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Reverse Mathematics and the Coloring Number of Graphs

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Abstract We use methods of reverse mathematics to analyze the proof theoretic strength of a theorem involving the notion of coloring number. Classically, the coloring number of a graph $G = (V, E)$ is the least cardinal κ such that there is a well-ordering of V for which below any vertex in V there are fewer than κ many vertices connected to it by E. We will study a theorem due to Komjáth and Milner, stating that if a graph is the union of n forests, then the coloring number of the graph is at most $2n$. We focus on the case when $n = 1$.

1 Introduction

We assume the reader is familiar with the general program of reverse mathematics, in which we study the proof-theoretic strength of theorems of ordinary, "essentially countable" mathematics. For more on reverse mathematics, we refer the reader to Simpson [6]; for background in computability theory, we refer the reader to Soare [7]; for background in graph theory, see Diestel [1]. Within this paper, we will only be working within the subsystems RCA_0, WKL_0, and ACA_0.

We will use the following lemma from [6] extensively.

Lemma 1.1 (Simpson) *The following are pairwise equivalent over* RCA_0*:*

1. ACA_0;
2. *For all one-to-one functions* $f : \mathbb{N} \to \mathbb{N}$ *there exists a set* $X \subseteq \mathbb{N}$ *such that* $(\forall n)[n \in X \leftrightarrow \exists m(f(m) = n)]$*; that is,* X *is the range of* f.

First we clarify some of the notation used in this paper. Note that within RCA_0, every finite set can be encoded as a unique natural number and we denote the set of all codes for finite subsets of $A \subseteq \mathbb{N}$ by Fin_A. Similarly, every finite sequence can be encoded as a unique natural number, and we denote the set of all codes for finite

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sequences of elements of $A \subseteq \mathbb{N}$ by Seq_A (sometimes written $A^{<\mathbb{N}}$). We identify elements of Seq_A with partial functions $\sigma : \mathbb{N} \to A$.

For $\sigma \in \mathrm{Seq}_A$, let $|\sigma|$ denote the length of σ. For $\sigma, \tau \in \mathrm{Seq}_A$, write $\tau \subseteq \sigma$ to say that τ *is an initial segment of* σ; that is,

$$|\tau| \leq |\sigma| \wedge \big(\forall i < |\tau|\big)\big[\sigma(i) = \tau(i)\big],$$

which we could also write as $\tau = \sigma \upharpoonright \mathrm{dom}(\tau)$.

Definition 1.2 Say that $T \subseteq 2^{<\mathbb{N}}$ (or $T \subseteq \mathbb{N}^{<\mathbb{N}}$) is a *tree* if

$$(\forall \sigma \in T)(\forall \tau \in T)[\tau \subseteq \sigma \to \tau \in T].$$

In words, the above is equivalent to saying that T *is closed under initial segments*.

A *path* in a tree T is a function $f : \mathbb{N} \to 2$ such that $(\forall n)[f \upharpoonright n \in T]$.

We use $\langle a, b\rangle$ to denote the standard pairing of natural numbers a and b.

Now we turn to graph theory and formulate a few definitions within the base theory RCA_0.

Definition 1.3 (RCA_0) A *graph* G is a pair (V, E), where V is the set of vertices and E is an irreflexive, symmetric binary relation on V. (Note that our graphs are undirected.)

Definition 1.4 (RCA_0) Let $G = (V, E)$ be a graph, and $u, v \in V$, $u \neq v$.

A *path* in the graph G is a nonempty sequence $\sigma \in \mathrm{Seq}_V$ such that

$$\big(\forall i \neq j < |\sigma|\big)\big[\sigma(i) \neq \sigma(j)\big] \wedge \big(\forall i < |\sigma| - 1\big)\big[\sigma(i) E \sigma(i+1)\big].$$

The *collection of all paths in* G is given by

$$\mathrm{Path}_G := \{\sigma \in \mathrm{Seq}_V : \sigma \text{ is a path in } G\}.$$

The *collection of all paths from* u *to* v *in* G is given by

$$\mathrm{Path}_G^{u,v} := \big\{\sigma \in \mathrm{Path}_G : \sigma(0) = u \wedge \sigma\big(|\sigma| - 1\big) = v\big\}.$$

Definition 1.5 (RCA_0) An *acyclic graph* is a graph $F = (V, E)$ such that

$$(\forall u, v \in V)\big[|\,\mathrm{Path}_F^{u,v}\,| < 2\big].$$

A *forest* is an acyclic graph. A *tree* is a forest $T = (V, E)$ such that

$$(\forall u, v \in V)[\mathrm{Path}_T^{u,v} \neq \emptyset].$$

Notice that this definition of a tree is different from what we have already described. Nevertheless, which definition of tree we intend should always be clear from the context in which we are working.

Definition 1.6 Let $G = (V, E)$ be a graph. The *component of* G *with representative vertex* v is the subgraph of G that is induced by the set of vertices given by $\{u \in V : \mathrm{Path}_T^{u,v} \neq \emptyset\}$.

Therefore components are necessarily connected. Note that v is indeed in the component with representative vertex v as we have defined it, since the path from v to itself is contained in the set $\{u \in V : \mathrm{Path}_T^{u,v} \neq \emptyset\}$.

Definition 1.7 A graph $G = (V, E)$ *has finitely many components* if there is a finite set $X \in \mathrm{Fin}_V$ such that X contains exactly one vertex from each component.

When we say "component of G" we mean a component of G with representative vertex v for some $v \in V$.

Proposition 1.8 ($\mathsf{ACA_0}$) *Let G be a graph. Then a set of component representatives for G exists.*

Proposition 1.9 ($\mathsf{RCA_0}$) *Let G be a graph with only finitely many components. Then a set of component representatives for G exists.*

Definition 1.10 ($\mathsf{RCA_0}$) Let $T = (V, E)$ be a tree. For all $X \in \mathrm{Fin}_V$ and all $y \in V \setminus X$ we can form the set of all paths from the induced subgraph on X to the vertex y

$$\mathrm{Path}_T^{X,y} := \big\{\sigma \in \mathrm{Path}_T : \sigma(0) \in X \wedge \sigma\big(|\sigma| - 1\big) = y\big\}.$$

Because T is a tree (and hence acyclic), for each $x \in X$ there is a unique path from x to y. It follows that $\mathrm{Path}_T^{X,y}$ is a finite set because X is a finite set. Let $n = \min\{|\sigma| : \sigma \in \mathrm{Path}_T^{X,y}\}$. For any $\sigma \in \mathrm{Path}_T^{X,y}$ with $|\sigma| = n$, we have $(\forall i)[1 \leq i < |\sigma| \to \sigma(i) \notin X]$. We call such a σ with $|\sigma| = n$ a *path from X to y*. Since the induced subgraph on X need not be connected, there may be more than one such path, so choose the one with the least code to define the function

$$P : \mathrm{Fin}_V \times V \to \mathrm{Path}_T$$

such that

$$P(X, y) = \begin{cases} \emptyset & \text{if } y \in X \\ \sigma & \text{if } y \in V \setminus X, \text{ where } \sigma \text{ is a path from } X \text{ to } y \text{ with least code.} \end{cases}$$

Notice that if the induced subgraph on X is connected, then there is a unique path from X to y for any $y \in V \setminus X$. The existence of the function P in $\mathsf{RCA_0}$ will be useful to us later.

2 Different Notions of Coloring Number

We begin this section with the classical definition of coloring number.

Definition 2.1 (Classical) The *coloring number* of a graph G, written $\mathrm{Col}(G)$, is the least cardinal κ for which there is a well-ordering of the vertex set in which every vertex is joined by an edge to fewer than κ smaller vertices.

The reader is almost surely familiar with the notion of the chromatic number of a graph G, denoted $\mathrm{chr}(G)$, which is somewhat related to the coloring number of G. (To find reverse mathematics results relating to theorems involving chromatic number, we direct the reader to Gasarch and Hirst [3].) If there is a well-ordering that witnesses coloring number κ in a graph, then this well-ordering could actually give us a proper coloring of the graph (using at most κ colors) if we color greedily in a certain way along the ordering (although this process will not give us the chromatic number of the graph in general). For an example of a greedy algorithm that would succeed, consider the following. Suppose we are given a well-ordering that witnesses a certain coloring number. Then for a vertex v labeled by α in the well-ordering, we consider the set of colors of the neighbors of v that have label less than α in the well-ordering. We color v with the least color that is not in this set.

We do know that given a graph G, we have $\mathrm{chr}(G) \leq \mathrm{Col}(G)$. To show that we indeed have inequality, we give the easy example of $G = K_{3,3}$. In this example,

$\mathrm{chr}(G) = 2$ since G is a complete bipartite graph. On the other hand, we have $\mathrm{Col}(G) = 4$, because no matter what ordering of the vertices of G that we choose, one of the vertices, say v, must be greatest in that ordering. Then since G is complete bipartite, v is connected to three other, necessarily lower vertices in that ordering. Thus we have $\mathrm{Col}(G) = 4$. In fact, if we consider the example $G = K_{n,n}$, then it is easy to see by an argument similar to the above that we still have $\mathrm{chr}(G) = 2$, but $\mathrm{Col}(G) = n + 1$. So as we can see, the notion of coloring number, while related to chromatic number, has a somewhat different flavor. Coloring number is a very natural and interesting notion because it lends itself so well to set theory and recursion. Many of the results having to do with coloring number are set-theoretic. For example, consider the following.

Lemma 2.2 (Erdös, Hajnal) *Let $G = (V, E)$ be a graph. If $|V| = \lambda$ and $\mathrm{Col}(G) = \kappa$, then there exists a well-ordering of V with the order type λ witnessing $\mathrm{Col}(G) = \kappa$.*

We restrict ourselves to work with only countable graphs. (So from now on, when we say "infinite graph," we really mean "countably infinite graph.") Considering the above lemma, we are particularly interested in well-orderings of the vertex set V that have order type ω. Of course, to get such a well-ordering of type ω for an arbitrary G given that $\mathrm{Col}(G) = \kappa$ may require nontrivial axioms in the sense of reverse mathematics, and it is not immediately clear which subsystem is actually necessary for the lemma. We think this question is interesting, but we leave it open.

Now we give some definitions. (Note that we use the usual definition of *linear order* as our RCA_0 definition.)

Definition 2.3 (RCA_0) Let $G = (V, E)$ be a graph, and let $k \in \mathbb{N}$, $k \geq 1$. A *k-order* of V is a linear order $\leq_V$ of V such that for every $x \in V$ there are at most $k - 1$ many $y \in V$ such that $y \leq_V x$ and $E(x, y)$ holds.

If $G = (V, E)$ is a graph, then the existence of a k-order which is a well-order on V classically implies that $\mathrm{Col}(G) \leq k$. We now restate the classical definition of coloring number for countably infinite graphs.

Definition 2.4 For $k \geq 1$, $\mathrm{Col}_\omega(G) \leq k$ if there is a k-order of V of type ω.

In many ways the classical definitions of coloring number given above are unsatisfactory in terms of reverse mathematics. For instance, how do we define (in RCA_0) what it means for a linear order of V to be of type ω? This leads us to formulate a few new definitions.

The following definition gives a strong way of saying that a linear order $\leq_V$ on a set V has order type ω by specifying, for each element $v \in V$, exactly how many elements are below v in the order $\leq_V$.

Definition 2.5 (RCA_0) We say that a linear order $\leq_V$ of a set $V = \{v_0, v_1, v_2, \dots\}$ has *strong ω-type* if there is a bijection $f : \mathbb{N} \to V$ such that

$$i \leq_{\mathbb{N}} j \iff f(i) \leq_V f(j).$$

In other words, f explicitly gives the order $\leq_V$, by specifying $f(0) =$ the first element of V in the order $\leq_V, \dots, f(n) =$ the element of V in the $n + 1$ position in the ordering $\leq_V$.

The following definition gives a weaker way of saying that a linear order $\leq_V$ on a set V has order type ω. Under this definition, we cannot tell exactly how many elements are below a given vertex v in the order $\leq_V$, only that there is some finite bound on the number of elements below v in the order $\leq_V$.

Definition 2.6 ($\mathsf{RCA_0}$) We say that a linear order $\leq_V$ of a set $V = \{v_0, v_1, v_2, \dots\}$ has *weak ω-type* if

$$(\forall i)(\exists j)(\forall m \geq_{\mathbb{N}} j)[v_i \leq_V v_m].$$

Here are some variations on the reverse mathematics definition of coloring number. For the following, let $G = (V, E)$ be a graph, and $k \in \mathbb{N}$ with $k \geq 2$.

Definition 2.7 (***(Linear order coloring number)*** ($\mathsf{RCA_0}$)) We say that $\mathrm{Col}_{LO}(G) \leq k$ if there is a k-order of V.

Definition 2.8 (***(Strong ω coloring number)*** ($\mathsf{RCA_0}$)) For an infinite graph G we say that $\mathrm{Col}^S_\omega(G) \leq k$ if there is a k-order of V of strong ω-type.

Definition 2.9 (***(Weak ω coloring number)*** ($\mathsf{RCA_0}$)) For an infinite graph G we say that $\mathrm{Col}^W_\omega(G) \leq k$ if there is a k-order of V of weak ω-type.

It is not hard to see we have the following string of classical implications:

$$\mathrm{Col}^S_\omega(G) \leq k \iff \mathrm{Col}^W_\omega(G) \leq k \implies \mathrm{Col}_{LO}(G) \leq k.$$

The converse of the last implication above is false in general. Classically, $\mathrm{Col}_{LO}(G)$ and $\mathrm{Col}(G)$ are not the same. Consider the following to see this fact.

Lemma 2.10 $\mathrm{Col}_{LO}(G) \leq k$ *if and only if* $\mathrm{Col}_{LO}(H) \leq k$ *for every finite subgraph* $H \subseteq G$.

To show (classically) that $\mathrm{Col}_{LO}(G) \leq k$ does not imply $\mathrm{Col}^W_\omega(G) \leq k$, we direct the reader to examples constructed by Erdös and Hajnal [2]. These examples were originally used to show that the following result is sharp.

Theorem 2.11 (Erdös, Hajnal) *If every finite subgraph of a graph G has coloring number at most n $(2 \leq n < \omega)$, then the coloring number of G is at most $2n - 2$.*

That is, for each $n \geq 2$, Erdös and Hajnal constructed a graph G such that for every finite subgraph H of G, $\mathrm{Col}(H) = n$, but $\mathrm{Col}(G) > 2n - 3$ (and so by the theorem it must be the case that $\mathrm{Col}(G) = 2n - 2$).

Notice that, together with Lemma 2.10, Theorem 2.11 proves that if $\mathrm{Col}_{LO}(G) \leq n$, then classically we have that $\mathrm{Col}_\omega(G) \leq 2n - 2$ (where $\mathrm{Col}_\omega(G)$ denotes the classical coloring number where we consider only well-orderings of V of type ω). So classically, linear order coloring number and omega coloring number are not entirely different. At least they are either both finite or both infinite.

While it is evident classically that $\mathrm{Col}^S_\omega(G) \leq n \iff \mathrm{Col}^W_\omega(G) \leq n$, we note that the equivalence between strong and weak ω-type linear orders requires nontrivial axioms in the sense of reverse mathematics analysis, as illustrated by the following theorem.

Theorem 2.12 ($\mathsf{RCA_0} + \mathbf{\Sigma^0_2}$ **Induction**) *The following are equivalent:*

1. $\mathsf{ACA_0}$;
2. *Every linear order of weak ω-type has strong ω-type.*

We omit the proof of the above theorem, but note that Σ^0_2 induction is indeed used in our proof. It would be a nicer result if we could eliminate the need for Σ^0_2 induction.

3 Summary of Results

One of the main theorems we wish to study is the following. We first note for clarity that classically, a graph $G = (V, E)$ is a union of n forests when we can write $E = E_1 \cup E_2 \cup \cdots E_n$ such that each subgraph (V, E_i) of G is a forest.

Theorem 3.1 (Komjáth, Milner ($\mathsf{ACA_0}$)) *If a graph G is a union of $n < \omega$ forests, then* $\mathrm{Col}(G) \leq 2n$.

Proof The proof given by Komjáth and Milner [4] can be carried out in $\mathsf{ACA_0}$. □

Throughout this paper, we focus on the special case of when $n = 1$, that is, when G is a forest. In this case the above theorem says that $\mathrm{Col}(G) \leq 2$ for every forest G. Of course, it is classically much easier to prove this special case. Actually, when G is a forest, this fact can be proved classically in a similar way that one could prove that the chromatic number of a forest is at most 2. In Section 4 we will go through a brief sketch of a proof of the case that G is a forest.

In Section 4 we show that if G is a countably infinite tree, then Theorem 3.1 can be proven in $\mathsf{RCA_0}$. In Section 5 we go on to show that Theorem 3.1 can also be proven in $\mathsf{RCA_0}$ if G is a forest with finitely many components. In Section 6, we show that if G is a forest ($n = 1$), then Theorem 3.1, using the linear order coloring number, is equivalent to $\mathsf{WKL_0}$. Even better, for any $k \in \omega$ with $k \geq 2$, the statement, "$\mathrm{Col}_{LO}(G) \leq k$ for every forest G" is equivalent to $\mathsf{WKL_0}$. As a corollary, we obtain the existence of a computable graph G such that no computable linear ordering realizes $\mathrm{Col}_{LO}(G) \leq k$ for any $k \in \omega$. In Section 7, we demonstrate that for any $k \in \omega$ with $k \geq 2$, the statement "for any forest $G = (V, E)$, $\mathrm{Col}^S_\omega(G) \leq k$" is equivalent to $\mathsf{ACA_0}$. In Section 8, we turn our attention to the weak coloring number, as we prove that the statement "for any forest $G = (V, E)$, $\mathrm{Col}^W_\omega(G) \leq 2$" is equivalent to $\mathsf{ACA_0}$. It remains open whether we can replace the 2 with a k in the previous result. In Section 9, we demonstrate that REC models the existence of a graph G that has weak omega coloring number bounded by 2 and linear order coloring number bounded by 2, but REC does not model that the strong omega coloring number of G is bounded by any $k \in \omega$.

4 Countably Infinite Trees

As we mentioned in the previous section, every forest classically has coloring number at most 2. The proof of this fact is indeed quite simple. If G is a tree, then it is connected, so in this case, the idea is to order the vertices of G by levels. That is, pick a starting vertex v, and put it at the beginning of the ordering. Now let N_1 denote the set of neighbors of v. Let N_2 denote the set of the neighbors of the neighbors of v (not including v). Let N_i denote the set of all neighbors of the vertices in N_{i-1} (not including any vertices that were in any N_k with $k < i-1$). To order the vertices of G by levels means order them by $\{v\} < N_1 < N_2 < \cdots$, with the ordering within any given N_i chosen arbitrarily. This kind of ordering will be a 2-order regardless of the choice of the starting vertex v. (Since G is a tree and therefore has no cycles, there is no danger of any vertex being connected to more than one vertex that is smaller than it in the ordering we just described.) One could also interleave the vertices from $\{v\}$, $N_1, N_2, \ldots$ to obtain a 2-order with (possibly) smaller order type.

It is slightly harder to show that $\mathrm{Col}(G) \leq 2$ if G is a forest. In this case, G could very well be a countably infinite disjoint union of trees. Therefore, to obtain a 2-order of the vertices of G, we first need to choose a vertex representative from each of the connected components of G. Now that we have chosen a set of vertex representatives, we can order each of the connected components of G exactly the same way as described above for a tree, with the role of v being played by the chosen vertex representative. Once we have an ordering for each component, we can define an ordering for all of G by either interleaving the orderings (giving an ordering of smaller order type), or just lining them all up in a row. Of course, if one of the vertices has infinitely many neighbors, then the order we obtain by simply lining up the N_i's will have order type larger than ω, but we can always appeal to Lemma 2.2 to get one with order type ω if our graph is countable.

The critical step in the proof that $\mathrm{Col}(G) \leq 2$ if G is a forest, was the choice of a set of vertex representatives. The subsystem ACA_0 is strong enough to prove this fact in general, and in the restricted case of trees or a finite disjoint union of trees, RCA_0 suffices. First we need a definition.

Definition 4.1 ((RCA_0) ***(End Extension, Komjáth, Milner [4])***) Suppose that $A \subseteq V$ is a finite subset of vertices from V, and let $\leq_A$ be a linear order of A. We call a linear order $\leq_B$ on a finite set $B \supset A$ an *end extension* of $\leq_A$ if $\leq_B \restriction_A = \leq_A$ and

$$(\forall a \in A)(\forall b \in B \setminus A)[a \leq_B b].$$

If $A \subseteq V$ is finite and $\leq_A$ is a linear order on A, then we say that $\leq_A$ can be *end extended* to an linear order $\leq_B$ of a finite $B \supset A$ if $\leq_B$ is an end extension of $\leq_A$.

Theorem 4.2 (RCA_0) *If $G = (V, E)$ is a countably infinite tree, then* $\mathrm{Col}^S_\omega(G) \leq 2$.

Proof Assume RCA_0, and let $G = (V, E)$ be a tree. Furthermore suppose that $V = \{v_0, v_1, v_2, \dots\}$. We wish to define a sequence of finite subsets $V_0 \subseteq V_1 \subseteq V_2 \subseteq \dots$ of V and a sequence of linear orders $\leq_0 \subseteq \leq_1 \subseteq \leq_2 \subseteq \cdots$ on the finite sets of vertices $V_0, V_1, V_2, \dots$, respectively, such that

1. Each V_i is finite, connected, and $\{v_0, \dots, v_i\} \subseteq V_i$ (so that $V = \bigcup_{i \in \mathbb{N}} V_i$);
2. Each $\leq_i$ is a 2-order of V_i;
3. $\leq_{i+1}$ is an end extension of $\leq_i$.

Stage 0: Define $V_0 = \{v_0\}$ and $v_0 \leq_0 v_0$.

Stage $s+1$: Suppose that we have already defined V_s and $\leq_s$. To get V_{s+1} and $\leq_{s+1}$, we do the following:

1. If $v_{s+1} \in V_s$, then let $V_{s+1} = V_s$ and $\leq_{s+1} = \leq_s$;
2. If $v_{s+1} \notin V_s$, then consider the path $P(V_s, v_{s+1})$ from v_{s+1} to V_s (the function P was defined in Definition 1.10). Let $\sigma = P(V_s, v_{s+1})$. Say that the vertices in this path given by σ are $\sigma(0) = u_0, \sigma(1) = u_1, \dots, \sigma(k) = u_k$. Then, by definition of P, $u_0 \in V_s$ and $\{u_1, u_2, \dots, u_k\} \cap V_s = \emptyset$, while $E(u_i, u_{i+1})$ holds for each $i < k$ and $u_k = v_{s+1}$. Now define $V_{s+1} = V_s \cup \{u_1, \dots, u_k\}$ and extend $\leq_s$ to $\leq_{s+1}$ by taking $\leq_{s+1}$ to be an end-extension of $\leq_s$, where additionally,

$$u_1 \leq_{s+1} u_2 \leq_{s+1} \cdots \leq_{s+1} u_k = v_{s+1}.$$

The fact that each V_s is finite, connected, and contains $\{v_0, \dots, v_s\}$ follows by induction.

Define $\leq$ to be $\bigcup_s \leq_s$. The bijection $f : \mathbb{N} \to V$ that gives a 2-order of V of strong ω-type is determined in the following way: let $f(0) = v_0$. Now consider the induction step in the above. Suppose that we have f for the set V_s, and that the last number on which f has been defined is $m - 1$. If we are in the first case, we do not extend the definition of f. If we are in the second case, we let $f(m) = u_1$, $f(m+1) = u_2, \ldots, f(m+k-2) = u_{k-1}$, and $f(m+k-1) = u_k = v_{s+1}$. So basically we are defining f along the path from V_s to v_{s+1} in increasing order of the indices of the u_i vertices. (This path exists because G is a tree, and therefore connected.) We should also note that because G is a tree, G contains no cycles, so there is never any danger that any of the vertices from $u_1, \ldots, u_k$ will ever form a cycle in G (which would prevent our function f from being a 2-order). Therefore, the function f is a 2-order of V. □

5 Forests with a Set of Component Representatives

Theorem 5.1 (RCA$_0$) *If $G = (V, E)$ is a forest and there exists a set of component representatives for G, then $\mathrm{Col}^S_\omega(G) \leq 2$.*

Proof Let X be a set of component representatives for G. Define the set

$$\begin{aligned} B &:= \{\langle x, v\rangle \in X \times V : \mathrm{Path}^{x,v}_G \neq \emptyset\} \\ &= \{\langle x, v\rangle \in X \times V : (\exists \sigma \in \mathrm{Path}_G)[\sigma(0) = x \wedge \sigma(|\sigma| - 1) = v]\} \\ &= \{\langle x, v\rangle \in X \times V : (\forall y \in X) \\ &\qquad [y \neq x \to \neg\exists\sigma \in \mathrm{Path}_G[\sigma(0) = y \wedge \sigma(|\sigma| - 1) = v]]\}. \end{aligned}$$

Notice that we have found a form of B which is Σ^0_1 and a form which is Π^0_1. Thus B is Δ^0_1, and so RCA$_0$ proves it is a set.

Now we define

$$T_i := \{v \in V : \langle x_i, v\rangle \in B\}.$$

Then each T_i is Δ^0_1, and therefore exists in RCA$_0$. Note that T_i gives us the component of G with representative $x_i \in X$.

Now by Theorem 4.2, we have that $\mathrm{Col}^S_\omega(T_i) \leq 2$ for each i. Fix orderings $\leq_{T_i}$ which witness the previous statement. To define a strong ω 2-order of G, interleave the orderings $\leq_{T_i}$ of the component trees T_i. Since none of the vertices in T_i are adjacent to any of the vertices in T_j when $i \neq j$, it does not matter how we interleave the orders. This can be done in RCA$_0$. □

As a special case of a more general result which we will prove later, we will see that ACA$_0$ suffices to show $\mathrm{Col}^S_\omega(G) \leq 2$, where G is a forest with infinitely many components. Later, we will give a reversal to show that ACA$_0$ is actually necessary for that result.

6 Linear Order Coloring Number and WKL$_0$

In this section we show the connection between linear order coloring number and the subsystem WKL$_0$, but first we state a couple lemmas.

Lemma 6.1 (RCA$_0$) *Every finite forest F has $\mathrm{Col}_{LO}(F) \leq 2$.*

Proof Fix a finite forest $F = (V_F, E_F)$. Suppose that $V_F = \{v_0, \ldots, v_k\}$. To define a 2-order on F, first let $X = \{x_0, \ldots, x_j\}$ be a finite set of component representatives, and proceed as in the proof of Theorem 5.1. □

The following lemma will be extremely useful to us in the proof of the theorem that follows.

Lemma 6.2 (Lemma 2.2 from Schmerl [5]) *Let $2 \le n \in \omega$. Then the following statement is provable from $\mathsf{RCA}_0 + \neg\,\mathsf{WKL}_0$: there are pairwise disjoint Σ^0_1 subsets $A_0, A_1, \dots, A_{n-1} \subseteq \mathbb{N}$ such that whenever $f : \mathbb{N} \to n$ is a function, there is x such that $x \in A_{f(x)}$.*

Theorem 6.3 (RCA_0) *The following are equivalent;*

1. WKL_0;
2. *For any forest $G = (V, E)$, $\mathrm{Col}_{LO}(G) \le 2$.*

Proof (1 → 2) Assume WKL_0. Let $G = (V, E)$ be a forest with $V = \{v_0, v_1, v_2, \dots\}$. Let $T \subseteq \omega^{<\omega}$ be the bounded tree defined by

$$\sigma \in T \iff \big(\forall n < |\sigma|\big)\big[\sigma(n) \le n + 1\big].$$

Now define an ordering $\le_\sigma$ on Fin_V in the following way:

1. Let $v_0 \le_\emptyset v_0$;
2. Let $\le_{\sigma * k}$ be the ordering of $\{v_0, \dots, v_{|\sigma|}, v_{|\sigma|+1}\}$ which agrees with the ordering defined by $\le_\sigma$ on $\{v_0, \dots, v_{|\sigma|}\}$ and inserts $v_{|\sigma|+1}$ into the kth position in the ordering defined by $\le_\sigma$.

For example, $\le_{\langle 0\rangle}$ is the ordering given by $v_1 \le v_0$, while $\le_{\langle 1\rangle}$ is the ordering given by $v_0 \le v_1$. Also, $\le_{\langle 0,1\rangle}$ is the ordering given by $v_1 \le v_0$ with v_2 placed into the 1 position, obtaining $v_1 \le v_2 \le v_0$.

Here is a property of $\le_\sigma$, and a definition:

1. $\sigma \subseteq \tau \implies \le_\tau \restriction \{v_0, \dots, v_{|\sigma|}\} = \le_\sigma$;
2. If g is an infinite path in T, then we define $\le_g$ by

$$x \le_g y \iff (\exists \sigma \in T)[\sigma \subset g \wedge x \le_\sigma y].$$

Property 1 is clear from the definition of $\le_\tau$. If g is an infinite path in T, then it is a routine verification of the axioms to show that $\le_g$ defines a linear order on V.

Now we define another tree $S \subseteq T$ by

$$\sigma \in S \iff \text{the ordering } \le_\sigma \text{ on } \{v_0, \dots, v_{|\sigma|}\} \text{ is a 2-order.}$$

Formally, S is defined using Σ^0_0 comprehension by

$$\sigma \in S \iff \big(\forall n < |\sigma|\big)\big(\neg \exists i \ne j < |\sigma|\big) \\ \big[v_i \le_\sigma v_n \wedge v_j \le_\sigma v_n \wedge E(v_i, v_n) \wedge E(v_j, v_n)\big].$$

Since T is a bounded tree, we must also have that S is a bounded tree. By Lemma 6.1, S is infinite, and by WKL_0, S has a path. Let g be such a path in S. We verify that $\le_g$ is a 2-order.

Suppose that g is not a 2-order. Then there are distinct i, j, k such that

$$(v_i \le_g v_k) \wedge (v_j \le_g v_k) \wedge E(v_i, v_k) \wedge E(v_j, v_k).$$

Let $\sigma \in S$ be such that $\sigma \subset g$ and $(v_i \le_g v_k) \wedge (v_j \le_g v_k)$. (That is, σ is a witness to both $(v_i \le_g v_k)$ and $(v_j \le_g v_k)$—we can use the single string σ to witness both inequalities.) Thus $v_i \le_\sigma v_k$ and $v_j \le_\sigma v_k$, but this is a contradiction, as $\sigma \in S$ implies that $\le_\sigma$ is a 2-order on V.

(2 $\rightarrow$ 1) We work in RCA_0. Assume that for any forest $G = (V, E)$, $\mathrm{Col}_{LO}(G) \leq 2$. In other words, we assume that for any forest G, there is a 2-order of G.

It is sufficient to prove the negation of the statement in Lemma 6.2. We will use the formulation of the lemma for $n = 3$. That is, we will end up showing that for all pairwise disjoint Σ^0_1 subsets $A_0, A_1, A_2 \subseteq \mathbb{N}$, there is a function $f : \mathbb{N} \rightarrow 3$ such that for all x, $x \notin A_{f(x)}$. Since we are working over RCA_0, we cannot actually talk about Σ^0_1 sets as if they exist, because they might not. Talking about them as sets in this context is really shorthand for talking about the corresponding collections of numbers defined by Σ^0_1 formulas.

Fix Σ^0_1 formulas

$$(\exists s)\big[\varphi_i(x,s)\big] \quad \text{for } 0 \leq i < 3,$$

which are disjoint. That is, for each $0 \leq i < 3$, we have

$$(\forall x)\big[\exists s\varphi_i(x,s) \rightarrow \big(\neg\exists s\varphi_{i+1}(x,s) \wedge \neg\exists s\varphi_{i+2}(x,s)\big)\big].$$

(The addition in the subscripts for the formula above is done modulo 3.) The formulas above correspond to pairwise disjoint Σ^0_1 sets $A_0, A_1, A_2 \subseteq \mathbb{N}$, respectively, from Lemma 6.2.

We define the graph $G = (V, E)$ in the following way. Let the set of vertices V be defined by

$$V := \{u^i_x : 0 \leq i < 3, x \in \mathbb{N}\} \cup \{a_{\langle x,s\rangle} : x, s \in \mathbb{N}\} \cup \{b_{\langle x,s\rangle} : x, s \in \mathbb{N}\}.$$

Let the edge relation E be defined in the following way. For $0 \leq i < 3$, $x, s \in \mathbb{N}$:

$$\begin{aligned} &E(u^i_x, a_{\langle x,s\rangle}) \wedge E(u^i_x, b_{\langle x,s\rangle}) \wedge E(u^{i+1}_x, a_{\langle x,s\rangle}) \wedge E(u^{i+2}_x, b_{\langle x,s\rangle}) \\ &\quad\iff \varphi_i(x,s) \wedge (\forall t < s)\big[\neg\varphi_i(x,t)\big], \end{aligned}$$

where the addition $i+1$ and $i+2$ is modulo 3. We should note here (to be clear) that if $\varphi_i(x,s) \wedge (\forall t < s)[\neg\varphi_i(x,t)]$ does not hold, then we do not define any of the edges from $E(u^i_x, a_{\langle x,s\rangle})$, $E(u^i_x, b_{\langle x,s\rangle})$, $E(u^{i+1}_x, a_{\langle x,s\rangle})$, or $E(u^{i+2}_x, b_{\langle x,s\rangle})$ in G. We see that the edge relation E is definable in RCA_0, as only bounded quantifiers were used in its definition.

Figure 1 will aid the reader in seeing exactly what the edge connections look like in the graph G.

We can also see that if $\leq_V$ witnesses $\mathrm{Col}_{LO}(G) \leq 2$ and $(\exists s)[\varphi_i(x,s)]$ holds, then

$$u^i_x \neq \max\{u^0_x, u^1_x, u^2_x\},$$

where the maximum is taken relative to $\leq_V$. For suppose that $\leq_V$ witnesses $\mathrm{Col}_{LO}(G) \leq 2$ and $(\exists s)[\varphi_i(x,s)]$ holds, but $u^i_x = \max\{u^0_x, u^1_x, u^2_x\}$, where the maximum is taken relative to $\leq_V$ (and the addition is modulo 3). Then since $(\exists s)[\varphi_i(x,s)]$ holds, we have edges $E(u^i_x, a_{\langle x,s\rangle})$, $E(u^i_x, b_{\langle x,s\rangle})$, $E(u^{i+1}_x, a_{\langle x,s\rangle})$, and $E(u^{i+2}_x, b_{\langle x,s\rangle})$ in G as we have defined it. Then since $u^i_x = \max\{u^0_x, u^1_x, u^2_x\}$, without loss of generality assume that $\leq_V$ satisfies $u^{i+1} \leq_V u^{i+2}_x \leq_V u^i_x$. Now if both $a_{\langle x,s\rangle}$ and $b_{\langle x,s\rangle}$ are below u^i_x in $\leq_V$, then, as u^i_x is larger than both $a_{\langle x,s\rangle}$ and $b_{\langle x,s\rangle}$ in $\leq_V$ and connected to them both, the linear order $\leq_V$ only witnesses $\mathrm{Col}_{LO}(G) \leq 3$. So suppose that $a_{\langle x,s\rangle}$ is above u^i_x and $b_{\langle x,s\rangle}$ is below u^i_x in $\leq_V$. Then $a_{\langle x,s\rangle}$ is above both u^i_x and u^{i+1}_x, while at the same time being connected to both. The resulting linear order similarly does not witness coloring number at

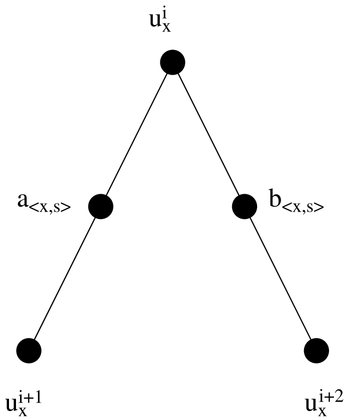

Figure 1 The edge connections in G for fixed $0 \leq i < 3$, $x, s \in \mathbb{N}$.

most 2. Supposing that $a_{\langle x,s\rangle}$ is below u_x^i and $b_{\langle x,s\rangle}$ is above u_x^i in $\leq_V$ yields another linear order not witnessing coloring number 2. By the same argument above, we certainly cannot have both $a_{\langle x,s\rangle}$ and $b_{\langle x,s\rangle}$ be above u_x^i in $\leq_V$, as it would yield a similar result (from both the case for $a_{\langle x,s\rangle}$ and $b_{\langle x,s\rangle}$). Since there is no other possibility, we have a contradiction, and therefore if $\leq_V$ witnesses $\mathrm{Col}_{LO}(G) \leq 2$ and $(\exists s)[\varphi_i(x,s)]$ holds, then $u_x^i \neq \max\{u_x^0, u_x^1, u_x^2\}$.

Now we define the function $f : \mathbb{N} \to 3$ by $f(x) = j$, where $u_x^j = \max\{u_x^0, u_x^1, u_x^2\}$, and the maximum is taken relative to $\leq_V$. By hypothesis, the graph G we have constructed satisfies $\mathrm{Col}_{LO}(G) \leq 2$. Let $\leq_V$ witness this fact. Then we cannot have $(\exists s)[\varphi_{f(x)}(x,s)]$, since $u_x^{f(x)} = \max\{u_x^0, u_x^1, u_x^2\}$ by definition, and $(\exists s)[\varphi_{f(x)}(x,s)]$ holding would imply that $u_x^{f(x)} \neq \max\{u_x^0, u_x^1, u_x^2\}$, as shown above. Therefore we have that $\neg(\exists s)[\varphi_{f(x)}(x,s)]$, and therefore that means (in the terminology of Lemma 6.2) that for all $x \in \mathbb{N}$, $x \notin A_{f(x)}$, and we are done. □

In light of the following theorem, we observe that the previous theorem is superfluous, albeit useful not only to the extent that it essentially already contains a proof of the forward direction, but also because it illustrates the simplest case of the reversal, giving us a better understanding of the general case.

Theorem 6.4 *For any $k \in \omega$ such that $k \geq 2$, RCA_0 proves that the following are equivalent:*

1. WKL_0*;*
2. *for any forest $G = (V, E)$, $\mathrm{Col}_{LO}(G) \leq k$.*

Proof (1 → 2) As noted above, by Theorem 6.3, we have in WKL_0 that for any forest $G = (V, E)$, $\mathrm{Col}_{LO}(G) \leq 2$. Thus it is clear that for any forest $G = (V, E)$, $\mathrm{Col}_{LO}(G) \leq k$ also holds in WKL_0 for $k \geq 2$.

(2 → 1) By Schmerl's lemma it suffices to show that for all pairwise disjoint Σ_1^0 subsets $A_0, A_1, \ldots, A_{k^2-k} \subseteq \mathbb{N}$, there is a function $f : \mathbb{N} \to k^2 - k + 1$ such that $(\forall x)[x \notin A_{f(x)}]$. Again, the collections above we call sets do not necessarily exist as sets in RCA_0.

Fix disjoint Σ^0_1 formulas

$$(\exists s)\big[\varphi_i(x,s)\big] \quad \text{for } 0 \le i < k^2 - k + 1.$$

That is, for each $0 \le i < k^2 - k + 1$, we have

$$(\forall x)\Big[(\exists s)\varphi_i(x,s) \to \bigwedge_{0 \le \ell < k^2-k+1, \ell \ne i} \neg(\exists s)\varphi_\ell(x,s)\Big].$$

The formulas above correspond to pairwise disjoint Σ^0_1 sets $A_0, A_1, \dots, A_{k^2-k} \subseteq \mathbb{N}$, respectively, from Lemma 6.2.

We define the graph $G = (V, E)$ in the following way. Let the set of vertices V be defined by

$$V := \{u^i_x : 0 \le i < k^2 - k + 1, x \in \mathbb{N}\} \cup \bigcup_{0 \le i < k} \{a^i_{\langle x,s\rangle} : x, s \in \mathbb{N}\}.$$

Let the edge relation E be defined in the following way. For $0 \le i < k^2 - k + 1$, $x, s \in \mathbb{N}$:

$$\bigwedge_{0 \le \ell \le k} E(u^i_x, a^\ell_{\langle x,s\rangle}) \wedge \bigwedge_{0 \le j < k} \Big(\bigwedge_{\ell = i + j(k-1)+1}^{i+(j+1)(k-1)} E(u^\ell_x, a^j_{\langle x,s\rangle})\Big)$$
$$\iff \varphi_i(x,s) \wedge (\forall t < s)\big[\neg\varphi_i(x,t)\big],$$

where all of the addition and multiplication is done modulo $k^2 - k + 1$. We see that the edge relation E is definable in RCA_0, as only bounded quantifiers were used in its definition.

Figure 2 will aid the reader in seeing exactly what the edge connections look like in the graph G for $k = 3$, for instance. It might also be helpful at this point to notice how we obtain the term $k^2 - k + 1$. This term comes from the fact that our "unsprung" gadget has k sets of $k - 1$ vertices, plus one central vertex, and therefore has a total of $k(k-1) + 1 = k^2 - k + 1$ total vertices.

We can see that if $\le_V$ witnesses $\mathrm{Col}_{LO}(G) \le k$ and $(\exists s)[\varphi_i(x,s)]$ holds, then

$$u^i_x \ne \max\{u^j_x : 0 \le j < k^2 - k + 1\},$$

where the maximum is taken relative to $\le_V$.

Now we define the function $f : \mathbb{N} \to k^2 - k + 1$ by

$$f(x) = i, \quad \text{where } u^i_x = \max\{u^j_x : 0 \le j < k^2 - k + 1\},$$

and the maximum is taken in the order $\le_V$. Then, since

$$u^i_x \ne \max\{u^j_x : 0 \le j < k^2 - k + 1\},$$

and $(\exists s)[\varphi_i(x,s)]$ holding corresponds to (in the sense of Lemma 6.2) x entering (or already being in) the Σ^0_1 set A_i at stage s, we see that, for all $x \in \mathbb{N}$, $x \notin A_{f(x)}$ (by an argument similar to that of the proof in Theorem 6.3), and we are done. □

Corollary 6.5 *For any $k \in \omega$, there is a computable forest $G = (V, E)$ such that no computable linear ordering realizes $\mathrm{Col}_{LO}(G) \le k$.*

Corollary 6.6 *For any computable forest $G = (V, E)$, there is a linear ordering of low Turing degree that realizes $\mathrm{Col}_{LO}(G) \le 2$.*

We can actually do slightly better than these corollaries to show the following.

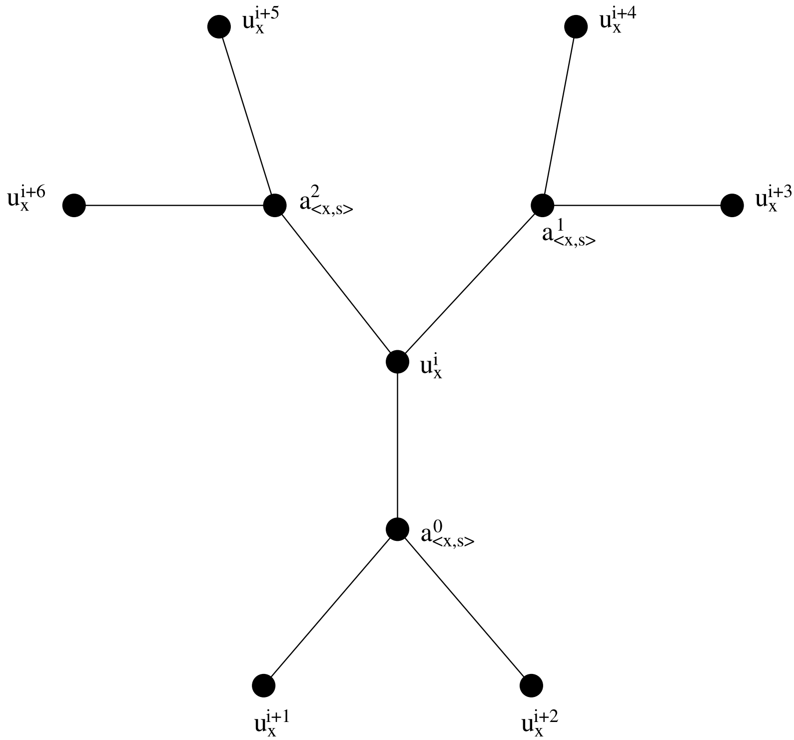

Figure 2 The edge connections in G in the case $k = 3$ for fixed $0 \leq i < 7$, $x, s \in \mathbb{N}$, where any addition is modulo 7.

Theorem 6.7 *There is a computable forest $G = (V, E)$ such that no computable linear ordering realizes $\mathrm{Col}_{LO}(G) \leq k$ for any $k \in \omega$. We say the computable linear order coloring number of G is ω.*

Such a computable forest is constructed by satisfying requirements

$$\mathcal{R}_{\langle e,k\rangle} : \text{ that } \varphi_e \text{ is not a } k\text{-order of } V,$$

where $\varphi_0, \varphi_1, \varphi_2, \ldots$ is an effective enumeration of all partial computable functions. The construction is a straightforward diagonalization. We can even improve this result to the following.

Theorem 6.8 *There is a computable forest $G = (V, E)$ such that any linear ordering realizing the fact that $\mathrm{Col}_{LO}(G)$ is finite must have PA degree.*

7 Strong ω Coloring Number and $\mathsf{ACA_0}$

Theorem 7.1 ($\mathsf{RCA_0}$) *For each $k \in \mathbb{N}$, $k \geq 2$, the following are equivalent:*

1. $\mathsf{ACA_0}$*;*
2. *For any forest $G = (V, E)$, $\mathrm{Col}^S_\omega(G) \leq k$.*

Proof (1 → 2) Assume $\mathsf{ACA_0}$, and let $G = (V, E)$ be a forest, that is, a disjoint union of infinitely many trees. By Proposition 1.8, we can form a set of component

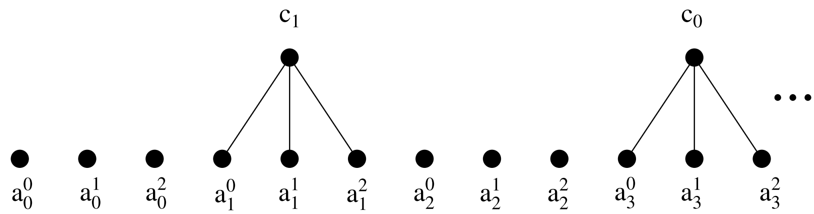

Figure 3 Edge connections in G for $k = 3$ if $f(0) = 3$, $f(1) = 1$, but 0 and 2 are not in the range of f.

representatives of G. Now by Theorem 5.1, we have $\mathrm{Col}^S_\omega(G) \leq 2$, and we are done with this direction.

(2 → 1) Fix a one-to-one function $f : \mathbb{N} \to \mathbb{N}$. Also fix $k \in \mathbb{N}$ with $k \geq 2$. We build a forest $G = (V, E)$ in RCA_0. Let

$$V := \{a^i_n : n \in \mathbb{N}, 0 \leq i < k\} \cup \{c_n : n \in \mathbb{N}\}.$$

The only edge relations that hold are $E(c_n, a^i_{f(n)})$ for $0 \leq i < k$ and $n \in \mathbb{N}$. Note that this is equivalent to making connections $E(c_{f^{-1}(m)}, a^i_m)$ for $0 \leq i < k$ and $n \in \mathbb{N}$, where $f(n) = m$. This ends the construction.

Figure 3 illustrates an example of what the edge connections in G will be if, for instance, $k = 3$ and $f(0) = 3$, $f(1) = 1$, but 0 and 2 are not in the range of f.

Note that if m never appears in the range of f, then we will never connect any of the vertices from $\{a^i_m : 0 \leq i < k\}$ to any of the vertices from $\{c_n : n \in \mathbb{N}\}$ (also note that none of the a's are connected by an edge).

Let $g : \mathbb{N} \to V$ be a bijection witnessing that $\mathrm{Col}^S_\omega(G) \leq k$ for the graph G we just constructed. Thus g defines a k-order $\leq_V$ on the vertex set V, where

$$g(0) \leq_V g(1) \leq_V g(2) \leq_V \cdots .$$

By the construction and the above argument,

$$m \in \mathrm{ran}(f) \iff (\exists c \in V)\Big[\bigwedge_{0 \leq i < k} E(c, a^i_m)\Big] \iff (\exists c \in V)\big[E(c, a^0_m)\big].$$

We claim that

$$(\exists c \in V)\big[E(c, a^0_m)\big] \iff \big(\exists j \leq \max\{g^{-1}(a^\ell_m) : 0 \leq \ell < k\}\big)\big[E\big(g(j), a^0_m\big)\big]$$

and therefore

$$m \in \mathrm{ran}(f) \iff \big(\exists j \leq \max\{g^{-1}(a^\ell_m) : 0 \leq \ell < k\}\big)\big[E\big(g(j), a^0_m\big)\big].$$

The last of this string can be checked in RCA_0 due to the bounded quantifier.

To show the forward direction of the claim, suppose that $(\exists c \in V)[E(c, a^0_m)]$, but $\neg(\exists j \leq \max\{g^{-1}(a^\ell_m) : 0 \leq \ell < k\})[E(g(j), a^0_m)]$. Fix j such that $g(j) = c$. Then since $j > \max\{g^{-1}(a^\ell_m) : 0 \leq \ell < k\}$, we have $a^\ell_m <_V c$ for $0 \leq \ell < k$. However, if $E(c, a^0_m)$ holds, then $E(c, a^\ell_m)$ holds for all $0 \leq \ell < k$, contradicting that $\leq_V$ is a k-order.

Conversely, suppose that $\neg(\exists c \in V)[E(c, a^0_m)]$. Then $\neg(\exists j \in \mathbb{N})[E(g(j), a^0_m)]$ and hence $\neg(\exists j \leq \max\{g^{-1}(a^\ell_m) : 0 \leq \ell < k\})[E(g(j), a^0_m)]$, completing the proof of the claim and the theorem. □

8 Weak ω Coloring Number

Theorem 8.1 (RCA$_0$) *The following are equivalent:*

1. ACA$_0$;
2. *For any forest* $G = (V, E)$, $\mathrm{Col}^W_\omega(G) \leq 2$.

Proof (1 → 2) This direction follows from Theorem 7.1 since $\mathrm{Col}^S_\omega(G) \leq 2$ implies $\mathrm{Col}^W_\omega(G) \leq 2$ over RCA$_0$.

(2 → 1) Suppose that for any forest $G = (V, E)$, $\mathrm{Col}^W_\omega(G) \leq 2$. Fix a one-to-one function $f : \mathbb{N} \to \mathbb{N}$. We wish to show that the range of f exists.

We construct a forest $G = (V, E)$ as follows. The vertex set is

$$V := \{a^e_n : e \in \mathbb{N} \wedge (\forall m < n)[f(m) \neq e]\} \cup \{b^e_n : e \in \mathbb{N} \wedge (\forall m < n)[f(m) \neq e]\}.$$

The edge relation is given by

$$E(a^e_n, a^e_{n+1}) \wedge E(b^e_n, b^e_{n+1}) \iff \neg(\exists m \leq n)[f(m) = e],$$

and

$$E(a^e_n, b^e_n) \iff f(n) = e.$$

This ends the construction of G.

Now fix a 2-order $\leq_V$ witnessing $\mathrm{Col}^W_\omega(G) \leq 2$. We claim that

$$e \notin \mathrm{ran}(f) \iff (\exists k)[a^e_k <_V a^e_{k+1} \wedge b^e_k <_V b^e_{k+1}].$$

Notice that this suffices to get the range of f, since we also have

$$e \notin \mathrm{ran}(f) \iff (\forall m)[f(m) \neq e],$$

which is a Π^0_1 condition, and thus there is a Δ^0_1 way to define the range of f. Hence by Δ^0_1 comprehension, the range of f exists.

For the forward direction of the claim, assume that $e \notin \mathrm{ran}(f)$. Notice V contains every element from $\{a^e_n : n \in \mathbb{N}\}$ and $\{b^e_n : n \in \mathbb{N}\}$. If $(\forall k)[a^e_{k+1} <_V a^e_k]$, then every a^e_k for $k \geq 1$ is below a^e_0 in the ordering $\leq_V$, which contradicts the fact that $\leq_V$ is a weak ω-type order. Thus $\neg(\forall k)[a^e_{k+1} <_V a^e_k]$. Thus we can fix $k \in \mathbb{N}$ such that $a^e_k <_V a^e_{k+1}$.

Now, we also have $a^e_\ell <_V a^e_{\ell+1}$ for all $\ell \geq k$. For if $\ell > k$ were least such that $a^e_\ell >_V a^e_{\ell+1}$, then we would have $E(a^e_{\ell+1}, a^e_\ell) \wedge E(a^e_\ell, a^e_{\ell-1})$ with $a^e_\ell >_V a^e_{\ell-1}$ (whether e is in the range of f or not) and $a^e_\ell >_V a^e_{\ell+1}$, contradicting the fact that $\leq_V$ is a 2-order.

The case for b^e_k is analogous to the case for the a^e_k. Therefore the forward direction of the claim holds.

Conversely, assume that $(\exists k)[a^e_k <_V a^e_{k+1} \wedge b^e_k <_V b^e_{k+1}]$. For a contradiction, suppose that $e \in \mathrm{ran}(f)$. So we can let n be such that $f(n) = e$. Notice we must have $n \geq k + 1$, for otherwise a^e_{k+1} and b^e_{k+1} would not be defined as vertices in V.

Then, using the fact that $a^e_\ell <_V a^e_{\ell+1}$ and $b^e_\ell <_V b^e_{\ell+1}$ for all $\ell \geq k$ (by an argument that is analogous to the forward direction), we have

$$(a^e_{n-1} <_V a^e_n) \wedge (b^e_{n-1} <_V b^e_n) \wedge E(a^e_{n-1}, a^e_n) \wedge E(b^e_{n-1}, b^e_n) \wedge E(a^e_n, b^e_n).$$

We have two cases: either $a^e_n <_V b^e_n$ or $b^e_n <_V a^e_n$. Either case violates the fact that $\leq_V$ is a 2-order. Hence $e \notin \mathrm{ran}(f)$, and we have proven the claim. Thus the theorem follows. □

An interesting open question involves the classification of Theorem 8.1 for values of $k \geq 2$. In other words, can we get a reversal from the statement, "for any $k \in \mathbb{N}$, $k \geq 2$, and any forest $G = (V, E)$, $\mathrm{Col}_\omega^W(G) \leq k$" to one of the major subsystems? At the very least, we already know that this statement is provable in ACA_0, by Theorem 8.1. It would appear as though the method of proof used for Theorem 8.1, however, does not translate into a reversal to ACA_0 for any case when $k > 2$.

9 Separating Computable Strong and Weak ω Coloring Number

Theorem 9.1 *There is a computable forest $G = (V, E)$ such that*

$$REC \models \mathrm{Col}_\omega^W(G) \leq 2, \text{ but } REC \not\models \mathrm{Col}_\omega^S(G) \leq k \text{ for any } k \in \omega.$$

That is, $REC \models \mathrm{Col}_\omega^S(G) = \omega$.

Proof The construction essentially employs the idea of the proof of Theorem 7.1 for each instance of k in the statement of that theorem. We define a graph $G = (V, E)$. First we place as vertices all of the even numbers in increasing order

$$a_0 < a_1 < a_2 < a_3 < a_4 < \cdots .$$

We want to satisfy the infinitely many requirements

$$\mathcal{R}_{\langle e,k\rangle} : \varphi_e \text{ does not witness } \mathrm{Col}_\omega^S(G) \leq k.$$

Formally, the requirement $\mathcal{R}_{\langle e,k\rangle}$ is that (assuming φ_e is a bijection from $\mathbb{N}$ onto V) there is an $n_k \in V$ and $\ell_0, \ldots, \ell_{k-1} \in V$ such that $E(n_k, \ell_i)$ holds for all $0 \leq i < k$ and $\varphi_e^{-1}(n_k) > \varphi_e^{-1}(\ell_i)$ for $0 \leq i < k$.

We claim that if all of the requirements are satisfied, then no computable well-ordering realizes $\mathrm{Col}_\omega^S(G) \leq k$, for any $k \in \mathbb{N}$. Suppose that there were such a computable strong ω-type k-order. Then it must be a computable bijection φ_e for some $e < \omega$. Since, for each $k < \omega$, $\mathcal{R}_{\langle e,k\rangle}$ is satisfied, we have that there is an $n_k \in V$ and $\ell_0, \ldots, \ell_{k-1} \in V$ such that $E(n_k, \ell_i)$ holds for all $0 \leq i < k$ and $\varphi_e^{-1}(n_k) > \varphi_e^{-1}(\ell_i)$ for $0 \leq i < k$. Thus φ_e fails to be a k-order of V for all k, which is exactly what we want.

Fix a well-ordering of the requirements $\mathcal{R}_{\langle e,k\rangle}$ given by

$$\mathcal{R}_{\langle e_0,k_0\rangle} < \mathcal{R}_{\langle e_1,k_1\rangle} < \mathcal{R}_{\langle e_2,k_2\rangle} < \cdots ,$$

and say that $\mathcal{R}_{\langle e_i,k_i\rangle}$ *has higher priority than* $\mathcal{R}_{\langle e_j,k_j\rangle}$ if and only if $\langle e_i, k_i\rangle < \langle e_j, k_j\rangle$.

To ensure that a single requirement $\mathcal{R}_{\langle e,k\rangle}$ is satisfied, do the following to construct the forest $G = (V, E)$. Assign the first k many even numbers $a_{i_0}, a_{i_1}, \ldots, a_{i_{k-1}}$, which have so far not been assigned to any requirement, to the highest priority requirement without an assignment. Wait for $a_{i_0}, a_{i_1}, \ldots, a_{i_{k-1}}$ to enter the range of φ_e. If we wait forever, then $\mathcal{R}_{\langle e,k\rangle}$ is satisfied trivially, since in that case φ_e fails to be a bijection. Suppose that

$$\varphi_e(\ell_0) = a_{i_0}, \qquad \varphi_e(\ell_1) = a_{i_1}, \qquad \ldots, \qquad \varphi_e(\ell_{k-1}) = a_{i_{k-1}}.$$

Next, we wait for a stage s by which φ_e has converged on all numbers in $\mathbb{N}$ which are $\leq_{\mathbb{N}} \max\{\ell_0, \ldots, \ell_{k-1}\}$. If φ_e fails to converge on any of these numbers, then $\mathcal{R}_{\langle e,k\rangle}$ is satisfied for all k, as φ_e is not total, and therefore not a bijection. Once we have found this stage s, let $c_{\langle e,k\rangle}$ be the least odd number greater than s and greater than all numbers in the range of φ_e on the domain $\mathbb{N} \upharpoonright \max\{\ell_0, \ldots, \ell_{k-1}\}$. Thus if $\varphi_e(m) = c_{\langle e,k\rangle}$, then m is greater than each of $\ell_0, \ldots, \ell_{k-1}$.

Put $c_{\langle e,k\rangle}$ into V, and make the edge connections $\bigwedge_{0\le j<k} E(c_{\langle e,k\rangle}, a_{i_j})$. With these edge connections, if φ_e is a bijection and $\varphi_e(m) = c_{\langle e,k\rangle}$, then there are $\ell_0, \ldots, \ell_{k-1}$ such that $E(c_{\langle e,k\rangle}, \ell_j)$ and $\varphi_e(\ell_j) < \varphi_e(m)$ for each $0 \le i < k$. Therefore φ_e is not a k-order.

Notice that the vertex set V that we have defined for our graph $G = (V, E)$ is computable, as V contains all the even numbers, and if an odd number c is in V, then we will know by stage c of the construction.

We can define a computable 2-order that has weak ω-type in the following way. Let $A_{\langle e,k\rangle}$ be the set of even numbers assigned to the requirement $\mathcal{R}_{\langle e,k\rangle}$. Define the weak ω-type 2-order $\le_V$ by

$$A_{\langle e_0,k_0\rangle} \le_V A_{\langle e_1,k_1\rangle} \le_V A_{\langle e_2,k_2\rangle} \le_V \cdots$$

(what essentially amounts to the natural ordering on the even numbers) with the addition of placing the odd number $c_{\langle e,k\rangle}$ as an immediate predecessor to $A_{\langle e,k\rangle}$ (that is, if we ever put the odd number $c_{\langle e,k\rangle}$ into V). □

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