Notre Dame Journal of Formal Logic
Volume 57, Number 1, 2016

Closed Normal Subgroups of the Automorphism Group of a Saturated Model of Peano Arithmetic

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Abstract In this paper we discuss automorphism groups of saturated models and boundedly saturated models of PA. We show that there are saturated models of PA of the same cardinality with nonisomorphic automorphism groups. We then show that every saturated model of PA has short saturated elementary cuts with nonisomorphic automorphism groups.

1 Introduction and Basics

The study of automorphism groups of models of PA usually requires one to consider these groups as topological groups. In some cases, one may use the topology for which the basic open subgroups are the stabilizers of finite subsets. Since finite subsets of models of PA are coded by a single element, we will name this topology the *point topology*. On the other hand, one may use the topology for which the basic open subgroups are the stabilizers of subsets of cardinality less than the cardinality of the model. We will name this topology the *set topology*. When a saturated model of PA is of cardinality $\aleph_1$, these topologies coincide. However, when the cardinality of the model is greater than $\aleph_1$, these topologies are different.

Using the point topology, we have shown in [14] that there are saturated (and boundedly saturated) models of PA of the same cardinality which have topologically nonisomorphic automorphism groups. In this paper, we use the set topology to show that there are saturated models of PA of the same cardinality which have topologically nonisomorphic automorphism groups. Since the set topology has the small index property, it implies that the automorphism groups of some saturated models of PA are not isomorphic as abstract groups. A similar result is then proved for short saturated models of PA.

Received August 8, 2013; accepted October 9, 2013
First published online November 24, 2015
2010 Mathematics Subject Classification: Primary 03C62
Keywords: saturation, bounded saturation, automorphism groups, models of Peano Arithmetic
 10.1215/00294527-3339587

We will assume that the reader is familiar with the general properties of models of Peano Arithmetic. We will follow standard basic notations, definitions, and facts.

Recall that PA has a saturated model of cardinality λ if and only if λ is uncountable, regular, and $2^\kappa \leq \lambda$ whenever $\kappa < \lambda$. For the rest of this paper we fix M, a saturated model of Peano Arithmetic of cardinality λ and $G = \mathrm{Aut}(M)$.

If $A \subset M$, we use the notation $G_{(A)} = \{g \in G | \forall x \in A : g(x) = x\}$ and $G_{\{A\}} = \{g \in G | g(A) = A\}$. We also define $\sup(A) = \{x \in M | \exists a \in A : x < a\}$ and $\inf(A) = \{x \in M | \forall a \in A : x < a\}$.

A set $I \subseteq M$ is called an *initial segment of* M, denoted by $I \subseteq_e M$, if $I \neq \emptyset$ and for any $x \in I$, whenever $y < x$, $y \in I$. We say that I is a *cut* if I is an initial segment of M which is closed under the successor function.

Let $I \subset M$ be a cut. We say that I is an *exponentially closed cut* if for every $a \in M$, if $a \in I$, then $2^a \in I$. By $\mathrm{cf}(I)$ we denote the least cardinality μ of a cofinal set $A \subset I$. By $\mathrm{dcf}(I)$ we denote the least cardinality κ of a downward cofinal set $B \subset M \setminus I$. If $g \in G$, let

$$I_{\mathsf{fix}}(g) = \big\{x \in M : \forall y < x\big(g(y) = y\big)\big\}.$$

Thus, $I_{\mathsf{fix}}(g)$ is the largest cut pointwise fixed by g.

Lemma 1.1 ([14, Theorem 1.1]) *A cut $I \subset M$ is $I_{\mathsf{fix}}(f)$ for some $f \in G$ if and only if $I \subset M$ is an exponentially closed cut such that* $\mathrm{dcf}(I) = \lambda$.

In fact, a stronger version is proved in [14].

Lemma 1.2 ([14, Lemma 3.6]) *Let $I \subset M$ be an exponentially closed cut such that* $\mathrm{dcf}(I) = \lambda$, *let $A \subseteq M$ have cardinality less than λ, and let $h : A \to M$ be such that $(M, x, a)_{a \in A} \equiv (M, x, h(a))_{a \in A}$ for all $x \in I$. Then there is $f \in G$ such that $f \supseteq h$ and $I_{\mathsf{fix}}(f) = I$.*

2 Closed Normal Subgroups

We use the notation $\mathrm{Aut}^\omega(M)$ when we consider $\mathrm{Aut}(M)$ with the point topology.

Let I be a cut in a model M. We say that I is *invariant* if for every $f \in G$, $f(I) = I$. Since M is saturated, a cut $I \subset M$ is invariant if and only if there is a sequence of definable elements in M which is cofinal in I, or a sequence of definable elements in M which is downward cofinal in $M \setminus I$.

It is not difficult to see that if I is an invariant cut, then $\mathrm{Aut}(M)_{(I)}$ is a closed normal subgroup of $\mathrm{Aut}^\omega(M)$. Kaye [6] showed that in countable recursively saturated models the converse is true (another proof can be found in Schmerl [15]). In [14] we proved a similar result for saturated models of PA.

Theorem 2.1 ([14, Theorem 1.2]) *Let $H \leq G$. Then H is a closed normal subgroup of* $\mathrm{Aut}^\omega(M)$ *if and only if there exists an invariant cut $I \subset M$ such that $H = G_{(I)}$.*

For $f, g \in G$, let $g^f = f^{-1}gf$ and $g^{-f} = f^{-1}g^{-1}f$. The proof of Theorem 2.1 is based on the next theorem.

Theorem 2.2 ([14, Theorem 4.7], [15, Theorem 2]) *Let $g \in G$. Then either the closure in the point topology of the set $\{g^{-f_1}g^{f_2} : f_1, f_2 \in G\}$ or the closure in the point topology of the set $\{g^{f_1}g^{-f_2} : f_1, f_2 \in G\}$ is a normal subgroup.*

We now define the set topology on G for which a basis of open sets is

$$\{g \cdot G_{(A)} : g \in G, A \subset M \text{ and } |A| \leq \lambda^-\}.$$

We use the notation $\mathrm{Aut}^\lambda(M)$ when we consider G with this topology and call this topology the set topology.

It is easy to see that any open subgroup in this topology has index at most λ. In general, for saturated models, the converse is true.

Theorem 2.3 (Small index property; see Lascar and Shelah [11, Theorem 1]) *Assume that N is a saturated model of cardinality λ, and let H be a subgroup of* $\mathrm{Aut}(N)$ *of index at most λ. Then H is open in* $\mathrm{Aut}^\lambda(N)$.

Lemma 2.4 *If $\lambda = \aleph_1$, then the point and set topologies on G coincide.*

Proof It easily follows from the definitions that every open set in the point topology is open in the set topology.

Let $A \subset M$ be of cardinality less than $\aleph_1$; that is, A is countable. Then by $\aleph_1$ saturation there exists $a \in M$ such that $A \subseteq \{(a)_i : i < \omega\}$. Hence $G_{(A)} \geq G_a$. Therefore, $G_{(A)}$ is open in $\mathrm{Aut}^\omega(M)$. So we have shown that every open set in $\mathrm{Aut}^\lambda(M)$ is open in $\mathrm{Aut}^\omega(M)$. □

It is not difficult to prove that Lemma 2.4 is not true for saturated models of higher cardinality.

Lemma 2.5 *If $\lambda > \aleph_1$, then the point and set topologies on G do not coincide.*

By combining Lemma 2.5 with Theorem 2.3, we obtain the following result.

Corollary 2.6 *Let N be a saturated model of Peano Arithmetic of cardinality $\lambda > \aleph_1$. Then there exists H, a subgroup of* $\mathrm{Aut}(N)$ *of index at most λ such that H is not open in* $\mathrm{Aut}^\omega(N)$.

Our goal is to prove Theorem 2.1 for saturated models in the set topology.

Theorem 2.7 *Let $H \leq G$. Then H is a closed normal subgroup of* $\mathrm{Aut}^\lambda(M)$ *if and only if there exists an invariant cut $J \subset M$ such that $H = G_{(J)}$.*

Proof Suppose that $g \in G$, and let $I = I_{\mathrm{fix}}(g)$. Let $J \subseteq I$ be the largest invariant exponentially closed cut (such J is well defined). The subgroup $G_{(J)}$ is a closed normal subgroup containing g, so it is sufficient to show that $G_{(J)}$ is in the closure of the subgroup generated by the conjugates of g.

First, assume that $J \neq \omega$. Let $A \subset M$ be of cardinality less than λ. Then, using saturation, we find $a \in M$ with the property that for every $a' \in A$ there exists $i \in J$ such that $a' = (a)_i$. Let $h \in G_{(J)}$ be such that $h(A) = B$. Then $h(a) = b$ for some $b \in M$, and for every $a' \in A$ and $h(a') = b' \in B$ there exists $i \in J$ such that $a' = (a)_i$ and $b' = (b)_i$. By Theorem 2.2, there exist $f_1, f_2 \in G$ such that either $g^{-f_1} g^{f_2}(a) = b$ or $g^{f_1} g^{-f_2}(a) = b$. Without loss of generality, consider the case when $g^{-f_1} g^{f_2}(a) = b$. Let $a' \in A$. By the choice of a, there exists $i \in J$ such that $a' = (a)_i$. Hence, $g^{-f_1} g^{f_2}(a') = g^{-f_1} g^{f_2}((a)_i) = (g^{-f_1} g^{f_2}(a))_{g^{-f_1} g^{f_2}(i)} = (b)_i$ and $h(a') = h((a)_i) = (h(a))_{h(i)} = (b)_i$. Therefore, for every $a' \in A$, $g^{-f_1} g^{f_2}(a') = h(a')$.

Next, assume that $J = \omega$ and $I = I_{\mathrm{fix}}(g) \neq \omega$. Let $A \subset M$ be of cardinality less than λ. Let $h \in G_{(\omega)}$. Because the cardinality of A is less than λ, we can find $h' \in G$

such that for every $a \in A$, $h(a) = h'(a)$ and $I_{\mathsf{fix}}(h') = I' \neq \omega$. Let $c \in M$ be undefinable, and let $c \in I \cap I'$. Now we expand the language by adding c to it. By applying the argument from the previous paragraph, we obtain $f_1, f_2 \in \mathrm{Aut}(M, c)$ such that for every $a \in A$: $g^{-f_1} g^{f_2}(a) = h(a)$.

Finally, assume that $I = I_{\mathsf{fix}}(g) = \omega$. For every cut $I' \neq \omega$, we can find $f \in G$ (using Lemma 1.2) such that $\omega \neq I_{\mathsf{fix}}(g^{-1}g^{f}) \subset I'$ and apply the previous paragraph. $\square$

By Ω_ω we denote the set of all elements greater than the standard cut and smaller than any nonstandard definable element. In some models of PA, Ω_ω is empty. However, since here M is saturated, one can show that $\Omega_\omega \neq \emptyset$. The next lemma easily follows from Theorem 2.7.

Lemma 2.8 *If* $\mathrm{Th}(M) \neq$ TA, *then* $\mathrm{Aut}(M)_{(\Omega_\omega)}$ *is the largest closed, normal proper subgroup of* $\mathrm{Aut}^\lambda(M)$.

Here True Arithmetic (TA) is $\mathrm{Th}(\mathbb{N})$. From Theorem 2.7, we obtain a corollary.

Corollary 2.9 *Let* M_1, M_2 *be two saturated models of Peano Arithmetic of cardinality* λ *such that* $M_1 \models$ TA *and* $M_2 \not\models$ TA. *Then their automorphism groups are not isomorphic.*

Proof Because $\mathbb{N} \models$ TA, M_1 does not have any nonstandard definable elements. Since $\mathrm{Th}(M_2) \neq$ TA, there are nonstandard definable elements in M_2. Thus, by Theorem 2.7, $\mathrm{Aut}(M_1)$ has no nontrivial closed normal subgroups and $\mathrm{Aut}(M_2)$ has nontrivial closed normal subgroups (consider for example a subgroup $\mathrm{Aut}(M_2)_{(\Omega_\omega)}$ from Lemma 2.8). Therefore, $\mathrm{Aut}^\lambda(M_1)$ cannot be topologically isomorphic to $\mathrm{Aut}^\lambda(M_2)$. Finally, by Theorem 2.3, $\mathrm{Aut}(M_1)$ and $\mathrm{Aut}(M_2)$ are not isomorphic. $\square$

In the next section we will use the fact that Theorem 2.7 remains true if we expand the language by adding less than λ constants to it.

3 Nice Subgroups

This section is similar to one from [12]. It also owes much to Kossak and Schmerl [9]. Lemmas 3.9 and 3.10, Definitions 3.3 and 3.4, and Corollary 3.11 are similar to those in [9].

We will start with a few definitions. Let $a \in M$ be a nondefinable element. Following Kaye, Kossak, and Kotlarski [7], we define two cuts for such a in M: $I_a^+ = \inf\{u \in M : u > a$ and u is definable$\}$ and $I_a^- = \sup\{u \in M : u < a$ and u is definable$\}$. The set difference of these cuts $\Omega_a = I_a^+ \setminus I_a^-$ is called the *interstice around* a. In other words, an interstice of M is a convex subset of M that is maximal with the property of having no definable elements.

One example of interstices is the last interstice Ω_∞, the set of all elements greater than any definable element. Another example is the smallest interstice Ω_ω (defined in Section 2). For saturated models of Peano Arithmetic, neither Ω_ω nor Ω_∞ is empty.

Let $\Omega \subset M$ be an interstice. We say that a definable function g is *nondecreasing on* Ω if for every $b, c \in \Omega$, if $b \leq c$, then $g(b) \leq g(c)$. The definition of *nonincreasing on* Ω is similar. We let

$$\begin{aligned}\Re_\Omega = \{g | g : M \longrightarrow M &\text{ is a definable function such that } a \in \Omega \longleftrightarrow g(a) \in \Omega\\ &\text{and } g \text{ is either nondecreasing or nonincreasing function on } \Omega\}.\end{aligned}$$

The notion of an interstitial gap is introduced in Bamber and Kotlarski [1] as well as in Bigorajska, Kotlarski, and Schmerl [2]. If $a \in \Omega$, we define $I_g^+(a) = \sup\{t(a) : t \in \Re_\Omega\}$ and $I_g^-(a) = \inf\{t(a) : t \in \Re_\Omega\}$. Let $\mathrm{igap}(a) = I_g^+(a) \setminus I_g^-(a)$. We call $\mathrm{igap}(a)$ an *interstitial gap* of the interstice Ω. If $\Omega \neq \mathrm{igap}(a)$ for every $a \in \Omega$, we say that Ω is *full.*

Theorem 3.1 ([2, Corollary 1.6]) *Every interstice of M is full.*

Let f be a definable function. We define $f^x(x)$ to be xth iteration of f applied to x. Following [1], we say that an interstice Ω is *very good* if for every $f \in \Re_\Omega$, the function $x \mapsto f^x(x)$ is in $\Re_\Omega$ as well. Both Ω_∞ and Ω_ω are examples of very good interstices.

A generalization of the Moving Gaps Lemma can be found in [1, Theorem 3.8] and in [7, Lemma 6.6].

Lemma 3.2 (Moving interstitial gaps lemma) *Let $\Omega \subset M$ be a very good interstice, and let $g \in G$, $a \in M$, and $b, d \in \Omega$ be such that $a \neq g(a) < \mathrm{igap}(b) < \mathrm{igap}(d)$. Then there exists $c \in M$ such that $\mathrm{igap}(b) < c < \mathrm{igap}(d)$ and $g(c) \notin \mathrm{igap}(c)$.*

For the rest of this section, the topology on M is the set topology $\mathrm{Aut}^\lambda(M)$. In addition, we fix Ω to be a very good interstice of M.

Definition 3.3 Given subgroups H, K of G, we say that H *precedes* K if $\bigcap_{h\in H} K^h \leq G_{(\Omega)}$ (where $K^h = h^{-1}Kh$).

Later Lemma 3.9 will clarify Definition 3.3.

Definition 3.4 Given a subgroup H of G, we define two subgroups of G:

1. H_* is the intersection of all conjugates of H which precede H;
2. H^* is the closure of the subgroup generated by the union of all K_*, where K is a conjugate of H which is preceded by H.

Our goal is to identify subgroups which are stabilizers of elements from Ω. With this goal in mind, we give the definition of a nice subgroup.

Definition 3.5 A subgroup H of G is called Ω-*nice* (or just *nice* if it is clear what Ω is) if it satisfies the following properties:

1. H is a maximal open subgroup of G;
2. $G_{(\Omega)}$ is the maximal closed normal subgroup of G such that $G_{(\Omega)} < H$;
3. for every $f \in G$, if $H^f \neq H$, then either H^f precedes H or H precedes H^f;
4. whenever K is an open nontrivial subgroup of H, then $K \cap H_* > K \cap G_{(\Omega)}$;
5. H_* is a closed normal subgroup of H.

Proposition 3.6 *If H is a nice subgroup of G, then $H = G_{\{J\}}$ for some icut $J \subseteq_e \sup(\Omega)$.*

We need a couple of lemmas to prove Proposition 3.6.

Lemma 3.7 *Let H be a nice subgroup of G. If $A \subset M$ is such that $|A| < \lambda$ and $G_{(A)} < H$, then $G_{(A)} \cap H_* = (G_{(A)})_{(I)}$ for some exponentially closed, invariant cut $I \subset \sup(\Omega)$ in $(M, a)_{a\in A}$ with $\mathrm{dcf}(I) = \lambda$.*

Proof We have that $G_{(A)} \cap H_*$ is closed normal in $G_{(A)} = \mathrm{Aut}(M, a)_{a \in A}$ and, by Theorem 2.7, $G_{(A)} \cap H_* = (G_{(A)})_{(I)}$, where I is an exponentially closed cut that is invariant in $(M, a)_{a \in A}$ with $\mathrm{dcf}(I) = \lambda$. (Since we expand the language by less then λ constants, we can apply Theorem 2.7 to $(M, a)_{a \in A}$.) By part 4 of Definition 3.5, $(G_{(A)})_{(I)} > G_{(A)} \cap G_{(\Omega)}$. Therefore, by Lemma 1.1, $I \subset \sup(\Omega)$. □

Lemma 3.8 *Let H be a nice subgroup of G. If $A, B \subset M$ are such that $|A|, |B| < \lambda$ and $G_{(A)}, G_{(B)} < H$, then $G_{(A)} \cap H_* = (G_{(A)})_{(I)}$ and $G_{(B)} \cap H_* = (G_{(B)})_{(I)}$ for the same exponentially closed cut $I \subset \sup(\Omega)$, $\mathrm{dcf}(I) = \lambda$.*

Proof Assume not. By Lemma 3.7, $G_{(A)} \cap H_* = (G_{(A)})_{(J_1)}$ and $G_{(B)} \cap H_* = (G_{(B)})_{(J_2)}$, where $J_1, J_2 \subset_e \sup(\Omega)$. We can assume that $J_1 \subset_e J_2$. Then by Lemma 1.1 applied to $(M, a, b)_{a \in A, b \in B}$, there exists $f \in G_{(A)} \cap G_{(B)} = \mathrm{Aut}(M, a, b)_{a \in A, b \in B}$ such that $J_1 \subseteq_e I_{\mathrm{fix}(f)} \subset_e J_2$. Therefore, $f \in (G_{(A)})_{(J_1)} = G_{(A)} \cap H_*$. Thus, $f \in H_*$ and by the choice of f we have $f \in G_{(B)} \cap H_*$. Because $G_{(B)} \cap H_* = (G_{(B)})_{(J_2)}$, we should have $I_{\mathrm{fix}(f)} \supseteq J_2$. That contradicts the assumption that $I_{\mathrm{fix}(f)} \subset_e J_2$. □

Now we are ready to prove Proposition 3.6.

Proof Let $f \in H$. Since H is open, there is $A \subset M$ such that $|A| < \lambda$ and $G_{(A)} < H$. Let $f(A) = B$; then $G_{(B)} < H$. By Lemma 3.8, $G_{(A)} \cap H_* = (G_{(A)})_{(I)}$ and $G_{(B)} \cap H_* = (G_{(B)})_{(I)}$ for the same exponentially closed cut I in $\sup(\Omega)$ with $\mathrm{dcf}(I) = \lambda$:

$$f(G_{(A)} \cap H_*)f^{-1} = f(G_{(A)})f^{-1} \cap f(H_*)f^{-1} = G_{(B)} \cap H_* = (G_{(B)})_{(I)}$$

and

$$f\big((G_{(A)})_{(I)}\big)f^{-1} = (G_{(B)})_{(f(I))}.$$

So $(G_{(B)})_{(I)} = (G_{(B)})_{(f(I))}$. Then by Lemma 1.1 applied to $(M, b)_{b \in B}$, $I = f(I)$ and hence $f \in G_{\{I\}}$. Then again by Lemma 1.1 and Theorem 3.1, $G_{\{I\}} \neq G$. So $H \leq G_{\{I\}}$, and because H is a maximal subgroup we have $H = G_{\{I\}}$.

To finish the proof of Proposition 3.6, define J to be the maximal icut in I. Then $G_{\{J\}} \geq G_{\{I\}}$ and, by Theorem 3.1 and Lemma 1.1, $G_{\{J\}} \neq G$. Because $G_{\{I\}}$ is maximal, we conclude that $G_{\{J\}} = G_{\{I\}}$. □

Lemma 3.9 *Let H be a nice subgroup, and let $K \neq H$ be a subgroup conjugate to H. Assume that $H = G_{\{I\}}$ and $K = G_{\{J\}}$, where I, J are icuts in* $\sup(\Omega)$. *Then H precedes K if and only if $I \subset J$.*

Proof $\Longrightarrow$ Assume that $I \supseteq J$. If $f \in G_{(I)}$, $h \in H$, then $hfh^{-1} \in G_{(I)} \leq G_{(J)} \leq K$, so that $f \in h^{-1}Kh$. Hence, for all $h \in H$ we have $K^h \geq G_{(I)}$. By Lemma 1.1, $G_{(I)} > G_{(\sup(\Omega))}$. Therefore, $\bigcap_{h \in H} K^h \geq G_{(I)} > G_{(\sup(\Omega))}$ and by Definition 3.3, H does not precede K.

$\Longleftarrow$ By the definition of a nice subgroup, if $H \neq K$, then either H precedes K or K precedes H. Because $I \subset J$, by the $\Rightarrow$ part of this lemma K cannot precede H. So it is enough to show that $H \neq K$. We can find $f \in G$ with $b, d \in M$ such that $I \subseteq I_{\mathrm{fix}}(f) \subset \sup(\mathrm{igap}(b)) \subset \sup(\mathrm{igap}(d)) \subset J$. By the Moving Interstitial Gaps Lemma 3.2 and Theorem 3.1, there is $a \in J \setminus I$ such that $\mathrm{igap}(a) \neq \mathrm{igap}(f(a))$. Without loss of generality we can assume that $a < f(a)$. Then there exists an icut I' such that $I \subset I' \subset J$, $(M, I) \cong (M, I')$, $a \in I'$, and $f(a) \notin I'$. We have $f \in G_{\{I\}}$ and $f \notin G_{\{I'\}}$, so $G_{\{I\}} \neq G_{\{I'\}}$. By the definition of a nice subgroup and by the $\Rightarrow$

part of this lemma, $G_{\{I\}}$ precedes $G_{\{I'\}}$. Now if $H = K$, then $G_{\{J\}}$ precedes $G_{\{I'\}}$, but $I' \subset J$, which contradicts the $\Rightarrow$ part of this lemma. $\square$

Lemma 3.10 *Let H be a nice subgroup of G. If $H = G_{\{J\}}$, where J is an icut in* $\sup(\Omega)$ *such that $J \neq \sup(D)$ for any igap D, then $H_* = G_{(J)}$.*

Proof First, suppose that $f \in G_{(J)}$ and that K is a conjugate of H which precedes H. Then $K = G_{\{I\}}$, where I is an icut, and according to Lemma 3.9, $I \subset J$. Clearly then, $f \in K$. Thus, $G_{(J)} \leq H_*$.

Next, suppose that $f \notin G_{(J)}$, so that $f(a) \neq a$ for some $a \in J$. By the Moving Interstitial Gaps Lemma 3.2 and the fact that $J \neq \sup(D)$ for any igap D, we can assume that $\mathrm{igap}(f(a)) \neq \mathrm{igap}(a)$ and, without loss of generality, that $\mathrm{igap}(a) < \mathrm{igap}(f(a))$. Then there is an icut I such that $I \subset J$, $(M, I) \cong (M, J)$, $a \in I$, and $f(a) \notin I$. By Lemma 3.9, $K = G_{\{I\}}$ precedes H. We have $f \notin K$, so $f \notin H_*$. Therefore, $H_* \leq G_{(J)}$. $\square$

Corollary 3.11 *Let H be a nice subgroup of G.*

1. *If $D \subset \Omega$ is an igap and $H = G_{\{D\}}$, then $H^* = G_{(\sup D)} < G_{(\inf D)} = H_*$.*
2. *If $H = G_{\{J\}}$, where J is an icut in* $\sup(\Omega)$ *such that for each igap D neither $J = \sup D$ nor $J = \inf D$, then $H^* = H_* = G_{(J)}$.*

Proof (1) Clearly, $G_{(\sup D)} \leq G_{(\inf D)}$, and it follows from Lemma 1.1 that $G_{(\sup D)} \neq G_{(\inf D)}$. Theorem 3.1 and Lemma 3.10 imply that $H_* = G_{(\inf D)}$, and with Lemma 3.9 imply that H^* is the closure of $G_{(>\sup D)}$, so $H^* = G_{(\sup D)}$.

(2) Lemma 3.10 implies that $H_* = G_{(J)}$. Theorem 3.1 implies that H^* is the closure of $G_{(>J)}$ which is $G_{(J)}$. Therefore, $H^* = H_* = G_{(J)}$. $\square$

Our next goal is to show that there is a way to recognize if a nice group is a stabilizer of a point. To obtain this goal we need a few lemmas and definitions.

We say $a \in \Omega$ realizes a *rare type* if a is the only element realizing $\mathrm{tp}(a)$ in $\mathrm{igap}(a)$. (This definition is a generalization of the definition of a rare type for gaps given in Kossak, Kotlarski, and Schmerl [8, Proposition 5.17]. These definitions coincide for unbounded elements.) An igap D is called *nonlabeled* if for every $b \in D$, $\mathrm{tp}(b)$ is not a rare type.

Lemma 3.12 (Kaye [5]) *Let D be a nonlabeled igap, and let $a \in D$. Then* $\inf\{b : b \in D$ *and* $\mathrm{tp}(a) = \mathrm{tp}(b)\} = \inf D$.

Lemma 3.13 *Let H be a nice subgroup, and let $H = G_{\{D\}}$, where D is a nonlabeled igap. If $N < H$ is a normal, closed subgroup of H and $H^* \leq N \leq H_*$, then either $N = H_*$ or $N = H^*$.*

Proof Let $d \in D$. Then $G_d = \mathrm{Aut}(M, d) < H$ and $N' = N \cap G_d < G_d$. The subgroup N' is a closed normal subgroup of G_d. By Theorem 2.7, $N' = G_{(I)}$, where I is an invariant cut in (M, d). By Lemma 1.1, $H^* \cap G_d = (G_d)_{(\sup D)} < H_* \cap G_d = (G_d)_{(\inf D)}$. Because

$$H^* \cap G_d \leq N' \leq H_* \cap G_d,$$

we have $\inf D \subseteq I \subseteq \sup D$. Assume that $I \neq \sup D$, that is, $N' > H^* \cap G_d$. Then by Lemma 3.12, we can find $f_i \in G_{\{D\}}$, $i < \omega$ such that

$$I \supset f_1(I) \supset f_2(I) \supset \cdots$$

and

$$\bigcap_{i<\omega} f_i(I) = \inf D.$$

Because N is a normal subgroup of H and each $f_i \in H$, $i < \omega$, we have $N > G_{(f_i(I))}$, $i < \omega$. Hence, $N > G_{(>\inf D)}$. Since N is closed, N contains the closure of $G_{(>\inf D)}$, which is $G_{(\inf D)} = H_*$. □

The situation is different when H is a nice subgroup and $H = G_{\{D\}}$, where D is a *labeled* igap, that is, with elements realizing rare types. The next lemma shows this.

Lemma 3.14 *Let H be a nice subgroup, and let $H = G_{\{D\}}$, where D is a labeled igap. Then there is a normal closed subgroup N of H such that $H^* < N < H_*$.*

Proof Define $t(x)$ to be the least z such that

$$\forall i \leq x\big[(z)_0 = x \wedge (z)_i = 2^{(z)_{i-1}}\big].$$

Since Ω is very good and $t(x)$ is nondecreasing, if $c \in \Omega$, then $t(c) \in \mathrm{igap}(c)$.

Now let $a \in D$ realize a rare type. Then $G_{\{D\}} = G_a$. Consider

$$I = \sup\{2^a_n : n < \omega\} \quad \text{where } 2^a_0 = 1, 2^a_1 = a, 2^a_n = 2^{2^a_{n-1}}.$$

Then I is closed under exponentiation, I is invariant in (M, a), and $\inf D \subset_e I \subset_e \sup D$, because $I < t(a) \in D$. Then by Theorem 2.7, $G_{(I)}$ is closed normal in $\mathrm{Aut}(M, a) = G_{\{D\}} = G_a$, and by Lemma 1.1, $H_* < G_{(I)} < H^*$. □

Combining Lemmas 3.13 and 3.14, we obtain the main theorem of this section.

Theorem 3.15 *Let H be a nice subgroup of G such that there is a normal closed subgroup N of H with $H^* < N < H_*$. Then H is a stabilizer of a point from Ω.*

Let M be a saturated model of Peano Arithmetic. Theorem 3.15 shows that if $\mathrm{Aut}(M)$ recognizes that Ω is a very good interstice, then $\mathrm{Aut}(M)$ can recognize if a nice subgroup is a stabilizer of a nonstandard element from Ω.

4 Nonisomorphic Automorphism Groups

By using arguments similar to [12, Theorem 3.8], we could show the following result.

Theorem 4.1 *Let M_1, M_2 be saturated models of* PA *such that* $\mathrm{Aut}(M_1) \cong \mathrm{Aut}(M_2)$. *Then for every $n \in \omega$,*

$$\big(\omega, \mathrm{Rep}\big(\mathrm{Th}(M_1)\big)\big) \models \mathrm{RT}^n_2 \quad \textit{iff } \big(\omega, \mathrm{Rep}\big(\mathrm{Th}(M_2)\big)\big) \models \mathrm{RT}^n_2 .$$

Here RT^n_2 is infinite Ramsey's theorem stating that every 2-coloring of $[\omega]^n$ has an infinite homogeneous set. As in [12, Corollary 3.15], we can obtain the following corollary from Theorem 4.1.

Corollary 4.2 *There are saturated models M_1, M_2, M_3, M_4 of* PA *of cardinality λ such that* $\mathrm{Aut}(M_i) \ncong \mathrm{Aut}(M_j)$, *whenever* $1 \leq i < j \leq 4$.

We note that Nurkhaidarov and Schmerl [13] have recently used the results from the previous sections to improve on the last corollary and show that there are continuum many models whose automorphism groups are pairwise nonisomorphic.

5 Boundedly saturated models

A type $p(v, A)$ over a model K is *bounded* if it contains the formula $v < t(a)$ for some $a \in A \subset K$ and some Skolem term t. A model K is *boundedly saturated* if every finitely realized bounded type with parameters from a set $A \subset K$ such that $|A| < |K|$ is realized in K. Clearly, every saturated model is boundedly saturated. Moreover, any elementary cut of a saturated model is boundedly saturated. In addition, any boundedly saturated model of PA is an initial segment of a saturated model of PA, and it is saturated if and only if its cofinality equals the cardinality of the saturated extension. (For proofs of the above facts and for references, see [14].)

For the rest of the section, let N be an elementary cut of M such that $\mathrm{cf}(N) < |M|$. That is, N is a nonsaturated, boundedly saturated elementary cut of M.

Proposition 5.1 ([14, Theorem 5.5]) *Let $f \in \mathrm{Aut}(N)$. Then f can be extended to an automorphism of M.*

We now show that any boundedly saturated model of PA has the small index property in the set topology.

Theorem 5.2 *Let H be a subgroup of* $\mathrm{Aut}(N)$. *Then H is open in the set topology if and only if* $[\mathrm{Aut}(N) : H] \leq \lambda$.

Proof Clearly, every open subgroup has a small index.

Conversely, suppose that $[\mathrm{Aut}(N) : H] \leq \lambda$. Let

$$\bar{H} = \{g \in G : g > h \text{ for some } h \in H\}.$$

Let $G_{\{N\}} = \{g \in G : g(N) = N\}$. By using the previous proposition it is not hard to show that $G_{\{N\}}|_N = \mathrm{Aut}(N)$.

Claim *We have* $[G_{\{N\}} : \bar{H}] \leq \lambda$.

Proof Let g_1 and g_2 be in $G_{\{N\}}$. Suppose that $g_1\bar{H} \neq g_2\bar{H}$. Then $g_2^{-1}g_1 \notin \bar{H}$. Let $h_1 = g_1|N$ and $h_2 = g_2|N$. Notice that $g_2^{-1}g_1|_N = h_2^{-1}h_1$. Since $g_2^{-1}g_1 \notin \bar{H}$ and since $\bar{H}$ is the collection of all automorphisms of M which extend H, $h_2^{-1}h_1 \notin H$. This implies that $h_1H \neq h_2H$.

Thus, whenever two cosets of $\bar{H}$ are different, $g_1\bar{H} \neq g_2\bar{H}$, then the restrictions of their automorphisms to N give different cosets of H. Since there are at most λ cosets of H in $\mathrm{Aut}(N)$, $\bar{H}$ has at most λ cosets in $G_{\{N\}}$. This proves the claim. □

Now since N is boundedly saturated but not saturated, there is a cofinal set $A \subset N$ of cardinality less than λ. Note that $g \in G_{\{N\}}$ if and only if $g(A)$ is cofinal in N. Thus, $G_{\{N\}}$ is an open subgroup since it is the union of all the open sets of the form

$$S_{A,B} = \left\{g \in G : g(A) = B \text{ for some } B \subseteq_{\mathrm{cof}} N\right\}.$$

Since $G_{\{N\}}$ is open, $[G : G_{\{N\}}] \leq \lambda$. Combining this with the claim, we get that $[G : \bar{H}] \leq \lambda$. □

6 Short Saturated Models

A model K is *short* if $K = K(a) = \sup(\mathrm{Scl}(a))$ for some $a \in K$. Otherwise, K is *tall*. Clearly, all saturated models are tall.

A model is *short saturated* if it is short and boundedly saturated. Thus, for any $a \in M$, $M(a)$ is a short saturated model of PA. Note that the standard model is short saturated.

In this section we show that Theorem 2.7 is true for short saturated models as well. We then show that any complete theory of PA has short saturated models with nonisomorphic automorphism groups.

The following result, which follows from the Blass–Gaifman lemma (see Blass [3] and Gaifman [4]), can be found in Kossak and Schmerl [10, Proposition 8.3.1].

Proposition 6.1 ([10, Proposition 8.3.1]) *Let K be a model of* PA*, and let $f \in \mathrm{Aut}(K)$. If $a \in K$ and $f(a) \in \mathrm{gap}(a)$, then there is $c \in \mathrm{gap}(a)$ such that $f(c) = c$.*

Also, it follows from the Blass–Gaifman lemma that whenever $f, g \in \mathrm{Aut}(K)$ and $c, d \in \mathrm{gap}(a) \subseteq K$ are such that $f(c) = c$ and $g(d) = d$, then there is $e \in \mathrm{gap}(a)$ such that $f(e) = g(e) = e$. Thus, we get the following.

Lemma 6.2 *Let $a \in M$. Then for any $g, h \in \mathrm{Aut}(M(a))$, there is $e \in \mathrm{gap}(a)$ such that $g(e) = h(e) = e$.*

Recall that $\mathrm{gap}(a)$ is labeled if it has elements realizing rare types. That is, there is $d \in \mathrm{gap}(a)$ such that no other element in $\mathrm{gap}(a)$ realizes $\mathrm{tp}(d)$. In this case, since any automorphism of $M(a)$ fixes $\mathrm{gap}(a)$ setwise, any automorphism of $M(a)$ must fix d. Therefore, any cut $I \subset M(a)$ for which either $\mathrm{Scl}(d) \cap I$ is cofinal in I or $\mathrm{Scl}(d) \cap M(a) \setminus I$ is downward cofinal in $M(a) \setminus I$ is an invariant cut. Since $\mathrm{Scl}(d)$ is cofinal in $M(a)$, $M(a)$ has no largest proper invariant cut.

On the other hand, if $\mathrm{gap}(a)$ is nonlabeled, every element d in $\mathrm{gap}(a)$ realizes a ubiquitous type (see [8]); that is, there are elements cofinally high and cofinally low in $\mathrm{gap}(a)$ which realize $\mathrm{tp}(d)$. Therefore, any proper cut of $M(a)$ which intersects $\mathrm{gap}(a)$ is not invariant, and hence $M(a)$ has a largest proper invariant cut, namely, $M(a) \setminus \mathrm{gap}(a)$.

The following is the analogue of Theorem 2.7 for short saturated models.

Theorem 6.3 *Let $a \in M$, and let $H \leq \mathrm{Aut}(M(a))$. Then H is a closed normal subgroup of $\mathrm{Aut}^{\lambda}(M(a))$ if and only if there exists an invariant cut $J \subset M(a)$ such that $H = \mathrm{Aut}(M(a))_{(J)}$.*

Proof The proof is essentially the same as the proof of Theorem 2.7. Therefore, we only show the first case. The second and third cases follow in the same fashion as in the saturated case.

Let $g \in G(a) = \mathrm{Aut}(M(a))$, and let $I = I_{\mathsf{fix}}(g)$. Let $J \subseteq I$ be the largest invariant exponentially closed cut (such J is well defined). The subgroup $G(a)_{(J)}$ is a closed normal subgroup containing g, so it is sufficient to show that $G(a)_{(J)}$ is in the closure of the subgroup generated by the conjugates of g.

As mentioned above, we consider only the first case. That is, we assume that $J \neq \omega$. Let $A \subset M(a)$ be of cardinality less than λ. Then, using saturation, we find $d \in M$ (note: d is not necessarily in $M(a)$) with the property that for every $d' \in A$ there exists $i \in J$ such that $d' = (d)_i$. Let $h \in G(a)_{(J)}$ be such that $h(A) = B$. By Lemma 6.2, there is $e \in \mathrm{gap}(a)$ such that $g(e) = h(e) = e$. We will work in the expanded saturated extension (M, e). By Proposition 5.1, there exist $g_1, h_1 \in \mathrm{Aut}((M, e))$ extending g and h, respectively. Then $h_1(a) = b$ for some $b \in M$, and for every $d' \in A$ and $h(d') = b' \in B$, there exists $i \in J$ such that $d' = (d)_i$ and $b' = (b)_i$. By Theorem 2.2, there exist $f_3, f_4 \in \mathrm{Aut}((M, e))$ such that either $g_1^{-f_3} g_1^{f_4}(d) = b$ or $g_1^{f_3} g_1^{-f_4}(d) = b$. We consider the case when $g_1^{-f_3} g_1^{f_4}(d) = b$. Let $d' \in A$. By the choice of d, there exists $i \in J$ such that $d' = (d)_i$. Hence,

$$g_1^{-f_3} g_1^{f_4}(d') = g_1^{-f_3} g_1^{f_4}\big((d)_i\big) = \big(g_1^{-f_3} g_1^{f_4}(d)\big)_{g_1^{-f_3} g_1^{f_4}(i)} = (b)_i$$

and

$$h_1(d') = h_1\big((d)_i\big) = \big(h_1(d)\big)_{h_1(i)} = (b)_i.$$

Therefore, for every $d' \in A$: $g_1^{-f_3} g_1^{f_4}(d') = h_1(d')$. Let $f_1 = f_3|_{M(a)}$ and $f_2 = f_4|_{M(a)}$. Thus, since $A \subset M(a)$, for all $d' \in A$,

$$g^{-f_1} g^{f_2}(d') = h(d'). \qquad \square$$

Since automorphisms of short saturated models with a labeled last gap have no largest proper invariant cut, while short saturated models with a nonlabeled last gap have a largest proper invariant cut, it follows from the above theorem that the automorphism groups of such models are not isomorphic as topological groups. Since these models have the small index property (Theorem 5.2), we get the following corollary.

Corollary 6.4 *Let M be a saturated model of* PA*, and let $a, b \in M \setminus M(0)$ be such that* $\mathrm{gap}(a)$ *is labeled and* $\mathrm{gap}(b)$ *is nonlabeled. Then* $\mathrm{Aut}(M(a)) \not\cong \mathrm{Aut}(M(b))$.

We remark that whenever the *extremely short* model $M(0)$ is nonstandard, it has no largest invariant cut. Thus, for any $a > M(0)$ such that $\mathrm{gap}(a)$ is nonlabeled, $\mathrm{Aut}(M(0))$ is not isomorphic to $\mathrm{Aut}(M(a))$. On the other hand, we do not know if the same is true when $\mathrm{gap}(a)$ is labeled. In fact, if a realizes a rare type, then $(M(a), a)$ is an extremely short model with the same automorphisms as $M(a)$. Another question which remains open is whether $\mathrm{Aut}(M(a))$ and $\mathrm{Aut}(M(b))$ are isomorphic when $\mathrm{gap}(a)$ is labeled and has elements realizing a minimal type and $\mathrm{gap}(b)$ is labeled but has no elements realizing minimal types.

In [14] we have shown that in the point topology, the closed normal subgroups of the automorphism group of any boundedly saturated model of PA are the stabilizers of the invariant cuts. We would like to know if this is true in the set topology as well.

Question 6.5 Let N be an elementary cut of M. Let $H \leq \mathrm{Aut}(N)$. Is it true that H is a closed normal subgroup of $\mathrm{Aut}^{\lambda}(N)$ if and only if there exists an invariant cut $J \subset M$ such that $H = \mathrm{Aut}(M)_{(J)}$?

As we have shown in Theorem 2.7 and in Theorem 6.3, respectively, the answer is positive for saturated elementary cuts and short saturated elementary cuts. However, it is still unknown for boundedly saturated cuts that are neither saturated nor short. A positive answer to this question will imply, using the same argument as in Corollary 2.9, that whenever $M_1 \models$ TA and $M_2 \not\models$ TA are boundedly saturated models which are not short, their automorphism groups are nonisomorphic.

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Acknowledgment

The authors thank Roman Kossak for his useful comments on an earlier version of this paper.

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