assume (for the sake of contradiction) $H(f \upharpoonright (n+1)) \geq H(f \upharpoonright n)$. By Claim 2, $H(f \upharpoonright (n+1)) = H(f \upharpoonright n)$. By definition of H this implies that $\forall \eta < \alpha$, $[f \upharpoonright (n+1)] \subseteq A_\eta$ if and only if $[f \upharpoonright n] \subseteq A_\eta$. This implies $G(f \upharpoonright (n+1)) = G(f \upharpoonright n)$, a contradiction.

By Claims 1–3, G and H witness that S is guessable with fewer than $\alpha + 1$ mind changes. □

Corollary 4.5 *Let* $\alpha > 0$. *If* $S \in D_\alpha$ *or* $S^c \in D_\alpha$, *then* S *is guessable with fewer than* $\alpha + 1$ *mind changes.*

Proof If $S \in D_\alpha$, this is immediate by Proposition 4.4. If $S^c \in D_\alpha$, then Proposition 4.4 says that S^c is guessable with fewer than $\alpha + 1$ mind changes, and this clearly implies that S is too. □

Lemma 4.6 *Suppose* S *is guessable with fewer than* α *mind changes. Let* $G : \mathbb{N}^{<\mathbb{N}} \to \{0,1\}$, $H : \mathbb{N}^{<\mathbb{N}} \to \alpha$ *be a pair of functions witnessing as much (see Definition 3.1). There is an* $H' : \mathbb{N}^{<\mathbb{N}} \to \alpha$ *such that* G, H' *also witness that* S *is guessable with fewer than* α *mind changes, with* $H'(\emptyset) = H(\emptyset)$, *and with the additional property that for every* $f : \mathbb{N} \to \mathbb{N}$ *and every* $n \in \mathbb{N}$,

$$H'(f \upharpoonright (n+1)) \equiv H'(f \upharpoonright n) \text{ if and only if } G(f \upharpoonright (n+1)) = G(f \upharpoonright n).$$

Proof Define $H'(\sigma)$ by induction on the length of σ as follows. Let $H'(\emptyset) = H(\emptyset)$. If $\sigma \neq \emptyset$, write $\sigma = \sigma_0 \frown n$ for some $n \in \mathbb{N}$ ($\frown$ denotes concatenation). If $G(\sigma) = G(\sigma_0)$, let $H'(\sigma) = H'(\sigma_0)$. Otherwise, let $H'(\sigma)$ be either $H(\sigma)$ or $H(\sigma) + 1$, whichever has parity opposite to $H'(\sigma_0)$.

By construction, H' has the desired parity properties. A simple inductive argument shows that $(*)$ $\forall \sigma \in \mathbb{N}^{<\mathbb{N}}$, $H(\sigma) \leq H'(\sigma) < \alpha$. I claim that for all $f : \mathbb{N} \to \mathbb{N}$ and $n \in \mathbb{N}$, $H'(f \upharpoonright (n+1)) \leq H'(f \upharpoonright n)$, and if $G(f \upharpoonright (n+1)) \neq G(f \upharpoonright n)$, then $H'(f \upharpoonright (n+1)) < H'(f \upharpoonright n)$.

If $G(f \upharpoonright (n+1)) = G(f \upharpoonright n)$, then by definition $H'(f \upharpoonright (n+1)) = H'(f \upharpoonright n)$ and the claim is trivial. Now assume $G(f \upharpoonright (n+1)) \neq G(f \upharpoonright n)$. If $H'(f \upharpoonright (n+1)) = H(f \upharpoonright (n+1))$, then $H'(f \upharpoonright (n+1)) < H(f \upharpoonright n) \leq H'(f \upharpoonright n)$, and we are done. Assume

$$H'\big(f \upharpoonright (n+1)\big) \neq H\big(f \upharpoonright (n+1)\big),$$

which forces that $(**)$ $H'(f \upharpoonright (n+1)) = H(f \upharpoonright (n+1)) + 1$. To see that

$$H'\big(f \upharpoonright (n+1)\big) < H'(f \upharpoonright n),$$

assume not $(***)$. By Definition 3.1, $H(f \restriction (n+1)) < H(f \restriction n)$, so

$$\begin{aligned} H(f \restriction n) &\geq H\big(f \restriction (n+1)\big) + 1 && \text{(basic arithmetic)} \\ &= H'\big(f \restriction (n+1)\big) && \text{(by } (**)) \\ &\geq H'(f \restriction n) && \text{(by } (***)) \\ &\geq H(f \restriction n). && \text{(by } (*)) \end{aligned}$$

Equality holds throughout, and $H'(f \restriction (n+1)) = H'(f \restriction n)$. Contradiction: we chose $H'(f \restriction (n+1))$ with parity opposite to $H'(f \restriction n)$. □

Definition 4.7 For all G, H as in Definition 3.1, $f \in \mathbb{N}^{\mathbb{N}}$, write $G(f)$ for $\lim_{n\to\infty} G(f \restriction n)$ (so $G(f) = \chi_S(f)$), and write $H(f)$ for $\lim_{n\to\infty} H(f \restriction n)$. Write $G \equiv H$ to indicate that $\forall f \in \mathbb{N}^{\mathbb{N}}$, $G(f) \equiv H(f)$; write $G \not\equiv H$ to indicate that $\forall f \in \mathbb{N}^{\mathbb{N}}$, $G(f) \not\equiv H(f)$ (we pronounce $G \not\equiv H$ as "G is anticongruent to H").

Lemma 4.8 *Suppose $G : \mathbb{N}^{<\mathbb{N}} \to \{0,1\}$ and $H : \mathbb{N}^{<\mathbb{N}} \to \alpha$ witness that S is guessable with fewer than α mind changes. There is an $H' : \mathbb{N}^{<\mathbb{N}} \to \alpha$ such that G, H' witness that S is guessable with fewer than α mind changes, and such that the following hold.*

If $G(\emptyset) \equiv \alpha$, then $H' \not\equiv G$. *If $G(\emptyset) \not\equiv \alpha$, then $H' \equiv G$.*

Proof I claim that without loss of generality, we may assume the following $(*)$.

If $G(\emptyset) \equiv \alpha$, then $H(\emptyset) \not\equiv G(\emptyset)$. If $G(\emptyset) \not\equiv \alpha$, then $H(\emptyset) \equiv G(\emptyset)$.

To see this, suppose not: either $G(\emptyset) \equiv \alpha$ and $H(\emptyset) \equiv G(\emptyset)$, or else $G(\emptyset) \not\equiv \alpha$ and $H(\emptyset) \not\equiv G(\emptyset)$. In either case, $H(\emptyset) \equiv \alpha$. If $H(\emptyset) \equiv \alpha$, then $H(\emptyset) + 1 \not\equiv \alpha$, and so, since $H(\emptyset) < \alpha$, $H(\emptyset) + 1 < \alpha$, meaning we may add 1 to $H(\emptyset)$ to enforce the assumption.

Having assumed $(*)$, we may use Lemma 4.6 to construct $H' : \mathbb{N}^{<\mathbb{N}} \to \alpha$ such that G, H' witness that S is guessable with fewer than α mind changes, $H'(\emptyset) = H(\emptyset)$, and H' changes parity precisely when G changes parity. The latter facts, combined with $(*)$, prove the lemma. □

Proposition 4.9 *Suppose $G : \mathbb{N}^{<\mathbb{N}} \to \{0,1\}$ and $H : \mathbb{N}^{<\mathbb{N}} \to \alpha + 1$ witness that S is guessable with fewer than $\alpha + 1$ mind changes. If $G(\emptyset) = 0$, then $S \in D_\alpha$.*

Proof By Lemma 4.8, we may safely assume the following.

If $G(\emptyset) \equiv \alpha + 1$, then $H \not\equiv G$. If $G(\emptyset) \not\equiv \alpha + 1$, then $H \equiv G$.

In other words, we have the following.

$(*)$ If $G(\emptyset) \equiv \alpha$, then $H \equiv G$. $(**)$ If $G(\emptyset) \not\equiv \alpha$, then $H \not\equiv G$.

For each $\eta < \alpha$, let

$$A_\eta = \big\{f \in \mathbb{N}^{\mathbb{N}} : H(f) \leq \eta\big\} \qquad (H(f) \text{ as in Definition 4.7}).$$

I claim that $S = D_\alpha((A_\eta)_{\eta<\alpha})$, which will prove the proposition since each A_η is clearly open.

Suppose $f \in S$. I will show $f \in D_\alpha((A_\eta)_{\eta<\alpha})$. Since $f \in S$, $H(f) \neq \alpha$, because if $H(f)$ were $= \alpha$, this would imply that G never changes its mind on f, forcing $\lim_{n\to\infty} G(f \upharpoonright n) = \lim_{n\to\infty} G(\emptyset) = 0$, contradicting the fact that G guesses S.

Since $H(f) \not\equiv \alpha$, $H(f) < \alpha$. It follows that for $\eta = H(f)$, we have $f \in A_\eta$ and η is minimal with this property.

Case 1: $G(\emptyset) \equiv \alpha$. By $(*)$, $H \equiv G$. Since $f \in S$, $\lim_{n\to\infty} G(f \upharpoonright n) = 1$, so $\eta = \lim_{n\to\infty} H(f \upharpoonright n) \equiv 1$. Since $\alpha \equiv G(\emptyset) = 0$, this shows $\eta \not\equiv \alpha$, putting $f \in D_\alpha((A_\eta)_{\eta<\alpha})$.

Case 2: $G(\emptyset) \not\equiv \alpha$. By $(**)$, $H \not\equiv G$. Since $f \in S$, $\lim_{n\to\infty} G(f \upharpoonright n) = 1$, so $\eta = \lim_{n\to\infty} H(f \upharpoonright n) \equiv 0$. Since $\alpha \not\equiv G(\emptyset) = 0$, this shows $\eta \not\equiv \alpha$, so $f \in D_\alpha((A_\eta)_{\eta<\alpha})$.

Conversely, suppose $f \in D_\alpha((A_\eta)_{\eta<\alpha})$. I will show $f \in S$. Let η be minimal such that $f \in A_\eta$ (by definition of A_η, $\eta = H(f)$). By definition of $D_\alpha((A_\eta)_{\eta<\alpha})$, $\eta \not\equiv \alpha$.

Case 1: $G(\emptyset) \equiv \alpha$. By $(*)$, $H \equiv G$. Since $\lim_{n\to\infty} H(f \upharpoonright n) = H(f) = \eta \not\equiv \alpha \equiv G(\emptyset) = 0$, we see $\lim_{n\to\infty} H(f \upharpoonright n) = 1$. Since $H \equiv G$, $\lim_{n\to\infty} G(f \upharpoonright n) = 1$, forcing $f \in S$ since G guesses S.

Case 2: $G(\emptyset) \not\equiv \alpha$. By $(**)$, $H \not\equiv G$. Since

$$\lim_{n\to\infty} H(f \upharpoonright n) = H(f) = \eta \not\equiv \alpha \not\equiv G(\emptyset) = 0,$$

we see $\lim_{n\to\infty} H(f \upharpoonright n) = 0$. Since $H \not\equiv G$, $\lim_{n\to\infty} G(f \upharpoonright n) = 1$, again showing $f \in S$. □

Corollary 4.10 *If S is guessable with fewer than $\alpha+1$ mind changes, then $S \in D_\alpha$ or $S^c \in D_\alpha$.*

Proof Let G, H witness that S is guessable with fewer than $\alpha + 1$ mind changes. If $G(\emptyset) = 0$, then $S \in D_\alpha$ by Proposition 4.9. If not, then $(1 - G), H$ witness that S^c is guessable with fewer than $\alpha + 1$ mind changes, and $(1 - G)(\emptyset) = 0$, so $S^c \in D_\alpha$ by Proposition 4.9. □

Combining Corollaries 4.5 and 4.10 proves Theorem 4.2.

5 Higher-Order Guessability

In this section we introduce a notion that generalizes guessability to provide a characterization for $\mathbf{\Delta}^0_{\mu+1}$ $(1 \le \mu < \omega_1)$. We will show that $S \in \mathbf{\Delta}^0_{\mu+1}$ if and only if S is μth-order guessable. Throughout this section, μ denotes an ordinal in $[1, \omega_1)$.

Definition 5.1 Let $\mathcal{S} = (S_0, S_1, \ldots)$ be a countably infinite tuple of subsets $S_i \subseteq \mathbb{N}^\mathbb{N}$.

(i) For every $f \in \mathbb{N}^\mathbb{N}$, write $\mathcal{S}(f)$ for the sequence $(\chi_{S_0}(f), \chi_{S_1}(f), \ldots) \in \{0,1\}^\mathbb{N}$.

(ii) We say that S is *guessable based on* $\mathcal{S}$ if there is a function

$$G : \{0,1\}^{<\mathbb{N}} \to \{0,1\}$$

(called an *S-guesser based on* $\mathcal{S}$) such that $\forall f \in \mathbb{N}^\mathbb{N}$,

$$\lim_{n\to\infty} G\big(\mathcal{S}(f) \upharpoonright n\big) = \chi_S(f).$$

Game theoretically, we envision a game where I (the sequence chooser) has zero information and where II (the guesser) has possibly *better-than-perfect* information: II is allowed to ask (once per turn) whether I's sequence lies in various S_i. For each S_i, player I's act (by answering the question) of committing to play a sequence in S_i or in S_i^c is similar to the act (described in Martin [6, p. 366]) of choosing a I-imposed subgame.

Example 5.2 If $\mathcal{S}$ enumerates the sets of the form $\{f \in \mathbb{N}^{\mathbb{N}} : f(i) = j\}$, $i, j \in \mathbb{N}$, then it is not hard to show that S is guessable (in the sense of Definition 1.1) if and only if S is guessable based on $\mathcal{S}$.

Definition 5.3 We say that S is *μth-order guessable* if there is some $\mathcal{S} = (S_0, S_1, \ldots)$ as in Definition 5.1 such that the following hold:

(i) S is guessable based on $\mathcal{S}$;
(ii) $\forall i$, $S_i \in \mathbf{\Delta}^0_{\mu_i+1}$ for some $\mu_i < \mu$.

Theorem 5.4 *The set S is μth-order guessable if and only if $S \in \mathbf{\Delta}^0_{\mu+1}$.*

To prove Theorem 5.4 we will assume the following result, which is a specialization and rephrasing of [5, Exercise 22.17, pp. 172–73] (attributed to Kuratowski).

Lemma 5.5 *The following are equivalent.*

(i) $S \in \mathbf{\Delta}^0_{\mu+1}$.
(ii) There is a sequence $(A_i)_{i\in\mathbb{N}}$, each $A_i \in \mathbf{\Delta}^0_{\mu_i+1}$ for some $\mu_i < \mu$, such that

$$S = \bigcup_n \bigcap_{m \geq n} A_m = \bigcap_n \bigcup_{m \geq n} A_m.$$

Proof of Theorem 5.4 ($\Rightarrow$) Let $\mathcal{S} = (S_0, S_1, \ldots)$, and let G witness that S is μth-order guessable (so each $S_i \in \mathbf{\Delta}^0_{\mu_i+1}$ for some $\mu_i < \mu$). For all $a \in \{0,1\}$ and $X \subseteq \mathbb{N}^{\mathbb{N}}$, define

$$X^a = \begin{cases} X, & \text{if } a = 1; \\ \mathbb{N}^{\mathbb{N}} \backslash X, & \text{if } a = 0. \end{cases}$$

For notational convenience, we will write "$G(\vec{a}) = 1$" as an abbreviation for "$0 \leq a_0, \ldots, a_{m-1} \leq 1$ and $G(a_0, \ldots, a_{m-1}) = 1$," provided m is clear from context. Observe that for all $f \in \mathbb{N}^{\mathbb{N}}$ and $m \in \mathbb{N}$, $G(\mathcal{S}(f) \upharpoonright m) = 1$ if and only if

$$f \in \bigcup_{G(\vec{a})=1} \bigcap_{j=0}^{m-1} S_j^{a_j}.$$

Now, given $f : \mathbb{N} \to \mathbb{N}$, $f \in S$ if and only if $G(\mathcal{S}(f) \upharpoonright n) \to 1$, which is true if and only if $\exists n \forall m \geq n$, $G(\mathcal{S}(f) \upharpoonright m) = 1$. Thus

$$\begin{aligned} f \in S &\text{ iff } \exists n \forall m \geq n,\ G\big(\mathcal{S}(f) \upharpoonright m\big) = 1 \\ &\text{ iff } \exists n \forall m \geq n,\ f \in \bigcup_{G(\vec{a})=1} \bigcap_{j=0}^{m-1} S_j^{a_j} \\ &\text{ iff } f \in \bigcup_n \bigcap_{m \geq n} \bigcup_{G(\vec{a})=1} \bigcap_{j=0}^{m-1} S_j^{a_j}. \end{aligned}$$

So

$$S = \bigcup_n \bigcap_{m \geq n} \bigcup_{G(\vec{a})=1} \bigcap_{j=0}^{m-1} S_j^{a_j}.$$

At the same time, since $G(\mathcal{S}(f) \upharpoonright m) \to 0$ whenever $f \notin S$, we see $f \in S$ if and only if $\forall n \exists m \geq n$ such that $G(\mathcal{S}(f) \upharpoonright m) = 1$. Thus by similar reasoning to the above,

$$S = \bigcap_n \bigcup_{m \geq n} \bigcup_{G(\vec{a})=1} \bigcap_{j=0}^{m-1} S_j^{a_j}.$$

For each m, $\bigcup_{G(\vec{a})=1} \bigcap_{j=0}^{m-1} S_j^{a_j}$ is a finite union of finite intersections of sets in $\mathbf{\Delta}^0_{\mu'+1}$ for various $\mu' < \mu$, thus $\bigcup_{G(\vec{a})=1} \bigcap_{j=0}^{m-1} S_j^{a_j}$ itself is in $\mathbf{\Delta}^0_{\mu_m+1}$ for some $\mu_m < \mu$. Letting $A_m = \bigcup_{G(\vec{a})=1} \bigcap_{j=0}^{m-1} S_j^{a_j}$, Lemma 5.5 says $S \in \mathbf{\Delta}^0_{\mu+1}$.

($\Leftarrow$) Assume $S \in \mathbf{\Delta}^0_{\mu+1}$. By Lemma 5.5, there are $(A_i)_{i \in \mathbb{N}}$, each $A_i \in \mathbf{\Delta}^0_{\mu_i+1}$ for some $\mu_i < \mu$, such that

$$S = \bigcup_n \bigcap_{m \geq n} A_m = \bigcap_n \bigcup_{m \geq n} A_m. \qquad (*)$$

I claim that S is guessable based on $\mathcal{S} = (A_0, A_1, \ldots)$. Define $G : \{0,1\}^{<\mathbb{N}} \to \{0,1\}$ by $G(a_0, \ldots, a_m) = a_m$. I will show that G is an S-guesser based on $\mathcal{S}$.

Suppose $f \in S$. By $(*)$, $\exists n$ s.t. $\forall m \geq n$, $f \in A_m$ and thus $\chi_{A_m}(f) = 1$. For all $m \geq n$,

$$\begin{aligned} G\big(\mathcal{S}(f) \upharpoonright (m+1)\big) &= G\big(\chi_{A_0}(f), \ldots, \chi_{A_m}(f)\big) \\ &= \chi_{A_m}(f) \\ &= 1, \end{aligned}$$

so $\lim_{n \to \infty} G(\mathcal{S}(f) \upharpoonright n) = 1$. A similar argument shows that if $f \notin S$, then $\lim_{n \to \infty} G(\mathcal{S}(f) \upharpoonright n) = 0$. □

Combining Theorems 1.2 and 5.4, we see that S is guessable if and only if S is 1st-order guessable. It is also not difficult to give a direct proof of this equivalence, and having done so, Theorem 5.4 provides yet another proof of Theorem 1.2.

Notes

1. A third independent usage of the term *guessable*, with similar but not the same meaning, appears in Tsaban and Zdomskyy [8, p. 1280], where a subset $Y \subseteq \mathbb{N}^{\mathbb{N}}$ is called *guessable* if there is a function $g \in \mathbb{N}^{\mathbb{N}}$ such that for each $f \in Y$, $g(n) = f(n)$ for infinitely many n.

2. In general, there seems to be a correspondence between remainders on $\mathbb{N}^{\mathbb{N}}$ and remainders on $\mathbb{N}^{<\mathbb{N}}$ that take trees to trees; in the future we might publish more general work based on this observation.

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