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From Closure Games to Strong Kleene Truth

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Abstract In this paper, we study *the method of closure games*, a game-theoretic valuation method for languages of self-referential truth developed by the author. We prove two theorems which jointly establish that the method of closure games characterizes all 3- and 4-valued strong Kleene fixed points in a novel, informative manner. Among others, we also present closure games which induce the minimal and maximal intrinsic fixed point of the strong Kleene schema.

1 Introduction

Take a language of self-referential truth L_T, that is, a first-order language L with a truth predicate symbol T added to it and in which Liar sentences and their ilk are expressible. Which sentences of L_T are assertible? Which are deniable? These questions are answered by a *theory of truth*, by which we mean

> a theory that purports to explain for a first-order language L_T what sentences are assertible [and deniable] in a [ground] model M. Gupta [2, p. 19]

The *ground model* M that Gupta is alluding to is a classical model for L, the truth-free fragment of L_T. A ground model M equips the sentences of L with a classical valuation $\mathcal{C}_M : Sen(L) \rightarrow \{\mathbf{a}, \mathbf{d}\}$ which determines which sentences of L are **a**ssertible and **d**eniable and which is defined as usual.[1] A theory of truth extends $\mathcal{C}_M$ to a valuation of L_T and exploits the extended valuation to specify which sentences of L_T are assertible/deniable in M.

In his seminal paper on truth, Kripke [4] specified an inductive method which allows one to define 3-valued *fixed-point theories of truth*. The valuations associated with these theories, called 3-*valued fixed points*, satisfy the *identity of truth*.

$$\text{For all } \sigma \in Sen(L_T)\text{: } V_M\big(T(t)\big) = V_M(\sigma)\text{, whenever } t \text{ denotes } \sigma \text{ in } M.$$

Satisfying the identity of truth formally ensures that

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> we are entitled to assert (or deny) of any sentence that it is true precisely under the circumstances when we can assert (or deny) the sentence itself. Kripke [4, p. 701]

Kripke showed how to define 3-valued fixed points for various 3-valued valuation schemas, which differ in the way in which they evaluate logically complex sentences. The fixed points associated with the strong Kleene (SK) schema, 3-*valued SK fixed points*, have arguably attracted the most attention in the literature on truth. For both philosophical and technical reasons, Visser [6] and Woodruff [11] have generalized Kripke's 3-valued *SK* fixed points to 4-valued ones. A 4-*valued SK fixed point* satisfies the identity of truth and evaluates logically complex sentences according to the Dunn–Belnap (or extended SK) schema, which naturally generalizes the SK schema to 4-valued logic.

In this paper, we present *the method of closure games* which allows us to construct 3- and 4-valued *SK* fixed points in a novel and informative manner. As suggested by its name, the method of closure games is a *game-theoretic* framework in which 3- and 4-valued *SK* fixed points can be characterized and studied. A game-theoretic approach to theories of truth is not completely novel. For instance, Martin [5] has shown how the *minimal* (3-valued) *SK* fixed point over a ground model M, which we denote as $\mathcal{K}_M$, can be obtained as the outcome of a *game*, meaning a *2-player perfect information game*.[2] Martin shows the following.

Martin's game for $\mathcal{K}_M$ There is a game G_σ for the minimal *SK* fixed-point $\mathcal{K}_M$ over a ground model M so that player I has a winning strategy in G_σ if and only if $\mathcal{K}_M(\sigma) = \mathbf{a}$; player *II* has a winning strategy if and only if $\mathcal{K}_M(\sigma) = \mathbf{d}$; if neither player has a winning strategy, then $\mathcal{K}_M(\sigma) \notin \{\mathbf{a}, \mathbf{d}\}$ and the game can be declared a draw.

Welch [7] extends Martin's work and shows how games can be used to induce the minimal fixed point associated with the *supervaluation* schema, and he also shows how certain Herzbergian style revision sequences can be characterized via such games. However, a systematic study of the relation between 2-player perfect information games and (the class of all) 3- and 4-valued *SK* fixed points is lacking in the literature. In this paper, we will fill this lacuna.

Our *closure games* differ in some important aspects from the game considered by Martin. Martin associates a *single* game G_σ with each sentence σ of L_T. Just as in Hintikka's game-theoretic semantics (cf. [3]), the players in Martin's game[3] fulfill, at each stage of the game, either the role of *verifier* or of *falsifier*. The game G_σ starts with the sentence σ and player I in the role of verifier (and so with *II* as falsifier) and, as the game proceeds, the players list *sentences* of L_T in accordance with rules that reflect the SK schema. Whose turn it is does not only depend on the last sentence listed but also on who is, at that stage, the verifier (falsifier). For instance, if the last sentence listed is γ, Martin's rules specify among others that

1. If γ is $\alpha \vee \beta$, then the verifier must list α or list β.
2. If γ is $\alpha \wedge \beta$, then the falsifier must list α or list β.
3. If γ is $T(t)$ and t denotes a sentence α of L_T, then the falsifier must list α.

Role changes (between verifier and falsifier) only occur when negated sentences are listed:[4]

1. If γ is $\neg\alpha$, then the falsifier must list α and the players change roles.

The roles of the players are important not only in the rules of Martin's game, but also in its winning conditions.

Winning conditions of Martin's game If γ is an atomic sentence of L, then play terminates; the player who is verifier wins if $\mathcal{C}_M(\gamma) = \mathbf{a}$ and loses if $\mathcal{C}_M(\gamma) = \mathbf{d}$. The verifier also loses (and play terminates) if γ is $T(t)$ with t denoting a nonsentence. If play never terminates, then the game is a draw.

In sharp contrast, the rules and winning conditions of closure games do not involve player roles. This gain in simplicity is achieved by associating *two* games with each sentence σ of L_T: the game G_{A_σ}, where player I tries to show that σ is assertible and the game G_{D_σ}, where player I tries to show that σ is deniable. Before we describe how the closure games G_{A_σ} and G_{D_σ} jointly induce a valuation of σ, we first describe the rules, strategies, and winning conditions of our closure games.

Rules. The game G_{A_σ} (G_{D_σ}) starts with the signed sentence A_σ (D_σ) and, as a closure game proceeds, the players produce a list of *signed* (with Assertible or Deniable) *sentences* of L_T. Whose turn it is only depends on the last (signed) sentence X_γ that is listed: X_γ is either *controlled* by player I or by player II. The moves that are available to the player who controls X_γ are described by (*assertoric*) *rules* that include:

1. If X_γ is $A_{\alpha\vee\beta}$, then player I must list A_α or list A_β.
2. If X_γ is $D_{\alpha\vee\beta}$, then player II must list D_α or list D_β.
3. If X_γ is $A_{\alpha\wedge\beta}$, then player II must list A_α or list A_β.
4. If X_γ is $D_{\alpha\wedge\beta}$, then player I must list D_α or list D_β.
5. If X_γ is $A_{T(t)}$ and t denotes a sentence α of L_T, then player II must list α.
6. If X_γ is $D_{T(t)}$ and t denotes a sentence α of L_T, then player I must list α.

Negated sentences are not associated with role changes but rather with sign changes:[5]

1. If X_γ is $A_{\neg\alpha}$, then player II must list D_α.
2. If X_γ is $D_{\neg\alpha}$, then player I must list A_α.

Although our signs can be interpreted as implicitly encoding the two-player roles[6] of Martin's game, we feel that doing so is only confusing. More importantly, our (*assertoric*) rules have a clear rationale of their own. For instance, when one (i.e., player I) wants to show that a disjunction is assertible, one must be able to show that one of the disjuncts (up to one's choice) is assertible. When one (i.e., player I) wants to show that a disjunction is deniable, one must be able to show that both disjuncts are deniable; so no matter which disjunct is picked by player II, player I must be able to show that it is deniable. The other assertoric rules receive a similar justification.

Strategies. Another difference with Martin's game concerns the notion of a *strategy*. In Martin's game, a strategy of a player is a function from the class of all *histories* of the game to the set of sentences of L_T, where a history is any finite sequence of sentence–role pairs that can be generated via the game rules. A strategy is thus a quite complicated object. However, it is not hard to show that for Martin's game, a player has a winning strategy just in case he has a *memoryless strategy*, that is, a strategy that does not depend on the history of the game, but only on the last sentence–role pair of the game. The method of closure games *restricts itself from the outset to memoryless strategies*. That is, a strategy for a player of a closure game is a function from the set of signed sentences that are in his control to the set of signed sentences. For instance, a strategy of player I maps $A_{\alpha\vee\beta}$ to A_α or A_β and a strategy of player II maps $D_{\alpha\vee\beta}$ to D_α or D_β. Strategies in the method of closure games are thus, per definition, quite simple objects. The results of this paper testify that (with respect to characterizing *SK* fixed points) nothing is lost by our restriction to memoryless strategies.

Winning conditions. When, in a closure game G_{X_σ}, players I and II pick their strategies, they realize an *expansion of* X_σ, a sequence of signed sentences with X_σ as its first element and with a successor relation that is determined by the strategies of the players. Expansions are classified into those that result in player I winning the game in a ground model M (these expansions are called *open* in M) and into those that do not (these expansions are called *closed* in M). Such a bipartition of the set of all expansions into those that are open and closed we call a *closure condition*. Intuitively, a closure condition may be thought of as an assertoric norm: closed (open) expansions contain assertoric actions that are forbidden (allowed). Player I has a winning strategy in the closure game G_{X_σ} that is played under closure condition † in a ground model M just in case he can *ensure* that an expansion of X_σ is realized that is open in M according to †. That is, a strategy f for player I is winning just in case, no matter which strategy g is picked by player II; the expansion of X_σ that is realized by f and g is open in M according to †.

Inducing L_T valuations by closure games. We say that sentence σ is *assertible* (*deniable*) *in ground model M according to closure condition* † just in case player I has a winning strategy for G_{A_σ} (G_{D_σ}) that is played in M under †. So, relative to a ground model M and closure condition †, the method of a closure game induces a valuation function $\mathcal{V}_M^\dagger$ that evaluates L_T sentences as **a**ssertible only, **b**oth assertible and deniable, **n**either assertible nor deniable, or as **d**eniable only. Depending on † and M, $\mathcal{V}_M^\dagger$ is either a 2-, 3-, or 4-valued function with a range that is a subset of $\{\mathbf{a},\mathbf{b},\mathbf{n},\mathbf{d}\}$. Of course, not all closure conditions will induce *SK* fixed points. One of the main results of this paper is a characterization of those closure conditions that do. In some more detail, the main results of this paper are (organized) as follows.

Structure of the paper Section 2 presents some general preliminaries.

Section 3 starts with a rigorous presentation of the method of closure games. Then, in Section 3.3, we present two conditions—the *world respecting constraint* WRC and the *stable judgment constraint* SJC—and show that whenever closure conditions satisfy our conditions, they induce an *SK* fixed point (cf. Theorem 3.5 and Corollary 3.6).

In Section 3.4, we define some intuitively appealing closure conditions that satisfy WRC and SJC and study the (3- and 4-valued) *SK* fixed points that they induce. In particular, we will present closure conditions that induce two versions of the minimal fixed point, having range $\{\mathbf{a},\mathbf{n},\mathbf{d}\}$ and $\{\mathbf{a},\mathbf{b},\mathbf{d}\}$, respectively.

In Section 3.5 we show, conversely, that any *SK* fixed point can be induced from closure conditions that satisfy WRC and SJC (cf. Theorem 3.14, Corollary 3.15). To do so, we take an *SK* fixed point V_M and define closure conditions that satisfy WRC and SJC in terms of V_M. To induce an *SK* fixed point in this way is, in some sense, "cheating," as we put "V_M in to get V_M out." In contrast, the closure conditions presented in Section 3.4 are clear examples of "noncheating closure conditions." Section 4 will be, among others, devoted to finding noncheating closure conditions for the *maximal intrinsic SK fixed point* (cf. [4]).

In Section 4.1, we define a slight modification of the method of closure games, which we call *assertoric semantics*. Whereas the method of closure games induces L_T valuations by putting closure conditions on *sequences* of signed sentences (expansions), assertoric semantics does so by putting closure conditions on *sets* of signed sentences.

In Section 4.2, we show how assertoric semantics induces the minimal *SK* fixed point (with range $\{\mathbf{a}, \mathbf{b}, \mathbf{d}\}$) and also how it induces Kripke's 4-valued "modal theory of truth" $\mathcal{K}^4$, which he defined (implicitly) in [4] by quantifying over all 3-valued *SK* fixed points; for instance, according to $\mathcal{K}^4$ the Liar is *paradoxical* as there is no 3-valued *SK* fixed point in which it is evaluated as **a**, and also there is no 3-valued *SK* fixed point in which it is evaluated as **d**.

In Section 4.3 we show how our characterization of $\mathcal{K}^4$ (via assertoric semantics) allows us to define noncheating closure conditions that induce (via the method of closure games) two versions of the maximal intrinsic *SK* fixed point, having range $\{\mathbf{a}, \mathbf{n}, \mathbf{d}\}$ and $\{\mathbf{a}, \mathbf{b}, \mathbf{d}\}$, respectively.

Section 5 is the conclusion.

2 Preliminaries

L_T will denote a first-order language with *identity* ($\approx$), a *truth predicate* (T) and with a *quotational name* ($[\sigma]$) for each sentence σ of L_T. L will denote the language that is exactly like L_T, except for the fact that it does not contain the truth predicate T. A *ground model* $M = (D, I)$ is a classical model for L such that $Sen(L_T) \subseteq D$ and such that $I([\sigma]) = \sigma$ for all $\sigma \in Sen(L_T)$. A sentence may be denoted in various ways; $\overline{\sigma}$ will be used to denote any closed term, quotational name or not, that denotes σ in M. For each ground model $M = (D, I)$, we will (tacitly) expand our language L_T to a language $L_T + M$ which has, in addition to the vocabulary of L_T, constant symbols available to refer to all the members of the domain of M. This (tacit, we will always simply speak of L_T) expansion has the advantage that quantification can be treated substitutionally, so that we do not need to be bothered with variable assignments.

Observe that a ground model may, but need not, define self-referential sentences such as *the Liar*; that is, a sentence that says, *of itself* that it is not true. It will turn out to be convenient to fix some notation pertaining to some canonical self-referential sentence such as *the Liar* and *the Truthteller*.

Definition 2.1 (Some notational conventions) In this paper, the nonquotational constants λ, τ, η, θ, and μ will be used as follows, where I is some interpretation function.

1. $I(\lambda) = \neg T(\lambda)$. We say that $\neg T(\lambda)$ is a *Liar*.
2. $I(\tau) = T(\tau)$. We say that $T(\tau)$ is a *Truthteller*.
3. $I(\eta) = T(\eta) \vee \neg T(\eta)$. We say that $T(\eta) \vee \neg T(\eta)$ is a *Tautologyteller*.
4. $I(\theta) = T(\theta) \wedge \neg T(\theta)$. We say that $T(\theta) \wedge \neg T(\theta)$ is a *Contradictionteller*.
5. $I(\mu) = T(c_0)$, where, for each n, $I(c_n) = \neg T(c_{n+1})$. We say that $T(\mu)$ is an *Unstabilityteller*.

To be sure, the notational convention does not imply that every ground model contains a Liar: given an interpretation function I a constant λ satisfying $I(\lambda) = \neg T(\lambda)$ may not exist. Similar remarks apply to the Truthteller, Tautologyteller, Contradictionteller, and Unstabilityteller.

Given a ground model M, $\mathcal{C}_M : Sen(L) \to \{\mathbf{a}, \mathbf{d}\}$ denotes the *classical valuation* of L based on M and is defined as usual. A *theory of truth* $\mathbf{T}$ takes a ground model M as input and outputs a valuation $\mathbf{T}_M$ of the sentences of L_T. That is, $\mathbf{T}$ outputs a function $\mathbf{T}_M : Sen(L_T) \to \mathbf{V}$, where $\mathbf{V}$ contains *the (semantic) values recognized*[7] *by* $\mathbf{T}$. We assume, without loss of generality, that when $\mathbf{T}$ is a theory of truth, $\mathbf{a}$ and $\mathbf{d}$ are always among the semantic values recognized by $\mathbf{T}$. Not any semantic valuation of the sentences of L_T qualifies as the valuation of a theory of truth. In this paper, we assume that for $\mathbf{T}$ to qualify as a theory of truth, $\mathbf{T}_M$ should *respect the world* and the *identity of truth*, as defined below. Besides these two familiar conditions we impose one further arbitrary, but technically convenient, condition: every truth ascription to an object that is not a sentence is to be evaluated as $\mathbf{d}$ by a theory of truth $\mathbf{T}$.

Definition 2.2 (Theory of truth) Let $\mathbf{T}$ be a valuation method which, given a ground model $M = (D, I)$, outputs a valuation function $\mathbf{T}_M : Sen(L_T) \to \mathbf{V}$. We say that $\mathbf{T}$ is a theory of truth just in case, for every ground model M, we have

$$\forall \sigma \in Sen(L) : \mathcal{C}_M(\sigma) = \mathbf{T}_M(\sigma), \tag{1}$$

$$\forall \sigma \in Sen(L_T) : \mathbf{T}_M\big(T(\overline{\sigma})\big) = \mathbf{T}_M(\sigma), \tag{2}$$

$$\mathbf{T}_M\big(T(t)\big) = \mathbf{d} \text{ whenever } I(t) \notin Sen(L_T). \tag{3}$$

That is, $\mathbf{T}_M$ should (1) *respect the world* and (2) *the identity of truth*, while (3) *all truth ascriptions to nonsentences are evaluated as* $\mathbf{d}$.

We will be particularly interested in theories of truth that output *SK valuations*. We distinguish between 2-valued, 3-valued, and 4-valued *SK* valuations, where a 2-valued *SK* valuation is just a classical valuation. In a 3-valued *SK* valuation, logically complex sentences are evaluated according to the SK schema and in a 4-valued *SK* valuation, these are evaluated according to the extended SK (or Dunn–Belnap) schema. It will be convenient to distinguish between two types of 3-valued *SK* valuations: those with range $\{\mathbf{a}, \mathbf{n}, \mathbf{d}\}$ and those with range $\{\mathbf{a}, \mathbf{b}, \mathbf{d}\}$. *SK* valuations of the first type we call 3**n**-*valued SK valuations*, and the second type we call 3**b**-*valued SK valuations*.

Definition 2.3 (SK valuations) Let $V_M : Sen(L_T) \to \mathbf{V}$ be a valuation of L_T in M such that $\mathbf{V}$ is $\{\mathbf{a}, \mathbf{d}\}$, $\{\mathbf{a}, \mathbf{n}, \mathbf{d}\}$, $\{\mathbf{a}, \mathbf{b}, \mathbf{d}\}$, or $\{\mathbf{a}, \mathbf{b}, \mathbf{n}, \mathbf{d}\}$. We say that V_M is a 2-*valued* (3**n**-*valued,* 3**b**-*valued,* 4-*valued*) *SK valuation* just in case, with $V_{\leq}$ the lattice associated with its range as in Figure 1, we have

1. $\neg$ swaps $\mathbf{a}$ for $\mathbf{d}$ and vice versa and leaves other values unchanged.
2. $\wedge$ and $\vee$ act, respectively, as meet and join on $V_{\leq}$.
3. $\forall$ and $\exists$ act, respectively, as generalized meet and join on $V_{\leq}$.

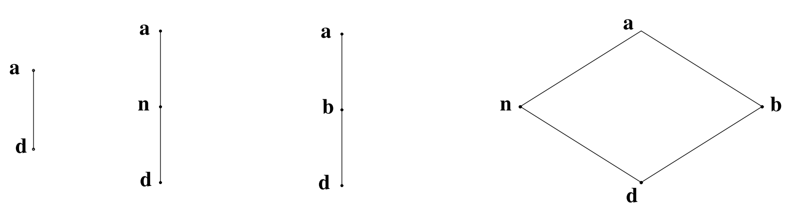

Figure 1 Hasse diagrams of lattices on $\{\mathbf{a}, \mathbf{d}\}$, $\{\mathbf{a}, \mathbf{n}, \mathbf{d}\}$, $\{\mathbf{a}, \mathbf{b}, \mathbf{d}\}$, and $\{\mathbf{a}, \mathbf{b}, \mathbf{n}, \mathbf{d}\}$.

Observe that the notion of an SK valuation does not mention the semantic behavior of the truth predicate or the relation with the valuation of L as induced by the ground model M. It will turn out to be convenient to separate the notion of an SK valuation from the notion of an *SK fixed point*, by which we mean an SK valuation that respects the defining clauses of a theory of truth.

Definition 2.4 (*SK* fixed points, $\mathbf{FP^n}(M)$ and $\mathbf{FP^b}(M)$) Let $V_M : Sen(L_T) \to \mathbf{V}$ be an *SK* valuation of L_T in M. We say that V_M is an *SK fixed point over M* just in case V_M satisfies clauses (1), (2), and (3) of Definition 2.2. We will use $\mathbf{FP^n}(M)$ to denote the set of all 2- and 3**n**-valued *SK* fixed points over M, whereas $\mathbf{FP^b}(M)$ will denote the set of all 2- and 3**b**-valued *SK* fixed points over M.

An SK theory of truth (*SK theory*) is a theory of truth that assigns an *SK* fixed point to each ground model M.

Definition 2.5 (*SK* theory of truth) Let **T** be a theory of truth. We say that **T** is an *SK theory* just in case, for every ground model M, $\mathbf{T}_M$ is an *SK* fixed point. An *SK* theory that recognizes 3 or 4 semantic values is called an SK_3 *theory* or SK_4 *theory*, respectively.

Note that there are no *SK theories* that recognize only two semantic values, as is attested to by a ground model $M = (D, I)$ that contains a Liar $\neg T(\lambda)$. On the other hand, some ground models allow L_T to be valued by a 2-valued *SK* fixed point. Also, note that the definition of an *SK* theory is quite liberal. A "genuine" *SK* theory **T** must, arguably, consist of a *systematic way* in which an arbitrary ground model M is converted into an *SK* fixed-point $\mathbf{T}_M$, and the notion of a "systematic conversion" does not appear in our definition. However, the definition as given is just fine for our purposes.

Two interesting and well-known SK_3 theories are Kripke's *minimal fixed-point theory* and his *maximal intrinsic fixed-point theory*. In line with Definition 2.3, we distinguish a 3**n**-valued and a 3**b**-valued version of both theories. In order to define the 3**n**-valued versions of those theories, we define the following partial order on $\mathbf{FP^n}(M)$. With $V_M, V'_M \in \mathbf{FP^n}(M)$, we let

$$V_M \leq V'_M \Leftrightarrow \forall \sigma \in Sen(L_T) : V_M(\sigma) = \mathbf{a} \Rightarrow V'_M(\sigma) = \mathbf{a}.$$

When $V_M \leq V'_M$ and $V_M \neq V'_M$, we write $V_M < V'_M$. We say that V_M is *maximal* just in case for no V'_M do we have $V_M < V'_M$, *minimal* just in case for no V'_M do we have $V'_M < V_M$. We say that V_M and V'_M are *compatible* just in case there exists a $V^*_M \in \mathbf{FP^n}(M)$ which extends them both: $V_M \leq V^*_M$ and $V'_M \leq V^*_M$. V_M is called *intrinsic* just in case it is compatible with every other fixed point in $\mathbf{FP^n}(M)$. For any ground model M, we let $\mathbf{I^n}(M)$ be the set of all intrinsic fixed points over M. As [4] observes, $\mathbf{I^n}(M)$ has a maximum element and $\mathbf{FP^n}(M)$ has a least element with respect to the relation $\leq$. The 3**n**-valued minimal fixed-point $\mathcal{K}$ and the 3**n**-valued maximal intrinsic fixed-point $\mathcal{K}^+$ can now be defined as follows.

Definition 2.6 ($\mathcal{K}$ and $\mathcal{K}^+$) Let M be a ground model. According to the theory $\mathcal{K}$, the valuation of L_T in M is given by $\mathcal{K}_M : Sen(L_T) \to \{\mathbf{a}, \mathbf{n}, \mathbf{d}\}$, where $\mathcal{K}_M$ is the minimum of $\mathbf{FP^n}(M)$. According to the theory $\mathcal{K}^+$, the valuation of L_T in M is given by $\mathcal{K}^+_M : Sen(L_T) \to \{\mathbf{a}, \mathbf{n}, \mathbf{d}\}$, where $\mathcal{K}^+_M$ is the maximum of $\mathbf{I^n}(M)$.

The 3**b**-valued versions of the minimal and maximal intrinsic fixed point will be denoted as $\overline{\mathcal{K}}$ and $\overline{\mathcal{K}}^+$, respectively. For sake of definiteness, their definitions are as follows.

Definition 2.7 ($\overline{\mathcal{K}}$ and $\overline{\mathcal{K}}^+$) Let M be a ground model. According to the theory $\overline{\mathcal{K}}$, the valuation of L_T in M is given by $\overline{\mathcal{K}}_M : Sen(L_T) \to \{\mathbf{a}, \mathbf{b}, \mathbf{d}\}$, where $\overline{\mathcal{K}}_M(\sigma) = \mathbf{b}$ if and only if $\mathcal{K}_M(\sigma) = \mathbf{n}$ and $\overline{\mathcal{K}}_M(\sigma) = \mathcal{K}_M(\sigma)$ otherwise. According to the theory $\overline{\mathcal{K}}^+$, the valuation of L_T in M is given by $\overline{\mathcal{K}}^+_M : Sen(L_T) \to \{\mathbf{a}, \mathbf{b}, \mathbf{d}\}$, where $\overline{\mathcal{K}}^+_M(\sigma) = \mathbf{b}$ if and only if $\mathcal{K}^+_M(\sigma) = \mathbf{n}$ and $\overline{\mathcal{K}}^+_M(\sigma) = \mathcal{K}^+_M(\sigma)$ otherwise.

3 The Method of Closure Games

3.1 Defining the method of closure games In the Introduction we sketched the method of closure games and its central notions such as assertoric rules, strategies, expansions, and closure conditions. In this section, we turn the previous sketches into precise definitions.

Some preliminary notions. A signed (with A or D) sentence of L_T will be called an *AD sentence*. $\mathcal{X}$ denotes the set of all *AD* sentences:

$$\mathcal{X} = \{X_\sigma \mid X \in \{A, D\}, \sigma \in Sen(L_T)\}.$$

With $At(L)$, we denote the set of atomic sentences of L. These sentences are assumed to receive their (classical) valuation from the ground model M and can be thought of as the “nonsemantic facts.” We will treat (atomic) truth ascriptions to nonsentential objects on a par with members of $At(L)$. Hence, it is convenient to define, with $M = (D, I)$, the set $At^*_M(L)$ as follows:

$$At^*_M(L) = At(L) \cup \{T(t) \mid I(t) \notin Sen(L_T)\}.$$

Assertoric rules. In a closure game the two players produce a list of *AD* sentences. Whose turn it is only depends on the last *AD* sentence X_γ that is listed and is determined by *assertoric rules* that include:

1. If X_γ is $A_{\alpha\vee\beta}$, then player I must list A_α or list A_β.
2. If X_γ is $D_{\alpha\vee\beta}$, then player II must list D_α or list D_β.

To present all assertoric rules in a uniform manner, it is convenient to first introduce some notation. We will say that player I *controls* $A_{\alpha\vee\beta}$ and that player II *controls* $D_{\alpha\vee\beta}$. More generally, each *AD* sentence X_γ is controlled by one of the players. With respect to an *AD* sentence X_γ that is in his control, a player always has to list a single element of the set $\Pi(X_\gamma)$, consisting of all *immediate AD subsentences* of X_γ. Thus, $\Pi(A_{\alpha\vee\beta}) = \{A_\alpha, A_\beta\}$ and $\Pi(D_{\alpha\vee\beta}) = \{D_\alpha, D_\beta\}$. The general form of an assertoric rule can then be depicted as follows:

$$\frac{X_\gamma}{\Pi(X_\gamma)} J \quad \big(\text{where } J \in \{I, II\}\big). \tag{4}$$

Thus, (4) states that player J controls X_γ and so, with respect to X_γ, player J has to pick an *AD* sentence in $\Pi(X_\gamma)$. By exploiting the notation just introduced, Figure 2 below states the assertoric rules.[8] With respect to Figure 2, observe that it does not matter which player, I or II, controls (signed) negations, truth ascriptions, and elements of $At^*_M(L)$. The actual allotment of player control to those sentences was chosen for sake of symmetry only: if player I controls A_σ, player II controls

$\neg$	$\dfrac{A_{\neg\alpha}}{\{D_\alpha\}}II$	$\dfrac{D_{\neg\alpha}}{\{A_\alpha\}}I$
$\vee$	$\dfrac{A_{(\alpha\vee\beta)}}{\{A_\alpha, A_\beta\}}I$	$\dfrac{D_{(\alpha\vee\beta)}}{\{D_\alpha, D_\beta\}}II$
$\wedge$	$\dfrac{A_{(\alpha\wedge\beta)}}{\{A_\alpha, A_\beta\}}II$	$\dfrac{D_{(\alpha\wedge\beta)}}{\{D_\alpha, D_\beta\}}I$
$\exists$	$\dfrac{A_{\exists x\varphi(x)}}{\{A_{\varphi(x/t)} \mid t \in CTerm(L_T)\}}I$	$\dfrac{D_{\exists x\varphi(x)}}{\{D_{\varphi(x/t)} \mid t \in CTerm(L_T)\}}II$
$\forall$	$\dfrac{A_{\forall x\varphi(x)}}{\{A_{\varphi(x/t)} \mid t \in CTerm(L_T)\}}II$	$\dfrac{D_{\forall x\varphi(x)}}{\{D_{\varphi(x/t)} \mid t \in CTerm(L_T)\}}I$
T	$\dfrac{A_{T(\overline{\sigma})}}{\{A_\sigma\}}II$	$\dfrac{D_{T(\overline{\sigma})}}{\{D_\sigma\}}I$
$\sigma \in At^*_M(L)$	$\dfrac{A_\sigma}{\{A_\sigma\}}II$	$\dfrac{D_\sigma}{\{D_\sigma\}}I$

Figure 2 The assertoric rules.

D_σ, and vice versa. The reason that there are also (trivial) assertoric rules pertaining to elements of $At^*_M(L)$ will be explained below, where we define the notion of an *expansion*.

Strategies. A player's strategy determines the moves that the player will make at any stage of the game. As was announced in the Introduction, the method of closure games restricts itself to *memoryless strategies*. This means that the moves of a player do not depend on the *history* of the game—that is, on the list of *AD* sentences that has been produced thus far—but only on the last element of the list. In other words, a strategy of a player in a closure game is a function that maps each *AD* sentence X_σ that is in his control to an element of $\Pi(X_\sigma)$.

Definition 3.1 (Strategies and strategy sets) A *strategy for player I* is a function f that maps each X_σ that is controlled by I to an element of $\Pi(X_\sigma)$. The set of all strategies of player I is denoted by $\mathcal{F}$.

A *strategy for player II* is a function g that maps each X_σ that is controlled by *II* to an element of $\Pi(X_\sigma)$. The set of all strategies of player *II* is denoted by $\mathcal{G}$.

Expansions. Let X_σ be an *AD* sentence. A pair of strategies f (for player I) and g (for player *II*) realizes an *expansion* of X_σ, that is, a sequence of *AD* sentences that has X_σ as its first term and with a successor relation that is determined by f and g. As an example, consider the *AD* sentence $A_{P(c_1)\wedge P(c_2)}$, where $P(c_1)$ and $P(c_2)$ are atomic sentences of L. A strategy g of player *II* according to which $g(A_{P(c_1)\wedge P(c_2)}) = A_{P(c_1)}$ (combined with any strategy f for player I) realizes the following expansion of $A_{P(c_1)\wedge P(c_2)}$:

$$A_{P(c_1)\wedge P(c_2)}, A_{P(c_1)}, A_{P(c_1)}, A_{P(c_1)}, \ldots. \tag{5}$$

Indeed, due to the (trivial) assertoric rules pertaining to elements of $At^*_M(L)$, an expansion is an *infinite* sequence of *AD* sentences.

Definition 3.2 (Expansions and the set EXP_M) With $f \in \mathcal{F}$, $g \in \mathcal{G}$ and $X_\sigma \in \mathcal{X}$, $\mathsf{exp}(X_\sigma, f, g)$ denotes *the expansion of X_σ by f and g*. The set of all expansions in M is[9] denoted by EXP_M.

Closure conditions. A closure game G_{X_σ} starts with *AD* sentence X_σ and, when the players pick their respective strategies f and g, results in the expansion $\mathsf{exp}(X_\sigma, f, g)$. A closure game is always played in, or relative to, a ground model M. In a closure game G_{A_σ}, player I tries to show that σ is assertible in M, whereas in the game G_{D_σ}, player I tries to show that σ is deniable in M. Player I succeeds in showing the assertibility (deniability) of σ in M just in case he has a *winning strategy* in G_{A_σ} (G_{D_σ}), where a strategy f for player I in G_{A_σ} (G_{D_σ}) is winning just in case f ensures that an expansion of A_σ (D_σ) is realized that is *open in M*. Which expansions are open in M (and which are closed, i.e., not open) depends on the *closure condition* under which a game is played. Intuitively, a closure condition may be thought of as describing an *assertoric norm*. Formally, a closure condition is defined as follows.

Definition 3.3 (Closure conditions) A *closure condition* $\dagger = \{O_M^\dagger, C_M^\dagger\}$ is a bipartition of EXP_M into the sets $O_M^\dagger \neq \emptyset$, consisting of the *open$_\dagger$ expansions in M*, and $C_M^\dagger \neq \emptyset$, containing the *closed$_\dagger$ expansions*[10] in M.

Let G_{X_σ} be a closure game played in ground model M under closure condition $\dagger = \{O_M^\dagger, C_M^\dagger\}$. When player I has a winning strategy in G_{X_σ}, we write $O_M^\dagger(X_\sigma)$:

$$O_M^\dagger(X_\sigma) \Leftrightarrow \exists f \in \mathcal{F}\, \forall g \in \mathcal{G} : \mathsf{exp}(X_\sigma, f, g) \in O_M^\dagger.$$

We also let

$$C_M^\dagger(X_\sigma) \Leftrightarrow \text{ not } O_M^\dagger(X_\sigma).$$

Inducing L_T valuations. Given a ground model M and closure condition $\dagger$, the method of closure games induces $\mathcal{V}_M^\dagger$, a valuation of L_T in M, as follows:

$$\mathcal{V}_M^\dagger(\sigma) = \begin{cases} \mathbf{a}, & O_M^\dagger(A_\sigma) \text{ and } C_M^\dagger(D_\sigma), \\ \mathbf{b}, & O_M^\dagger(A_\sigma) \text{ and } O_M^\dagger(D_\sigma), \\ \mathbf{n}, & C_M^\dagger(A_\sigma) \text{ and } C_M^\dagger(D_\sigma), \\ \mathbf{d}, & C_M^\dagger(A_\sigma) \text{ and } O_M^\dagger(D_\sigma). \end{cases}$$

3.2 Defining closure conditions: Classifying expansions The method of closure games induces L_T valuations by closure conditions, that is, by declaring expansions open or closed. This section introduces some classifications of expansions which facilitate the definition of closure conditions later on.

Here are six examples of expansions, which will be used to illustrate our classifications of expansions. The examples involve $P(c)$, an atomic sentence of L, and a Liar, a Truthteller, and an Unstabilityteller as defined in Definition 2.1. They are:

1. $A_{P(c)\vee T(\lambda)}, A_{P(c)}, A_{P(c)}, \ldots,$
2. $A_{\neg P(c)}, D_{P(c)}, D_{P(c)}, \ldots,$
3. $A_{P(c)\vee T(\tau)}, A_{T(\tau)}, A_{T(\tau)}, A_{T(\tau)}, \ldots,$
4. $D_{T(\tau)}, D_{T(\tau)}, D_{T(\tau)}, \ldots,$
5. $A_{T(\mu)}, A_{T(c_0)}, A_{\neg T(c_1)}, D_{T(c_1)}, D_{\neg T(c_2)}, A_{T(c_2)}, \ldots,$
6. $A_{T(\lambda)}, A_{\neg T(\lambda)}, D_{T(\lambda)}, D_{\neg T(\lambda)}, A_{T(\lambda)}, \ldots.$

First, observe that every expansion is either *stable*$_A$, *stable*$_D$, or *unstable*. The formal definition of these notions is clear from the remark that expansions 1 and 3 are stable$_A$, 2 and 4 are stable$_D$, and 5 and 6 are unstable. Next, observe that every expansion is either *grounded* or *ungrounded*, where an expansion is grounded just in case it contains, for some $\sigma \in At^*_M(L)$, X_σ; we say that X_σ is the *ground* of the expansion. Grounded expansions are either *correct in* M or *incorrect in* M. An expansion is correct in M just in case its ground is contained in the *world* w_M, which is defined as follows:

$$w_M = \{A_\sigma \mid \mathcal{C}_M(\sigma) = \mathbf{a}, \sigma \in At(L)\} \cup \{D_\sigma \mid \mathcal{C}_M(\sigma) = \mathbf{d}, \sigma \in At(L)\} \\ \cup \{D_{T(t)} \mid I(t) \notin Sen(L_T)\}.$$

The definition of w_M reveals that, in line with Definition 2.2, we assume that (atomic) sentences that ascribe truth to nonsentential objects always have to be denied. Now, when we assume a ground model M in which $\mathcal{C}_M(P(c)) = \mathbf{a}$, expansion 1 is grounded and correct (as $A_{P(c)} \in w_M$), whereas expansion 2 is grounded and incorrect (as $A_{P(c)} \notin w_M$). Expansions 3, 4, 5, and 6 are ungrounded. An (ungrounded) expansion is *vicious* just in case it contains a *vicious cycle*, or in other words, an expansion $\{y_n\}_{n\in\mathbb{N}}$ is vicious just in case

$$\exists\sigma\forall n\exists m, m' > n : y_m = A_\sigma \text{ and } y_{m'} = D_\sigma.$$

Indeed, expansion 6 is vicious. We now introduce the following abbreviations for subsets of EXP_M.

Definition 3.4 (Classifying expansions) We define the following subsets of EXP_M:

1. G_M: the set of all grounded expansions,
2. U_M: the set of all ungrounded expansions,
3. G_M^{cor}: the set of all grounded and correct expansions,
4. G_M^{inc}: the set of all grounded and incorrect expansions,
5. U_M^{vic}: the set of all (ungrounded) vicious expansions,
6. U_M^{nvi}: the set of all ungrounded nonvicious expansions,
7. US_M^A: the set of all ungrounded stable$_A$ expansions,
8. US_M^D: the set of all ungrounded stable$_D$ expansions,
9. UU_M: the set of all (ungrounded) unstable expansions.

3.3 The first stable judgment theorem In this section, we prove our *first stable judgment theorem*, which states that if closure conditions satisfy what we call the *stable judgment constraint* (SJC), they induce an *SK* valuation that respects the identity of truth. We further show that if, in addition, closure conditions satisfy the *world respecting constraint* (WRC), they induce an *SK* fixed point.

For any expansion exp, we let exp′ denote the *successor expansion* of exp, by which we mean the expansion that is obtained by removing the first term of exp. A closure condition $\dagger = \{O^\dagger_M, C^\dagger_M\}$ satisfies the SJC, just in case, for every expansion $\mathsf{exp} \in \mathsf{EXP_M}$, we have

$$\mathsf{SJC} : \mathsf{exp} \in C^\dagger_M \Leftrightarrow \mathsf{exp}' \in C^\dagger_M.$$

Note that, equivalently, SJC can be formulated in terms of openness:

$$\mathsf{SJC} : \mathsf{exp} \in O^\dagger_M \Leftrightarrow \mathsf{exp}' \in O^\dagger_M.$$

Below, we prove the first stable judgement theorem, in which we refer to the set of all *AD* subsentences of X_σ, denoted $\overline{\Pi}(X_\sigma)$. Formally, $\overline{\Pi}(X_\sigma)$ is defined by taking the transitive closure of the binary relation induced by the set of all *immediate AD* subsentences of X_σ:

1. $\Pi(\cdot,\cdot)$ is defined by $\Pi(X_\sigma, Y_\alpha) \Leftrightarrow Y_\alpha \in \Pi(X_\sigma)$.
2. $\overline{\Pi}(\cdot,\cdot)$ is defined as the transitive closure of $\Pi(\cdot,\cdot)$.
3. $\overline{\Pi}(\cdot)$ is defined by $\overline{\Pi}(X_\sigma) = \{Y_\alpha \mid \overline{\Pi}(X_\sigma, Y_\alpha)\}$.

Theorem 3.5 (First stable judgment theorem) *Let $M = (D, I)$ be a ground model, let $\dagger = \{O^\dagger_M, C^\dagger_M\}$ be a closure condition that satisfies* SJC*, and let $\mathcal{V}^\dagger_M$ be the valuation function induced by $\dagger$. We have:*

1. *$\mathcal{V}^\dagger_M$ is a* 2-*,* 3**n**-*,* 3**b***, or* 4-*valued SK valuation (see Definition 2.3).*
2. *For each $\sigma \in Sen(L_T)$ we have $\mathcal{V}^\dagger_M(T(\overline{\sigma})) = \mathcal{V}^\dagger_M(\sigma)$. That is $\mathcal{V}^\dagger_M$ respects the identity of truth.*

Proof Let $\dagger = \{O^\dagger_M, C^\dagger_M\}$ be a closure condition that satisfies SJC. Notice that, to show that $\mathcal{V}^\dagger_M$ is an SK valuation that respects the identity of truth, it suffices to show that for every *AD* sentence X_σ:

$$\text{player } I \text{ controls } X_\sigma \Rightarrow \big(O^\dagger_M(X_\sigma) \Leftrightarrow \exists Y_\alpha \in \Pi(X_\sigma) : O^\dagger_M(Y_\alpha)\big),$$

$$\text{player } II \text{ controls } X_\sigma \Rightarrow \big(O^\dagger_M(X_\sigma) \Leftrightarrow \forall Y_\alpha \in \Pi(X_\sigma) : O^\dagger_M(Y_\alpha)\big).$$

We illustrate for $A_{\alpha\wedge\beta}$. Other cases are similar and left to the reader.

$\Rightarrow$ Suppose that $O^\dagger_M(A_{\alpha\wedge\beta})$. This means that there is a strategy $f \in \mathcal{F}$ such that for all $g \in \mathcal{G}$, $\mathsf{exp}(A_{\alpha\wedge\beta}, f, g)$ is open$_\dagger$. Now $A_{\alpha\wedge\beta}$ is controlled by player *II*, and the strategies of player *II* can be bipartitioned into strategies of type g_α, which have $g(A_{\alpha\wedge\beta}) = A_\alpha$ and strategies of type g_β, which have $g(A_{\alpha\wedge\beta}) = A_\beta$. As f results in an open expansion, no matter whether player *II* plays a strategy of type g_α or g_β, it follows, as $\dagger$ satisfies SJC, that f is such that for all $g \in \mathcal{G}$ we have $\mathsf{exp}(A_\alpha, f, g) \in O^\dagger_M$ and that $\mathsf{exp}(A_\beta, f, g) \in O^\dagger_M$. Hence, $O^\dagger_M(A_\alpha)$ and $O^\dagger_M(A_\beta)$.

$\Leftarrow$ Suppose that $O^\dagger_M(A_\alpha)$ and $O^\dagger_M(A_\beta)$. This means that there exists a strategy $f_\alpha \in \mathcal{F}$ such that for all $g \in \mathcal{G}$ we have $\mathsf{exp}(A_\alpha, f_\alpha, g) \in O^\dagger_M$ and that there exists a strategy $f_\beta \in \mathcal{F}$ such that for all $g \in \mathcal{G}$ we have $\mathsf{exp}(A_\beta, f_\beta, g) \in O^\dagger_M$. Let $f \in \mathcal{F}$ be any strategy that satisfies:

1. $X_\sigma \in \overline{\Pi}(A_\alpha)$, player I controls $X_\sigma \Rightarrow f(X_\sigma) = f_\alpha(X_\sigma)$.
2. $X_\sigma \in (\overline{\Pi}(A_\beta) - \overline{\Pi}(A_\alpha))$, player I controls $X_\sigma \Rightarrow f(X_\sigma) = f_\beta(X_\sigma)$.

From the fact that $\dagger$ satisfies SJC, it follows that the constructed f is such that for all $g \in \mathcal{G}$ we have $\mathsf{exp}(A_{\alpha\wedge\beta}, f, g) \in O^\dagger_M$. □

Thus, picking a closure condition that satisfies SJC ensures that we induce an SK valuation that respects the identity of truth. As such, a closure condition that satisfies SJC does not guarantee that we induce an *SK* fixed point (as defined by Definition 2.4). However, by posing the following additional constraint on closure conditions, we ensure that they induce fixed-point valuations. Let $\dagger = \{O^\dagger_M, C^\dagger_M\}$ be any closure condition. We say that $\dagger$ satisfies the WRC, just in case

$$\mathsf{WRC} : G^{cor}_M \subseteq O^\dagger_M \text{ and } G^{inc}_M \subseteq C^\dagger_M.$$

We get the following corollary to Theorem 3.5.

Corollary 3.6 (First stable judgment corollary) *Let* $\dagger = \{O^{\dagger}_{M}, C^{\dagger}_{M}\}$ *be a closure condition that satisfies* WRC *and* SJC*. Then* $\mathcal{V}^{\dagger}_{M}$ *is an* (2-, 3-, *or* 4-*valued) SK fixed point over M (cf. Definition 2.4).*

Proof In light of Theorem 3.5, it suffices to show that if † satisfies WRC, then $\mathcal{V}^{\dagger}_{M}$ satisfies clauses (1) and (3) of Definition 2.2. A proof can be given by induction on the complexity of sentences of L, accounting for the nonsentential truth ascriptions in a straightforward way. □

3.4 Putting the first stable judgment theorem to work In this section, we put the first stable judgment theorem to work; we define closure conditions that satisfy SJC and WRC and that, accordingly, induce *SK* fixed points. Among others, we present closure conditions that induce the 3**n**- and 3**b**-valued version of the minimal *SK* fixed point.

Consider the following closure conditions:

$$\textbf{gr(oundedness) closure conditions}: O^{gr}_{M} = G^{cor}_{M};$$

$$\blacklozenge\ \textbf{closure conditions}: O^{\blacklozenge}_{M} = G^{cor}_{M} \cup U^{nvi}_{M}.$$

It is easily seen that those closure conditions satisfy SJC and WRC. Hence, by the first stable judgment theorem, $\mathcal{V}^{gr}$ and $\mathcal{V}^{\blacklozenge}$ are *SK* theories of truth. In fact, we have the following.

Proposition 3.7 $\mathcal{V}^{gr} = \mathcal{K}$*, whereas* $\mathcal{V}^{\blacklozenge}$ *is an* SK_4 *theory.*

Proof In [8] we showed that $\mathcal{V}^{gr} = \mathcal{K}$. The fact that $\mathcal{V}^{\blacklozenge}$ is an SK_4 theory follows from the observation (that the ♦ closure conditions satisfy SJC and WRC and) that $\mathcal{V}^{\blacklozenge}_{M}(\neg T(\lambda)) = \mathbf{n}$ while $\mathcal{V}^{\blacklozenge}_{M}(T(\tau)) = \mathbf{b}$. □

It is instructive to explain, in terms of the method of closure games, why $\mathcal{V}^{gr}_{M}$ is 3-valued, whereas $\mathcal{V}^{\blacklozenge}_{M}$ is 4-valued.[11] To do so, we will prove a useful lemma which requires the following definition of the *inverses* of *AD* sentences and (sets of) expansions.

Definition 3.8 (Inverses) For each *AD* sentence X_σ, we define its *inverse* X^{-1}_{σ} by stipulating that $A^{-1}_{\sigma} = D_\sigma$ and $D^{-1}_{\sigma} = A_\sigma$.

For each expansion $\mathsf{exp} = \{y_n\}_{n\in\mathbb{N}}$, we define its *inverse expansion* $\mathsf{exp}^{-1} = \{z_n\}_{n\in\mathbb{N}}$ by letting, for any $n \in \mathbb{N}$,

$$z_n = X_\sigma \Leftrightarrow y_n = X^{-1}_{\sigma}.$$

We define the *inverse* S^{-1} of a set of expansions S by stipulating that $S^{-1} = \{\mathsf{exp}^{-1} \mid \mathsf{exp} \in S\}$.

Here is the announced lemma.

Lemma 3.9 $\exists f \forall g\ \mathsf{exp}(X_\sigma, f, g) \in S \Leftrightarrow \exists g \forall f\ \mathsf{exp}(X^{-1}_{\sigma}, f, g) \in S^{-1}$.

Proof For each strategy f of player I, there is a *mirror strategy* for player II, call it g_f, that is defined as follows:

$$g_f(X_\alpha) = Y_\beta \Leftrightarrow f(X^{-1}_{\alpha}) = Y^{-1}_{\beta}.$$

Similarly, for each strategy g of player II, there is a mirror strategy for player I which may be called f_g. The lemma readily follows from an in inspection of the notion of a mirror strategy. □

To explain the 3-valuedness of $\mathcal{V}^{gr}_M$ in terms of the method of closure games, observe that, as the set of expansions G^{cor}_M is the inverse of G^{inc}_M, it follows from Lemma 3.9 that

$$\exists f \forall g \ \mathsf{exp}(X_\sigma, f, g) \in G^{cor}_M \Leftrightarrow \exists g \forall f \ \mathsf{exp}(X^{-1}_\sigma, f, g) \in G^{inc}_M. \quad (6)$$

From (6) it immediately follows that

$$O^{gr}_M(X_\sigma) \Rightarrow C^{gr}_M(X^{-1}_\sigma). \quad (7)$$

And from (7) it directly follows that $\mathcal{V}^{gr}$ can never evaluate a sentence as **b** and so (as there clearly are sentences that are evaluated as **n** by $\mathcal{V}^{gr}$) $\mathcal{V}^{gr}$ is a 3-valued theory of truth.

The principle that is underlying the 3-valuedness of $\mathcal{V}^{gr}$, that is, (7), breaks down for $\mathcal{V}^{\blacklozenge}$. We have

$$O^{\blacklozenge}_M(X_\sigma) \not\Rightarrow C^{\blacklozenge}_M(X^{-1}_\sigma). \quad (8)$$

The reason for this is that the set of expansions U^{nvi}_M, which is open$_\blacklozenge$, is its own inverse. Hence, the fact that player I can force an expansion of A_σ to end up in U^{nvi}_M implies that player II can force the expansion of D_σ to end up in $(U^{nvi}_M)^{-1} = U^{nvi}_M$. But the fact that player II can force the expansion of D_σ to end up in U^{nvi}_M does not preclude the possibility that player I may as well be able to force D_σ to end up in U^{nvi}_M. Hence, A_σ and D_σ may both be open$_\blacklozenge$. The previous remarks are illustrated by considering the following two expansions of the Truthteller:

$$A_{T(\tau)}, A_{T(\tau)}, A_{T(\tau)}, \ldots, \qquad D_{T(\tau)}, D_{T(\tau)}, D_{T(\tau)}, \ldots.$$

The gr(oundedness) closure conditions thus induce $\mathcal{K}$, the 3**n**-valued version of the minimal fixed point. By invoking Lemma 3.9 and a further lemma (Lemma 3.10) we will show that the $\overline{gr}$(*oundedness*) closure conditions induce $\overline{\mathcal{K}}$, the 3**b**-valued version of the minimal fixed point:

$\overline{gr}$**(oundedness) closure conditions**: $C^{\overline{gr}}_M = G^{inc}_M$.

Lemma 3.10 *If* $\dagger$ *satisfies* SJC, *then* $C^{\dagger}_M(X_\sigma) \Leftrightarrow \exists g \forall f \ \mathsf{exp}(X_\sigma, f, g) \in C^{\dagger}_M$.

Proof The $\Leftarrow$ direction is trivial. To prove the $\Rightarrow$ direction, observe that, per definition,

$$C^{\dagger}_M(X_\sigma) \Rightarrow \forall f \exists g \ \mathsf{exp}(X_\sigma, f, g) \in C^{\dagger}_M. \quad (9)$$

Now let Q be the set of all expansions of X_σ that end up in $C^{\dagger}_M$, and let

$$Q' = \{Y_\alpha \mid Y_\alpha \text{ occurs on some } \mathsf{exp} \in Q \text{ and is controlled by player } II\}.$$

Now define, for each $Y_\alpha \in Q'$, the set $Suc(Y_\alpha)$ as

$$Suc(Y_\alpha) = \{Z_\beta \mid Y_\alpha \text{ is succeeded by } Z_\beta \text{ on some } \mathsf{exp} \in Q\}.$$

Let h be a (choice) function that maps each $Y_\alpha \in Q'$ to an element of $Suc(Y_\alpha)$. Fix an arbitrary strategy g' for player II. We define a strategy g for player II as follows:

$$g(Y_\alpha) = \begin{cases} h(Y_\alpha), & Y_\alpha \in Q' \\ g'(Y_\alpha), & Y_\alpha \notin Q'. \end{cases}$$

From (9) and the fact that $\dagger$ satisfies SJC, it readily follows that the constructed strategy g is such that $\forall f \ \mathsf{exp}(X_\sigma, f, g) \in C^{\dagger}_M$. □

Proposition 3.11 $\mathcal{V}^{\overline{gr}} = \overline{\mathcal{K}}$.

Proof Let M be a ground model. Observe that, as $\mathcal{V}^{gr} = \mathcal{K}$, it suffices to show that for any $\sigma \in Sen(L_T)$:

(i) $\mathcal{V}^{gr}_M(\sigma) = \mathbf{a} \Leftrightarrow \mathcal{V}^{\overline{gr}}_M(\sigma) = \mathbf{a}$,
(ii) $\mathcal{V}^{gr}_M(\sigma) = \mathbf{d} \Leftrightarrow \mathcal{V}^{\overline{gr}}_M(\sigma) = \mathbf{d}$.

(i) $\Rightarrow$ Suppose that $\mathcal{V}^{gr}_M(\sigma) = \mathbf{a}$, that is, that $O^{gr}_M(A_\sigma)$ and $C^{gr}_M(D_\sigma)$. As $O^{gr}_M \subseteq O^{\overline{gr}}_M$, $O^{gr}_M(A_\sigma)$ implies that $O^{\overline{gr}}_M(A_\sigma)$. Further, $O^{gr}_M(A_\sigma)$ means that player I has a strategy that ensures that the expansion of A_σ will end up in G^{cor}_M. As the inverse of G^{cor}_M is G^{inc}_M, this implies, via Lemma 3.9, that player II has a strategy g that ensures that the expansion of D_σ will end up in G^{inc}_M. But this means that player I does *not* have a strategy that ensures that the expansion of D_σ will end up in $\mathsf{EXP}_M - G^{inc}_M = O^{\overline{gr}}_M$. Thus, $C^{\overline{gr}}_M(D_\sigma)$. Together with the already established $O^{\overline{gr}}_M(A_\sigma)$, we thus have $\mathcal{V}^{\overline{gr}}_M(\sigma) = \mathbf{a}$.

(i) $\Leftarrow$ Suppose that $\mathcal{V}^{\overline{gr}}_M(\sigma) = \mathbf{a}$, that is, that $O^{\overline{gr}}_M(A_\sigma)$ and that $C^{\overline{gr}}_M(D_\sigma)$. As $O^{gr}_M \subseteq O^{\overline{gr}}_M$, $C^{\overline{gr}}_M(D_\sigma)$ implies that $C^{gr}_M(D_\sigma)$. Further, from $C^{\overline{gr}}_M(D_\sigma)$ and as the $\overline{gr}$(oundedness) closure condition satisfies SJC, it follows from Lemma 3.10 that player II has a strategy g that ensures that the expansion of D_σ will end up in $C^{\overline{gr}}_M = G^{inc}_M$. This implies, via Lemma 3.9 and as G^{cor}_M is the inverse of G^{inc}_M, that player I has a strategy f that ensures that the expansion of A_σ will end up in G^{cor}_M. Hence $O^{gr}_M(A_\sigma)$. Together with the already established $C^{gr}_M(D_\sigma)$ we thus have $\mathcal{V}^{gr}_M(\sigma) = \mathbf{a}$.

(ii). Just like i. □

To get a better grip on the role of SJC in the construction of *SK* theories, it is instructive to compare the ♦ closure conditions with the ◊ closure conditions, which are defined below. Before we define the ◊ closure conditions, observe that the ♦ closure conditions allow for the following, equivalent, definition:

$$\blacklozenge \textbf{ closure conditions}\text{: } C^{\blacklozenge}_M = G^{inc}_M \cup U^{vic}_M.$$

This reformulation is convenient as it clearly lays bare the distinction in the ◊ closure conditions:

$$\lozenge \textbf{ closure conditions}\text{: } C^{\lozenge}_M = G^{inc}_M \cup \{\mathsf{exp} \mid \exists \sigma \in Sen(L_T) : A_\sigma, D_\sigma \text{ on } \mathsf{exp}\}.$$

So the only difference between the ♦ closure conditions and the ◊ closure conditions is that the former classifies all expansions that contain A_σ and D_σ *in a cycle* as closed, whereas the latter does away with the condition of cyclicality: whenever an expansion contains an "*AD* clash" it is closed, whether or not this clash occurs in a cycle. To illustrate the difference between the ♦ and the ◊ closure conditions, we consider the following expansion of a denial of the Tautologyteller (cf. Definition 2.1):

$$D_{T(\eta)\vee\neg T(\eta)}, D_{\neg T(\eta)}, A_{T(\eta)}, A_{T(\eta)\vee\neg T(\eta)}, A_{T(\eta)}, A_{T(\eta)\vee\neg T(\eta)}, \ldots . \qquad (10)$$

This expansion is open according to the ♦ closure conditions—as the "*AD* clash" does not occur in a cycle—while it is closed according to the ◊ closure conditions. The successor expansion of (10), however, is open according to the ◊ closure conditions, which establishes that these closure conditions do not satisfy SJC.

$\mathcal{V}^{\diamond}$ defines a 4-valued theory of truth, which can be shown[12] not to be an SK_4 theory. This raises the question whether satisfying SJC is, besides a sufficient condition, also a necessary condition for closure conditions to induce an SK valuation that respects the identity of truth. The answer to that question is "no," as attested to by the following proposition.

Proposition 3.12 *Violating* SJC *and inducing an* SK_3 *theory.*

Proof The $\star$ closure conditions, stated below, violate SJC while they define a 3-valued SK theory of truth. In the definition of the $\star$ closure conditions, c is an arbitrary nonquotational constant of L_T:

$$O^{\star}_M = G^{cor}_M \cup \big\{\mathsf{exp} \mid A_{T(c)\vee\neg T(c)} \text{ or } D_{T(c)} \text{ on } \mathsf{exp} \text{ and } I(c) = T(c)\big\}. \tag{11}$$

The $\star$ closure conditions are a (minimal) modification of the (gr)oundedness closure conditions. According to the $\star$ closure conditions, the expansions in G^{cor}_M are open and, besides those, all (and only) the expansions that contain $A_{T(c)\vee\neg T(c)}$ or $D_{T(c)}$ for some c, such that $I(c) = T(c)$, are open. A little reflection shows that this ensures that $\mathcal{V}^{\star}_M$ is just like $\mathcal{V}^{gr}_M$, apart from a valuation of Truthtellers—that is, sentences of the form $T(c)$ such that $I(c) = T(c)$—and compounds of Truthtellers. In particular, with $T(\tau)$ a Truthteller, we have:

$$\mathcal{V}^{\star}_M\big(T(\tau)\big) = \mathbf{d}, \qquad \mathcal{V}^{\star}_M\big(T(\tau)\vee\neg T(\tau)\big) = \mathbf{a}.$$

Being a minimal modification of $\mathcal{V}^{gr}_M$, $\mathcal{V}^{\star}_M$ is an SK_3 theory. However, the $\star$ closure conditions violate SJC, which is easily seen by inspecting the following expansion:

$$A_{T(\tau)\vee\neg T(\tau)}, A_{T(\tau)}, A_{T(\tau)}, A_{T(\tau)} \ldots.$$

Indeed, this expansion is open according to $\star$ closure conditions as it contains $A_{T(\tau)\vee\neg T(\tau)}$ and as $I(\tau) = T(\tau)$. Its successor expansion, which does not contain $A_{T(\tau)\vee\neg T(\tau)}$ or $D_{T(\tau)}$, is closed and so the $\star$ closure conditions violate SJC while they induce an SK_3 theory. □

Thus, Proposition 3.12 testifies that the first stable judgment theorem cannot be read in the converse direction. However, our *second stable judgment theorem* comes close to a converse reading of the first stable judgment theorem.

3.5 The second stable judgment theorem In this section, we present the *second stable judgment theorem*, which states that any SK valuation that respects the identity of truth can be induced from a closure condition that satisfies SJC. A corollary of the second stable judgment theorem states that any SK fixed point can be induced from a closure condition that satisfies SJC and WRC.

Before we state the second stable judgment theorem and its corollary, we define the notion of the *correctness* of an AD sentence with respect to a (2-, 3-, or 4- valued) valuation[13] V_M.

Definition 3.13 (V_M correctness) Let V_M be a (2-, 3-, or 4-valued) valuation for L_T whose range $\mathbf{V}$ is such that $\{\mathbf{a},\mathbf{d}\} \subseteq \mathbf{V} \subseteq \{\mathbf{a},\mathbf{b},\mathbf{n},\mathbf{d}\}$. The notion of V_M *correctness*, applicable to AD sentences, is defined as follows:

$$X_\sigma \text{ is } V_M \text{ correct } \Leftrightarrow \big(X = A, V_M(\sigma) \in \{\mathbf{a},\mathbf{b}\}\big) \text{ or } \big(X = D, V_M(\sigma) \in \{\mathbf{d},\mathbf{b}\}\big).$$

Intuitively, an AD sentence X_σ is V_M correct if and only if its judgment (*A*ssertible or *D*eniable) with respect to σ is correct from the standpoint of V_M.

Theorem 3.14 (Second stable judgment theorem) *Let M be a ground model, and let V_M be a 2-, 3-, or 4-valued SK valuation of L_T that respects the identity of truth. Then there is a closure condition † that satisfies* SJC *and such that $\mathcal{V}_M^\dagger = V_M$.*

Proof Let V_M be a 2-, 3-, or 4-valued *SK* valuation of L_T that respects the identity of truth. Using the notion of V_M correctness, we define a closure condition $\dagger = \{O_M^\dagger, C_M^\dagger\}$, and we will show that † satisfies SJC and is such that $\mathcal{V}_M^\dagger = V_M$. Let $\mathsf{exp} = \{y_n\}_{n \in \mathbb{N}}$ be an arbitrary expansion in EXP_M. We let

$$\mathsf{exp} \in O_M^\dagger \Leftrightarrow \exists n \forall m > n : y_m \text{ is } V_M \text{ correct.} \tag{12}$$

It is clear from the "limit behavior definition" of † that † satisfies SJC. Note that to show that $\mathcal{V}_M^\dagger = V_M$, it suffices to show that

$$X_\sigma \text{ is } V_M \text{ correct} \Leftrightarrow \exists f \in \mathcal{F} \forall g \in \mathcal{G} : \mathsf{exp}(X_\sigma, f, g) \in O_M^\dagger.$$

⇒ Suppose that X_σ is V_M correct. Observe that from the fact that V_M is *SK* and respects the identity of truth, we have:

$$X_\sigma \text{ is controlled by player } I \Rightarrow \exists Y_\alpha \in \Pi(X_\sigma) : Y_\alpha \text{ is } V_M \text{ correct;}$$
$$X_\sigma \text{ is controlled by player } II \Rightarrow \forall Y_\alpha \in \Pi(X_\sigma) : Y_\alpha \text{ is } V_M \text{ correct.}$$

From these two observations, it readily follows that if we start from an X_σ that is V_M correct, player I has a strategy, say f, that ensures that, for every $g \in \mathcal{G}$, *all* the terms of $\mathsf{exp}(X_\sigma, f, g)$ are V_M correct. Hence, player I can ensure that the expansion of X_σ ends up in $O_M^\dagger$.

⇐ Suppose that X_σ is V_M incorrect. Observe that from the fact that V_M is *SK* and respects the identity of truth, we have:

$$X_\sigma \text{ is controlled by player } I \Rightarrow \forall Y_\alpha \in \Pi(X_\sigma) : Y_\alpha \text{ is } V_M \text{ incorrect;}$$
$$X_\sigma \text{ is controlled by player } II \Rightarrow \exists Y_\alpha \in \Pi(X_\sigma) : Y_\alpha \text{ is } V_M \text{ incorrect.}$$

From these two observations, it readily follows that if we start from an X_σ that is V_M incorrect, player II has a strategy, say g, that ensures that, for every $f \in \mathcal{F}$, *all* the terms of $\mathsf{exp}(X_\sigma, f, g)$ are V_M incorrect. Hence, player II can ensure that the expansion of X_σ ends up in $C_M^\dagger$, from which it follows that player I cannot ensure that the expansion of X_σ ends up in $O_M^\dagger$. □

Corollary 3.15 (Second stable judgment corollary) *Let M be a ground model, and let V_M be an SK fixed point over M. Then there is a closure condition † that satisfies* SJC *and* WRC *and such that $\mathcal{V}_M^\dagger = V_M$.*

Proof As an *SK* fixed point is an *SK* valuation that respects the identity of truth *and the world*, the closure condition † that is defined in the proof of the second stable judgment theorem satisfies SJC and WRC and is such that $\mathcal{V}_M^\dagger = V_M$. □

So, to induce, say, Kripke's *SK* maximal intrinsic fixed-point $\mathcal{K}^+$, via the method of closure games, we may define closure conditions, via (12), in terms of $\mathcal{K}_M^+$ correctness. Closure conditions for $\mathcal{K}^+$ that are defined as such are parasitic on Kripke's framework for truth in a way that the gr(oundedness), $\overline{gr}$(oundedness), ♦, and ◊ closure conditions are not. We say that closure conditions for $\mathcal{K}^+$ that are defined via (12) are *cheating* closure conditions, whereas the gr(oundedness), $\overline{gr}$(oundedness), ♦, and ◊ are species of *noncheating* closure conditions. As $\mathcal{K}^+$ is an interesting theory of truth, it would be bad news for the method of closure games, as a framework for truth, if it had to rely, for the definition of $\mathcal{K}^+$, on notions that are borrowed from an alternative framework. Luckily, the method of closure games does have access to $\mathcal{K}^+$ via notions that are not borrowed from an alternative framework. In the next section, we see how this works out.

4 Assertoric semantics

4.1 Defining assertoric semantics In this section we present *assertoric semantics*,[14] which is a slight modification of the method of closure games. Whereas the method of closure games induces L_T valuations by putting closure conditions on *sequences* of signed sentences (expansions), assertoric semantics does so by putting closure conditions on *branches*, which are *sets* of signed sentences that are defined in terms of expansions.

For any expansion exp, [exp] will denote the set of terms of exp. For any AD sentence X_σ and strategy f of player I, $B_f(X_\sigma)$ denotes the set of terms that occur on some expansion of X_σ relative to f. We will call $B_f(X_\sigma)$ the *branch* of X_σ that is induced by f. To be sure, $B_f(X_\sigma)$ is defined as follows:

$$B_f(X_\sigma) = \bigcup_{g \in \mathcal{G}} \big[\mathsf{exp}(X_\sigma, f, g)\big].$$

We will use Branch_M to denote[15] the set of all branches relative to ground model M. The *(assertoric) tree* of X_σ, $\mathfrak{T}^\sigma_X$, is the set of all its branches:[16]

$$\mathfrak{T}^\sigma_X = \big\{B_f(X_\sigma) \bigm| f \in \mathcal{F}\big\}.$$

Branches are judged to be open or closed relative to closure conditions that are applicable to branches; a branch closure condition $\ddagger = \{O^\ddagger_M, C^\ddagger_M\}$ is a bipartition of Branch_M. An assertoric tree $\mathfrak{T}^\sigma_X$ is said to be open$_\ddagger$ in M just in case it contains a branch that is open$_\ddagger$ in M, that is, just in case $B_f(X_\sigma) \in O^\ddagger_M$ for some $B_f(X_\sigma) \in \mathfrak{T}^\sigma_X$. We write $O^\ddagger_M(X_\sigma)$ just in case $\mathfrak{T}^\sigma_X$ is open$_\ddagger$ in M, and $C^\ddagger_M(X_\sigma)$ if not $O^\ddagger_M(X_\sigma)$. In this sense, branch closure conditions induce closure conditions for AD sentences. These closure conditions can be used to define an L_T valuation $\mathbb{V}^\dagger_M$ in the expected manner. That is:

$$\mathbb{V}^\ddagger_M(\sigma) = \begin{cases} \mathbf{a} & O^\ddagger_M(A_\sigma) \text{ and } C^\ddagger_M(D_\sigma), \\ \mathbf{b} & O^\ddagger_M(A_\sigma) \text{ and } O^\ddagger_M(D_\sigma), \\ \mathbf{n} & C^\ddagger_M(A_\sigma) \text{ and } C^\ddagger_M(D_\sigma), \\ \mathbf{d} & C^\ddagger_M(A_\sigma) \text{ and } O^\ddagger_M(D_\sigma). \end{cases} \tag{13}$$

4.2 Inducing familiar theories of truth via assertoric semantics In this paper, we will only be concerned with two closure conditions for branches: the *tolerant* and *strict* closure condition. A branch B is *tolerantly closed in* M, that is, contained in C^{tol}_M, just in case

$$B \text{ contains } X_\sigma \text{ with } X_\sigma \in At^*_M \text{ and } X_\sigma \notin w_M.$$

A branch B is *strictly closed in* M, that is, contained in C^{strict}_M, just in case B is tolerantly closed in M *or*

$$B \text{ contains both } A_\sigma \text{ and } D_\sigma.$$

The tolerant closure condition induces a familiar theory of truth.

Proposition 4.1 *The following holds:* $\mathbb{V}^{tol} = \overline{\mathcal{K}}$.

Proof Let M be a ground model. Clearly, it suffices to show that for each AD sentence X_σ,

$$\exists B \in \mathfrak{T}_X^\sigma : B \in O_M^{tol} \Leftrightarrow \exists f \in \mathcal{F} \forall g \in \mathcal{G} : \mathsf{exp}(X_\sigma, f, g) \in O_M^{\overline{gr}}. \tag{14}$$

Now (14) follows immediately from a comparison of the tolerant closure condition for branches with the $\overline{gr}$(oundedness) closure condition for expansions and from the fact that $\mathcal{V}^{\overline{gr}} = \overline{\mathcal{K}}$. □

$\mathbb{V}^{strict}$, the theory of truth that is induced by the strict closure condition for branches, is also a, albeit less, familiar theory of truth. As we will show next, $\mathbb{V}_M^{strict}$ is equivalent to Kripke's "modal fixed-point valuation" $\mathcal{K}_M^{\mathbf{4}}$, which he defined implicitly by quantifying over $\mathbf{FP^n}(M)$: for instance, Kripke classified the Liar as a *paradoxical* sentence as *there is no* fixed point in which it evaluates as **a** and *there is no fixed point* in which it evaluates as **d**. More generally, Kripke used these quantifications (cf. [4, pp. 708–709]) over $\mathbf{FP^n}(M)$ to draw distinctions between the Liar, the Truthteller, the Tautologyteller, and the Contradictionteller (cf. Definition 2.1). These distinctions are captured by $\mathcal{K}_M^{\mathbf{4}}$, which is defined by quantifying over $\mathbf{FP^n}(M)$ as follows:

1. $\mathcal{K}_M^{\mathbf{4}}(\sigma) = \mathbf{a} \Leftrightarrow$ for some $V_M : V_M(\sigma) = \mathbf{a}$ and for no $V_M : V_M(\sigma) = \mathbf{d}$;
2. $\mathcal{K}_M^{\mathbf{4}}(\sigma) = \mathbf{b} \Leftrightarrow$ for some $V_M : V_M(\sigma) = \mathbf{a}$ and for some $V_M : V_M(\sigma) = \mathbf{d}$;
3. $\mathcal{K}_M^{\mathbf{4}}(\sigma) = \mathbf{n} \Leftrightarrow$ for no $V_M : V_M(\sigma) = \mathbf{a}$ and for no $V_M : V_M(\sigma) = \mathbf{d}$;
4. $\mathcal{K}_M^{\mathbf{4}}(\sigma) = \mathbf{d} \Leftrightarrow$ for no $V_M : V_M(\sigma) = \mathbf{a}$ and for some $V_M : V_M(\sigma) = \mathbf{d}$.

Although $\mathcal{K}_M^{\mathbf{4}}$ respects the ground model M and the identity of truth, $\mathcal{K}_M^{\mathbf{4}}$ is not an SK_4 valuation, which is attested to by the following observations:

$$\mathcal{K}_M^{\mathbf{4}}\big(T(\tau)\big) = \mathcal{K}_M^{\mathbf{4}}\big(\neg T(\tau)\big) = \mathbf{b}, \qquad \mathcal{K}_M^{\mathbf{4}}\big(T(\tau) \vee \neg T(\tau)\big) = \mathbf{a}.$$

To prove that $\mathbb{V}^{strict} = \mathcal{K}^{\mathbf{4}}$, we need some definitions, which are all modifications of notions defined, among others, in Fitting [1].

Definition 4.2 (Saturated sets, upward closure) Let S be a set of AD sentences. We say that S is *downward saturated* just in case:

$$\text{player } I \text{ controls } X_\sigma \Rightarrow \big(X_\sigma \in S \Rightarrow \Pi(X_\sigma) \cap S \neq \emptyset\big);$$
$$\text{player } II \text{ controls } X_\sigma \Rightarrow \big(X_\sigma \in S \Rightarrow \Pi(X_\sigma) \subseteq S\big).$$

This notion of an *upward saturated* set is defined dually. That is, S is upward saturated just in case:

$$\text{player } I \text{ controls } X_\sigma \Rightarrow \big(X_\sigma \in S \Leftarrow \Pi(X_\sigma) \cap S \neq \emptyset\big);$$
$$\text{player } II \text{ controls } X_\sigma \Rightarrow \big(X_\sigma \in S \Leftarrow \Pi(X_\sigma) \subseteq S\big).$$

Every set of AD sentences S has an upward closure $S^{\Uparrow}$, that is, a smallest set of AD sentences which extends S and which is upward saturated.[17]

Definition 4.3 ($\mathbf{FP^n}(M)$ sets and associated valuations) Let S be a set of AD sentences. We say that S is an $\mathbf{FP^n}(M)$ *set* just in case:

1. $\forall \sigma \in Sen(L_T)$: $A_\sigma \in S \Rightarrow D_\sigma \notin S$.
2. S is downward and upward saturated.
3. $w_M \subseteq S$.

An $\mathbf{FP^n}(M)$ set S is a notational variant of the associated fixed-point valuation, $V_M^S : Sen(L_T) \to \{\mathbf{a}, \mathbf{n}, \mathbf{d}\}$:

(i) $A_\sigma \in S \Leftrightarrow V_M^S(\sigma) = \mathbf{a}$.
(ii) $D_\sigma \in S \Leftrightarrow V_M^S(\sigma) = \mathbf{d}$.
(iii) $\{A_\sigma, D_\sigma\} \cap S = \emptyset \Leftrightarrow V_M^S(\sigma) = \mathbf{n}$.

On the other hand, every fixed-point valuation $V_M \in \mathbf{FP^n}(M)$ corresponds via (i), (ii), and (iii) to an $\mathbf{FP^n}(M)$ set S.

Before we prove that $\mathbb{V}^{strict} = \mathcal{K}^{\mathbf{4}}$, it is instructive to comment on our proof strategy, which is a modification of the soundness and completeness proofs for signed tableaux systems. Consider the assertoric rules for $\wedge$ and $\neg$ for a propositional language $\mathcal{L}_P$ under the usual closure conditions: a branch[18] is closed just in case it contains, for some sentence σ of $\mathcal{L}_P$, both A_σ and D_σ and a tableau for X_σ is closed just in case all its branches are closed. This specifies a sound and complete signed tableau proof system with respect to the classical semantics of $\mathcal{L}_P$: a sentence σ of $\mathcal{L}_P$ is true in every $\mathcal{L}_P$-valuation just in case D_σ has a closed tableau. Soundness is proved by observing that if D_σ has a closed tableau, there is no $\mathcal{L}_P$-valuation in which σ is false. Completeness is proved by showing that if D_σ does not have a closed tableau, we can take an open branch and transform it into an $\mathcal{L}_P$-valuation which renders σ false.

We use our branches and assertoric trees to induce semantic valuations; our strict closure conditions are defined relative to a ground model M. The role that is played by the classical valuation in the $\mathcal{L}_P$-case is, in our case, played by a fixed point of $\mathbf{FP^n}(M)$. If all the branches of $\mathfrak{T}_A^\sigma$ are strictly closed, there is no fixed point in which σ is evaluated as $\mathbf{a}$. Similarly, if all the branches of $\mathfrak{T}_D^\sigma$ are strictly closed, there is no fixed point in which σ is evaluated as $\mathbf{d}$. On the other hand, if $\mathfrak{T}_A^\sigma$ has a strictly open branch, we can convert this branch into a fixed point which evaluates σ as $\mathbf{a}$. Similarly for the case when $\mathfrak{T}_D^\sigma$ has a strictly open branch. Let us now turn to the proof that makes these remarks precise.

Theorem 4.4 *The following holds:* $\mathbb{V}^{strict} = \mathcal{K}^{\mathbf{4}}$.

Proof Let M be a ground model. Let B be a branch of A_σ that is strictly open in M. Then, $(B \cup w_M)^{\Uparrow}$, that is, the upward closure of $B \cup w_M$, is an $\mathbf{FP^n}(M)$ set that contains A_σ. From this, it follows that

$$O_M^{strict}(A^\sigma) \Rightarrow \exists V_M \in \mathbf{FP^n}(M) : V_M(\sigma) = \mathbf{a}.$$

And, similarly, we get that

$$O_M^{strict}(D^\sigma) \Rightarrow \exists V_M \in \mathbf{FP^n}(M) : V_M(\sigma) = \mathbf{d}.$$

On the other hand, let $\sigma \in Sen(L_T)$, and let $V_M \in \mathbf{FP^n}(M)$ be such that $V_M(\sigma) = \mathbf{a}$. Let S be the $\mathbf{FP^n}(M)$ set associated with V_M. By definition, $A_\sigma \in S$. Let f be any strategy for player I such that, for every $X_\sigma \in S$ that is controlled by player I, $f(X_\sigma) \in S$. It follows that $B_f(A_\sigma)$ is strictly open in M. Similar remarks apply to $V_M(\sigma) = \mathbf{d}$ and D_σ. Hence, we get

$$\exists V_M \in \mathbf{FP^n}(M) : V_M(\sigma) = \mathbf{a} \Rightarrow O_M^{strict}(A^\sigma),$$

$$\exists V_M \in \mathbf{FP^n}(M) : V_M(\sigma) = \mathbf{d} \Rightarrow O_M^{strict}(D^\sigma).$$

From the four established equations, it follows that $\mathbb{V}_M^{strict} = \mathcal{K}_M^{\mathbf{4}}$. □

Although $\mathbb{V}^{strict}$ is not defined via the method of closure games, it is clearly defined by using only notions that "belong" to the method of closure games. In the next subsection, we will use $\mathbb{V}^{strict}$ to define closure conditions that define $\mathcal{K}^+$ and $\overline{\mathcal{K}}^+$. In doing so, we obtain a definition of $\mathcal{K}^+$ that is, in an important sense, not parasitic on Kripke's framework for truth.

4.3 Using $\mathbb{V}^{strict}$ to define $\mathcal{K}^+$ and $\overline{\mathcal{K}}^+$ We will prove that $\mathcal{K}^+$ can be induced from closure conditions that are defined in terms of $\mathbb{V}^{strict}$. To do so, we define the notion of *strong correctness* in terms of $\mathbb{V}^{strict}_M$. We say that X_σ is *strongly correct* (in a ground model M) just in case

$$\big(X = A \text{ and } \mathbb{V}^{strict}_M(\sigma) = \mathbf{a}\big) \qquad \text{or} \qquad \big(X = D \text{ and } \mathbb{V}^{strict}_M(\sigma) = \mathbf{d}\big).$$

We use the strong correctness to define the + *closure condition* (for expansions) which, as we will eventually prove, induces $\mathcal{K}^+$. With $\mathsf{exp} = \{y_n\}_{n\in\mathbb{N}}$, we let

$$\mathsf{exp} \in O^+_M \Leftrightarrow \exists n \forall m > n : y_m \text{ is strongly correct.} \tag{15}$$

We will show that the valuation function induced by the strong closure condition, that is, $\mathcal{V}^+_M$, is equal to $\mathcal{K}^+_M$. Before we do so, however, we first sketch the rationale of the definition of $\mathcal{K}^+_M$ in terms of strong $\mathbb{V}^{strict}_M$ correctness.

For sure, if we have $\mathcal{K}^+_M(\sigma) = \mathbf{a}$, we have $\mathbb{V}^{strict}_M(\sigma) = \mathbf{a}$. For, if $\mathcal{K}^+_M(\sigma) = \mathbf{a}$, there is a (3-valued *SK*) fixed point that evaluates σ as $\mathbf{a}$ and also there is no fixed point that evaluates σ as $\mathbf{d}$. Similarly, $\mathcal{K}^+_M(\sigma) = \mathbf{d}$ implies that $\mathbb{V}^{strict}_M(\sigma) = \mathbf{d}$. The converses of these implications do not hold, however. For instance, we have

$$\mathbb{V}^{strict}_M\big(\neg T(\lambda) \vee T(\tau)\big) = \mathbf{a}, \qquad \mathcal{K}^+_M\big(\neg T(\lambda) \vee T(\tau)\big) = \mathbf{n},$$
$$\mathbb{V}^{strict}_M\big(\neg T(\tau) \wedge T(\tau)\big) = \mathbf{d}, \qquad \mathcal{K}^+_M\big(\neg T(\tau) \wedge T(\tau)\big) = \mathbf{n}.$$

Although $A_{\neg T(\lambda)\vee T(\tau)}$ and $D_{\neg T(\tau)\wedge T(\tau)}$ are strongly correct, none of their immediate *AD* subsentences is strongly correct. This ensures, as is readily noticed, that $\mathcal{V}^+_M(\neg T(\lambda) \vee T(\tau)) = \mathcal{V}^+_M(\neg T(\tau) \wedge T(\tau)) = \mathbf{n}$, mimicking the judgment of $\mathcal{K}^+_M$ with respect to these sentences. More generally, the definition of $\mathcal{V}^+_M$ ensures that, for *AD* sentences that are "unstable" strongly correct—that is, ultimately, they depend on a combination of *AD* sentences that are not strongly correct—player I does not have a strategy that ensures that his expansion ends up in O^+_M. To prove that $\mathcal{V}^+_M = \mathcal{K}^+_M$, we will evoke the following three lemmas.

Lemma 4.5 (Strict-openness is preserved downward in assertoric trees) *By the phrase "strict-openness is preserved downward in assertoric trees," we mean that*

$$\textit{player I controls } X_\sigma \Rightarrow \big(O^{strict}_M(X_\sigma) \Rightarrow \exists Y_\alpha \in \Pi(X_\sigma) : O^{strict}_M(Y_\alpha)\big);$$
$$\textit{player II controls } X_\sigma \Rightarrow \big(O^{strict}_M(X_\sigma) \Rightarrow \forall Y_\alpha \in \Pi(X_\sigma) : O^{strict}_M(Y_\alpha)\big).$$

Proof This follows immediately from an inspection of the strict closure conditions and the observation that the branches that constitute the tree of an immediate *AD* subsentence of X_σ are subsets of the branches that constitute the tree of X_σ. □

Lemma 4.6 $\mathcal{V}^+_M : Sen(L_T) \to \{\mathbf{a}, \mathbf{n}, \mathbf{d}\}$ *is an* SK_3 *theory.*

Proof It is clear that the strong closure conditions satisfy SJC and WRC, and so by the (corollary to the) first stable judgment theorem, they define an SK theory. The point of this lemma then is to show that $\mathcal{V}^+_M$ is 3-valued. To do so, we proceed as in the proof of Lemma 3.9. Suppose that $O^+_M(A_\sigma)$, and let f be a strategy of player I that ensures that the expansion of A_σ ends up in O^+_M. The *mirror strategy* of f, g_f (see Lemma 3.9) testifies that $C^+_M(D_\sigma)$. □

Our proof of the fact that $\mathcal{V}^+ = \mathcal{K}^+$ will exploit a further lemma, which invokes the notion of a *totally strongly correct expansion*. An expansion is said to be totally strongly correct just in case *all* its terms are strongly correct. Here is the lemma.

Lemma 4.7 *$\forall g \in \mathcal{G} : \mathsf{exp}(X_\sigma, f', g) \in O^+_M \Leftrightarrow \forall g \in \mathcal{G} : \mathsf{exp}(X_\sigma, f', g)$ is totally strongly correct.*

Proof The right-to-left direction is trivial. For the converse direction, let f' be a strategy testifying that $O^+_M(X_\sigma)$; that is,

$$\forall g \in \mathcal{G} : \mathsf{exp}(X_\sigma, f', g) \in O^+_M.$$

Let $g' \in \mathcal{G}$. We have to show that $\mathsf{exp}' = \mathsf{exp}(X_\sigma, f', g')$ is totally strongly correct. As $\mathsf{exp}' \in O^+_M$, exp' contains a first strongly correct term (after which all other terms are strongly correct). We will prove by contraposition that this first term is equal to X_σ. Thus, assume that exp' contains a first strongly correct term and that this term has a predecessor on exp' that is not strongly correct. We assume, without loss of generality, that the first strongly correct term has form A_α, the case where its form is D_α being similar. The predecessor of A_α on exp has one of the following six forms:

$$D_{\neg\alpha}, A_{\alpha\vee\beta}, A_{\alpha\wedge\beta}, A_{\forall x\varphi(x)}, A_{\exists x\varphi(x)}, A_{T(\overline{\alpha})}.$$

We only prove the claim for the cases where the predecessor of A_α is $A_{\alpha\vee\beta}$ or $A_{\alpha\wedge\beta}$, as the other four cases are either trivial or similar to the two cases that we will discuss.

Predecessor of A_α is $A_{\alpha\vee\beta}$. As A_α is strongly correct, we have $\mathbb{V}^{strict}_M(\alpha) = \mathbf{a}$. Hence, there is a (3-valued *SK*) fixed point in which α is evaluated as $\mathbf{a}$ and no fixed point in which α is evaluated as $\mathbf{d}$. In the fixed point in which α is evaluated as $\mathbf{a}$, $\alpha \vee \beta$ is also evaluated as $\mathbf{a}$. Thus, $\mathbb{V}^{strict}_M(\alpha \vee \beta) \in \{\mathbf{a}, \mathbf{b}\}$. Suppose that $\mathbb{V}^{strict}_M(\alpha \vee \beta) = \mathbf{a}$. This gives a contradiction with the assumption that A_α is the first strong $\mathbb{V}^{strict}_M$ correct element on exp'. Thus, suppose that $\mathbb{V}^{strict}_M(\alpha \vee \beta) = \mathbf{b}$. By definition of $\mathbb{V}^{strict}_M$, we get $O^{strict}_M(D_{\alpha\vee\beta})$. From Lemma 4.5, we get that $O^{strict}_M(D_\alpha)$ and $O^{strict}_M(D_\beta)$. From $O^{strict}_M(D_\alpha)$ it follows, by Theorem 4.4, that there is a fixed point in which α is evaluated as $\mathbf{d}$. This gives a contradiction with the strong correctness of A_α.

Predecessor of A_α is $A_{\alpha\wedge\beta}$. As A_α is strongly correct, we have $\mathbb{V}^{strict}_M(\alpha) = \mathbf{a}$. Further, strategy f' (by considering the mirror strategy of f' as in the proof of Lemma 4.6) testifies that $\mathcal{V}^+_M(\alpha \wedge \beta) = \mathcal{V}^+_M(\alpha) = \mathcal{V}^+_M(\beta) = \mathbf{a}$. From the fact that $\mathcal{V}^+_M(\alpha \wedge \beta) = \mathbf{a}$, it follows that there is a 3-valued fixed point (e.g., $\mathcal{V}^+_M$) in which $\alpha\wedge\beta$ is valuated as $\mathbf{a}$. Hence, from Theorem 4.4, it follows that $\mathbb{V}^{strict}_M(\alpha\wedge\beta) \in \{\mathbf{a}, \mathbf{b}\}$. Suppose that $\mathbb{V}^{strict}_M(\alpha \wedge \beta) = \mathbf{a}$. This gives a contradiction with the assumption that A_α is the first strongly correct element on exp'. Thus, suppose that $\mathbb{V}^{strict}_M(\alpha\wedge\beta) = \mathbf{b}$. From Lemma 4.5, we get that $O^{strict}_M(A_\beta)$. Further, from $\mathbb{V}^{strict}_M(\alpha) = \mathbf{a}$ it follows, by definition, that $C^{strict}_M(D_\alpha)$. Similarly, from $\mathbb{V}^{strict}_M(\alpha \wedge \beta) = \mathbf{b}$ we get, by definition, that $O^{strict}_M(D_{\alpha\wedge\beta})$. From $O^{strict}_M(D_{\alpha\wedge\beta})$ and $C^{strict}_M(D_\alpha)$ it follows that $O^{strict}_M(D_\beta)$ and so $\mathbb{V}^{strict}_M(\beta) = \mathbf{b}$. Hence A_β is not strongly correct. Now, let $g'' \in \mathcal{G}$ be the strategy that is defined just like g' except for the fact that $g'(A_{\alpha\wedge\beta}) = A_\alpha$, whereas $g''(A_{\alpha\wedge\beta}) = A_\beta$. Let $\mathsf{exp}'' = \mathsf{exp}(X_\sigma, f', g'')$ be the expansion of X_σ induced by

f' and g, and note that $A_{\alpha\wedge\beta}$ occurs on exp$''$. Let Y_γ be the first element controlled by player I that occurs on exp$''$ after $A_{\alpha\wedge\beta}$ such that $|\Pi(Y_\gamma)| > 1$. If there is no such element, it follows from Lemma 4.5 that for every element Z_θ on exp$''$, we have $\mathbb{V}_M^{strict}(Z_\theta) = \mathbf{b}$. Observe that this contradicts the assumption that strategy f' guarantees that for *every* g, exp(X_σ, f', g) ends up in O_M^+. Thus, let Y_γ be as indicated. From Lemma 4.5, it follows that $\Pi(Y_\gamma)$ contains at least one element, say Y_δ, such that $O_M^{strict}(Y_\delta)$. Moreover, from the definition of f', it follows that f' has to pick a $Y_\delta \in \Pi(Y_\gamma)$ such that $O_M^{strict}(Y_\delta)$. For suppose not, that is, suppose that $f'(Y_\gamma) = Y_{\delta'}$ such that $C_M^{strict}(Y_{\delta'})$. According to Theorem 4.4, this means that there is no 3-valued fixed point that contains $Y_{\delta'}$. On the other hand, from the definition of f' and the assumption that $f'(Y_\gamma) = Y_{\delta'}$, it follows that there is a 3-valued fixed point (e.g., $\mathcal{V}_M^+$) that contains $Y_{\delta'}$. Thus, $f'(Y_\gamma) = Y_\delta$ for some Y_δ such that $O_M^{strict}(Y_\delta)$. From Lemma 4.5, the fact that $\mathbb{V}_M^{strict}(\alpha \wedge \beta) = \mathbf{b}$ and the fact that Y_γ is the first element on exp$''$ after $A_{\alpha\wedge\beta}$ for which player I has to make a genuine choice, it follows that $O_M^{strict}(Y_\delta^{-1})$. Hence, we have $\mathbb{V}_M^{strict}(\delta) = \mathbf{b}$, and so Y_δ is not strongly correct. We are now back were we started, with δ playing the role of β. We can repeat the argument by looking at the first element that occurs on exp$''$ after Y_γ for which player I has to make a genuine choice. By a similar argument, f' cannot allot a strongly correct element to it. Hence, f' does not guarantee that for every g, exp(X_σ, f', g) ends up in O_M^+. □

Before we (finally) show that $\mathcal{V}^+ = \mathcal{K}^+$, we first recall the definition of $\mathcal{K}_M^+$ in terms of the $\mathcal{K}^+$ closure conditions that are associated with the second stable judgment theorem. With exp $= \{y_n\}_{n\in\mathbb{N}}$, these closure conditions are defined as follows:

$$\text{exp} \in O_M^{\mathcal{K}^+} \Leftrightarrow \exists n \forall m > n : y_m \text{ is } \mathcal{K}_M^+ \text{ correct.}$$

Theorem 4.8 *The following holds:* $\mathcal{V}^+ = \mathcal{K}^+$.

Proof Let M be a ground model. It suffices to show that for every *AD* sentence X_σ, we have

$$O_M^{\mathcal{K}^+}(X_\sigma) \Leftrightarrow O_M^+(X_\sigma).$$

The left to right direction is immediate from the definition of $O_M^{\mathcal{K}^+}$ and O_M^+. Thus, assume that $O_M^+(X_\sigma)$. This means that there exists an $f \in \mathcal{F}$ such that for every $g \in \mathcal{G}$, exp$(X_\sigma, f, g) \in O_M^+$. By Lemma 4.7, this means that every term that occurs on an expansion of X_σ that is induced by f, is strongly correct. Hence, all elements of $B_f(X_\sigma)$, the branch of X_σ as induced by f, are strongly correct. From this, it follows that the (3-valued SK) fixed-point valuation induced by $B_f(X_\sigma)^\Uparrow$, that is, by the upward closure of $B_f(X_\sigma)$, is compatible (see Section 2) with every fixed-point valuation over M; hence it is an intrinsic fixed-point valuation, that is, a member of $\mathbf{I^n}(M)$ (see Definition 2.6). With S the $\mathbf{FP^n}(M)$ set corresponding to $\mathcal{K}_M^+$, we get that $B_f(X_\sigma)^\Uparrow \subseteq S$, as $\mathcal{K}_M^+$ is *maximal* intrinsic. From $B_f(X_\sigma) \subseteq S$, it follows that $O_M^{\mathcal{K}^+}(X_\sigma)$. □

We end this section by defining a closure condition that induces $\overline{K}^+$, the 3**b**-version of the maximal intrinsic fixed point. To do so, we first define the notion of an *AD* sentence being *strongly incorrect*. We say that X_σ is *strongly incorrect* (in a ground model M) just in case

$$\big(X = A \text{ and } \mathbb{V}^{strict}_M(\sigma) = \mathbf{d}\big) \qquad \text{or} \qquad \big(X = D \text{ and } \mathbb{V}^{strict}_M(\sigma) = \mathbf{a}\big).$$

The $\overline{+}$ *closure conditions* are defined in terms of the notion of strong incorrectness as follows:

$$\mathsf{exp} \in C^{\overline{+}}_M \Leftrightarrow \exists n \forall m > n : y_m \text{ is strongly incorrect.}$$

Proposition 4.9 *The following holds:* $\mathcal{V}^{\overline{+}} = \overline{\mathcal{K}}^{+}$.

Proof Let M be a ground model. Observe that, as $\mathcal{V}^{+} = \mathcal{K}^{+}$, it suffices to show that for any $\sigma \in Sen(L_T)$:

(i) $\mathcal{V}^{+}_M(\sigma) = \mathbf{a} \Leftrightarrow \mathcal{V}^{\overline{+}}_{\overline{M}}(\sigma) = \mathbf{a}$,

(ii) $\mathcal{V}^{+}_M(\sigma) = \mathbf{d} \Leftrightarrow \mathcal{V}^{\overline{+}}_{M}(\sigma) = \mathbf{d}$.

To prove (i) and (ii), proceed just as in the proof of Proposition 3.11: observe that $O^{+}_M \subseteq O^{\overline{+}}_M$ and that the inverse of O^{+}_M is equal to $C^{\overline{+}}_M$, and use these observations, together with Lemma 3.9 and Lemma 3.10, to obtain the desired result. □

5 Concluding remarks

We presented the method of closure games, a novel game-theoretic valuation method for languages of self-referential truth. We illustrated how our two stable judgment theorems (and their corollaries) allow us to study and define 3- and 4-valued *SK* theories of truth in a uniform manner. By doing so, the method of closure games sheds new light on *SK* fixed points. In particular, the method gives us a great understanding of the interrelatedness of the various *SK* fixed points, which is attested to, among others, by our characterization of the 3**n**- and 3**b**-valued versions of the minimal and maximal intrinsic fixed point by means of closure games.

In future work, we hope to show that the method of closure games is also a fruitful framework to shed light on fixed points associated with valuation schemas other than the SK one (or on "nonfixed point" theories of truth). Can we also use (a modified version of) the method of closure games to characterize the fixed points of the *weak Kleene schema* or the *supervaluation schema*? Although Welch [7] characterized the *minimal* fixed point of the supervaluation schema by game-theoretic means, the last question—which is about the class of *all* supervaluation fixed points—is still open.

Finally, our (intuitive) assertoric interpretation of the constituent notions of the method of closure games—closure conditions as assertoric norms, game rules as assertoric rules—stems from certain philosophical intuitions concerning the notions of assertion and denial. To spell out these intuitions in any detail is far beyond the scope of this paper but to do so rigorously is ongoing work.

Notes

1. Modulo our symbolism which reflects that we interpret the semantic values (directly) in assertoric terms.

2. A game is one of perfect information when a player who is about to make his move in the game can see all the moves that have been made before.

3. For simplicity, Martin describes his game for a first-order language whose connectives are—besides a truth predicate—$\vee$, $\neg$, and $\exists$. For such a language, there is no need to incorporate the roles of verifier and falsifier in the game. However, Martin remarks that when the language contains, in addition, $\wedge$ and $\forall$, the games become more complicated. In a footnote, Martin indicates how the rules and winning conditions of his game have to be modified for such languages. We state the game which Martin describes in this footnote with one terminological difference: Martin speaks of a player "being responsible for a sentence" which is interchangeable with a player being the verifier of that sentence. We have chosen to use the verifier/falsifier terminology as this terminology is well known from Hintikka's influential work on game-theoretic semantics.

4. The fact that the falsifier has to list a sentence when γ is $T(t)$ and t denotes a sentence of L_T or when γ is $\neg\alpha$ is arbitrary: we could also let the verifier do the listing.

5. The actual assignment of player control to $A_{\neg\alpha}$, $D_{\neg\alpha}$, $A_{T(t)}$, and $D_{T(t)}$ was chosen for the sake of symmetry only: for those sentences it does not matter whether they are controlled by player I or by player II.

6. The sign A can be taken to indicate that player I is the verifier, D that player II is the verifier.

7. The range of $\mathbf{T}_M$ may depend on M; that is, for some M, the range of $\mathbf{T}_M$ may be a strict subset of $\mathbf{V}$.

8. In the rules for T, $\overline{\sigma} \in CTerm(L_T)$ is a closed term (quotational constant or not) that denotes σ in M. In the rules for the quantifiers, $\varphi(x/t)$ denotes the result of the uniform replacement of variable x by constant t in $\varphi(x)$. As attested to by, among others, the rules for the truth predicate T, the assertoric rules depend on the details of sentential reference and are, accordingly, defined relative to a ground model M.

9. The assertoric rules for truth testify that the set of all expansions depends on the ground model under consideration.

10. The condition that $C_M^\dagger$ and $O_M^\dagger$ are nonempty rules ensures that we do not have to consider the possibility that $\mathcal{V}_M^\dagger$ valuates all L_T sentences as $\mathbf{n}$ ($O_M^\dagger = \emptyset$) or as $\mathbf{b}$ ($C_M^\dagger = \emptyset$), ensuring that $\mathcal{V}_M^\dagger$ is at least 2-valued. This feature will be convenient for the formulation of theorems that follow.

11. In fact, one can show that $\mathcal{V}_M^{gr}$ is 3-valued for every ground model M, whereas $\mathcal{V}_M^\blacklozenge$ is, depending on the ground model, either 3- or 4-valued.

12. The reader may verify this by considering the sentence $I(c) = \neg T(c) \vee T(\tau)$, where $T(\tau)$ is the Truthteller.

13. Note: V_M does not need to be SK.

14. A version of assertoric semantics that is closely related to the present one was defined in [10] and further studied in [9]. However, none of these papers mentions the close relation between assertoric semantics and the method of closure games.

15. The definition of Branch_M depends on M for the same reasons as EXP_M does.

16. Note: $\mathfrak{T}_X^\sigma$ is not a tree in the mathematical sense of this notion.

17. The notions of downward and upward saturation are closely related to the notions of downward and upward saturation as defined by Fitting [1]. However, an important (and the only) difference between Fitting's notions and ours is that Fitting's notions are defined with respect to the assertoric rules for L only; that is, in his definition Fitting does not treat the rules for truth not on par with the other rules. Likewise, the other notions defined in this section are inspired by [1] and differ from Fitting's notions only in the aspect just indicated. For a proof of the claim that every set of AD sentences has an upward closure, see [1].

18. The notion of a branch in this setting is slightly different from our definition of a branch. In fact, we use 'branch' to denote what is more commonly called 'completed branch.' Likewise, the notion of an assertoric tree differs from that of a tableau.

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