Notre Dame Journal of Formal Logic
Volume 57, Number 1, 2016

Reverse Mathematics and Ramsey Properties of Partial Orderings

Jared Corduan and Marcia Groszek

Abstract A partial ordering $\mathbb{P}$ is n-Ramsey if, for every coloring of n-element chains from $\mathbb{P}$ in finitely many colors, $\mathbb{P}$ has a homogeneous subordering isomorphic to $\mathbb{P}$. In their paper on Ramsey properties of the complete binary tree, Chubb, Hirst, and McNicholl ask about Ramsey properties of other partial orderings. They also ask whether there is some Ramsey property for pairs equivalent to ACA_0 over RCA_0.

A characterization theorem for finite-level partial orderings with Ramsey properties has been proven by the second author. We show, over RCA_0, that one direction of the equivalence given by this theorem is equivalent to ACA_0 (for $n \geq 3$), and the other is provable in ATR_0.

We answer Chubb, Hirst, and McNicholl's second question by showing that there is a primitive recursive partial ordering $\mathbb{P}$ such that, over RCA_0, "$\mathbb{P}$ is 2-Ramsey" is equivalent to ACA_0.

1 Introduction

A fruitful branch of inquiry in reverse mathematics has been combinatorics, in particular, Ramsey's theorem [17] and variants thereof.

The infinitary Ramsey theorem for colorings of n-tuples, for standard $n \geq 3$, is equivalent to ACA_0 over RCA_0 (Simpson [19]). For colorings of 1-tuples, Ramsey's theorem becomes the infinite pigeonhole principle, which was shown by Hirst [11] to be equivalent to $B\Sigma^0_2$ over RCA_0. Ramsey's theorem for pairs is a more complicated case. It is strictly weaker than ACA_0 (Seetpun, see [18] and [12]), and not equivalent to any of the standard second-order systems. Some recent results in the ongoing investigation of the strength of Ramsey's theorem for pairs and of related, generally weaker, combinatorial results, can be found in Cholak, Jockusch, and Slaman [1],

Received June 11, 2012; accepted July 23, 2013
First published online September 9, 2015
2010 Mathematics Subject Classification: Primary 03B30; Secondary 05C55, 03F35
Keywords: reverse mathematics, Ramsey, partitions, partial orderings
 10.1215/00294527-3314771

Chong, Slaman, and Yang [2], Dzhafarov [6], Hirschfeldt et al. [9], Hirschfeldt and Shore [10], and Kach, Lerman, and Solomon [13].

Structural Ramsey theory is centered around the following variant: Let F be some finite structure, color substructures of $\mathfrak{A}$ isomorphic to F in finitely many colors, and look for a monochromatic substructure $\mathfrak{B}$ of a given type; to say that $\mathfrak{A}$ has a *Ramsey property* is to say that, for any coloring [in k colors], such a $\mathfrak{B}$ exists. If $\mathfrak{A}$ is countably infinite, we often look for $\mathfrak{B} \approxeq \mathfrak{A}$. Nešetřil gives some overview of this program (see [15], [16]). A classic, but still not completely understood, case is the study of edge colorings of graphs.

An example of a structural Ramsey theorem is the following: The complete binary tree $\mathbb{T}$, viewed as a partial ordering, is n-Ramsey for every natural number n. That is, if the n-element chains of $\mathbb{T}$ are colored in finitely many colors, then there is a monochromatic subset of $\mathbb{T}$ isomorphic to $\mathbb{T}$ as a partial ordering. This is a consequence of a theorem of Milliken [14]. Chubb, Hirst, and McNicholl [3] investigated the proof-theoretic strength of "$\mathbb{T}$ is n-Ramsey" in the context of reverse mathematics. In that paper, they asked about Ramsey properties of other partial orderings.

A first step toward an answer was to consider other trees. A nontrivial countable rooted tree is n-Ramsey if and only if it contains a copy of the complete binary tree. Corduan, Groszek, and Mileti [5] investigated the proof-theoretic strength of this statement.

It seemed, however, that an ideal answer to Chubb, Hirst, and McNicholl's question would be first to prove a general theorem about n-Ramsey partial orderings, and then to determine the proof-theoretic strength of that theorem. The second author has characterized the finite-level n-Ramsey partial orderings for $n \geq 3$ (see [8]). Here we investigate the proof-theoretic strength of this characterization theorem.

Before giving a more detailed outline, we give one additional definition. Partial orderings $\mathbb{P}$ and $\mathbb{Q}$ are *biembeddable* ($\mathbb{P} \rightleftarrows \mathbb{Q}$) if each contains a copy of the other. In this case, if $\mathbb{P}$ is n-Ramsey, then so is $\mathbb{Q}$. (This is because a coloring on $\mathbb{Q}$ induces a coloring on an embedded copy of $\mathbb{P}$, which must contain a monochromatic copy of $\mathbb{P}$, which then contains a monochromatic copy of $\mathbb{Q}$.) Hence n-Ramseyness can be viewed as a property of biembeddability classes.

An easy argument in RCA_0 shows that it is sufficient to consider well-founded partial orderings of height ω. Such a partial ordering is called *finite-level* if there are finitely many elements of any given height, and *bounded-level* if there is a function g such that the elements of height n are all less than $g(n)$. Given ACA_0 these notions are the same.

For the moment we will consider partial orderings with least element. This is because a 1-Ramsey bounded-level partial ordering is biembeddable with a partial ordering with least element, and for partial orderings with least element, n-Ramsey implies m-Ramsey for all $m < n$.

The characterization theorem states that, for $n \geq 3$, a bounded-level partial ordering $\mathbb{P}$ with least element is n-Ramsey if and only if

1. $\mathbb{P}$ is *weakly proto-Ramsey*; that is, $\mathbb{P}$ has certain structural properties detailed in Section 2;
2. the collection $\mathscr{G}(\mathbb{P})$ of finite bipartite graphs naturally associated with $\mathbb{P}$ is *edge-Ramsey* and has the *joint embedding property* (defined in Section 2); and

3. $\mathbb{P}$ is biembeddable with a *densely self-embeddable* weakly proto-Ramsey partial ordering ($\mathbb{P}$ is densely self-embeddable if there is a copy of $\mathbb{P}$ above every point in $\mathbb{P}$) (see [8]).

In Section 2, we show that RCA_0 suffices to prove that if $\mathbb{P}$ is a 3-Ramsey bounded-level partial ordering with least element, then $\mathbb{P}$ is weakly proto-Ramsey and $\mathcal{G}(\mathbb{P})$ satisfies the joint embedding property and is edge-Ramsey. These proofs use some tricks with colorings, and an analysis of how colorings and embeddings of $\mathbb{P}$ correspond to colorings and embeddings of graphs in $\mathcal{G}(\mathbb{P})$.

In Section 3, we show ATR_0 proves that every 2-Ramsey weakly proto-Ramsey partial ordering is biembeddable with a densely self-embeddable, weakly proto-Ramsey partial ordering. This proof is similar to the proof in ATR_0 that every binary tree with uncountably many branches has a perfect subtree (see [19]); the analogue of a complete Cantor–Bendixson decomposition provides a counterexample to 2-Ramseyness, and the analogue of a perfect subtree is the desired self-embeddable partial ordering.

In Section 4, we show that, for standard $n \geq 3$, ACA_0 is equivalent over RCA_0 to "If $\mathbb{P}$ is weakly proto-Ramsey and densely self-embeddable and $\mathcal{G}(\mathbb{P})$ is edge-Ramsey and has the joint embedding property, then $\mathbb{P}$ is n-Ramsey, which we denote $R^n(\mathbb{P})$." This is a theorem from the first author's thesis [4]. The general idea of the proof is similar to that for Ramsey's theorem [19] and n-Ramseyness of the complete binary tree [3].

As noted above, one of the most interesting results about Ramsey's theorem is that while $R^n(\omega)$ (the infinitary Ramsey's theorem for colorings of n-tuples) is equivalent to ACA_0 for standard $n \geq 3$, $R^2(\omega)$ is strictly weaker than ACA_0. Whether $R^2(\mathbb{T})$ is weaker than ACA_0 is unknown. ($R^3(\mathbb{T})$ is equivalent to $R^3(\omega)$, and hence to ACA_0 [3], but $R^1(\mathbb{T})$ is strictly stronger than $R^1(\omega)$ [5].) Chubb, Hirst, and McNicholl [3] asked whether there is a Ramsey property for pairs on some class of partial orderings equivalent to ACA_0. Corduan, Groszek, and Mileti [5] showed that there is an arithmetically definable class of trees $\mathcal{T}$ such that "every tree in $\mathcal{T}$ is 2-Ramsey" is equivalent to ACA_0, but this is for various reasons not a completely satisfying answer. In Section 5 we give a better answer by showing that there is a primitive recursive $\mathbb{P}$ such that "$\mathbb{P}$ is 2-Ramsey" is equivalent to ACA_0.

In Section 6, we mention some open questions.

2 Structural Properties from Ramsey Properties in RCA_0

In this section, we show that RCA_0 suffices to prove that if $\mathbb{P}$ is bounded-level and n-Ramsey for $n = 1, 2$, and 3, then $\mathbb{P}$ must be equivalent to a weakly proto-Ramsey partial ordering. That is, $\mathbb{P}$ has certain structural properties, which will be specified in this section.

Then we show (again in RCA_0) that a weakly proto-Ramsey partial ordering $\mathbb{P}$ determines a collection $\mathcal{G}(\mathbb{P})$ of finite bipartite graphs and that Ramsey properties of $\mathbb{P}$ imply Ramsey properties of $\mathcal{G}(\mathbb{P})$: if $\mathbb{P}$ is 2-Ramsey, then $\mathcal{G}(\mathbb{P})$ must be edge-Ramsey and have the joint embedding property.

2.1 Weak proto-Ramseyness

Definition 2.1 For a partial ordering $\mathbb{P}$ and natural numbers n and k, we say that $\mathbb{P}$ has the n,k-Ramsey property, $R^n_k(\mathbb{P})$, if for every coloring of n-element chains from $\mathbb{P}$ in k colors, there is a monochromatic subordering of $\mathbb{P}$ isomorphic to $\mathbb{P}$.

If $R^n_k(\mathbb{P})$ for all k, then $\mathbb{P}$ is n-Ramsey, $R^n(\mathbb{P})$.

Definition 2.2 If $\mathbb{P} = \langle P, \leq_{\mathbb{P}} \rangle$, then $\mathbb{P}^* = \langle P, \geq_{\mathbb{P}} \rangle$.

From the known proof that the rationals do not have the 2, 2-Ramsey property, we can derive the following fact (see [5]).

Proposition 2.3 (*RCA*$_0$) *If $\mathbb{P}$ is a countably infinite partial ordering and $R^2(\mathbb{P})$ holds, then $\mathbb{P}$ is either ω-linearizable or ω^*-linearizable.*

Proof Assume $\mathbb{P} \subseteq \omega$. Color 2-element chains from $\mathbb{P}$ according to whether or not the $<_{\mathbb{P}}$ ordering agrees with the usual ordering $<$ on ω; for $p <_{\mathbb{P}} q$,

$$c(p,q) = \begin{cases} 0 & p < q; \\ 1 & p > q. \end{cases}$$

Let $\mathbb{Q}$ be a monochromatic subordering isomorphic to $\mathbb{P}$. Suppose that $\mathbb{Q}$ is monochromatic in color 0; that is, for $p <_{\mathbb{Q}} q$ we have $p < q$. Then the enumeration of $\mathbb{Q}$ in its natural ordering shows that $\mathbb{Q}$ is ω-linearizable. If $\mathbb{Q}$ is monochromatic in color 1, then $\mathbb{Q}$ is ω^*-linearizable. Since $\mathbb{Q}$ is isomorphic to $\mathbb{P}$, this shows that $\mathbb{P}$ is either ω-linearizable or ω^*-linearizable. □

If $\mathbb{P}$ is ω^*-linearizable, then $\mathbb{P}^*$ is ω-linearizable. As $\mathbb{P}^*$ is n-Ramsey if and only if $\mathbb{P}$ is, we will restrict our attention to ω-linearizable partial orderings. We will further assume that $\mathbb{P} \subseteq \omega$ and that the natural ordering $\leq$ on ω is a linearization of $\leq_{\mathbb{P}}$; that is, $p <_{\mathbb{P}} q \implies p < q$.

Definition 2.4 If $\mathbb{P}$ is an ω-linearizable partial ordering, and $p \in \mathbb{P}$, then

$\operatorname{lev}(p) = n$ if the height of p in $\mathbb{P}$ is n,
$\mathbb{P}_n = \{p \in \mathbb{P} \mid \operatorname{lev}(p) = n\}$,
$\mathbb{P}_{<n} = \bigcup_{m<n} \mathbb{P}_m$,
$\operatorname{Pred}_m(p) = \{q \in \mathbb{P}_m \mid q \leq_{\mathbb{P}} p\}$.

If p belongs to more than one relevant partial ordering and there is some possibility of ambiguity, then for $\operatorname{Pred}_n(p)$ we may write $\operatorname{Pred}^{\mathbb{P}}_n(p)$.

Definition 2.5 $\mathbb{P}$ is *finite-level* if each $\mathbb{P}_n$ is finite, and *bounded-level* if there is a bounding function g such that $\mathbb{P}_n \subseteq \{i \mid i < g(n)\}$.

Given ACA_0, all finite-level partial orderings are bounded-level. In RCA_0 it makes sense to restrict our attention to bounded-level partial orderings. In particular, if $\mathbb{P}$ is bounded-level and infinite, then $\mathbb{P}$ has infinite height; if $\varphi : \mathbb{P} \to \mathbb{P}$ is an embedding, then the range of φ is a set.

Proposition 2.6 (*RCA*$_0$) *If $\mathbb{P}$ is an infinite, bounded-level partial ordering with least element and $k < n$, then $R^n(\mathbb{P}) \implies R^k(\mathbb{P})$.*

Proof Assume $R^n(\mathbb{P})$, and let c be a coloring of k-element chains of $\mathbb{P}$. First color n-element chains according to the color of their final k elements. Let $\mathbb{Q}$ be a monochromatic subordering isomorphic to $\mathbb{P}$. Now color n-element chains according to whether their least element is above level $n-k$ of $\mathbb{Q}$. A subordering monochromatic for this coloring is also monochromatic for c.

The assumption that $\mathbb{P}$ has a least element and is infinite ensures that a subordering monochromatic for the second embedding is entirely contained above level $n-k$, as the minimal element is the least element of some n-element chain. For example, a disconnected partial ordering consisting of a copy of ω plus a discrete set of incomparable minimal elements is 2-Ramsey but not 1-Ramsey, simply because all 2-element chains are contained in the copy of ω: assuming a minimal element (and infinite height) rules out similar trivial counterexamples. $\square$

Proposition 2.7 (*RCA*$_0$) *Suppose that $\mathbb{P}$ is an infinite, bounded-level partial ordering, and $R^1(\mathbb{P})$ holds. Then $\mathbb{P}$ is biembeddable with a bounded-level partial ordering $\mathbb{Q}$ with a least element and an infinite chain.*

Proof By assumption, $\mathbb{P}$ has finitely many minimal points. Color the points of $\mathbb{P}$ according to the least (in the sense of the ordering on ω) minimal point below them. By $R^1(\mathbb{P})$ there are a minimal point p_0 and an embedding φ of $\mathbb{P}$ into $\{p \mid p \geq_{\mathbb{P}} p_0\}$. Let $\mathbb{Q} = \text{range}(\varphi) \cup \{p_0\}$. Then $\mathbb{P} \rightleftarrows \mathbb{Q}$.

To show that $\mathbb{Q}$ has an infinite chain, note that $\mathbb{Q} \rightleftarrows \mathbb{P}$, and so $R^1(\mathbb{Q})$ holds. By coloring the least element p_0 of $\mathbb{Q}$ red and all other elements blue, we get from $R^1(\mathbb{Q})$ an embedding $\varphi : \mathbb{Q} \to \mathbb{Q}$ such that $p_0 <_{\mathbb{P}} \varphi(p_0)$. Then the downward closure of $\{\varphi(p_0), \varphi(\varphi(p_0)), \dots, \varphi^n(p_0), \dots\}$ is an infinite chain in $\mathbb{Q}$. (This chain is a set in the model, since p is on the chain if, for some n, $p \leq_{\mathbb{P}} \varphi^n(p_0)$, and p is not on the chain if, for some n, $\text{lev}(\varphi^n(p_0)) > \text{lev}(p)$ and $p \not\leq_{\mathbb{P}} \varphi^n(p_0)$.) $\square$

In light of this proposition, we can restrict our attention to partial orderings that have a least element. Without loss of generality, this least element is 0.

So far we have justified restricting our attention to ω-linearizable partial orderings with least element. Now, assuming that $\mathbb{P}$ is bounded-level and 3-Ramsey, we develop some deeper structural properties.

Definition 2.8 Points p and q in $\mathbb{P}$ are *compatible* if they have a common upper bound r (that is, $p \leq_{\mathbb{P}} r$ and $q \leq_{\mathbb{P}} r$), and *incompatible* otherwise. If p and q are incompatible, then we write $p \perp q$.

Lemma 2.9 (*RCA*$_0$) *Suppose that $\mathbb{P}$ is a bounded-level partial ordering with least element and $R^3(\mathbb{P})$ holds. Then, for any p in $\mathbb{P}$, there is an $n \in \omega$ such that if $p_0 <_{\mathbb{P}} p_1 <_{\mathbb{P}} \cdots <_{\mathbb{P}} p_n$ is any increasing chain, either $p \leq_{\mathbb{P}} p_n$ or $p \perp p_n$.*

Proof Loosely, moving up a chain $p_0 <_{\mathbb{P}} p_1 <_{\mathbb{P}} p_2 \cdots$ adds new elements in the cone below p_i at each step; we use 3-Ramseyness to show that we may assume the least new element added below p_{i+1} is greater (in the usual ordering on ω) than the least new element added below p_i. Then (viewing p as both an element of ω and an element of $\mathbb{P}$) the least new element added below p_{p+2} must be larger than p, and so p is either below or incomparable to p_n for $n \geq p+2$.

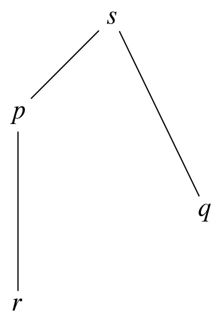
In more detail, color 3-chains $p <_{\mathbb{P}} q <_{\mathbb{P}} r$ as follows: For $q <_{\mathbb{P}} r$, let $i(q,r)$ be the least i (in the sense of the usual ordering $<$ on ω) such that $i \not\leq_{\mathbb{P}} q$ but $i \leq_{\mathbb{P}} r$. Color the chain $\langle p,q,r \rangle$ color 0 if $i(p,q) < i(q,r)$ and color 1 if $i(p,q) > i(q,r)$.

Let φ be an isomorphism of $\mathbb{P}$ onto a monochromatic isomorphic subordering $\mathbb{Q}$. As $\mathbb{Q}$ must have an infinite increasing chain $q_0 <_{\mathbb{P}} q_1 <_{\mathbb{P}} \cdots$, $\mathbb{Q}$ must be monochromatic in color 0; we must have $i(q_k, q_{k+1}) < i(q_{k+1}, q_{k+2})$ rather than the reverse.

Now let $q = \varphi(p)$ and $n = q+2$. Suppose toward a contradiction that $p \not\leq_{\mathbb{P}} p_n$ and $p \not\perp p_n$ for some increasing chain $p_0 <_{\mathbb{P}} p_1 <_{\mathbb{P}} \cdots <_{\mathbb{P}} p_n$. Let $q_k = \varphi(p_k)$

for $k \leq n$. Then $q \not\leq_{\mathbb{P}} q_n$ and $q \not\perp q_n$ in $\mathbb{Q}$. Because the function i increases along chains in $\mathbb{Q}$, $i(q_{n-1}, q_n) \geq n-1 > q$. Since $q \not\perp q_n$, let r be a common upper bound in $\mathbb{Q}$. Then as $q \not\leq_{\mathbb{P}} q_n$ but $q \leq_{\mathbb{P}} r$, we have $i(q_n, r) \leq q < i(q_{n-1}, q_n)$. This means the chain $\langle q_{n-1}, q_n, r\rangle$ has color 1, a contradiction. $\square$

Proposition 2.10 (*RCA*$_0$) *Suppose that $\mathbb{P}$ is a bounded-level partial ordering with least element, and suppose that $R^3(\mathbb{P})$ (hence also $R^2(\mathbb{P})$) holds. Then the following forbidden configuration does not occur in $\mathbb{P}$: points p and q that are incomparable but compatible with common successor s, and a point r that is below p but not below q.*



Proof By Lemma 2.9, for every p in $\mathbb{P}$ there is a level $h(p)$ such that any point on level $h(p)$ or above must be either above or incompatible with p; furthermore, such a function h can be computed from $\mathbb{P}$, a bounding function for $\mathbb{P}$, and the embedding φ in the proof of Lemma 2.9. Because $\mathbb{P}$ is bounded-level, we can define a function f as follows: $f(0) = 0$; given $f(n)$, choose $f(n+1) > f(n)$ so that, for any point p on or below level $f(n)$, we have $h(p) < f(n+1)$. That is, any point on or below level $f(n)$ is either below or incompatible with any point on or above level $f(n+1)$.

Color chains $p <_{\mathbb{P}} q$ as follows: Let n be least such that p is below level $f(n)$. Then the chain $\langle p, q\rangle$ has color 0 if q is above level $f(n+2)$, and color 1 otherwise.

Let $\mathbb{Q}$ be an isomorphic monochromatic subordering. Because $\mathbb{Q}$ has infinite height, it must be monochromatic in color 0. Suppose toward a contradiction that p, q, s, r realize the forbidden configuration in $\mathbb{Q}$, and let n be least such that the level of r is below $f(n)$. By definition of f, as r and q are incomparable but compatible, the level of q is below $f(n+1)$, and as p and q are incomparable but compatible, the level of p is below level $f(n+2)$. But this means that the chain $\langle r, p\rangle$ has color 1, a contradiction. $\square$

This forbidden configuration, when augmented by a bottom point, becomes a copy of the pentagon lattice $N5$.

Definition 2.11 A partial ordering with least element in which this forbidden configuration does not occur is called *$N5$-omitting*.

Definition 2.12 The partial ordering $\mathbb{P}$ is *densely self-embeddable* if, for every $p \in \mathbb{P}$, there is an embedding φ of $\mathbb{P}$ into $\{q \mid p \leq_{\mathbb{P}} q\}$.

Definition 2.13 A partial ordering $\mathbb{P}$ is *proto-Ramsey* if $\mathbb{P}$ is ω-linearizable, $N5$-omitting, and densely self-embeddable, and has a least element.

A partial ordering $\mathbb{P}$ is *weakly proto-Ramsey* if $\mathbb{P}$ is ω-linearizable and $N5$-omitting and has a least element (but is not necessarily densely self-embeddable).

Theorem 2.14 (*RCA_0*) *Let n be any standard number with $n \geq 3$.*

If a bounded-level partial ordering with least element is n-Ramsey, then it is biembeddable with a weakly proto-Ramsey, bounded-level partial ordering.

Proof This follows from Propositions 2.6, 2.7, and 2.10. □

Noting some structural properties of weakly proto-Ramsey partial orderings will be useful.

Proposition 2.15 (*RCA_0*) *Suppose that $\mathbb{P}$ is a weakly proto-Ramsey partial ordering. Since the forbidden configuration consists exactly of incomparable but compatible points p and q such that p has a predecessor not shared by q, any two incomparable but compatible points of $\mathbb{P}$ must have exactly the same predecessors.*

It follows that if p and q are incomparable but compatible, then p and q are on the same level. Also, p and q have a common successor on the next level; otherwise, they have compatible but different successors on the next level, which are compatible but incomparable points with different predecessors, realizing the forbidden configuration.

It follows that predecessors of p on different levels, being compatible and not on the same level, must be comparable. Thus the predecessors of p are in a sense almost linearly ordered: if $q <_{\mathbb{P}} p$ has level n, and $r <_{\mathbb{P}} p$ has level $m < n$, then $r <_{\mathbb{P}} q$.

This means that if p is above level n, and $p \leq_{\mathbb{P}} q$, then p and q have the same predecessors on level n.

It also follows that if p is above level n, then $\mathrm{Pred}_n(p) = \mathrm{Pred}_n(q)$ for some $q \in \mathbb{P}_{n+1}$; in fact, this holds for any $q <_{\mathbb{P}}$ in $\mathbb{P}_{n+1}$.

Also, the ordering $<_{\mathbb{P}}$ is the transitive closure of $\{\langle p, q\rangle \mid p <_{\mathbb{P}} q \;\&\; \mathrm{lev}(q) = \mathrm{lev}(p) + 1\}$: if $r <_{\mathbb{P}} p$ is on level n, p is above level $n + 1$, and $q <_{\mathbb{P}} p$ is on level $n + 1$, then since $\mathrm{Pred}_n(p) = \mathrm{Pred}_n(q)$, we have $r <_{\mathbb{P}} q <_{\mathbb{P}} p$; inductively, we can build a chain from r to p moving up one level at each step.

In general, the structure and complexity of a weakly proto-Ramsey partial ordering are completely determined by the ordering between successive levels. The general picture of a weakly proto-Ramsey partial ordering is as follows: Put a single point on level 0. Once level n is defined, place points on level $n + 1$; for each p on level $n + 1$, choose as predecessors of p finitely many (but not zero) points on level n, all of which must have the same predecessors on level $n - 1$. Having defined each level and the connections between adjacent levels, take the transitive closure.

2.2 Bipartite graphs associated with $\mathbb{P}$ Suppose that $\mathbb{P}$ is weakly proto-Ramsey. For any n, we create a bipartite graph by taking level n of $\mathbb{P}$ and level $n + 1$ of $\mathbb{P}$ as the two sets of vertices and putting an edge between $\leq_{\mathbb{P}}$-comparable elements. We then take a quotient, by identifying elements on level $n + 1$ if they have the same predecessors on level n. The connected components of this graph are in a sense the basic building blocks of $\mathbb{P}$; in fact, for proto-Ramsey bounded-level $\mathbb{P}$ (with the exception of some trivial, or at least uncomplicated, cases), these building blocks determine $\mathbb{P}$ up to biembeddability. Furthermore, Ramsey properties of $\mathbb{P}$ correspond to Ramsey properties of the associated collection of graphs. In this section we make this statement precise.

Definition 2.16 A bipartite graph $G = \langle M(G), S(G), E(G)\rangle$ consists of disjoint sets of vertices $M(G)$ and $S(G)$ and a set of edges $E(G) \subseteq M(G) \times S(G)$.

Our bipartite graphs have distinguished parts, so an embedding of G into H must send $M(G)$ to $M(H)$ and $S(G)$ to $S(H)$.

Definition 2.17 For bipartite graphs G and H, if H contains a copy of G we write $G \hookrightarrow H$.

If $\mathscr{G}$ and $\mathscr{H}$ are collections of bipartite graphs, and for every $G \in \mathscr{G}$ there is $H \in \mathscr{H}$ such that $G \hookrightarrow H$, then we write $\mathscr{G} \hookrightarrow \mathscr{H}$.

If $\mathscr{G} \hookrightarrow \mathscr{H}$ and $\mathscr{H} \hookrightarrow \mathscr{G}$, then $\mathscr{G} \rightleftarrows \mathscr{H}$.

Definition 2.18 If G and H are bipartite graphs and $k \in \omega$, then

$$G \to (H)^e_k$$

if, for every coloring of the edges of G in k colors, G contains a monochromatic copy of H.

Definition 2.19 A collection of bipartite graphs $\mathscr{G}$ is *edge-Ramsey*, if for every $H \in \mathscr{G}$ and every $k \in \omega$, there is $G \in \mathscr{G}$ such that $G \to (H)^e_k$.

A collection of bipartite graphs $\mathscr{G}$ has the *joint embedding property* if, for every $G, H \in \mathscr{G}$, there is some $K \in \mathscr{G}$ such that $G \hookrightarrow K$ and $H \hookrightarrow K$.

Edge-Ramseyness of collections of finite graphs has been much studied in its own right.

Throughout the remainder of this section, $\mathbb{P}$ will denote a bounded-level weakly proto-Ramsey partial ordering. From $\mathbb{P}$ we define a collection $\mathscr{G}(\mathbb{P})$ of bipartite graphs. We will show that if $\mathbb{P}$ is 2-Ramsey, then $\mathscr{G}(\mathbb{P})$ is edge-Ramsey and has the joint embedding property.

Definition 2.20 Define $\equiv$ to be the transitive closure of the compatibility relation on $\mathbb{P}_n$ (where p and q are compatible if they have a common successor); denote the equivalence class of p by $[p]$.

Because $\mathbb{P}$ is $N5$-omitting, equivalent points must have the same predecessors. By definition, each set $s = \mathrm{Pred}_n(p)$ is contained within a single $\equiv$-class.

Definition 2.21 For each $\equiv$-class $a = [p]$ for $p \in \mathbb{P}_n$, define a connected bipartite graph G_a. Set $M(G_a) = a$ and $S(G_a) = \{s \subseteq a \mid (\exists q)\,(\mathrm{Pred}_n(q) = s)\}$. The edge relation of G_a is membership.

If p belongs to more than one relevant partial ordering and there is some possibility of ambiguity, then for $G_{[p]}$ we may write $G^{\mathbb{P}}_{[p]}$.

Note that the function taking p to $G_{[p]}$ can be computed from $\mathbb{P}$ and a bounding function for $\mathbb{P}$. This is because each $\mathrm{Pred}_n(q)$ is equal to $\mathrm{Pred}_n(r)$ for some $r \in \mathbb{P}_{n+1}$.

Definition 2.22 $\mathscr{G}(\mathbb{P}) = \{G_{[p]} \mid p \in \mathbb{P}\}$.

We say that $\mathbb{P}$ is *edge-Ramsey*, or *j.e.p.*, just in case $\mathscr{G}(\mathbb{P})$ is.

Now we show that if $\mathbb{P}$ is 2-Ramsey, then $\mathscr{G}(\mathbb{P})$ is edge-Ramsey and has the joint embedding property. We do this by associating colorings of the edges of the graphs $G_{[p]}$ with colorings of pairs in $\mathbb{P}$, and using a monochromatic embedded copy of $\mathbb{P}$ to produce monochromatic subgraphs.

Definition 2.23 If $\mathscr{G}$ is a collection of bipartite graphs, an edge coloring of $\mathscr{G}$ in k colors is a collection $\{c_G : E(G) \to \{0, 1, \ldots, k-1\} \mid G \in \mathscr{G}\}$.

In the above definition, $\mathcal{G}$ must be understood as a collection of individual graphs rather than isomorphism types. Isomorphic elements of $\mathcal{G}$ may be colored differently.

Definition 2.24 If c is an edge coloring of $\mathcal{G}(\mathbb{P})$, then c induces a coloring $\overline{c}$ of 2-element chains in $\mathbb{P}$ by

$$\overline{c}(p_1, p_2) = c_{G_{[p_1]}}\big(p_1, \mathrm{Pred}_{\mathrm{lev}(p_1)}(p_2)\big).$$

Such a coloring is called *graph-induced*.

If c is an edge coloring of $\mathcal{G}(\mathbb{P})$ such that, for $\langle p, s\rangle \in E(G_a)$, the color $c_{G_a}(p, s)$ depends only on p, and we denote that color by $c_{G_a}(p)$, then c induces a coloring $\overline{c}$ of 1-tuples in $\mathbb{P}$ by $\overline{c}(p) = c_{G_{[p]}}(p)$.

Remark 2.25 Any coloring $\overline{c}$ of 2-element chains in $\mathbb{P}$, such that $\overline{c}(p, q)$ depends only on p and $\mathrm{Pred}_{\mathrm{lev}(p)}(q)$, is graph-induced, and $\overline{c}$ determines the corresponding edge coloring c of $\mathcal{G}(\mathbb{P})$ by $c_{G_a}(p, s) = \overline{c}(p, q)$, where $\mathrm{Pred}_{\mathrm{lev}(p)}(q) = s$; by assumption, this depends only on s and not on the choice of q.

Proposition 2.26 (*RCA*$_0$) *An embedding of bounded-level, weakly proto-Ramsey partial orderings $\varphi : \mathbb{Q} \to \mathbb{P}$ induces a (not necessarily unique) embedding of $G_{[p]}$ into $G_{[\varphi(p)]}$. If the range of φ is monochromatic in color i for some coloring $\overline{c}$ induced by an edge coloring c of $\mathcal{G}(\mathbb{P})$, then the image of $G_{[p]}$ is a subgraph of $G_{[\varphi(p)]}$ monochromatic in color i for $c_{G_{[\varphi(p)]}}$. Conversely, if, for some i, each image of each $G_{[p]}$ is monochromatic for $c_{G_{[\varphi(p)]}}$ in color i, then the range of φ is monochromatic for $\overline{c}$ in color i.*

Proof Define $k : G_{[p]} \to G_{[\varphi(p)]}$ as follows: For $q \in M(G_{[p]}) = [p]$, let $k(q) = \varphi(q)$. For $s \in S(G_{[p]})$, choose r such that $s = \mathrm{Pred}_{\mathrm{lev}(p)}(r)$, and let $k(s) = \mathrm{Pred}_{\mathrm{lev}(\varphi(p))}(\varphi(r))$. This function k is an embedding: q is connected to s iff $q \in s = \mathrm{Pred}_{\mathrm{lev}(p)}(r)$ iff $q <_{\mathbb{Q}} r$ iff $\varphi(q) <_{\mathbb{P}} \varphi(r)$ iff $\varphi(q) \in \mathrm{Pred}_{\mathrm{lev}(\varphi(p))}(\varphi(r))$, iff $k(q) \in k(s)$.

(This embedding is not necessarily unique; if also $s = \mathrm{Pred}_{\mathrm{lev}(p)}(r')$, and $r \perp r'$, we can have $\mathrm{Pred}_{\mathrm{lev}(\varphi(p))}(\varphi(r)) \neq \mathrm{Pred}_{\mathrm{lev}(\varphi(p))}(\varphi(r'))$, as long as their difference lies outside the range of φ.)

The edge in $G_{[\varphi(p)]}$ between $k(p) = \varphi(p)$ and $k(s) = \mathrm{Pred}_{\mathrm{lev}(\varphi(p))}(\varphi(r))$ is assigned color $c_{G_{[\varphi(p)]}}(\varphi(p), \mathrm{Pred}_{\mathrm{lev}(\varphi(p))}(\varphi(r))) = \overline{c}(\varphi(p), \varphi(r))$, so if the range of φ is monochromatic (for $\overline{c}$) in color i, so is the range of k (for $c_{G_{[\varphi(p)]}}$). Similarly, if the range of k is monochromatic in color i for all k, then the range of φ is monochromatic in color i. $\square$

Theorem 2.27 (*RCA*$_0$) *Let $\mathbb{P}$ be a weakly proto-Ramsey partial ordering. If $\mathbb{P}$ is 2-Ramsey, then $\mathcal{G}(\mathbb{P})$ is edge-Ramsey and has the joint embedding property.*

Proof If $\mathcal{G}(\mathbb{P})$ fails to have the joint embedding property, choose $G_{[p]}$ and $G_{[q]}$ such that no element of $\mathcal{G}(\mathbb{P})$ contains copies of both $G_{[p]}$ and $G_{[q]}$. Edge color $\mathcal{G}(\mathbb{P})$ by coloring all edges of G color 0 if $G_{[p]} \hookrightarrow G$, and color 1 otherwise, which induces a coloring $\overline{c}$ of pairs in $\mathbb{P}$. Any embedding φ of $\mathbb{P}$ into itself must send $G_{[p]}$ to a graph with color 0 edges, and $G_{[q]}$ to a graph with color 1 edges, and so the range of φ cannot be monochromatic. Hence $R^2(\mathbb{P})$ fails.

If $\mathcal{G}(\mathbb{P})$ fails to be edge-Ramsey, choose a counterexample $G_{[p]}$ and k such that, for each $G \in \mathcal{G}(\mathbb{P})$, there is a coloring c_G of the edges of G in k colors such that G contains no monochromatic copy of $G_{[p]}$. The collection of c_G is an edge coloring of $\mathcal{G}(\mathbb{P})$, which induces a coloring $\overline{c}$ of pairs in $\mathbb{P}$. Any embedding φ of $\mathbb{P}$ into itself must send $G_{[p]}$ to a graph that is not monochromatic, and so the range of φ cannot be monochromatic. Hence $R^2(\mathbb{P})$ fails. □

Definition 2.28 Points p and q in $\mathbb{P}$ are strongly incompatible, written $p \perp_{st} q$, if, for all $p' \equiv p$ and $q' \equiv q$, we have $p' \perp q'$.

Remark 2.29 Equivalently, if $\mathrm{lev}(p) \leq \mathrm{lev}(q)$ and $r \leq_{\mathbb{P}} q$ has the same level as p, then $p \perp_{st} q$ if and only if $p \not\equiv r$.

If $p \perp q$, $p <_{\mathbb{P}} p'$ and $q <_{\mathbb{P}} q'$, then $p' \perp_{st} q'$.

Proposition 2.30 (*RCA*$_0$) *If $\mathbb{P}$ and $\mathbb{Q}$ are bounded-level weakly proto-Ramsey partial orderings, then there are incompatible points above every point in $\mathbb{P}$, and every $G \in \mathcal{G}(\mathbb{Q})$ is realized above every point $p \in \mathbb{P}$ (i.e., there is $q \geq_{\mathbb{P}} p$ such that $G \hookrightarrow G_{[q]}$), then $\mathbb{Q} \hookrightarrow \mathbb{P}$.*

Furthermore, $\mathbb{Q}$ can be embedded above any point in $\mathbb{P}$.

The embedding technique in the proof of this proposition will be used again in the proof of Theorem 4.2. The proposition itself is used in the proof of Theorem 3.1.

Defining an embedding φ by recursion on levels is fairly straightforward in the case that $\mathbb{Q}$ and $\mathbb{P}$ are trees; this is the case in which $\mathcal{G}(\mathbb{Q}) = \mathcal{G}(\mathbb{P})$ contains only the graph consisting of a single edge. If level n of $\mathbb{Q}$ is embedded in $\mathbb{P}$ in any way at all, then we can extend the embedding to level $n+1$ by sending the (necessarily incompatible) immediate successors of σ to incompatible successors of $\varphi(\sigma)$, which by assumption on $\mathbb{P}$ must have infinitely many incompatible successors. That is, extending the embedding from one level to the next is merely a local problem.

In the general case, if we have embedded points p, q, and r of $\mathbb{Q}_n$, and now wish to embed a point s that is above p and q but not r, then our embedding on level n must be such that there is some point in $\mathbb{P}$ above $\varphi(p)$ and $\varphi(q)$ but not $\varphi(r)$; that is, when embedding level n we must look ahead. However, because the structure of the partial ordering is determined by the ordering between adjacent levels, this is still a local problem; it merely requires looking ahead one level. We do this by using the graphs in $\mathcal{G}(\mathbb{Q})$, whose edges capture the ordering between levels. We know by Proposition 2.26 that if φ is an embedding, then it must induce an embedding of $G_{[p]}$ into $G_{[\varphi(p)]}$, and we will see that choosing such an embedding as we define φ on $\mathbb{Q}_n$ suffices to guarantee that we can extend φ to the next level.

Definition 2.31 An embedding φ from $\mathbb{Q}_{\leq n}$ into $\mathbb{P}$ is *extendible* if it satisfies the following properties:

(1) If p and q are in $\mathbb{Q}_n$ and $p \not\equiv q$, then $\varphi(p) \perp_{st} \varphi(q)$.
(2) If p and q are in $\mathbb{Q}_n$ and $p \equiv q$, then $\varphi(p) \equiv \varphi(q)$.
(3) For each equivalence class $a = [p] \subseteq \mathbb{Q}_n$, there is an embedding $j_a : G_a^{\mathbb{Q}} \to G_b^{\mathbb{P}}$ such that, for $q \in a$, $j_a(q) = \varphi(q)$. This implies $b = [\varphi(p)]$. (It also implies condition (2), but, in practice, we will first guarantee condition (2) and then act to satisfy condition (3).)

The key property of $\mathbb{P}$, that every $G_a^{\mathbb{Q}}$ is realized above every point in $\mathbb{P}$, guarantees that an extendible embedding of $\mathbb{Q}_{\leq n}$ can be extended to an extendible embedding of $\mathbb{Q}_{\leq n+1}$, which allows us to recursively embed $\mathbb{Q}$ into $\mathbb{P}$.

Proof of Proposition 2.30 Choose any $p \in \mathbb{P}$. We will recursively define an embedding $\varphi : \mathbb{Q} \to \mathbb{P}$ with range above p as the limit of extendible embeddings of $\mathbb{Q}_n$.

For the least element $0 \in \mathbb{Q}$, the graph $G_{[0]}$ consists of a single edge $(0, \{0\})$. By assumption, we can choose $q \geq_{\mathbb{P}} p$ and an embedding $j : G_{[0]} \to G_{[q]}$. Choose $\varphi(0) = j(0)$, and set $j_{[0]} = j$.

Now, suppose we have defined an embedding φ of $\mathbb{Q}_{\leq n}$ that is extendible on $\mathbb{Q}_{\leq m}$ for all $m \leq n$. We must show that we can extend j to level $n+1$ preserving extendibility.

As every point of $\mathbb{P}$ has incompatible extensions, we can find above any point in $\mathbb{P}$ infinitely many incompatible extensions, and by further extending each, infinitely many strongly incompatible extensions. We will use this to find strongly incompatible points above which to embed the graphs G_d for equivalence classes $d \subseteq \mathbb{Q}_{n+1}$.

For any equivalence class $d \subseteq \mathbb{Q}_{n+1}$, suppose $p \in d$, $s = \mathrm{Pred}_n(p)$, $q \in s$, $a = [q]$. (By the weakly proto-Ramsey structure of $\mathbb{P}$, as in Proposition 2.15, d uniquely determines s and a.) We have $j_a : G_a \to G_b$ where $b \subseteq \mathbb{P}_{\overline{n}}$, and φ maps a into b. Let $j_a(s) = \mathrm{Pred}_{\overline{n}}(r)$, and choose $p_d >_{\mathbb{P}} r$. Since $r <_{\mathbb{P}} p_d$ and $\overline{n} < \mathrm{lev}(r)$, we have $\mathrm{Pred}_{\overline{n}}(p_d) = \mathrm{Pred}_{\overline{n}}(r) = j_a(s)$.

By further extending the p_d if necessary, guarantee that if $d \neq \overline{d}$, then $p_d \perp_{st} p_{\overline{d}}$.

If we define φ on $\mathbb{Q}_{n+1}$ to embed each d into an equivalence class above p_d, then (1) and (2) of the definition of extendible embedding will be satisfied. We will show that φ is still an embedding. Note that, for $p \in d$, since $p_d \leq_{\mathbb{P}} \varphi(p)$, we have $\mathrm{Pred}_{\overline{n}}(\varphi(p)) = \mathrm{Pred}_{\overline{n}}(p_d) = j_a(s)$.

Suppose as above that $p \in \mathbb{P}_{n+1}$, $[p] = d$, $\mathrm{Pred}_n(p) = s \in G_a$, and $a = [q]$. First, consider $\overline{q} \in \mathbb{P}_n$.

If $\overline{q} <_{\mathbb{P}} p$, then $\overline{q} \in a$ and $\overline{q} \in s$. Therefore, since j_a is a graph embedding, $\varphi(\overline{q}) = j_a(\overline{q}) \in j_a(s) = \mathrm{Pred}_{\overline{n}}(\varphi(p))$, so $\varphi(\overline{q}) <_{\mathbb{P}} \varphi(p)$.

If $\overline{q} \in a$ but $\overline{q} \not<_{\mathbb{P}} p$, then $\overline{q} \notin s$; so, in the same way, $\varphi(\overline{q}) \not<_{\mathbb{P}} \varphi(p)$.

If $\overline{q} \notin a$ (in which case $\overline{q} \not<_{\mathbb{Q}} p$), then $\overline{q} \not\equiv q$, so $\varphi(\overline{q}) \perp_{st} \varphi(q)$. We have seen that $\varphi(q) <_{\mathbb{P}} \varphi(p)$, so we must have that $\varphi(\overline{q}) \not<_{\mathbb{P}} \varphi(p)$.

Now, suppose $\overline{q} \in \mathbb{Q}_m$ for $m < n$, and choose $\overline{r} \in \mathbb{Q}_m$ with $\overline{r} <_{\mathbb{Q}} q$, so $\varphi(\overline{r}) <_{\mathbb{P}} \varphi(q) <_{\mathbb{P}} \varphi(p)$. If $\overline{r} \not\equiv \overline{q}$ (in which case $\overline{q} \not<_{\mathbb{Q}} p$), then $\varphi(\overline{r}) \perp_{st} \varphi(\overline{q})$, and so $\varphi(\overline{q}) \not<_{\mathbb{P}} \varphi(p)$.

If $\overline{r} \equiv \overline{q}$, then $\varphi(\overline{r})$ and $\varphi(\overline{q})$ are on the same level of $\mathbb{P}$ below the level of $\varphi(q)$; so by the structure of $\mathbb{P}$ we have $\varphi(\overline{q}) <_{\mathbb{P}} \varphi(p)$ iff $\varphi(\overline{q}) <_{\mathbb{P}} \varphi(q)$; because φ is an embedding on $\mathbb{Q}_{\leq n}$ we have $\varphi(\overline{q}) <_{\mathbb{P}} \varphi(q)$ iff $\overline{q} <_{\mathbb{P}} q$; and by the structure of $\mathbb{Q}$ we have $\overline{q} <_{\mathbb{P}} q$ iff $\overline{q} <_{\mathbb{P}} p$.

In any case, $\overline{q} <_{\mathbb{Q}} p$ iff $\varphi(\overline{q}) <_{\mathbb{P}} \varphi(p)$. It remains only to show that we can find a suitable graph embedding j_d into a graph above p_d.

Because every graph $G^{\mathbb{Q}}_d$ is realized above every point of $\mathbb{P}$, there is some $p \geq p_d$ with an embedding $j_d : G_d \to G_{[p]}$. Choose such p and j_d, and for $q \in d$, define $\varphi(q) = j_d(q)$. Do this for every equivalence class $d \subseteq \mathbb{Q}_{n+1}$; this extends φ to be an extendible embedding of $\mathbb{Q}_{n+1}$ and completes the proof. □

Notice that the embeddings j_a used in this construction are exactly the embeddings of G_a induced by φ as in Proposition 2.26. This is because if $\mathrm{Pred}_n(p) = s \subseteq a$, we always choose $\varphi(p)$ so that $\mathrm{Pred}_{\overline{n}}(\varphi(p)) = j_a(s)$.

Corollary 2.32 *If $\mathbb{P}$ and $\mathbb{Q}$ are bounded-level proto-Ramsey partial orderings and have incompatible points, and $\mathcal{G}(\mathbb{P}) \rightleftarrows \mathcal{G}(\mathbb{Q})$, then $\mathbb{P} \rightleftarrows \mathbb{Q}$.*

Proof Since $\mathbb{P}$ is proto-Ramsey and has incompatible points, there are incompatible points above every $p \in \mathbb{P}$: Suppose that p_0 and p_1 are incompatible. Because $\mathbb{P}$ is densely self-embeddable, there are $q_0 >_{\mathbb{P}} p_0$ and $q_1 >_{\mathbb{P}} p_1$ in $\mathbb{P}$, and (by incompatibility of p_0 and p_1) no other relations hold. Now, because $\mathbb{P}$ is densely self-embeddable, above *every* $p \in \mathbb{P}$, there are $\overline{q}_0 >_{\mathbb{P}} \overline{p}_0$ and $\overline{q}_1 >_{\mathbb{P}} \overline{p}_1$ such that no other relations hold. Since $\overline{q}_0$ and $\overline{q}_1$ are incomparable points with different predecessors, and $\mathbb{P}$ is $N5$-omitting, $\overline{q}_0$ and $\overline{q}_1$ are incompatible.

Now Proposition 2.30 shows that $\mathbb{Q} \hookrightarrow \mathbb{P}$, and the same argument shows $\mathbb{P} \hookrightarrow \mathbb{Q}$. $\square$

The assumption of incompatible points is necessary in both Proposition 2.30 and Corollary 2.32. (We can drop this assumption in Proposition 2.30 if $\mathbb{Q}$ has no incompatible points, and Corollary 2.32 also holds if neither $\mathbb{P}$ nor $\mathbb{Q}$ has any incompatible points.) We will use these results in later sections; therefore, we will need to understand the exceptional cases.

Suppose that $\mathbb{P}$ is bounded-level and weakly proto-Ramsey. If there is any $G_{[p]} \in \mathcal{G}(\mathbb{P})$ such that $S(G_{[p]})$ has more than one element, then necessarily there are incompatible points in $\mathbb{P}$: We know that $S(G_{[p]})$ contains at least two elements, s and t. If $\operatorname{lev}(p) = n$, then we can set $s = \operatorname{Pred}_n(p_0)$ and $t = \operatorname{Pred}_n(p_1)$ for $p_0, p_1 \in \mathbb{P}_{n+1}$. Then p_0 and p_1 are incomparable points with different predecessors, and hence incompatible.

Thus, by Proposition 2.26 and Corollary 2.32, for proto-Ramsey $\mathbb{P}$ and $\mathbb{Q}$ that fall into this case, $\mathbb{P} \rightleftarrows \mathbb{Q}$ iff $\mathcal{G}(\mathbb{P}) \rightleftarrows \mathcal{G}(\mathbb{Q})$.

If, on the other hand, for every $G_{[p]} \in \mathcal{G}(\mathbb{P})$ there is only one element of $S(G_{[p]})$, then by Proposition 2.30, $\mathbb{P}$ must be isomorphic to a subordering of the partial ordering $\mathbb{S}$ defined below.

Definition 2.33

$$\mathbb{S} = \{(\sigma, n) \mid \sigma \in 2^{<\omega} \ \&\ n \leq \operatorname{length}(\sigma)\},$$
$$(\sigma, n) \leq_{\mathbb{S}} (\tau, m) \iff \sigma \subseteq \tau.$$

Essentially, $\mathbb{S}$ is obtained from the full binary tree by replacing each point on level n with $n+1$ many points. Proposition 2.30 applies because any finite bipartite G with only one element of $S(G)$ is realized above every point in $\mathbb{S}$; if $M(G)$ has n elements, then $G \hookrightarrow G_p$ for any p above level n of $\mathbb{S}$.

For $\mathbb{P}$ that fall into this case, there are two biembeddability classes of proto-Ramsey $\mathbb{Q}$ such that $\mathcal{G}(\mathbb{Q}) \rightleftarrows \mathcal{G}(\mathbb{P})$, one with incompatible points (hence, by the above comments, incompatible points above every point) and one with no incompatible points. For example, if $\mathbb{P}$ is proto-Ramsey and every $G_{[p]}$ consists of two points joined by a single edge, then $\mathbb{P}$ may be biembeddable with either ω or the full binary tree.

3 Structural Properties from Ramsey Properties in ATR_0

In this section we show from ATR_0 that every weakly proto-Ramsey, 2-Ramsey bounded-level partial ordering is biembeddable with a proto-Ramsey bounded-level partial ordering.

Theorem 3.1 **(ATR_0)** *If $\mathbb{P}$ is a bounded-level, weakly proto-Ramsey, 2-Ramsey partial ordering, then $\mathbb{P}$ is biembeddable with a proto-Ramsey partial ordering.*

This proof uses the main technique of the proof from ACA_0 that any perfect tree with uncountably many branches has a perfect subtree (see [19, pp. 186–188]). Either $\mathbb{P}$ has the analogue of a Cantor–Bendixson decomposition that removes all of $\mathbb{P}$ in α-many steps for some α (which contradicts $R^2(\mathbb{P})$), or $\mathbb{P}$ has the analogue of a perfect subtree (a proto-Ramsey $\mathbb{Q} \subseteq \mathbb{P}$ with $\mathscr{G}(\mathbb{Q}) \rightleftarrows \mathscr{G}(\mathbb{P})$).

Proof If every $G_{[p]} \in \mathscr{G}(\mathbb{P})$ has only one element in $S(G_{[p]})$, then $\mathbb{P}$ is a subordering of $\mathbb{S}$ (Definition 2.33). In this case it requires only RCA_0 to show that $\mathbb{P}$ must be biembeddable with $\mathbb{S}$, $2^{<\omega}$, ω, or the partial ordering $\overline{\mathbb{S}}$ with $n+1$ elements on level n and $p <_{\overline{\mathbb{S}}} q$ iff $\operatorname{lev}(p) < \operatorname{lev}(q)$ (see [4, Proposition 2.17]; the arguments are similar to those for trees [5]). Each of these partial orderings is proto-Ramsey.

Therefore we may assume that some $S(G_{[p]})$ contains at least two elements.

For any (downward closed) subordering $\mathbb{Q}$ of $\mathbb{P}$, define an analogue of the Cantor–Bendixson derivative,

$$\mathbb{Q}' = \left\{ p \in \mathbb{Q} \mid \big(\forall G \in \mathscr{G}(\mathbb{P})\big)(\exists r \geq_{\mathbb{P}} p)\,(r \in \mathbb{Q} \;\&\; G \hookrightarrow G^{\mathbb{Q}}_{[r]}) \right\}.$$

Taking this Cantor–Bendixson derivative removes q from $\mathbb{Q}$ if some graph realized in $\mathbb{P}$ cannot be realized in $\mathbb{Q}$ above q.

Let $\langle X, \leq_X \rangle$ be any countable well-ordering with least element 0. By arithmetic transfinite recursion, we can iterate this derivative along X: there is a sequence

$$\big\langle \mathbb{P}(x) \mid x \in X \big\rangle$$

with $\mathbb{P}(0) = \mathbb{P}$, and for all $x \in X$,

$$\mathbb{P}(x) = \bigcap_{y <_X x} \big(\mathbb{P}(y)\big)'.$$

If for some $x \in X$ we have $\mathbb{P}(x) = \emptyset$, then we get a contradiction to $R^2(\mathbb{P})$: For $p \in \mathbb{P}$, let $d(p)$ be the unique $x \in \mathbb{X}$ such that $p \in \mathbb{P}(x)$ but $p \notin (P(x))'$; that is, $d(p)$ is the stage of the iteration at which p is removed. Note that if $p <_{\mathbb{P}} q$ then $d(p) \geq_X d(q)$; p cannot be removed at an earlier stage than q. Color pairs $p <_{\mathbb{P}} q$ by

$$c(p,q) = \begin{cases} 0 & d(p) = d(q); \\ 1 & d(p) >_X d(q). \end{cases}$$

Since $\mathbb{P}$ has a branch, a monochromatic copy of $\mathbb{P}$ in color 1 would produce an infinite descending sequence in X, contradicting the fact that X is well ordered. On the other hand, a monochromatic copy of $\mathbb{P}$ in color 0 would also yield a contradiction: Let $\varphi : \mathbb{P} \to \mathbb{P}$ have range of color 0, so that for some x and for all $\varphi(p)$ we have $d(\varphi(p)) = x$. In particular, $\varphi(\mathbb{P}) \subseteq \mathbb{P}(x)$. This means that if 0 is the least element of $\mathbb{P}$, then every graph $G_{[p]}$ is realized in $\mathbb{P}(x)$ above $\varphi(0)$, which contradicts $\varphi(0) \notin (P(x))'$.

Therefore, for every countable well-ordering X, there is such a sequence with all $\mathbb{P}(x)$ nonempty. Because ATR_0 proves that the collection of countable well-orderings is not $\boldsymbol{\Sigma}^1_1$-definable (see [19]), there must be a non-well-founded countable linear ordering X associated to which there is also such a sequence. Because X is not well-founded, there is an infinite descending sequence $\langle x(i) \mid i \in \omega \rangle$ in X.

Set $\mathbb{Q} = \bigcup_i \mathbb{P}(x(i))$. Every $G \in \mathscr{G}(\mathbb{P})$ is realized above every $p \in \mathbb{Q}$: If $p \in \mathbb{P}(x(i))$, then since $P(x_i) \subseteq (P(x_{i+1}))'$, G is realized above p in $\mathbb{P}(x(i+1)) \subseteq \mathbb{Q}$. Since also $\mathbb{Q} \subseteq \mathbb{P}$, we have that $\mathscr{G}(\mathbb{Q}) \rightleftarrows \mathscr{G}(\mathbb{P})$, and every

$G \in \mathcal{G}(\mathbb{Q})$ is realized above every point in $\mathbb{Q}$. Since some $G_{[p]}$ has more than one element in $S(G_{[p]})$ and is realized above every point in $\mathbb{Q}$, we know that $\mathbb{Q}$ has incompatible elements above every point. By Proposition 2.30, then, $\mathbb{Q}$ is densely self-embeddable and so is proto-Ramsey, and $\mathbb{P} \hookrightarrow \mathbb{Q}$. As $\mathbb{Q} \subseteq \mathbb{P}$, we have $\mathbb{P} \rightleftarrows \mathbb{Q}$, as required. □

4 Ramsey Properties from Structural Properties in ACA_0

We have shown that RCA_0 suffices to show every bounded-level 3-Ramsey partial ordering with least element is biembeddable with a weakly proto-Ramsey, j.e.p., edge-Ramsey partial ordering, and ATR_0 shows every bounded-level 3-Ramsey partial ordering with least element is biembeddable with a proto-Ramsey, j.e.p., edge-Ramsey partial ordering.

In this section, we look at the opposite direction of the characterization of n-Ramsey partial orderings. We show ACA_0 proves that every bounded-level, proto-Ramsey, j.e.p., edge-Ramsey partial ordering is n-Ramsey for every standard n. As ω is an example of such a partial ordering, and $R^3(\omega)$ implies ACA_0, over RCA_0 (see [19]), we have the following theorem.

Theorem 4.1 *The following are equivalent for any standard $n \geq 3$:*

1. *ACA_0.*
2. *Every bounded-level, proto-Ramsey, j.e.p., edge-Ramsey partial ordering is n-Ramsey.*
3. *The partial ordering ω is n-Ramsey.*
4. *There is an infinite, bounded-level, n-Ramsey partial ordering with least element.*

Proof It is clear that (2) $\implies$ (3) $\implies$ (4), and we are about to prove (1) $\implies$ (2); Simpson [19] showed that (3) $\implies$ (1). To see (4) $\implies$ (3), suppose that $\mathbb{P}$ is n-Ramsey and has a least element, and let c be a coloring of n-element chains in ω. Color n-element chains in $\mathbb{P}$ by $\overline{c}(p_1, \ldots, p_n) = c(\mathrm{lev}(p_1), \ldots, \mathrm{lev}(p_n))$. Let $\mathbb{Q}$ be a monochromatic suborder of $\mathbb{P}$, and let b be a branch through $\mathbb{Q}$ (guaranteed by Proposition 2.7); then $\{\mathrm{lev}^{\mathbb{P}}(p) \mid p \in b\}$ is a monochromatic subset of ω. □

The following theorem gives (1) $\implies$ (2).

Theorem 4.2 (ACA_0) *If $\mathbb{P}$ is a bounded-level, proto-Ramsey partial ordering and $\mathcal{G}(\mathbb{P})$ is edge-Ramsey and satisfies the joint embedding property, then $\mathbb{P}$ is n-Ramsey for all standard n.*

We deal first with the case that $\mathbb{P}$ has incompatible points.

The only use of the full strength of arithmetic comprehension here is in Proposition 4.6. Because the arithmetic recursions in Lemmas 4.4 and 4.7 are of finite length, weaker assumptions (in this case, $I\mathbf{\Sigma}^0_2$) actually suffice. We make use of Proposition 2.26, connecting embeddings and colorings of $\mathbb{P}$ to embeddings and colorings of graphs in $\mathcal{G}(\mathbb{P})$, in these proofs.

We prove $R^n(\mathbb{P})$ by induction on n. The base step, Proposition 4.3, shows that if c is a graph-induced 2-coloring of $\mathbb{P}$, then there is an embedding $\varphi : \mathbb{P} \to \mathbb{P}$ with monochromatic range. In particular, since every 1-coloring is equivalent to a graph-induced 2-coloring, $R^1(\mathbb{P})$ holds.

Proposition 4.3 (**ACA_0**) *If $\mathbb{P}$ satisfies the conditions of Theorem 4.2 and has incompatible points, and c is a graph-induced 2-coloring of $\mathbb{P}$ in m colors (i.e., $c(p_0, p_1)$ depends only on p_0 and $\mathrm{Pred}_{\mathrm{lev}(p_0)}(p_1)$), then there is an embedding $\varphi : \mathbb{P} \to \mathbb{P}$ with range $\mathbb{Q}$ monochromatic for c.*

The idea behind the proof is to use the fact that $\mathcal{G}(\mathbb{P})$ is edge-Ramsey to embed $\mathbb{P}$ in such a way that each graph $G_{[p]}$ is sent to a monochromatic subgraph of $G_{[\varphi(p)]}$. The following lemma guarantees that we can make all these graphs the same color.

Lemma 4.4 (**ACA_0**) *If $\mathbb{P}$ satisfies the conditions of Theorem 4.2, and c is a graph-induced 2-coloring of $\mathbb{P}$ in m colors, then there are a point $p \in \mathbb{P}$ and a color i such that, for every $G \in \mathcal{G}(\mathbb{P})$, color i copies of G are dense in $\mathbb{P}$ above p. That is, for every $q \geq_{\mathbb{P}} p$ there are an $r \geq_{\mathbb{P}} q$ and an embedding $j : G \to G_{[r]}$ such that the image of j is monochromatic in color i for $c_{G_{[r]}}$. (That is, for every edge (x, y) in G, and every $r' >_{\mathbb{P}} r$ such that $\mathrm{Pred}_{\mathrm{lev}(r)}(r') = j(y)$, we have $c(j(x), r') = i$.)*

Proof of Lemma 4.4 If not, by arithmetic recursion, define a sequence $\langle p_i, G_i \mid i \leq m\rangle$ such that $G_i \in \mathcal{G}(\mathbb{P})$, $p_i \in \mathbb{P}$, $p_i \leq_{\mathbb{P}} p_{i+1}$, and there is no color i copy of G_i in $\mathbb{P}$ above p_{i+1}. Hence, for all $i < m$, there is no color i copy of G_i in $\mathbb{P}$ above p_m.

Because $\mathcal{G}(\mathbb{P})$ satisfies the joint embedding property, there is a graph $G \in \mathcal{G}(\mathbb{P})$ such that for every $i < m$ we have $G_i \hookrightarrow G$. Thus there is no monochromatic copy of G, in any color $i < m$, in $\mathbb{P}$ above p_m.

Because $\mathcal{G}(\mathbb{P})$ is edge-Ramsey, there is an $H \in \mathcal{G}(\mathbb{P})$ such that $H \to (G)^e_m$. Because $\mathbb{P}$ is proto-Ramsey, there is $p \geq_{\mathbb{P}} p_m$ such that $H \hookrightarrow G_{[p]}$; hence $G_{[p]} \to (G)^e_m$.

This is now a contradiction: Color $G_{[p]}$ by $\bar{c}(q, \mathrm{Pred}_{\mathrm{lev}(q)}(r)) = c(q, r)$; because c is graph-induced, this coloring is well-defined. Choose an embedding $j : G \to G_{[p]}$ with monochromatic range of color i; because $G_{[p]} \to (G)^e_m$, this is possible. But since $G_i \hookrightarrow G$, this gives a color i copy of G_i in $\mathbb{P}$ above p_i, contradicting the choice of G_i and p_i. □

Proof of Proposition 4.3 Choose $p_0 \in \mathbb{P}$ and i as in Lemma 4.4. We will inductively define an embedding $\varphi : \mathbb{P} \to \mathbb{P}$ with range monochromatic of color i.

Construct an embedding of $\mathbb{P}$ into $\mathbb{P}$ above p_0 as a limit of extendible embeddings, as in the proof of Proposition 2.30; when choosing the $j_d : G_d \to G_{[p]}$, choose so that the image $G_{[p]}$ is monochromatic in color i. This is possible because of the choice of p_0 and i as in Lemma 4.4. Since the j_d are the graph embeddings induced by the embedding φ, and all their ranges are monochromatic in color i, by Proposition 2.26, the range of φ is monochromatic for c. □

The inductive step in the proof of Theorem 4.2 is contained in the following proposition. First we need a definition.

Definition 4.5 For $n > 2$, an n-coloring of $\mathbb{P}$ is *graph-induced* if $c(p_0, \ldots, p_{n-1})$ depends only on $(p_0, p_1, \ldots, p_{n-2})$ and $\mathrm{Pred}_{\mathrm{lev}(p_{n-2})}(p_{n-1})$.

That is, $c(p_0, \ldots, p_{n-1})$ depends only on $(p_0, p_1, \ldots, p_{n-3})$ and the edge in $G_{[p_{n-2}]}$ determined by (p_{n-2}, p_{n-1}). Globally, c is determined by a family of edge colorings $c_{(p_0,\ldots,p_{n-3})}$ of graphs $G_{[p]}$ for $p_{n-3} <_{\mathbb{P}} p$.

Proposition 4.6 (**ACA_0**) *If $\mathbb{P}$ satisfies the conditions of Theorem 4.2 and has incompatible points, and if c is a graph-induced n-coloring of $\mathbb{P}$ for any $n > 2$,* *then there is an embedding $\varphi : \mathbb{P} \to \mathbb{P}$ with range $\mathbb{Q}$ such that on $\mathbb{Q}$ the color $c(p_0, \dots, p_{n-1})$ depends only on $(p_0, \dots, p_{n-3}, \mathrm{Pred}_{\mathrm{lev}(p_{n-3})}(p_{n-2}))$; that is, $\mathbb{Q}$ is a copy of $\mathbb{P}$ on which the coloring c is equivalent to a graph-induced $(n-1)$-coloring, $\overline{c}(q_0, \dots, q_{n-2}) = c(q_0, \dots, q_{n-2}, q)$ for any successor q of q_{n-2} in $\mathbb{Q}$.*

To prove this proposition, we embed $\mathbb{P}$ into itself by nesting constructions like that in the proof of Proposition 4.3 for the family of 2-colorings $c_{(p_0,\dots,p_{n-3})}(p_{n-2}, p_{n-1}) = c(p_0, \dots, p_{n-1})$, in such a way that the color chosen for $c_{(p_0,\dots,p_{n-3})}(p_{n-2}, p_{n-1})$ depends only on $(p_0, \dots, p_{n-3})$ and $\mathrm{Pred}_{\mathrm{lev}(p_{n-3})}(p_{n-2})$.

The dependence on $\mathrm{Pred}_{\mathrm{lev}(p_{n-3})}(p_{n-2})$ is because the inductive definition of φ forces us to have a different version of the construction for $c_{(p_0,\dots,p_{n-3})}$ above each such set.

As before, we need a lemma to tell us that we can choose colors in this way.

Lemma 4.7 **(*ACA*$_0$)** *Suppose that $\overline{c}$ and c are graph-induced 2-colorings of $\mathbb{P}$, in k and m colors, respectively, and suppose that $p_0 \in \mathbb{P}$ and $\ell < k$ are such that, for every $G \in \mathcal{G}(\mathbb{P})$, color ℓ copies of G for $\overline{c}$ are dense in $\mathbb{P}$ above p_0. Then there are a point $p \geq_{\mathbb{P}} p_0$ and a color $i < m$ such that, for the coloring $(\overline{c}, c)(q, r) = (\overline{c}(q, r), c(q, r))$, for every $G \in \mathcal{G}(\mathbb{P})$, color (ℓ, i) copies of G are dense in $\mathbb{P}$ above p.*

The proof is essentially the same as that of Lemma 4.4, working above p_0 and inside graphs that are monochromatic for $\overline{c}$ in color ℓ.

Proof of Lemma 4.7 Suppose not. Then by arithmetic recursion define a sequence $\langle p_i, G_i \mid i \leq m\rangle$ such that p_0 is given in the statement of the lemma, $G_i \in \mathcal{G}(\mathbb{P})$, $p_i \in \mathbb{P}$, $p_i \leq_{\mathbb{P}} p_{i+1}$, and there is no color (ℓ, i) copy of G_i in $\mathbb{P}$ above p_{i+1}. Hence, for all $i < m$, there is no color (ℓ, i) copy of G_i in $\mathbb{P}$ above p_m.

Because $\mathcal{G}(\mathbb{P})$ satisfies the joint embedding property, there is a graph $G \in \mathcal{G}(\mathbb{P})$ such that, for every i, we have $G_i \hookrightarrow G$. Thus there is no monochromatic copy of G, in any color (ℓ, i) for $i < m$, in $\mathbb{P}$ above p_m.

Because $\mathcal{G}(\mathbb{P})$ is edge-Ramsey, there is an $H \in \mathcal{G}(\mathbb{P})$ such that $H \to (G)^e_m$. By assumption, there is $p \geq_{\mathbb{P}} p_m$ such that $G_{[p]}$ contains a copy $\overline{H}$ of H monochromatic for $\overline{c}$ in color ℓ.

This is now a contradiction: Color $\overline{H}$ by $\overline{\overline{c}}(q, \mathrm{Pred}_{\mathrm{lev}(q)}(r)) = (\overline{c}, c)(q, r)$; because $\overline{c}$ and c are graph-induced, this coloring is well-defined. Choose an embedding $j : G \to G_{[p]}$ with monochromatic range of color (ℓ, i); because $\overline{H} \to (G)^e_m$ and $\overline{H}$ is already monochromatic for $\overline{c}$ in color ℓ, this is possible. But since $G_i \hookrightarrow G$, this gives a color (ℓ, i) copy of G_i in $\mathbb{P}$ above p_i, contradicting the choice of G_i and p_i. □

Proof of Proposition 4.6 Define by arithmetic recursion an embedding $\varphi : \mathbb{P} \to \mathbb{P}$ and a function i such that $c(\varphi(p_0), \dots, \varphi(p_{n-1})) = i(p_0, \dots, p_{n-3}, \mathrm{Pred}_{\mathrm{lev}(p_{n-3})}(p_{n-2}))$. The definition of φ is again as a limit of extendible embeddings, as in the proof of Proposition 2.30; since our construction guarantees that $p_0, \dots, p_{n-3}, \mathrm{Pred}_{\mathrm{lev}(p_{n-3})}(p_{n-2})$ is determined by $(\varphi(p_0), \dots, \varphi(p_{n-3}), \mathrm{lev}_{\varphi(p_{n-3})}(\varphi(p_{n-2})))$ on the range of φ, and c will be a graph-induced coloring of n-element chains.

At the kth stage we define both φ on $\mathbb{P}_k$ and $i(p_0, \dots, p_{n-3}, \mathrm{Pred}_{\mathrm{lev}(p_{n-3})}(p_{n-2}))$ for $\mathrm{lev}(p_{n-3}) = k - 1$.

To define φ, we need only to specify how we choose the p_d and the graph embeddings j_d. We also need to give the definition of i.

For $d = [p] \subseteq \mathbb{P}_k$ and $s = \mathrm{Pred}_{k-1}(p) \subseteq [q] = a$, choose p_s with $\mathrm{Pred}_{\mathrm{lev}(\varphi(q))}(p_s) = j_a(s)$.

By abuse of notation, if either $q \in s$ or $\mathrm{lev}(q) < k-1$ and $q <_{\mathbb{P}} p$ for some (hence all) $p \in s$, we will say $q \leq_{\mathbb{P}} s$. For $i < k$, we shall use $\mathrm{Pred}_i(s)$ to denote $\mathrm{Pred}_i(r)$ for any r such that $\mathrm{Pred}_{k-1}(r) = s$. (This does not depend on r. By this definition, $\mathrm{Pred}_{k-1}(s) = s$.)

Further extend the p_s so that for some colors $i(q_0, \ldots, q_{n-3}, s)$ with $q_0 <_{\mathbb{P}} \cdots <_{\mathbb{P}} q_{n-3} <_{\mathbb{P}} s$ and for all $G \in \mathcal{G}(\mathbb{P})$, for the coloring

$$C(x, y) = \left\langle c_{(\varphi(q_0),\ldots,\varphi(q_{n-3}))}(x, y) \mid q_0 <_{\mathbb{P}} \cdots <_{\mathbb{P}} q_{n-3} \leq_{\mathbb{P}} s \right\rangle,$$

color $\langle i(q_0, \ldots, q_{n-3}, \mathrm{Pred}_{\mathrm{lev}(q_{n-3})}(s)) \mid q_0 <_{\mathbb{P}} \cdots <_{\mathbb{P}} q_{n-3} \leq_{\mathbb{P}} s\rangle$ copies of G are dense in $\mathbb{P}$ above p_s.

This is possible by Lemma 4.7, applied to the colorings

$$\left\langle c_{(\varphi(q_0),\ldots,\varphi(q_{n-3}))}(x, y) \mid q_0 <_{\mathbb{P}} \cdots <_{\mathbb{P}} q_{n-3} <_{\mathbb{P}} s \right\rangle,$$
$$\left\langle c_{(\varphi(q_0),\ldots,\varphi(q_{n-3}))}(x, y) \mid q_0 <_{\mathbb{P}} \cdots <_{\mathbb{P}} q_{n-3} \in s \right\rangle,$$

and color $\langle i(q_0, \ldots, q_{n-3}, \mathrm{Pred}_{\mathrm{lev}(q_{n-3})}(s)) \mid q_0 <_{\mathbb{P}} \cdots <_{\mathbb{P}} q_{n-3} <_{\mathbb{P}} s\rangle$, which satisfy the conditions of the lemma by induction. Note that this gives the kth stage definition of i.

Now further extend the p_s to choose p_d for $d \subseteq s$ as in the proof of Proposition 2.30. Finally, choose j_d so that the image of j_d is monochromatic for the coloring

$$C(x, y) = \left\langle c_{(\varphi(q_0),\ldots,\varphi(q_{n-3}))}(x, y) \mid q_0 <_{\mathbb{P}} \cdots <_{\mathbb{P}} q_{n-3} \leq_{\mathbb{P}} s \right\rangle$$

in color $\langle i(q_0, \ldots, q_{n-3}, \mathrm{Pred}_{\mathrm{lev}(q_{n-3})}(s)) \mid q_0 <_{\mathbb{P}} \cdots <_{\mathbb{P}} q_{n-3} \leq_{\mathbb{P}} s\rangle$. This is possible by choice of p_s.

This completes the construction. As in the proof of Proposition 4.6, the fact that the induced graph embeddings j_d have monochromatic range guarantees that φ is as desired. □

Proof of Theorem 4.2 If $\mathbb{P}$ does not have incompatible elements, then whenever $\mathrm{lev}(p) < \mathrm{lev}(q)$ we have $p <_{\mathbb{P}} q$. (Otherwise, p and q, being incomparable with different predecessors, must be incompatible.) Hence, for all $p \in \mathbb{P}$, each $S(G_{[p]})$ has only one element. Since $\mathcal{G}(\mathbb{P})$ is edge-Ramsey, either each $M(G_{[p]})$ has only one element or the sizes of the $M(G_{[p]})$ are unbounded. In the first case, $\mathbb{P}$ is isomorphic to ω, and in the second, $\mathbb{P}$ is biembeddable with a partial ordering $\overline{\mathbb{S}}$ with $n+1$ elements on level n and $p <_{\overline{\mathbb{S}}} q$ iff $\mathrm{lev}(p) < \mathrm{lev}(q)$. We know that ACA_0 proves ω is n-Ramsey (see [19]), and a very similar proof shows that $\overline{\mathbb{S}}$ is n-Ramsey.

If $\mathbb{P}$ has incompatible elements, then the theorem holds by induction on n, using Propositions 4.3 and 4.6. □

5 Ramsey for Pairs and ACA_0

In this section, we prove the following theorem.

Theorem 5.1 *There is a primitive recursive partial ordering $\mathbb{P}$ such that, over RCA_0, the statement $R^2_2(\mathbb{P})$ is equivalent to ACA_0.*

Definition 5.2 Define $\mathbb{P}$ as follows: the elements of $\mathbb{P}$ are the points of the complete ω-branching tree (i.e., finite sequences of natural numbers). The ordering on $\mathbb{P}$ is defined as follows:

For elements σ and τ, we have that $\tau <_{\mathbb{P}} \sigma$ iff $|\tau| = m < |\sigma|$ and one of the following:

(a) $\tau = \sigma \upharpoonright m$;

(b) $\tau \upharpoonright (m-1) = \sigma \upharpoonright (m-1)$ & $\tau(m-1) < \sigma(m-1)$.

One way to picture this partial ordering is to envision the usual drawing of the complete ω-branching tree (growing upward), and put in additional connections as follows: connect every immediate successor $\sigma^\frown i^\frown k$ of $\sigma^\frown i$ downward and to the left, to every $\sigma^\frown j$ for $j < i$.

As another helpful visualization, $\sigma <_{\mathbb{P}} \tau$ iff σ is shorter than τ, and either σ is a proper initial segment of τ or σ agrees with τ up to the final entry of σ and then branches off to the left.

This partial ordering is proto-Ramsey without being finite-level. Every element of $\mathcal{G}(\mathbb{P})$ (except the trivial one) is isomorphic to the graph G defined by $M(G) = \omega$, $S(G) = \omega - \{0\}$, $E(G) = \{(j, i) \mid j < i\}$. (For fixed σ, $\sigma^\frown i$ and $\sigma^\frown j$ are compatible, and no other nodes on the same level are compatible, so a typical equivalence class a consists of all $\sigma^\frown j$ for a fixed σ. The sets of G_a are all sets of the form $s_{i+1} = \{\sigma^\frown j \mid j < i + 1\}$; s_{i+1} consists of all predecessors $\sigma^\frown j$ of any extension of $\sigma^\frown i$.) From $R^2(\omega)$ one can easily see that $G \to (G)^e_k$ for all k, as an edge coloring of G is precisely a coloring of pairs from ω. Hence $\mathcal{G}(\mathbb{P})$ is edge-Ramsey, and since all elements of $\mathcal{G}(\mathbb{P})$ are isomorphic, $\mathcal{G}(\mathbb{P})$ trivially has the joint embedding property. The proof in ACA_0 that $\mathbb{P}$ is 2-Ramsey is similar to the proof of the finite-level case Theorem 4.2, although the fact that each $G_{[p]}$ is infinite induces an extra layer of complexity; the colorings in the conclusion of Proposition 5.11 below are precisely the graph-induced colorings.

The proof of ACA_0 from $R^2(\mathbb{P})$ uses the fact that the $\equiv$-classes of $\mathbb{P}$ (above the root) are infinite. Each element of $\mathbb{P}$ is associated with a pair of natural numbers, the height of p in $\mathbb{P}$ and the number of elements of the equivalence class of p to the left of p; we can first choose the height of p to be arbitrarily large, and then given this choice, choose p to be arbitrarily far to the right in its equivalence class. We use this to associate with each pair $p <_{\mathbb{P}} q$ a triple (x, s_0, s_1) of natural numbers, which by choice of $\langle p, q\rangle$ can be made arbitrarily large successively, each choice depending on the previous one. We then color pairs using the same coloring of triples that gives ACA_0 from $R^3(\omega)$. The fact that an embedding of $\mathbb{P}$ into $\mathbb{P}$ carries equivalence classes to equivalence classes allows the proof to go through.

Definition 5.3 By a *predecessor* or *successor* of σ we mean a point lying below or above σ in the ordering $\leq_{\mathbb{P}}$ on $\mathbb{P}$. By a *restriction* or *extension* of σ we mean a point lying below or above σ in the ordering $\subseteq$ on the complete ω-branching tree. (A restriction [extension] must also be a predecessor [successor], but not conversely.)

Definition 5.4 Define an equivalence relation on $\mathbb{P}$ by setting $\sigma \equiv \tau$ if and only if, for some ρ, we have $\sigma = \rho^\frown i$ and $\tau = \rho^\frown j$. In this case, we say that σ is *to the left* of τ if $i < j$. (This is actually the same equivalence relation as in Definition 2.20; points are equivalent iff they are on the same level and are compatible.) That is, the immediate extensions of ρ form an equivalence class, ordered from left to right in the natural way.

Note that if σ is to the left of τ, then all proper successors of τ are also successors of σ, but not conversely.

Lemma 5.5 (*RCA*$_0$) *If $\varphi : \mathbb{P} \to \mathbb{P}$ is an embedding (of partial orderings), then φ preserves the relations $\equiv$, $\not\equiv$, and "to the left of."*

This implies, in particular, that if φ is an embedding and $\rho^\frown i$ is in its range (but not the least element of its range), then infinitely many points $\rho^\frown j$ (infinitely many elements of the equivalence class of $\rho^\frown i$) are also in its range.

Proof These relations can be defined existentially in the ordering: $\sigma \equiv \tau$ iff σ and τ either are equal or are incomparable and have a common successor, $\sigma \not\equiv \tau$ iff σ and τ have different predecessors, and if $\sigma \equiv \tau$, then σ is to the left of τ iff there is a successor of σ that is not a successor of τ. □

Lemma 5.6 (*RCA*$_0$) *If φ is an injection of $\mathbb{P}$ to $\mathbb{P}$ that preserves the relations $\equiv$, $\not\equiv$, "to the left of," and $\subseteq$, then φ is an embedding.*

An embedding need not preserve $\subseteq$.

Definition 5.7 An embedding that preserves $\subseteq$ is called a *strong embedding*.

Remark 5.8 Because embeddings preserve $\equiv$, if φ is an embedding, then, for every σ, there is some $\psi(\sigma) \geq_{\mathbb{P}} \varphi(\sigma)$ such that if τ is an immediate extension of σ, then $\varphi(\tau)$ is an immediate extension of $\psi(\sigma)$. (The equivalence class of immediate extensions of σ maps into the equivalence class of immediate extensions of $\psi(\sigma)$.) The embedding φ is strong just in case we always have $\psi(\sigma) \supseteq \varphi(\sigma)$.

A consequence of this lemma is that to inductively produce a strong embedding φ, we must choose $\psi(\sigma) \supseteq \varphi(\sigma)$ and map the immediate extensions of σ to immediate extensions of $\psi(\sigma)$, in order from left to right. If the embedding need not be strong, it suffices to choose $\psi(\sigma) \geq_{\mathbb{P}} \varphi(\sigma)$ (or choose $\psi(\sigma)$ to the right of σ); we must then ensure that points to the right of σ embed to the right of $\psi(\sigma)$ (or its restriction to the appropriate level).

Proposition 5.9 (*RCA*$_0$) *If $R^2_2(\mathbb{P})$, then ACA_0.*

Proof As usual, we show that, for any set X, we can color pairs from $\mathbb{P}$ in two colors so that a monochromatic copy of $\mathbb{P}$ allows us to recover X'.

Given a pair $\tau <_{\mathbb{P}} \sigma$, suppose $\sigma = \rho^\frown i$. Let $x = |\tau|$, and let $s_0 = |\sigma|$ and $s_1 = i$. Color the pair $\langle \tau, \sigma \rangle$ color 0 if $s_0 \leq s_1$ and $X' \upharpoonright x[s_0] = X' \upharpoonright x[s_1]$, and color 1 otherwise.

A monochromatic copy $\mathbb{Q} \subseteq \mathbb{P}$ must be of color 0: To see this, choose some $\tau \in \mathbb{Q}$, and set $x = |\tau|$. Choose s large enough so that $X' \upharpoonright x[s] = X' \upharpoonright x$, and choose $\sigma >_{\mathbb{P}} \tau$ in $\mathbb{Q}$ with $|\sigma| = s_0 \geq s$. If $\sigma = \rho^\frown j$, then there is some $i > s_0$ with $\rho^\frown i$ in $\mathbb{Q}$ (since, as embeddings preserve equivalence classes, there must be infinitely many elements of $\mathbb{Q}$ equivalent to σ, and those elements are exactly the $\rho^\frown i$). Then the pair $\langle \tau, \rho^\frown i \rangle$ has color 0.

Now from such a $\mathbb{Q}$ we can compute X': To find $X' \upharpoonright x$, choose τ in $\mathbb{Q}$ with $|\tau| \geq x$, and choose $\rho^\frown i >_{\mathbb{P}} \tau$ in $\mathbb{Q}$. Then we must have $X' \upharpoonright x = X' \upharpoonright x[|\rho|]$. If not, then there is some $j > i$ with $\rho^\frown j$ in $\mathbb{Q}$ such that $X' \upharpoonright x[j] \neq X' \upharpoonright x[|\rho|]$, from which it follows that $\langle \tau, \rho^\frown j \rangle$ has color 1, contradicting the assumption that $\mathbb{Q}$ is monochromatic of color 0. □

Proposition 5.10 (*RCA_0*) *If ACA_0, then $R^2_2(\mathbb{P})$.*

In fact, we can replace $R^2_2(\mathbb{P})$ with $R^n_k(\mathbb{P})$ for any standard n and any k. However, we give the proof only for $n = k = 2$.

It is clear that this follows immediately from Propositions 5.11, 5.12, and 5.13.

Proposition 5.11 (*ACA_0*) *If c is a coloring of pairs from $\mathbb{P}$ in two colors, then there is an embedding of $\mathbb{P}$ into itself with range on which the color of $\langle\tau, \sigma\rangle$ depends only on τ and $\sigma \upharpoonright |\tau|$.*

Proposition 5.12 (*ACA_0*) *If c is a coloring of pairs from $\mathbb{P}$ in two colors such that the color of $\langle\tau, \sigma\rangle$ depends only on τ and $\sigma \upharpoonright |\tau|$, then there is an embedding of $\mathbb{P}$ into itself with range on which the color of $\langle\tau, \sigma\rangle$ depends only on τ.*

Proposition 5.13 (*ACA_0*) *If c is a coloring of single points from $\mathbb{P}$ in two colors, then there is an embedding of $\mathbb{P}$ into itself with monochromatic range.*

The proofs of these propositions make use of the fact that ACA_0 suffices for carrying out constructions by arithmetic recursion. We assume an underlying ω-ordering on $\mathbb{P}$ that respects both the ordering on $\mathbb{P}$ and the relation "to the left of"; our definitions are by recursion on this ordering.

Proof of Proposition 5.13 Given a coloring c of single points of $\mathbb{P}$ in two colors, we define a strong embedding with monochromatic range, by cases.

Intuitively, in case 1, there is a point above which every node has infinitely many red immediate extensions. In this case, define the range of the embedding by choosing any red node above that point as the root, and once any node has been included in the range, include all its red immediate extensions. Otherwise, in case 2, there is a dense set of nodes with cofinitely many blue extensions. In this case, we choose any blue node as the root, and once any node has been included in the range, find an extension τ with cofinitely many blue immediate extensions, and include all the blue immediate extensions of τ.

Case 1: There is a point σ, such that for every extension $\tau \supseteq \sigma$, there are infinitely many i such that $c(\tau^\frown i) = 0$. In that case, choose the least such σ for which $c(\sigma) = 0$, let $\varphi(\langle\rangle) = \sigma$, and, in general, if $\varphi(\rho) = \tau$ and $\varphi(\rho^\frown i)$ have been defined for all $i < j$, define $\varphi(\rho^\frown j)$ to be the least $\tau^\frown k$ unequal to any $\varphi(\rho^\frown i)$ for $i < j$ such that $c(\tau^\frown k) = 0$.

Case 2: Otherwise, for every σ, there is $\tau \supseteq \sigma$ such that $c(\tau^\frown i) = 1$ for all but finitely many i. In this case, define φ and an auxiliary ψ, as in Remark 5.8. (The function ψ acts as guide to creating a strong embedding; the equivalence class of immediate extensions of ρ will be embedded into the equivalence class of immediate extensions of $\psi(\rho) \geq \varphi(\rho)$.)

Let $\varphi(\langle\rangle)$ be the least σ with $c(\sigma) = 1$, and let $\psi(\langle\rangle)$ be the least $\tau \supseteq \sigma$ such that $c(\tau^\frown i) = 1$ for all but finitely many i. In general, if $\varphi(\rho)$, $\psi(\rho)$, $\varphi(\rho^\frown i)$, and $\psi(\rho^\frown i)$ have been defined for all $i < j$, and $\psi(\rho) = \tau$, then define $\varphi(\rho^\frown j)$ to be the least $\tau^\frown k$ unequal to any $\varphi(\rho^\frown i)$ for $i < j$, such that $c(\tau^\frown k) = 1$, and let $\psi(\rho^\frown j)$ be the least $\pi \supseteq \varphi(\rho^\frown j)$ such that $c(\pi^\frown i) = 1$ for all but finitely many i.

This constructs a monochromatic embedding of $\mathbb{P}$ by Lemma 5.6. □

Proof of Proposition 5.11 Given a coloring c of pairs from $\mathbb{P}$ in two colors, we produce an embedding of $\mathbb{P}$ with range on which the color of $\langle\tau, \rho\rangle$ depends only on

τ and $\rho \upharpoonright |\tau|$. Note that when $\tau <_{\mathbb{P}} \rho$ we must have $\tau = \sigma^\frown j$ and $\rho \supset \sigma^\frown i$ for some σ and $i \leq j$; we are demanding that the color of $\langle \tau, \rho \rangle$ depend only on $\sigma^\frown i$ (i.e., $\rho \upharpoonright |\tau|$) and j (i.e., the additional piece of information needed to determine τ).

Intuitively, we construct the desired suborder $\mathbb{Q}$ more or less as follows, using the previous proposition that $\mathbb{P}$ is Ramsey for colorings of single points: Include the minimal element $\langle\rangle$ of $\mathbb{P}$ in $\mathbb{Q}$. Choose an isomorphic suborder of points that all realize the same color, say red, when paired with $\langle\rangle$, and work within that suborder. Within that suborder, above the first immediate extension σ_0 of $\langle\rangle$, restrict to an isomorphic suborder of points that all form (say) blue pairs with σ_0. Above the next immediate extension σ_1 of $\langle\rangle$, restrict to an isomorphic suborder of points that form (say) red pairs with σ_0, and then working within that suborder, further restrict to an isomorphic suborder of points that form (say) blue pairs with σ_1. Continue in this way, nesting copies of the construction in the previous proposition.

We don't literally nest copies of the construction, as each iteration would increase in complexity, and the end result would not be arithmetic. Instead, when working above σ_1, we choose the color red for σ_0, and a node above which we are guaranteed to always have sufficiently many nodes that form red pairs with σ_0 (and red pairs with $\langle\rangle$); then we choose the color blue for σ_1, and a node above which we are guaranteed to always have sufficiently many nodes that form red pairs with σ_0 and blue pairs with σ_1 (and red pairs with $\langle\rangle$). When working at higher levels, we restrict ourselves to including only extensions of σ_1 that form correctly colored pairs with all three of $\langle\rangle$, σ_0, and σ_1.

The formal construction follows.

Again, we use Lemma 5.6 and arithmetic induction to produce a strong embedding with the desired properties. By Proposition 5.13, we may assume without loss of generality that, for all $\sigma >_{\mathbb{P}} \langle\rangle$, we have $c(\langle\rangle, \sigma) = 0$.

Recursively define the embedding φ and associated functions $\psi : \mathbb{P} \to \mathbb{P}$, and Δ on $\mathbb{P}$. The function ψ acts as in Case 2 of the previous proposition, as a guide to making φ a strong embedding. The function Δ chooses colors: each $\Delta(\sigma^\frown i)$ will be a function from $\{j \mid j \leq i\}$ to $\{0, 1\}$; we will guarantee that, for $\rho \supset \sigma^\frown i$, the pair $\langle \varphi(\sigma^\frown j), \varphi(\rho)\rangle$ has color $\Delta(\sigma^\frown i)(j)$. This suffices to ensure that φ has the desired property.

First we set $\varphi(\langle\rangle) = \psi(\langle\rangle) = \langle\rangle$, and $\Delta(\langle\rangle) = \emptyset$.

Now suppose that we have defined $\varphi(\gamma)$, $\psi(\gamma)$, and $\Delta(\gamma)$ for all γ below or to the left of $\sigma^\frown i$. We will say that $\pi \supset \psi(\sigma)$ is *acceptable* if, for all $\gamma^\frown k \subseteq \sigma$ and all $j < k$, we have $c(\varphi(\gamma^\frown j), \pi) = \Delta(\gamma^\frown k)(j)$. (Intuitively, π is a possible element of the range of φ, in light of commitments made by our definition of Δ so far; pairs $\langle \delta, \pi\rangle$ are correctly colored for all predecessors δ already in the range of φ.) We say that $\rho \supseteq \psi(\sigma)$ is *good* if infinitely many $\rho^\frown k$ are acceptable. (Similarly, ρ is a possible element of the range of ψ; it has infinitely many immediate extensions that are possible elements of the range of φ.)

Assume as inductive hypothesis that $\psi(\sigma)$ is good and every extension of $\psi(\sigma)$ has a further extension that is also good.

Since $\psi(\sigma)$ is good, there are infinitely many acceptable $\psi(\sigma)^\frown k$. Let $\varphi(\sigma^\frown i)$ be the least acceptable $\psi(\sigma)^\frown k$ unequal to any previously defined $\varphi(\sigma^\frown j)$.

For $d : \{j \mid j \leq i\} \to \{0, 1\}$, let $\pi \supseteq \varphi(\sigma^\frown i)$ be d-acceptable if π is acceptable and for all $j \leq i$ we have $c(\varphi(\sigma^\frown j), \pi) = d(j)$. Here d represents a potential choice for $\Delta(\sigma^\frown i)$, and π is d-good if the colors of pairs $\langle\varphi(\sigma^\frown j), \pi\rangle$ are as determined by Δ and d. That is, if we define $\Delta(\sigma^\frown i) = d$, π will still be acceptable, a possible element of the range of φ in light of these new commitments.

Let ρ be d-good if infinitely many $\rho^\frown k$ are d-acceptable. That is, if we define $\Delta(\sigma^\frown i) = d$, ρ will still be good, a possible element of the range of ψ.

Every acceptable extension π of $\varphi(\sigma^\frown i)$ is d-acceptable for a unique d (namely, the d that chooses the colors actually realized by π). Hence, every good extension of $\varphi(\sigma^\frown i)$ is d-good for at least one d. (The infinitely many acceptable immediate extensions are each d-acceptable for one of the finitely many values of d, so for at least one d there must be infinitely many d-acceptable immediate extensions.) Hence, for some d and some $\rho \supseteq \varphi(\sigma^\frown i)$, we must have that ρ is d-good, and every extension of ρ has a further extension that is d-good. (The proof of this is like that of Lemma 4.4; if not, then we can build an ascending chain of nodes above $\varphi(\sigma^\frown i)$, the ith of which has no extension d-good for the ith d. Since there are finitely many d's we eventually reach a node with no extension d-good for any d. This contradicts the inductive hypothesis that every extension of $\psi(\sigma)$ has further extensions that are good, and hence d-good for some d.)

Choose the least such ρ and d. Set $\psi(\sigma^\frown i) = \rho$ and $\Delta(\sigma^\frown i) = d$. Since we choose ρ to be d-good and have densely many d-good extensions, we now have that $\psi(\sigma^\frown i)$ is good and has densely many good extensions, so the inductive hypothesis is satisfied.

This completes the construction.

In the range of φ, the color of $\langle\delta^\frown j, \varepsilon\rangle$ for $\delta^\frown i \subseteq \varepsilon$, $j < i$, depends only on $\delta^\frown i$ and j, as desired: Let $\delta = \psi(\sigma)$ (since δ has more than one immediate extension in the range of φ, there must be such a σ), $\delta^\frown j = \varphi(\sigma^\frown \bar{j})$, $\delta^\frown i = \varphi(\sigma^\frown \bar{i})$, and $\varepsilon = \varphi(\rho)$. Because we chose $\varepsilon = \varphi(\rho)$ to be acceptable, we must have $c(\delta^\frown j, \varepsilon) = \Delta(\sigma^\frown \bar{i}, \bar{j})$. Since δ, i, and j determine σ, $\bar{i}$, and $\bar{j}$, they also determine $c(\delta^\frown j, \varepsilon)$. □

Proof of Proposition 5.12 Given a coloring c of pairs of $\mathbb{P}$ in two colors such that the color of $\langle\sigma, \tau\rangle$ depends only on σ and $\tau \restriction |\sigma|$, we produce an embedding of $\mathbb{P}$ with range on which the color of $\langle\sigma, \tau\rangle$ depends only on σ.

As noted above, setting $\sigma = \rho^\frown i$, we have $\tau \supset \rho^\frown j = \tau \restriction |\sigma|$ for some $j \geq i$; for fixed ρ, the color of $\langle\sigma, \tau\rangle$ depends only on i and j. That is, for each ρ, the color $\langle\rho^\frown i, \tau\rangle$ for $\rho^\frown j \subset \tau$ is actually a coloring of pairs (i, j); we would like it to depend only on the first coordinate i, that is, only on σ.

As a first approximation, to produce $\mathbb{Q}$ on which this holds, we will carry out the inductive step of the proof of (ordinary) Ramsey's theorem for pairs to prune the immediate extensions of ρ so that, on $\{k \mid \rho^\frown k \in \mathbb{Q}\}$, this coloring depends only on i. Simply pruning immediate extensions in this way would build a strong embedding. We cannot quite do this because the coloring is defined for $i \leq j$, not merely $i < j$, and it is possible that the color of (i, i) is different from that of (i, j) for $i < j$.

We may suppose, though, that we have made the color depend only on i; so, for example, any pair $\langle\rho^\frown 0, \pi\rangle$ is red unless π is actually an extension, not merely a successor, of $\rho^\frown 0$. Now we can remove all proper extensions of $\rho^\frown 0$ from $\mathbb{Q}$, and we can also remove $\rho^\frown 1$ (but not its proper extensions) from $\mathbb{Q}$; so if the embedding maps γ to $\rho^\frown 0$, then it maps the proper extensions of γ to proper extensions of $\rho^\frown 1$.

Since proper extensions of $\rho^\frown 1$ are successors of $\rho^\frown 0$, we still have an embedding. Now, as there are no extensions of $\rho^\frown 0$ left in $\mathbb{Q}$, any pair $\langle \rho^\frown 0, \pi \rangle$ must be red; that is, it depends only on $\rho^\frown 0$, as desired.

Actually, we won't quite do this, but we will combine the pruning strategies to make the color depend only on i for $j > i$ and to eliminate the problem case $i = j$.

The formal construction follows.

First we set $\varphi(\langle\rangle) = \psi(\langle\rangle) = \langle\rangle$, and $h(\langle\rangle)$ to be the color of pairs $\langle \langle\rangle, \sigma \rangle$, which by assumption is the same for all σ. The function ψ will now be a guide to building an embedding that is not strong; we will still embed the immediate extensions of σ into the immediate extensions of $\psi(\sigma)$, but now we will choose $\psi(\sigma)$ to the right of $\varphi(\sigma)$. The function h will be our choice of colors; any ordered pair $\langle \varphi(\sigma), \varphi(\tau) \rangle$ will have color $h(\sigma)$.

Suppose that we have defined $\varphi(\delta)$, $\psi(\delta)$, and $h(\delta)$ for all δ below or to the left of $\rho^\frown \ell$.

We assume the following inductive hypotheses: $\varphi(\gamma^\frown j)$ and $\psi(\gamma^\frown j)$ are immediate extensions of $\psi(\gamma)$; and $\varphi(\gamma^\frown (j+1))$ is to the right of $\psi(\gamma^\frown j)$, which is to the right of $\varphi(\gamma^\frown j)$. (Hence extensions of $\psi(\gamma^\frown j)$ are successors of $\varphi(\gamma)$ but not extensions of $\varphi(\gamma)$.) For all $\sigma \supset \psi(\gamma^\frown j)$ and all $i \leq j$, the pair $\langle \varphi(\gamma^\frown i), \sigma \rangle$ has color $h(\gamma^\frown i)$.

We have a further inductive assumption allowing us to keep the coloring commitments made by h, as in the last proof: Say that k is good if, first, k is large enough so that for all $i < \ell$ (all i for which $\varphi(\rho^\frown i)$ and $\psi(\rho^\frown i)$ have been defined), $\psi(\rho)^\frown k$ is to the right of $\psi(\rho^\frown i)$ and $\varphi(\rho^\frown i)$ (this makes sense because by inductive hypothesis $\psi(\rho^\frown i)$ and $\varphi(\rho^\frown i)$ immediately extend $\psi(\rho)$); and, second, for all $i < \ell$, $\langle \varphi(\rho^\frown i), \sigma \rangle$ for $\sigma \supset \psi(\rho)^\frown k$ has color $h(\rho^\frown i)$. (By the hypothesis of the theorem, this color is independent of σ, depending only on $\psi(\rho)^\frown k$ and $\varphi(\rho^\frown i)$.) The final inductive hypothesis is that there are infinitely many good k. Note that if $\ell = 0$, all k are good, and this final inductive hypothesis holds trivially.

Let $\varphi(\rho^\frown \ell) = \psi(\rho)^\frown k_0$ where k_0 is the least good k. This preserves the inductive hypothesis that $\varphi(\rho^\frown \ell)$ is an immediate extension of $\psi(\rho)$ and to the right of $\varphi(\rho^\frown i)$ and $\psi(\rho^\frown i)$ for $i < \ell$.

Let $h(\rho^\frown \ell)$ be 0 if there are infinitely many good $k > k_0$ such that $\langle \varphi(\rho^\frown \ell), \sigma \rangle$ for $\sigma \supset \psi(\rho)^\frown k$ has color 0, and $h(\rho^\frown \ell) = 1$ otherwise. Finally, let $\psi(\rho^\frown \ell) = \psi(\rho)^\frown k_1$ where k_1 is the least good $k > k_0$ such that $\langle \varphi(\rho^\frown \ell), \sigma \rangle$ for $\sigma \supset \psi(\rho)^\frown k$ has color $h(\rho^\frown \ell)$. This preserves the inductive hypothesis that $\psi(\rho^\frown \ell)$ is an immediate extension of $\psi(\rho)$ and to the right of $\varphi(\rho^\frown \ell)$.

It also preserves the inductive hypothesis that $\langle \varphi(\rho^\frown i), \sigma \rangle$ for $\sigma \supset \psi(\rho)^\frown k$ has color $h(\rho^\frown i)$ for $i \leq \ell$, for $i < \ell$ because k_1 is good, and for $i = \ell$ by choice of k_1 to realize $h(\rho^\frown \ell)$. Finally, since those formerly good $k > k_1$ such that $\langle \varphi(\rho^\frown \ell), \sigma \rangle$ for $\sigma \supset \psi(\rho)^\frown k$ has color $h(\rho^\frown \ell)$ are still good, it preserves the inductive hypothesis that there are infinitely many good k.

This completes the construction.

The embedding φ is as required; that is, if $\delta <_{\mathbb{P}} \gamma$, then $\langle \varphi(\delta), \varphi(\gamma) \rangle$ has color $h(\delta)$. If $\delta = \langle\rangle$, then this is by assumption. Otherwise, $\delta = \rho^\frown i$ and $\gamma \supset \rho^\frown k$ for some $k \geq i$. By construction, $\varphi(\gamma) = \sigma \supset \psi(\rho^\frown k)$, and since $\psi(\rho^\frown k)$ was chosen to be good, $\langle \varphi(\rho^\frown i), \sigma \rangle$ has color $h(\rho^\frown i)$, that is, $h(\delta)$. $\square$

6 Questions

Theorem 4.1 shows that one direction of the characterization of countably infinite, bounded-level, n-Ramsey partial orderings ($n \geq 3$) is equivalent to ACA_0 over RCA_0. The other direction, by Theorem 3.1, is provable from ATR_0. Can it be proven from less? In particular, can Theorem 3.1 be proven from ACA_0? We know that the special case of Theorem 3.1 restricted to trees is provable from RCA_0 (see [5]). We also know that we cannot have a reverse mathematics result (proving, for example, ATR_0 from Theorem 3.1), since in the ω-model consisting of the recursive sets there are no infinite, bounded-level, n-Ramsey partial orderings for any $n \geq 2$, and so the theorem holds vacuously.

Theorem 5.1 says that there is a primitive recursive partial ordering $\mathbb{P}$ such that, over RCA_0, the statement $R^2_2(\mathbb{P})$ is equivalent to ACA_0. The proof makes essential use of the fact that the levels of $\mathbb{P}$ are infinite. Is this necessary? That is, is there a bounded-level partial ordering $\mathbb{P}$ such that $R^2(\mathbb{P})$ is equivalent to ACA_0? More specifically, to repeat a question of Chubb, Hirst, and McNicholl [3], is $R^2(\mathbb{T})$, where $\mathbb{T}$ denotes the complete binary tree, equivalent to ACA_0? Dzhafarov, Hirst, and Lakins [7] have investigated aspects of $R^3(\mathbb{T})$ related to relevant aspects of $R^2(\omega)$.

References

[1] Cholak, P., C. Jockusch, and T. Slaman, "On the strength of Ramsey's theorem for pairs," *Journal of Symbolic Logic*, vol. 66 (2001), pp. 1–55. Zbl 0977.03033. MR 1825173. DOI 10.2307/2694910. 1

[2] Chong, C. T., T. Slaman, and Yue Yang, "Π^1_1-conservation of combinatorial principles weaker than Ramsey's theorem for pairs," *Advances in Mathematics*, vol. 230 (2012), pp. 1060–1077. MR 2921172. DOI 10.1016/j.aim.2012.02.025. 2

[3] Chubb, J., J. Hirst, and T. McNicholl, "Reverse mathematics, computability, and partitions of trees," *Journal of Symbolic Logic*, vol. 74 (2009), pp. 201–215. MR 2499427. DOI 10.2178/jsl/1231082309. 2, 3, 24

[4] Corduan, J., "Coloring posets and reverse mathematics," Ph.D. dissertation, Dartmouth College, Hanover, N.H., 2011. MR 2949706. 3, 13

[5] Corduan, J., M. Groszek, and J. Mileti, "Reverse mathematics and Ramsey's property for trees," *Journal of Symbolic Logic*, vol. 75 (2010), pp. 945–954. MR 2723776. DOI 10.2178/jsl/1278682209. 2, 3, 4, 13, 24

[6] Dzhafarov, D. D., "Stable Ramsey's theorem and measure," *Notre Dame Journal of Formal Logic*, vol. 52 (2011), pp. 95–112. MR 2747165. DOI 10.1215/00294527-2010-039. 2

[7] Dzhafarov, D. D., J. Hirst, and T. J. Lakins, "Ramsey's theorem for trees: The polarized tree theorem and notions of stability, *Archive for Mathematical Logic*, vol. 49 (2010), pp. 399–415. MR 2609990. DOI 10.1007/s00153-010-0179-6. 24

[8] Groszek, M., "Ramsey properties of countably infinite partial orderings," *Electronic Journal of Combinatorics*, vol. 20 (2013), paper no. 50, 27 pp. MR 3040612. 2, 3

[9] Hirschfeldt, D. R., C. G. Jockusch, Jr., B. Kjos-Hanssen, S. Lempp, and T. A. Slaman, "The strength of some combinatorial principles related to Ramsey's theorem for pairs," pp. 143–161 in *Computational Prospects of Infinity: Part II—Presented Talks*, vol. 15 of Lecture Notes Series, Institute for Mathematical Sciences, National University of Singapore, World Scientific, Hackensack, N.J., 2008. MR 2449463. DOI 10.1142/9789812796554_0008. 2

[10] Hirschfeldt, Denis R., and R. Shore, "Combinatorial principles weaker than Ramsey's theorem for pairs," *Journal of Symbolic Logic*, vol. 72 (2007), pp. 171–206. MR 2298478. DOI 10.2178/jsl/1174668391. 2

[11] Hirst, J., "Combinatorics in subsystems of second order arithmetic," Ph.D. dissertation, Pennsylvania State University, State College, Pa., 1987. MR 2635978. 1

[12] Hummel, T., "Effectiveness of Ramsey's theorem—avoiding the cone above $\emptyset'$," *Journal of Symbolic Logic*, vol. 59 (1994), pp. 1301–1305. MR 1312312. DOI 10.2307/2275707. 1

[13] Kach, A. M., M. Lerman, and R. Solomon, "Cappable CEA sets and Ramsey's theorem," pp. 114–127 in *Proceedings of the 11th Asian Logic Conference*, World Scientific, Hackensack, N.J., 2012. Zbl 1280.03041. MR 2868509. DOI 10.1142/9789814360548_0007. 2

[14] Milliken, K. R., "A Ramsey theorem for trees," *Journal of Combinatorial Theory, Series A*, vol. 26 (1979), pp. 215–237. MR 0535155. DOI 10.1016/0097-3165(79)90101-8. 2

[15] Nešetřil, J., "Ramsey theory," pp. 1331–1403 in *Handbook of Combinatorics*, edited by R. L. Graham, M. Grötschel, and L. Lovász, North-Holland, Amsterdam, 1995. MR 1373681. 2

[16] Nešetřil, J., "Ramsey classes and homogeneous structures," *Combinatorics, Probability, and Computing*, vol. 14 (2005), pp. 171–189. MR 2128088. DOI 10.1017/S0963548304006716. 2

[17] Ramsey, F. P., "On a problem of formal logic," *Proceedings of the London Mathematical Society, Series 2*, vol. 30 (1930), pp. 264–286. Zbl 55.0032.04. MR 1576401. DOI 10.1112/plms/s2-30.1.264. 1

[18] Seetapun, D., and T. A. Slaman. "On the strength of Ramsey's theorem," *Notre Dame Journal of Formal Logic*, vol. 36 (1995), pp. 570–582. MR 1368468. DOI 10.1305/ndjfl/1040136917. 1

[19] Simpson, S., *Subsystems of Second Order Arithmetic*. Perspectives in Mathematical Logic, Springer, Berlin, 1999. MR 1723993. DOI 10.1007/978-3-642-59971-2. 1, 3, 13, 14, 17

Acknowledgments

The authors would like to thank Jeff Hirst, Joe Mileti, and Pete Winkler for interesting and helpful discussions. We also thank the referee for careful reading and helpful suggestions.

Corduan
jared.corduan@gmail.com

Groszek
Department of Mathematics
Dartmouth College
6188 Kemeny Hall
Hanover NH 03755
USA
marcia.groszek@dartmouth.edu