Rule-Irredundancy and
the Sequent Calculus for Core Logic

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Abstract   We explore the consequences, for logical system-building, of taking seriously (i) the aim of having *irredundant* rules of inference, and (ii) a preference for proofs of *stronger* results over proofs of weaker ones. This leads one to reconsider the structural rules of Reflexivity, Thinning, and Cut.

Reflexivity survives in the minimally necessary form $\varphi : \varphi$. Proofs have to get started.

Cut is subject to a Cut-elimination theorem, to the effect that one can always make do without applications of Cut. So Cut is redundant, and should not be a rule *of* the system.

Cut-elimination, however, in the context of the usual forms of logical rules, requires the presence, in the system, of Thinning. But Thinning, it turns out, is not really necessary. Provided only that one liberalizes the statement of certain logical rules in an appropriate way, one can make do without Cut *or* Thinning. From the methodological point of view of this study, the logical rules ought to be framed in this newly liberalized form. These liberalized logical rules determine the system of core logic.

Given any intuitionistic Gentzen proof of $\Delta : \varphi$, one can determine from it a Core proof of some subsequent of $\Delta : \varphi$. Given any classical Gentzen proof of $\Delta : \varphi$, one can determine from it a classical Core proof of some subsequent of $\Delta : \varphi$. In both cases the Core proof is of a result at least as strong as that of the Gentzen proof; and the only structural rule used is $\varphi : \varphi$.

1   A Foundational Maxim

The foundationalist in mathematics has to furnish two important *desiderata*. One of them is specific to the branch of mathematics under investigation (such as, say,

arithmetic or geometry). The other one is invariant across, but adequate unto, all such branches.

The first *desideratum*, for a given branch, is an effectively decidable set of "first principles", or *axioms*. Ideally these axioms are self-evident, or obvious, or otherwise epistemologically grounded or secure. They may be cited without proof, in proving other results that *do* stand in need of proof, in order to ensure that any rational thinker can be convinced that these other results are true.

The second *desideratum* is a canon of deductive reasoning, whereby those less self-evident or obvious results, or mathematical *theorems*, may be deduced from those axioms. They are deduced as conclusions of *proofs* that are formed by means of the primitive *rules of inference* of the deductive system chosen. Proofs render their conclusions as grounded and secure as the axioms that serve as premises within them. One and the same proof system serves all branches of mathematics, since their respective languages all involve the same logical vocabulary.

The foundationalist's project is ultimately one of adequate *regimentation*, or *formalization*, of the various branches of mathematical theorizing. One can treat each branch *sui generis*; or one can seek a single, all-embracing system, such as set theory, within which each branch can be reconstructed.

Whichever of these two approaches the foundationalist favors, there is one methodological aim that is worth bearing in mind: one should always seek to *do more with less*. This maxim of theoretical economy, or economy of thought, is as apt for mathematics as it is for any other theoretical endeavor, including physics. There is no need to explicate it here, since its applications in the course of these investigations are intuitively compelling.

The aim of this study is to apply this simple maxim in pursuit of a "most economical possible" system of deductive proof—one that can be shown to be adequate for the project of "naturally" regimenting or formalizing mathematical reasoning. Our investigations will uncover a system of *core proof*. Core proofs serve to formalize, in a natural fashion, the intuitively "good" informal deductions that one finds in ordinary mathematical discourse. These deductions establish arguments of the form *Axioms*:*Theorem*. They show, convincingly, that if the *Axioms* are true, then the *Theorem* is true also.

There is more, however, to being a *good* argument than being *(barely) valid* in the usual semantic or model-theoretic sense. To be sure, a good argument must be valid in this sense. But any proof establishing it must also do something more. It has to establish a connection between the premises and the conclusion of the argument in question. And therein lies the rub. The standard proof systems—for intuitionistic and for classical logic—allow for such connections to be overlooked, attenuated, obscured, or broken. Core logic, by contrast, keeps these connections always to the fore. And in doing so, *it never sacrifices any of the deductive power essential for the formalization of mathematical reasoning*.

2   Irredundant Sets of Axioms

The maxim of economy explains why mathematicians and logicians prefer irredundant axiomatizations of theories to redundant ones. That is, they prefer to frame theories by means of sets of axioms in which none follows logically from the rest. An axiom that *does* follow logically from its companions is *redundant*; so too, then,

is the axiom set in question. An *irredundant* set of axioms contains no redundant axioms.

There is a simple explanation for this preference in theory formulation. One wants each axiom to be making its own special contribution. It must be "doing some work". There is no point in listing a redundant axiom; for one might as well just prove it from the rest. By dropping it from the list of axioms, one reduces the axiomatic basis itself, thereby making it simpler and more efficient. One also makes the deductive arguments (the proofs) of one's theoretical conclusions *stronger*. This is an excellent thing.

2.1 Strength of argument At the risk of appearing captious, we rehearse in detail here some basic ideas that need to be borne in mind when one is dealing with logical deducibility in general—as opposed to logical theoremhood—and seeking to find a satisfactory formulation of the crucial requirement of transitivity of deduction. The heterodox contribution of this study turns on a thorough appreciation, reconsideration, and possibly surprising application of the basic ideas to be set out here. Such reconsideration seldom arises within a tradition of acquiescence with *one particular, very entrenched, but possibly mistaken* way of explicating the requirement of transitivity of deduction.

Let Δ and Γ be two sets of sentences. Suppose that Δ logically implies every sentence in Γ. Then we say that Δ is *at least as strong as* Γ. If, moreover, Γ is not at least as strong as Δ—that is, if some sentence in Δ does not follow logically from Γ—then we say that Δ is *stronger than* Γ.

If Δ is at least as strong as Γ, and *vice versa*, then we say that Δ has *the same logical strength* as Γ.

Let φ be a sentence. If Δ is at least as strong as Γ, then the *argument* (or *sequent*) $\Gamma : \varphi$ (Γ, *ergo* φ) is said to be *at least as strong as* the *argument* $\Delta : \varphi$. And if Δ is *stronger* than Γ, then the argument $\Gamma : \varphi$ is said to be *stronger than* the argument $\Delta : \varphi$. Note the reversal in order. The weaker the set of premises, the stronger the argument. An apt slogan might be that *more is less*—in the sense that the more (logical strength) one has in the *premises*, the less (logical strength) one has in any *argument from* those premises. Likewise, *less is more*—in the sense that the less (logical strength) one has in the *premises*, the more (logical strength) one has in any *argument from* those premises. Doing more with less, the reader will recall, is our overarching theoretical maxim.

If Γ is a subset of Δ, then of course Δ logically implies every sentence in Γ; whence Δ is at least as strong as Γ, and the argument $\Gamma : \varphi$ is at least as strong as $\Delta : \varphi$. If Γ logically implies all those members of Δ not in Γ, then Γ has the same logical strength as Δ; otherwise, Δ is stronger than Γ.

Any irredundant set is stronger than each of its proper subsets. So if Δ is irredundant and Γ is a proper subset of Δ, then the argument $\Gamma : \varphi$ is stronger than the argument $\Delta : \varphi$.

It is desirable, when stating an argument $\Delta : \varphi$, to have Δ irredundant. Mathematicians and logicians will not knowingly state an argument $\Delta : \varphi$ with redundant premises. Instead, they will discard premises in Δ found to follow from others in Δ, until the residual subset of Δ that results is irredundant. They might arrive by different routes at different irredundant subsets of the original set Δ; but those irre-

dundant subsets of Δ will at least have the same logical strength since each of them will logically imply all the members of any other.

We have seen that an argument can be strengthened by weakening its premises. If the original set of premises is irredundant, then discarding any of those premises will strengthen the argument. That is,

$$\left.\begin{array}{c} \Delta \text{ irredundant} \\ \Gamma \subset \Delta \end{array}\right\} \quad \Rightarrow \quad \Gamma : \varphi \text{ is stronger than } \Delta : \varphi.$$

When a valid argument $\Delta : \varphi$ has a satisfiable set Δ of premises, considerable interest attaches to the question of whether there are any *proper* subsets Γ of Δ for which $\Gamma : \varphi$ is still valid. (Note that this question can arise even when Δ is irredundant.) Most interesting among these proper subsets are the *inclusion-minimal* ones—those proper subsets Γ of Δ whose every proper subset does *not* logically imply φ. Such proper subsets Γ of Δ (which will obviously themselves be irredundant) yield the most ideal strengthenings $\Gamma : \varphi$ of the original argument $\Delta : \varphi$.

There is another way, however, to strengthen an argument, and that is to claim that its premises cannot all be true:

$\Delta : \bot$ is stronger than $\Delta : \varphi$.

Note that when we say that an argument has been strengthened, we are not implying that the new (stronger) argument is itself valid. One can strengthen an argument so much that it becomes invalid—for example, by discarding sufficiently many premises, or by putting $\bot$ in place of the original conclusion when the original premises form a satisfiable set.

In what follows we shall frequently suppress the absurdity constant "$\bot$" and just leave an empty space to indicate that there is no sentence on the right of the colon. In that case the sequent "$\Delta :$ " in question can be understood as claiming that its premise-set Δ is not satisfiable.

2.2 A good historical example of (almost) irredundant axiomatization Hilbert [5] provided the first formal axiomatization of Euclid's geometry, thereby fixing the problem of certain logical lacunae in Euclid's *Elements*, especially concerning continuity. Hilbert arranged his axioms in groups, each group concerning a particular geometrical concept or tightly knit group of concepts. There were groups of axioms governing

1. connection;
2. order;
3. parallelism;
4. congruence;
5. continuity.

Hilbert devoted Chapter II of his book to establishing the mutual independence of his axioms. It was this arrangement of which Carnap [1] took advantage, in trying to argue that the groups of axioms for congruence and parallelism had a different epistemological status from the other groups.

3   Irredundant Sets of Rules

The basic *rules of inference* in a system of proof have a similar foundational role as axioms. The same careful attention should be paid to the question of mutual

independence, or possible redundancy within the overall project, of the rules of inference that one chooses as basic for a deductive system. This holds not only for natural-deduction rules such as the introduction and elimination rules for the logical operators—and for their sequent-calculus equivalents called, respectively, the "right" and "left" logical rules—but also for what are known, in the *sequent-calculus* setting, as the *structural* rules. There are three of the latter: the rule of REFLEXIVITY; the rule of THINNING (also known as DILUTION, or WEAKENING); and the rule of CUT.

Our task here is to inquire how the sequent rules—logical and structural—might best be formulated in a *mutually independent* way, and whether, indeed, any of them might turn out to be redundant within the overall project. We seek, that is, an *irredundant* set of logical and structural rules, if such can be found. Each rule must be stated in such a way that its own unique contribution is laid bare and is not confused or conflated with anything extraneous that might be provided for by the other rules.

In our search for rule-independence we need to pay particular attention to the logical rules *in their relationship to the structural rules*. The logical rules should be so formulated as to be able to furnish proofs of all those results for which, intuitively, no recourse need be had to structural rules. A simple example of such a result is $A, B : A \wedge B$. In natural deduction, this argument is so immediate and primitive that its proof is a special case of the rule of $\wedge$-Introduction:

$$\frac{A \qquad B}{A \wedge B} \ .$$

One would therefore expect the corresponding sequent rule (for "$\wedge$ on the right") to afford a similarly direct and immediate proof. Ideally, that proof would be

$$\frac{A : A \qquad B : B}{A, B : A \wedge B}_{(:\wedge)} \ .$$

For this proof to be available, however, one needs the general form of the rule $(:\wedge)$ to be

$$\frac{\Delta : A \qquad \Gamma : B}{\Delta, \Gamma : A \wedge B}_{(:\wedge)} \ .$$

Gentzen's own form of the rule $(:\wedge)$ in *LJ*, however, was[1]

$$\frac{\Delta : A \qquad \Delta : B}{\Delta : A \wedge B}_{(:\wedge)} \ .$$

Thus Gentzen's *LJ*-proof of the argument $A, B : A \wedge B$ resorts to two applications of THINNING on the left:

$$\frac{\dfrac{A : A}{A, B : A}_{(T_L)} \quad \dfrac{B : B}{A, B : B}_{(T_L)}}{A, B : A \wedge B} \ .$$

It is intuitively clear that Gentzen missed the opportunity to state $(: \wedge)$ in a form that would relieve one of such recurring, but unnecessary, recourse to the structural rule of THINNING.

We need to get clear at the outset about the sense in which a given structural rule can be "used in" or can "hold for" a system of proof. The matter is subtle and hardly ever remarked on when sequent-calculus rules are stated. From now on, all sets Δ, Γ, and so forth are assumed to be finite. The reader is advised that we confine ourselves throughout this study to single-conclusion sequents and sequent calculi involving them. This holds even for the classical case. Famously, Gentzen classicized his

sequent calculi by allowing multiple succedents within sequents. We do not allow multiple succedents. Rather, we classicize our single-conclusion system by adopting a suitably formulated rule of Dilemma (see p.), which involves only empty or singleton succedents.

3.1 The intuitionistic sequent system for negation Let us take a "simplest possible" sequent calculus, for a propositional language whose only logical operator is negation ($\neg$). Recall that we are confining ourselves to single-conclusion sequents, that is, sequents of the form $\Delta : \varphi$, where Δ is a (finite) set of sentences, and φ is a sentence, or of the form $\Delta :$, with empty succedent. What is "the" *simplest* single-conclusion sequent calculus for this language?

A predictable answer is as follows. First, one has the rule of "initial sequents" or "reflexivity":

$$(R)\ \ \varphi : \varphi.$$

This rule (R) allows one to *get sequent proofs started.* REFLEXIVITY should not have any "thinning" in it. It should be stated in the weakest form that still gets the job done. That is why we eschew any statement of reflexivity such as

$$(R_T)\ \ \Delta : \varphi,\ \text{where}\ \varphi \in \Delta.$$

Next, one has the rules for inferring $\neg$ on the right, and on the left, of sequents:

$$(: \neg)\ \ \frac{\Delta, \varphi :}{\Delta : \neg\varphi}\ (\text{where}\ \varphi\ \text{is not in}\ \Delta) \qquad\qquad (\neg :)\ \ \frac{\Delta : \varphi}{\Delta, \neg\varphi :}.$$

(The notation Δ, ψ is short for $\Delta \cup \{\psi\}$. We shall also follow the convention that when we write Δ, ψ on the left of a *premise*-sequent, the sentence ψ is not in the set Δ. Hereafter, we shall no longer state this explicitly.)

Using the rules just given, here is a sequent-proof of the result known as Double-Negation Introduction:

$$\frac{\dfrac{\varphi : \varphi}{\varphi, \neg\varphi :}}{\varphi : \neg\neg\varphi}.$$

To help the reader follow the proof, one can annotate each step with the rule that is being applied:

$$\frac{\dfrac{\varphi : \varphi}{\varphi, \neg\varphi :}(\neg :)}{\varphi : \neg\neg\varphi}(: \neg).$$

But that is to omit mention of the application of the rule (R) at the very outset. If we are to treat all of our "proof-constitutive" rules on a par (the reason for which will emerge below), then the rule (R) should be stated as follows:

$$(R)\ \ \frac{\rule{3em}{0.4pt}}{\varphi : \varphi}.$$

This underscores the fact that the sequent $\varphi : \varphi$ does not depend on any (other) sequent for its own justification. REFLEXIVITY is a *zero-premise rule*. (THINNING is a one-premise rule; CUT is a two-premise rule.) The inference stroke placed over the

"initial" sequent can also be annotated in our foregoing proof of Double-Negation Introduction, in the same way that subsequent steps were annotated:

$$\frac{\dfrac{\rule{3em}{0.4pt}}{\varphi \, : \, \varphi}(R)}{\dfrac{\varphi, \neg\varphi \, :}{\varphi \, : \, \neg\neg\varphi}(\neg:)}(:\neg) \; .$$

This stratagem allows one to appreciate that (R) is a rule *of the system*, essentially involved in the construction of the proof just given. This is what we meant above by "proof-constitutive" rules. Without the rule (R), the proof just given *would not exist* as a proof in the system. Indeed, *no* proofs would exist in the system; for the rule (R) corresponds to the (only) *basis clause* in the *inductive definition* of the notion of sequent-proof (for this system).

This is a very important point, and we need to expand on it. The rigorous formal definition of the notion of proof (for this system) defines the relational notion "Π is a proof of the sequent $\mathscr{S}$". The definition is as follows.

1. (Basis clause) Any sequent of the form $\varphi : \varphi$ is a proof of the sequent $\varphi : \varphi$.

2. (Inductive clause) If Π is a proof of the sequent $\Delta, \varphi :$, then $\dfrac{\Pi}{\Delta \, : \, \neg\varphi}$ is a proof of the sequent $\Delta : \neg\varphi$.

3. (Inductive clause) If Π is a proof of the sequent $\Delta : \varphi$, then $\dfrac{\Pi}{\Delta, \neg\varphi \, :}$ is a proof of the sequent $\Delta, \neg\varphi :$.

4. (Closure clause) If Π is a proof of a sequent $\mathscr{S}$, then this can be shown by appeal to clauses (1)–(3).

The Basis clause (1) corresponds to the rule (R).
Inductive clause (2) corresponds to the Right-rule $(: \neg)$.
Inductive clause (3) corresponds to the Left-rule $(\neg :)$.
Each of these rules found application in our proof above of the Double-Negation Introduction result. *Each* of these rules is "proof-constitutive", because *it corresponds to a clause in the inductive definition of proof.*

Definition 1 $\Delta \vdash \varphi$ [resp., $\Delta \vdash$] means that there is a proof of the sequent $\Delta : \varphi$ [resp., $\Delta :$].

This definition is of what we call the "exact" construal of deducibility. The set Δ on the left must be the *exact* set of premises actually occurring on the left of the conclusion-sequent of whatever proof is in question. This reading is to be distinguished from the usual, laxer one according to which $\Delta \vdash \varphi$ holds just in case there is a proof of a sequent $\Gamma : \varphi$, for some subset Γ of Δ.

Notice that the notion of proof that we have defined allows that any substitution instance of a proof is a proof. If, for example, we substitute $\neg\neg\theta$ for φ in our foregoing proof of the sequent $\varphi : \neg\neg\varphi$, we obtain the proof

$$\frac{\dfrac{\rule{5em}{0.4pt}}{\neg\neg\theta \, : \, \neg\neg\theta}(R)}{\dfrac{\neg\neg\theta, \neg\neg\neg\theta \, :}{\neg\neg\theta \, : \, \neg\neg\neg\neg\theta}(\neg:)}(:\neg) \; .$$

Now substitute φ for θ in the last proof. Putting our two little proofs alongside one another:

$$\dfrac{\dfrac{\dfrac{\rule{2cm}{0.4pt}}{\varphi : \varphi}(R)}{\dfrac{\varphi, \neg\varphi :}{\varphi : \neg\neg\varphi}(\neg:)}(:\neg)}{}\qquad\qquad \dfrac{\dfrac{\dfrac{\rule{2.5cm}{0.4pt}}{\neg\neg\varphi : \neg\neg\varphi}(R)}{\dfrac{\neg\neg\varphi, \neg\neg\neg\varphi :}{\neg\neg\varphi : \neg\neg\neg\neg\varphi}(\neg:)}(:\neg)}{},$$

one is tempted by the thought that there ought, now, to be a proof of the sequent $\varphi : \neg\neg\neg\neg\varphi$. Two lots of Double-Negation Introduction *ought* to yield Quadruple-Negation Introduction, should it not?

Indeed it does. We have a proof of $\varphi : \neg\neg\varphi$. This can be abbreviated as $\varphi \vdash \neg\neg\varphi$. We have also a proof of $\neg\neg\varphi : \neg\neg\neg\neg\varphi$. This can likewise be abbreviated as $\neg\neg\varphi \vdash \neg\neg\neg\neg\varphi$. Now that our achievements can be registered on a single line:

$$\varphi \vdash \neg\neg\varphi \qquad\qquad \neg\neg\varphi \vdash \neg\neg\neg\neg\varphi;$$

the temptation of transitivity is overpowering.

Here is a proof that it is perfectly in order to give in to that temptation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\rule{2cm}{0.4pt}}{\varphi : \varphi}(R)}{\dfrac{\varphi, \neg\varphi :}{}(\neg:)}}{\varphi : \neg\neg\varphi}(:\neg)}{\dfrac{\varphi, \neg\neg\neg\varphi :}{}(\neg:)}}{\varphi : \neg\neg\neg\neg\varphi}(:\neg)\quad.$$

The existence of this proof justifies the claim

$$\varphi \vdash \neg\neg\neg\neg\varphi.$$

Our way of justifying this claim might have taken the reader by surprise. *Surely*, the reader might say, one could just infer this sought result *directly* from the two proofs that we already had? Could not one just finish off with a step of so-called CUT, thus?:

$$\dfrac{\dfrac{\dfrac{\rule{2cm}{0.4pt}}{\varphi : \varphi}(R)}{\dfrac{\varphi, \neg\varphi :}{\varphi : \neg\neg\varphi}(\neg:)}(:\neg)\qquad \dfrac{\dfrac{\rule{2.5cm}{0.4pt}}{\neg\neg\varphi : \neg\neg\varphi}(R)}{\dfrac{\neg\neg\varphi, \neg\neg\neg\varphi :}{\neg\neg\varphi : \neg\neg\neg\neg\varphi}(\neg:)}(:\neg)}{\varphi : \neg\neg\neg\neg\varphi}(\text{CUT})\quad.$$

Note that the suggestion here is that the rule of CUT:

$$\text{CUT}\quad \dfrac{\chi : \theta \qquad \theta : \psi}{\chi : \psi}$$

or, more generally:

$$\text{CUT}\quad \dfrac{\Delta : \theta \qquad \Gamma, \theta : \psi}{\Delta, \Gamma : \psi}$$

should be taken as a *proof-constitutive* rule. That is, the foregoing construction that "ends" with an application of CUT really *is* a proof of the formal system. For that to be so, the inductive definition of proof that we provided above would have to be expanded, with a new clause for proof-formation by application of CUT. The expanded definition would read as follows.

1. (Basis clause) Any sequent of the form $\varphi : \varphi$ is a proof of the sequent $\varphi : \varphi$.

2. (Inductive clause) If Π is a proof of the sequent $\Delta, \varphi : $, then $\dfrac{\Pi}{\Delta : \neg\varphi}$ is a proof of the sequent $\Delta : \neg\varphi$.

3. (Inductive clause) If Π is a proof of the sequent $\Delta : \varphi$, then $\dfrac{\Pi}{\Delta, \neg\varphi : }$ is a proof of the sequent $\Delta, \neg\varphi : $.

4. **(Inductive clause) If Π is a proof of the sequent $\Delta : \varphi$, and Σ is a proof of the sequent $\Gamma, \varphi : \psi$, then $\dfrac{\Pi \qquad \Sigma}{\Delta, \Gamma : \psi}$ is a proof of the sequent $\Delta, \Gamma : \psi$.**

5. (Closure clause) If Π is a proof of a sequent $\mathcal{S}$, then this can be shown by appeal to clauses (1)–**(4)**.

This is *very* different from saying the following:

>   If $\Delta \vdash \varphi$ and $\Gamma, \varphi \vdash \psi$, then $\Delta, \Gamma \vdash \psi$,

by way of "meta-comment" on the earlier proof system. For in this case the turnstile $\vdash$ of (exact) deducibility is to be understood by reference to proofs in the earlier, *un*expanded system of proof that did *not* contain the proposed rule CUT.

The issue that needs to be addressed is this: is CUT to be construed as a rule *of* the system, that is, as a proof-constitutive rule that actually finds application within proofs? Or is it rather to be understood as a (true) statement about (exact) deducibility-within-a-system, where the system in question does *not* contain it as a constitutive rule? We can call these, respectively, the "object-" and "meta-" versions of CUT, or, more succinctly, Object-CUT and Meta-CUT. We shall discuss them in the reverse order.

4 Meta-CUT

With Meta-CUT, we need to distinguish what we shall call the "proof-based" interpretation from any contrasting semantic one. Meta-CUT is the following statement about (exact) deducibility that we encountered earlier:

>   Meta-CUT: If $\Delta \vdash \varphi$ and $\Gamma, \varphi \vdash \psi$, then $\Delta, \Gamma \vdash \psi$.

Spelled out more fully, this says:

>   *If* there is a proof of the sequent $\Delta : \varphi$ and there is a proof of the sequent $\Gamma, \varphi : \psi$, *then* then there is a proof of the sequent $\Delta, \Gamma : \psi$.

It is important to realize what a *bare* guarantee this really is. For, suppose that we have a proof, say Π, of the sequent $\Delta : \varphi$, and a proof, say Σ, of the sequent $\Gamma, \varphi : \psi$. The assurance is only that there is consequently *some* proof of the sequent $\Delta, \Gamma : \psi$. Call such a proof Ω. We are not told *anything* about the relationship between Π and Σ, on the one hand, and Ω, on the other. For all we know, the proofs Π and Σ, as formal objects, might have nothing at all to do with the proof Ω. In particular, there is no guarantee, express or implied, that Π and Σ might be *subproofs* of Ω.

For epistemological purposes, this version of Meta-CUT is rather unhelpful. Far better would be a version (still of Meta-CUT rather than of Object-CUT) that would assure us that the sought proof Ω (of the "target sequent" $\Delta, \Gamma : \psi$) can always be effectively determined from the proofs Π and Σ. Such a version of Meta-CUT would be:

>   Given any proof Π of the sequent $\Delta : \varphi$ and any proof Σ of the sequent $\Gamma, \varphi : \psi$,
>   one can effectively determine a proof $[\Pi, \Sigma]$ of the sequent $\Delta, \Gamma : \psi$.

The formalization of this version of Meta-CUT would read:

There is an effective method [,] such that for any proof Π of the sequent $\Delta : \varphi$ and any proof Σ of the sequent $\Gamma, \varphi : \psi$, the object $[\Pi, \Sigma]$ is a proof of the sequent $\Delta, \Gamma : \psi$.

This does not imply that the proof $[\Pi, \Sigma]$ would have Π and Σ as *sub*proofs; all it says is that $[\Pi, \Sigma]$ *can be determined from* Π and Σ in some effective way. And this process of determination need not involve containing Π and Σ as subproofs. Rather, it could involve, say, rearranging internal bits and pieces of Π and of Σ in some methodical way.

The problem, however, with this formulation of Meta-CUT is that its truth depends very much on whether the system in question also contains THINNING as a proof-constitutive rule. *If the system does not contain* THINNING, then one cannot guarantee the existence of a CUT-free proof of the "target" sequent $\Delta, \Gamma : \psi$. Here is a simple example that makes this clear. Consider the two sequents

$$\varphi : \varphi \vee \psi \quad \text{and} \quad \neg\varphi, \varphi \vee \psi : \psi.$$

The first is proved by Π below, and the second is proved (let us suppose) by Σ below:[2]

$$\Pi \quad \frac{\varphi : \varphi}{\varphi : \varphi \vee \psi} \qquad \Sigma \quad \frac{\dfrac{\varphi : \varphi}{\neg\varphi, \varphi :} \quad \psi : \psi}{\neg\varphi, \varphi \vee \psi : \psi}(\vee:) .$$

But what would be the object $[\Pi, \Sigma]$ that would prove the *exact target sequent* $\neg\varphi, \varphi : \psi$? Without THINNING as a rule, the answer would have to be: *none*.

We would be well advised, therefore, to consider the prospect of being able always to prove, instead of the exact "target" sequent $\Delta, \Gamma : \psi$, some *subsequent* thereof. That is, Meta-CUT should be formulated in a less expansive, but still epistemologically adequate, way. In its present unrestricted formulation it appears to be overreaching. We suggest the following more moderate, but still adequate, reformulation:

CUT FOR EPISTEMIC GAIN:
There is an effective method [,] such that for any proof Π of the sequent $\Delta : \varphi$ and any proof Σ of the sequent $\Gamma, \varphi : \psi$, the object $[\Pi, \Sigma]$ is a proof of *some subsequent of* the sequent $\Delta, \Gamma : \psi$.

This is the main result of [8], for the system of *Core Logic*, given by rules of *natural deduction*. In that paper, CUT FOR EPISTEMIC GAIN was called CUT ELIMINATION FOR CORE PROOF, and was stated as follows:

There is an effective method [,] that transforms any two core proofs

$$
\begin{array}{ll}
\Delta & A, \Gamma \\
\Pi & \Sigma \qquad \text{(where } A \notin \Gamma \text{ and } \Gamma \text{ may be empty)} \\
A & \theta
\end{array}
$$

into a core proof $[\Pi \ \Sigma]$ *of θ or of* $\bot$ from (some subset of) $\Delta \cup \Gamma$.[3]

Core natural deductions are so defined as to be in normal form; and one of the main features of such proofs is that all their major premises of eliminations "stand proud", with no proof-work above them. This feature affords a very simple and direct translation between natural deductions and sequent proofs. Given any natural deduction, one can find an "isomorphic" sequent proof of its result; and vice versa. In the present paper we work with sequent systems, since the discussion focuses on

whether we need the structural rules of Thinning and Cut. For the logical right-
and left-rules of the sequent system for Core Logic, see Section 9 below.

5 Object-Cut

The proof-theoretic tradition begun by Gentzen teaches us the perhaps surprising
lesson that Object-Cut need not be on any list of deductive rules; and therefore
should not be on any list of such rules that aspires to irredundancy. For it is the
aim of any proof-theorist who follows the example originally set by Gentzen [4] and
includes Cut as a proof-constitutive rule, to establish a metatheorem to the effect that
all applications of Cut can be *eliminated* from within proofs. Gentzen's *Hauptsatz*,
or Cut-elimination theorem, states

> there is an effective method γ such that for every proof Π of any sequent $\Delta : \varphi$,
> the object $\gamma(\Pi)$ is a Cut-free proof of $\Delta : \varphi$.

From the vantage point of this study, it is a thought-provoking irony, inviting one to
reexamine the fundamentals, that a logician of Gentzen's caliber, who prized irredun-
dant sets of *axioms* for a formalized theory (see [3]), should have *included* in his list
of proof-constitutive rules one that he was then intent on establishing as *unnecessary*
for the logical purposes he had in mind.[4]

Concerning Gentzen's own Cut-elimination theorem, some further remarks are
in order. First, Gentzen was able to state (and prove) his result in the form above
only because he had in his system, apart from Cut, the structural rule of Thinning.
For, note the following Gentzenian sequent proof, whose last step is an application
of Cut:

$$\dfrac{\varphi : \varphi \qquad \dfrac{\dfrac{\dfrac{\varphi : \varphi}{\neg\varphi, \varphi :}}{\neg\varphi, \varphi : \psi}(\mathrm{T}_R) \quad \psi : \psi}{\neg\varphi, \varphi \vee \psi : \psi}(\vee:)}{\varphi, \neg\varphi : \psi}(\textsc{Cut})$$

Gentzen's Cut-elimination procedure would not be able to deliver a Cut-free proof
Ω of the final sequent $\varphi, \neg\varphi : \psi$ *unless one could appeal to* Thinning (on the right),
at the step marked (T_R):

$$\Omega \qquad \dfrac{\dfrac{\varphi : \varphi}{\neg\varphi, \varphi :}}{\neg\varphi, \varphi : \psi}(\mathrm{T}_R).$$

The same holds for a system that differs from Gentzen's own, in that its rule $(\vee :)$
allows one or both of its premise-sequents to be empty on the right. In such a system,
the proof from which the Cut needs to be eliminated is as follows:

$$\dfrac{\varphi : \varphi \quad \dfrac{\dfrac{\varphi : \varphi}{\neg\varphi, \varphi :} \quad \psi : \psi}{\neg\varphi, \varphi \vee \psi : \psi}(\vee:)}{\varphi, \neg\varphi : \psi}(\textsc{Cut})$$

Note that this last proof, unlike the one in Gentzen's system, contains no application
of Thinning at all; however, its reduct, upon Cut-elimination, has to be the same
proof Ω as above, and necessarily contains an application of Thinning, because of
the requirement that it should prove the *exact* sequent $\neg\varphi, \varphi : \psi$.

6 Thinning

There are two rules of THINNING. The rule (T_R), which we have already seen in action, is for thinning on the right, and the rule (T_L) is for thinning on the left:[5]

$$(\mathrm{T}_R) \quad \frac{\Delta :}{\Delta : \varphi} \qquad\qquad (\mathrm{T}_L) \quad \frac{\Delta : \varphi \qquad \Delta :}{\Delta, \psi : \varphi \qquad \Delta, \psi :} \, , \text{ where } \psi \notin \Delta.$$

The idea is really very simple. THINNING allows one to "tack on" fresh sentences on the left or on the right of any sequent proved thus far. Moreover, one tacks them on *one at a time*, since repeated application of that permitted move would achieve the effect of tacking on any finite set of sentences.

THINNING, obviously, can logically *weaken* the sequent proved. (That is, after all, why the rule itself is sometimes called WEAKENING.) Note that the rule (T_R) is the intuitionistic sequent-calculus analogue of Ex Falso Quodlibet in natural deduction.[6] As such, it is anathema to the relevance logician—who regards it not only as counterintuitive, but also as *incorrect*, and certainly not to be taken as a canon of formally correct inference. The rule (T_R), as we have seen, enables one to prove the First Lewis Paradox

$$\neg\varphi, \varphi : \psi$$

(by virtue of the proof Ω above). What is not often appreciated is that THINNING on the *left* (in either of the two forms available) produces a result just as objectionable from the relevance logician's point of view, namely

$$\neg\varphi, \varphi : \neg\psi.$$

The two possible CUT-free proofs of this result are as follows:

$$\cfrac{\cfrac{\cfrac{\varphi : \varphi}{\neg\varphi, \varphi :}}{\neg\varphi, \varphi, \psi :}(\mathrm{T}_L)}{\neg\varphi, \varphi : \neg\psi} \qquad\qquad \cfrac{\cfrac{\cfrac{\varphi : \varphi}{\varphi, \psi : \varphi}(\mathrm{T}_L)}{\neg\varphi, \varphi, \psi :}}{\neg\varphi, \varphi : \neg\psi} \, .$$

So we see that THINNING is a fraught matter, for the logician who is concerned to maintain an intuitive connection of relevance of premises to conclusions of good arguments. As already remarked, preserving truth from its premises to its conclusion is a necessary, but insufficient, condition for the proof of an argument to qualify, intuitively, as a *good* proof. We wish to capture or distil what is essential to *good* proofs of arguments. We ought, therefore, to frame the *other* rules of the sequent-calculus—particularly the right- and left-rules for the logical operators—in such a way that one will be able to do everything that one "ought" to be able to do with the logical operators, should the system be one in which THINNING is *not* available as a proof-constitutive rule.

This speaks in favor of a certain liberality in the framing of right- and left-rules. One needs to find forms for these rules in which a certain range of highly desirable results can be secured *without* any explicit appeal to a rule of THINNING. We shall consider just two such results.

The first is disjunctive syllogism:

$$\neg A, A \vee B : B.$$

The second is the inference to the truth of a conditional from the falsity of its antecedent:

$$\neg A : A \rightarrow B.$$

In due course we shall see that the former inference may be secured by liberalizing the rule of $(\vee :)$, while the latter may be secured by liberalizing the rule of $(: \rightarrow)$.

But first, we need a word or two as to *why* these two inferences are to be secured, especially when both of them have been challenged (and given up) by certain writers in the tradition of "relevance" logic. It is not necessary here to give necessary and sufficient conditions (more exigent than bare validity) for regarding an argument as acceptable, and hence deserving of proof. It is sufficient, in fact, to give only sufficient conditions (more exigent than bare validity) for so doing. Sufficient conditions for the acceptability of any sequent

$$A_1, \ldots, A_n : B$$

are these:

1. $A_1, \ldots, A_n : B$ is valid.
2. Deleting any A_i from $A_1, \ldots, A_n : B$ results in an invalid sequent.
3. Deleting B results in an invalid sequent; that is, $A_1, \ldots, A_n$ are jointly satisfiable.
4. $A_1, \ldots, A_n : B$ is not a proper, uniform substitution instance of a valid sequent; that is, for $A_1, \ldots, A_n : B$ to be valid, it needs every one of its displayed occurrences of logical operators, and it needs all its displayed repetitions of nonlogical expressions.

Our methodological stance, with regard to potential reforms, is extremely hospitable. Any sequent satisfying conditions (1)–(4) is *surely* one that we would wish to be able to prove.

It is an easy exercise to check that our two chosen sequents, $\neg A, A \vee B : B$ and $\neg A : A \rightarrow B$, satisfy conditions (1)–(4).

Moreover, we note that inferences of the form $\neg A, A \vee B : B$ crop up frequently in mathematics, especially when dealing with orderings, for example. One wants to be able to provide regimentations of inferences such as

$$x \nleq y, \;\; x \leq y \;\vee\; x > y \;\; : \;\; x > y.$$

We note also that according to the truth-table for $\rightarrow$:

φ	ψ	$\varphi \rightarrow \psi$
T	T	T
T	F	F
F	T	*T*
F	F	*T*

the **falsity of the antecedent** φ suffices for the *truth of the conditional* $\varphi \rightarrow \psi$. In inferential terms, this means that the sequent $\neg A : A \rightarrow B$ should have a proof. Note that this consideration invokes *only* the "left-to-right", row-by-row reading of the truth table; so it is a *constructive* consideration. It does *not* invoke the strictly *classical* assumption that the four rows together exhaust all the possibilities. Thus the consideration in question falls short of "justifying" any strictly classical law such as Excluded Middle ($A \vee \neg A$), say, or Peirce's Law ($[(A \rightarrow B) \rightarrow A] \rightarrow A$).

6.1 Liberalizing proof-by-cases To the first end (i.e., being able to prove disjunctive syllogism), it is wise to finesse the statement of the usual sequent rule ($\vee$:), which can be given in two halves as follows:[7]

$$\frac{\Delta, \varphi : \theta \qquad \Gamma, \psi : \theta}{\Delta, \Gamma, \varphi \vee \psi : \theta} \qquad\qquad \frac{\Delta, \varphi : \qquad \Gamma, \psi :}{\Delta, \Gamma, \varphi \vee \psi :} .$$

The common feature is that each of the premise-sequents is required to have the *same* succedent (right-hand side)—that is, either both premise-sequents in any application of the rule have the same sentence θ on the right, or both have the empty succedent. With this form of left-rule for $\vee$ ("proof by cases"), the proof of disjunctive syllogism is as follows:

$$\frac{\dfrac{\dfrac{A : A}{\neg A, A :}(\mathrm{T}_R)}{\neg A, A : B} \qquad B : B}{\neg A, A \vee B : B}(\vee:) .$$

Note the need for Thinning on the right, occasioned by the conventional formulation of the left-rule for $\vee$. A more liberal form of left-rule for $\vee$ would allow the succedents of the two premise-sequents to differ. Thus, in addition to the two possibilities given above, there would be a further two:

$$\frac{\Delta, \varphi : \theta \qquad \Gamma, \psi :}{\Delta, \Gamma, \varphi \vee \psi : \theta} \qquad\qquad \frac{\Delta, \varphi : \qquad \Gamma, \psi : \theta}{\Delta, \Gamma, \varphi \vee \psi :} .$$

The second of these two new possibilities affords the following "Thinning-free" proof of disjunctive syllogism, which we saw earlier:

$$\frac{\dfrac{A : A}{\neg A, A :} \qquad B : B}{\neg A, A \vee B : B}(\vee:) .$$

6.2 Liberalizing conditional proof To the second end (i.e., being able to prove $\neg A : A \rightarrow B$): the only other rule that needs some finessing, in the absence of Thinning as a proof-constitutive rule, is the right-rule for $\rightarrow$ (in natural-deduction terms: the rule of conditional proof). The usual form of this rule in a sequent calculus is as follows:

$$(:\rightarrow) \qquad \frac{\Delta : \psi}{\Delta \setminus \{\varphi\} : \varphi \rightarrow \psi} .$$

Two features of this form of the rule stand out. One is that it is required that the consequent ψ of the conditional be the conclusion of the premise-sequent. The other is that the antecedent φ of the conditional *need not be in* Δ. If φ is *in* Δ, then φ will not be among the sentences on the left of the conclusion-sequent. The natural-deduction analogue of this is to say that φ will be *discharged* as an assumption when the introduction rule for $\rightarrow$ is applied. But even if φ is *not* in Δ, one can still apply the rule, ending up with the same sentences on the left (i.e., the members of Δ). In natural-deduction terms, this would be a case of so-called *vacuous discharge* of an assumption: the assumption has not been used, so it is already absent on the left, just as it would be after being "discharged"!

The Gentzenian sequent-calculus furnishes the following proof:

$$\frac{\dfrac{\dfrac{\dfrac{A : A}{\neg A, A :}}{\neg A, A : B}\text{(T}_R)}{\neg A : A \rightarrow B}\text{(:$\rightarrow$)}}{} .$$

But this contains a conspicuous (and unavoidable) application of THINNING, which we are at pains to avoid (for reasons already discussed). So the question arises: how might one reformulate the rules for $\rightarrow$ so as to permit a proof of the sequent $\neg A : A \rightarrow B$ *without* having to resort to THINNING?

As with the rule $(\vee :)$, the answer is very simple. One wants to be able to give the following proof:

$$\frac{\dfrac{\dfrac{A : A}{\neg A, A :}}{\neg A : A \rightarrow B}\text{(:$\rightarrow$)}}{} .$$

So, all one need do is reformulate the rule $(:\rightarrow)$ so as to permit this. One form that does so is this:

$$(:\rightarrow)_{\text{NEW}} \qquad \frac{\Delta, \varphi :}{\Delta : \varphi \rightarrow \psi} \ \text{ where } \varphi \notin \Delta.$$

The orthodox logician cannot object to this form of rule, since it is, after all, derivable in the usual Gentzen systems, by appeal to THINNING:

$$\frac{\dfrac{\Delta, \varphi :}{\Delta, \varphi : \psi}\text{(T}_R)}{\Delta : \varphi \rightarrow \psi} .$$

We have therefore, in effect, acquired a new 'half' of conditional proof, which we did not have explicitly before. The basic idea, in natural-deduction terms, is that one may infer a conditional $\varphi \rightarrow \psi$ and discharge its antecedent φ from one's assumptions, when one has refuted φ (*modulo* the remaining undischarged assumptions). Of course, we still retain the older, "other" half of the rule of conditional proof:

$$(:\rightarrow)_{\text{OLD}} \qquad \frac{\Delta : \psi}{\Delta \setminus \{\varphi\} : \varphi \rightarrow \psi} .$$

So our liberalized rule $(:\rightarrow)$ of conditional proof consists now of both $(:\rightarrow)_{\text{OLD}}$ and $(:\rightarrow)_{\text{NEW}}$.

Note that we are not "smuggling in" any kind of THINNING when stating these rules for $\rightarrow$. For direct inspection reveals that these rules are perfectly in order. They can be seen to be valid on the basis of both the meanings of the logical operators involved, and the pattern of repetition of occurrences of extralogical expressions. The rules capture, succinctly, the intuitively compelling and "primitive" moves that any competent deductive reasoner would be happy to make. Certainly, anyone who is *not* happy with $(:\rightarrow)_{\text{NEW}}$ would be hard put to explain why they are nevertheless happy to accept the truth table for $\rightarrow$ as directly encapsulating the latter's own, proprietary, contribution to the truth conditions of sentences in which it occurs.

7 Thinning-Elimination

We now address the question: if one can prove a *Cut*-elimination theorem for sequent calculus, then surely one should also be able to prove a *Thinning*-elimination theorem (at least of a suitably related kind)? Indeed, given Cut-elimination as already secured, it would be enough to prove the eliminability of Thinnings from arbitrary Cut-free proofs. Of course, one could not hope to prove such a theorem in the following form, which is a straightforward adaptation of the Cut-elimination theorem:

> there is an effective method γ such that for every Cut-free proof Π of any sequent $\Delta : \varphi$, the object $\gamma(\Pi)$ is a Cut-free, Thinning-free proof of *the exact same sequent* $\Delta : \varphi$,

for this is immediately counterexemplified by the now familiar little proof Ω:

$$\frac{\dfrac{\varphi : \varphi}{\neg\varphi, \varphi :}}{\neg\varphi, \varphi : \psi}(\text{T}_R)\ .$$

What this little proof clearly suggests, however, is that one might be able to state and prove a corrected Thinning-elimination result in the following more subtle, and epistemologically arresting, form:[8]

> there is an effective method γ such that for every Cut-free proof Π of any sequent $\Delta : \varphi$, the object $\gamma(\Pi)$ is a Cut-free, Thinning-free proof of *some subsequent of the sequent* $\Delta : \varphi$.

(For, note that Ω is a Cut-free proof of the sequent $\neg\varphi, \varphi : \psi$, and the Cut-free, Thinning-free proof

$$\frac{\varphi : \varphi}{\neg\varphi, \varphi :}$$

proves a *subsequent*, namely $\neg\varphi, \varphi : $, of the sequent $\neg\varphi, \varphi : \psi$.)

And indeed one can state and prove such a result. The Thinning-elimination result just stated was formulated and proved in [7].[9]

8 Taking Stock

We are interpreting $\Delta \vdash \varphi$ as meaning that there is a formal sequent-proof of the sequent $\Delta : \varphi$ (*exactly* as stated—so that *all* of the premises in Δ are used in arguing for the conclusion φ). We have ruled out incorporating Object-Cut as a proof-constitutive rule of a system of sequent-proof, because, as the ensuing Cut-elimination theorem always shows, it is a redundant rule. We have stressed the importance of formulating right- and left-logical rules for the logical operators in a manner that enables them to achieve their desired deductive effects all by themselves. One should not have to rely on the ancillary services rendered in Gentzen's sequent systems by the rule of Thinning. We have seen the desirability of being able to eliminate from proofs not just Cuts, but also Thinnings. But, to achieve this end, we have found it necessary to reformulate the result so that it claims only the availability of a proof, of the sought kind, of some *subsequent* of the original "target" sequent.

The logician should always prefer a stronger result to a weaker one. Thus a proof system made up *only* of (suitably formulated) right- and left-logical rules, but *not* containing either Cut or Thinning, proves to be perfectly adequate for the logician's

purposes. *We have no methodological need for the structural rules of* CUT *or* THIN-NING. The resulting system of "logical rules only" we call *core logic*. The right- and left-logical rules of core logic have had every trace of CUT or THINNING expunged from them. They form a truly irredundant set of deductive rules.

Our single-conclusion sequent system of core logic, which lacks the two rules CUT and THINNING, is still perfectly adequate for the foundational demands of (intuitionistic) mathematics and of theory testing in the natural sciences. Likewise, the "classicization" of core logic is still perfectly adequate for the foundational demands of classical mathematics (and, a fortiori, of theory testing in the natural sciences). The classical extension of the sequent system for core logic, which also lacks the two rules CUT and THINNING, is obtained by appending the rule of classical dilemma (still in single-conclusion form, hence with Ω at most a singleton):

$$\text{Dilemma:}\qquad \frac{\Delta, \varphi : \Omega \qquad \Gamma, \neg\varphi : \Omega}{\Delta, \Gamma : \Omega}\qquad \frac{\Delta, \varphi : \psi \qquad \Gamma, \neg\varphi :}{\Delta, \Gamma : \psi}.$$

It is no objection to refusing to have (Object-)CUT as a rule in the proof system that it would frequently render formal proof of the target sequent unfeasible (because of exponential blowup on "eliminating CUTS").[10] That is to succumb to the fantasy that mathematicians are in the business of giving formal proofs. *Of course* our deductive progress will always be piecemeal and stepwise. All we ever need is the assurance that *there exists* a formal proof of (some subsequent of) the target sequent and that we have an effective method that would, in principle, deliver it. This we can do, without Object-CUT.

Meanwhile, our formal systems (intuitionistic or classical) of CUT-free, THINNING-free deductive reasoning suffice to regiment the reasoning that mathematicians actually undertake, as they effect their logical transitions from their axioms to their lemmas, from their axioms and lemmas to their theorems, and from their (axioms and) theorems to their corollaries. Their actual *passages* of reasoning are faithfully represented by CUT-free, THINNING-free proofs. And their overall deductive progress—the belief that their theorems (and their corollaries) really do follow, ultimately, from their axioms—is underwritten by the principle above, which has been established as a metatheorem, called CUT FOR EPISTEMIC GAIN.[11]

9   Sequent Calculus for Core Logic

We use Δ, Γ for (possibly empty) sets of sentences. *Sequents* are of the form $\Delta : \Gamma$, where Γ has at most one member. Instead of $\{\varphi\}$ we shall write φ when there can be no confusion. Instead of $\Delta \cup \Gamma$ we shall write Δ, Γ. Whenever we write $\Delta, \varphi_1, \ldots, \varphi_n : \Gamma$ $(n \geq 1)$ in a premise-sequent, it is to be understood that $\varphi_i \notin \Delta$ $(1 \leq i \leq n)$. Instead of writing $\emptyset$ on the right side of a colon, we simply leave a blank space.

The sequent rules for any logical operator @ are of two kinds: those for introducing a dominant occurrence of @ on the *right* of the colon of the sequent and those for introducing it on the *left*. Right-rules, denoted (:@), correspond to Introduction rules, denoted (@-I), in natural deduction. Left-rules, denoted (@:), correspond to Elimination rules, denoted (@-E), in natural deduction.

$(: \neg)$ $\quad \dfrac{\Delta, \varphi :}{\Delta : \neg\varphi}$

$(\neg :)$ $\quad \dfrac{\Delta : \varphi}{\Delta, \neg\varphi :}$

$(: \wedge)$ $\quad \dfrac{\Delta_1 : \varphi_1 \qquad \Delta_2 : \varphi_2}{\Delta_1, \Delta_2 : \varphi_1 \wedge \varphi_2}$

$(\wedge :)$ $\quad \dfrac{\Delta, \varphi : \Gamma}{\Delta, \varphi \wedge \psi : \Gamma} \quad \dfrac{\Delta, \psi : \Gamma}{\Delta, \varphi \wedge \psi : \Gamma} \quad \dfrac{\Delta, \varphi, \psi : \Gamma}{\Delta, \varphi \wedge \psi : \Gamma}$

$(: \vee)$ $\quad \dfrac{\Delta : \varphi}{\Delta : \varphi \vee \psi} \quad \dfrac{\Delta : \psi}{\Delta : \varphi \vee \psi}$

$(\vee :)$ $\quad \dfrac{\Delta_1, \varphi_1 : \Gamma \quad \Delta_2, \varphi_2 : \Gamma}{\Delta_1, \Delta_2, \varphi_1 \vee \varphi_2 : \Gamma} \quad \dfrac{\Delta_1, \varphi_1 : \psi \quad \Delta_2, \varphi_2 :}{\Delta_1, \Delta_2, \varphi_1 \vee \varphi_2 : \psi} \quad \dfrac{\Delta_1, \varphi_1 : \quad \Delta_2, \varphi_2 : \psi}{\Delta_1, \Delta_2, \varphi_1 \vee \varphi_2 : \psi}$

$(:\rightarrow)$ $\quad \dfrac{\Delta, \varphi :}{\Delta : \varphi \rightarrow \psi} \quad \dfrac{\Delta : \psi}{\Delta \setminus \{\varphi\} : \varphi \rightarrow \psi}$

$(\rightarrow:)$ $\quad \dfrac{\Delta_1 : \varphi_1 \qquad \Delta_2, \varphi_2 : \Gamma}{\Delta_1, \Delta_2, \varphi_1 \rightarrow \varphi_2 : \Gamma}$

$(: \exists)$ $\quad \dfrac{\Delta : \psi_t^x}{\Delta : \exists x \, \psi}$

$(\exists :)$ $\quad \dfrac{\Delta, \psi_a^x : \Gamma}{\Delta, \exists x \, \psi : \Gamma}$ where the conclusion sequent has no occurrences of a

$(: \forall)$ $\quad \dfrac{\Delta : \psi}{\Delta : \forall x \, \psi_x^a}$ where a occurs in ψ but in no member of Δ

$(\forall :)$ $\quad \dfrac{\Delta, \psi_{t_1}^x, \ldots, \psi_{t_n}^x : \Gamma}{\Delta, \forall x \, \psi : \Gamma}$

$(:=)$ $$\overline{\varnothing : t = t}$$

$(=:)$ $$\frac{\Delta : \varphi}{\Delta, t = u : \psi}$$ where $\varphi_u^t = \psi_u^t$

Notes

1. See [4], Part I, p. 192, under 'Schemata für Logische-Zeichen-Schlußfiguren,' figure *UES* (short for 'U̲nd-Zeichen-E̲inführung im S̲ukzedens').

2. We shall see presently that there is a perfectly good formulation of Right- and Left-sequent rules in which Σ counts as a proof. The proof-system in question does not contain either Cut or Thinning as proof-constitutive rules, and has slightly liberalized formulations of proof-by-cases and conditional proof. Details will emerge in due course.

3. The corresponding result for Classical Core Logic is proved in [9].

4. Certainly, (Object-)Cut was a great help to Gentzen in proving to the satisfaction of his contemporaries that his sequent calculus was equivalent to their less natural but more familiar systems of predicate logic, whose completeness had already been established. But, for the purposes of a *direct* completeness proof for the sequent calculus itself, one needs only Meta-Cut, not Object-Cut. To be sure, there were other reasons Gentzen had, stemming from his earliest paper [3] (which did not treat of any logical operators) for "carrying Cut along" into his later sequent treatment, in [4], for the first time, of all the usual logical operators. Despite the interesting explanation offered by [2] of the "synthetic" role played by Cut in [3], a counterargument could be mounted for the view that a rule of "Cut" that allows for the excision *only of atomic sentences* is very different from one that allows for the excision of *logically compound* ones. In the "atomic" setting, what looks like a Cut-sentence A within a proof constructed by the rules of an atomic sequent calculus does not correspond to any unwanted maximal occurrence of A in the natural-deduction version of that sequent proof. In the natural-deduction setting, the two proof-trees would simply be grafted together, root A of the first proof-tree onto leaf A of the second proof-tree, to produce a new proof-tree with A now a grafting-point within it, and *without any violation of normality*. This does *not* happen when A, by contrast, is logically compound. Those who view the appearance of Object-Cut within the full sequent calculus as a harmless and understandable (let alone: necessitated) extrapolation from the earlier treatment of the atomic sequent calculus seem to be in need of more convincing reasons for condoning its appearance as Object-Cut, rather than being content merely to secure the truth of Meta-Cut. (The reflections in this footnote are in response to much-appreciated comments by an anonymous referee.)

5. If we follow a convention to the effect that in a sequent written as $\Delta : \Gamma$, the succedent Γ is either empty or a singleton, then the rule (T_L) can be written in the more unified form

$$(T_L) \qquad \frac{\Delta : \Gamma}{\Delta, \psi : \Gamma} \text{ , where } \psi \notin \Delta.$$

6. *Ex Falso Quodlibet* has also been called the *Absurdity Rule*, for example in the classic study [6], which awakened renewed interest in Genten's systems of natural deduction. It has also earned the moniker *Explosion*, among contemporary paraconsistent logicians.

7. If we follow the convention mentioned in footnote 5, then we can write the usual rule ($\vee$:) in the more unitary form

$$\frac{\Delta, \varphi : \Gamma \qquad \Gamma, \psi : \Gamma}{\Delta, \Gamma, \varphi \vee \psi : \Gamma}.$$

We shall follow the convention in question when stating the full set of sequent rules for Core Logic in §9.

8. It is interesting to note that [3], which concerned itself solely with sequents composed out of atomic sentences, established that any sequent-proof consisting of applications of the structural rules of REFLEXIVITY, THINNING and CUT could be turned into a normal form that relegated all its Thinnings to a single application of THINNING at the end. Thus the penultimate sequent in the proof would, as it were, be a potentially stronger result than the concluding sequent at the very end, albeit within a proof containing, higher up (nothing but) CUTS. Perhaps it was this latter feature that prevented Gentzen, subsequently, from considering turning CUT-free proofs into CUT-free, THINNING-free proofs of possibly stronger results. For further discussion of [3], see [10].

9. In that paper the rule in question was called Dilution rather than Thinning, and the result in question was accordingly called the Dilution-Elimination Theorem.

10. We need scare quotes here because we have to remember that there are no actual CUTS made within the formal proofs of Core Logic. Rather, we seek proofs of (possible strengthenings of) the target sequent that *would* result from applying (to two proofs) a rule of cut *if only* one had one!

11. The proof of this metatheorem is given in [8] for Core Logic, and in [9] for Classical Core Logic.

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