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Vaught's Conjecture Without Equality

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To Professor Gerald Sacks
on the occasion of his retirement from Harvard University

Abstract Suppose that $\sigma \in \mathcal{L}_{\omega_1,\omega}(\mathrm{L})$ is such that all equations occurring in σ are positive, have the same set of variables on each side of the equality symbol, and have at least one function symbol on each side of the equality symbol. We show that σ satisfies Vaught's conjecture. In particular, this proves Vaught's conjecture for sentences of $\mathcal{L}_{\omega_1,\omega}(\mathrm{L})$ without equality.

1 Introduction

Vaught's conjecture is one of the oldest open problems in model theory. It says (in its modern form) that for any countable language L and any sentence $\sigma \in \mathcal{L}_{\omega_1,\omega}(\mathrm{L})$, either σ has a perfect set of countable models or σ has countably many countable models. Vaught's conjecture is known to hold in many situations, such as for ω-stable theories (see Shelah, Harrington, and Makkai [7]), for o-minimal theories (see Mayer [4]), as well as many others. In this paper we add a new collection of sentences for which Vaught's conjecture is known to hold.

Call an equation $t_0(x_1, \ldots, x_n) = t_1(y_1, \ldots, y_m)$ *uniform* if the sets of variables $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_m\}$ are equal (as sets), and both t_0 and t_1 contain at least one function symbol. We will show (see Theorem 4.1) that if $\sigma \in \mathcal{L}_{\omega_1,\omega}(\mathrm{L})$ is any sentence in which all equations are uniform and occur positively, then σ satisfies Vaught's conjecture. As an immediate consequence we will see that Vaught's conjecture holds for any sentence of $\mathcal{L}_{\omega_1,\omega}(\mathrm{L})$ which does not contain equations. This will answer a question in Sági and Sziráki [6]. Our proof will also show that Martin's conjecture holds for sentences of this form.

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Our proof will proceed in three parts. First, in Section 2, we show that for each model there is a maximal equivalence relation whose quotient map is a homomorphism which reflects all non-equality relations. We then study these equivalence relations along with their quotients, which we call *cores*. In Section 3 we show that under certain conditions cores can be *blown up* to produce a perfect set of models all of which satisfy some of the same sentence of $\mathcal{L}_{\omega_1,\omega}(\mathrm{L})$. Finally, in Section 4, we use the results of Section 3 to show that both Vaught's conjecture and Martin's conjecture hold for any sentence which only contains equalities that occur positively and are uniform.

1.1 Background In this paper we will not treat equality as a logical symbol but rather as a relation which is in any language and which has a special interpretation in any structure. We will fix a countable language L along with a countable collection of variables, from which all variables will be drawn. In this paper σ and its variants will be elements of $\mathcal{L}_{\omega_1,\omega}(\mathrm{L})$. We let $\mathtt{Atomic}(\mathrm{L})$ be the collection all formulas which are either atomic or the negation of atomic formulas. If $F \subseteq \mathcal{L}_{\omega_1,\omega}(\mathrm{L})$, then we let $\mathcal{L}^c_{\omega_1,\omega}(F)$ be the smallest subset of $\mathcal{L}_{\omega_1,\omega}(\mathrm{L})$ containing F and closed under $\bigwedge, \bigvee, \exists$, and $\forall$. We will mainly be concerned with $\mathcal{L}^c_{\omega_1,\omega}(F)$ when $F \subseteq \mathtt{Atomic}(\mathrm{L})$. In particular, there are several subsets of $\mathtt{Atomic}(\mathrm{L})$ which will be important later and which we collect now:

- $\mathtt{Rel} = \{Q(\mathbf{x}) : Q$ does not contain $=\}$.
- $\mathtt{Uni} = \mathtt{Rel} \cup \{t_0(\mathbf{x}) = t_1(\mathbf{y})$: a uniform equation$\}$.
- $\mathtt{Func} = \mathtt{Rel} \cup \{t_0(\mathbf{x}) = t_1(\mathbf{y}) : t_0, t_1$ any terms each of which contains at least one function symbol$\}$.
- $\mathtt{Pos} = \mathtt{Rel} \cup \{t_0(\mathbf{x}) = t_1(\mathbf{y}) : t_0, t_1$ any terms$\}$.
- $\mathtt{Neg} = \mathtt{Rel} \cup \{t_0(\mathbf{x}) \neq t_1(\mathbf{y}) : t_0, t_1$ any terms$\}$.

Note that any sentence of $\sigma \in \mathcal{L}_{\omega_1,\omega}(\mathrm{L})$ is equivalent to one where negation only occurs in front of atomic formulas. Hence if σ is any sentence in which all equations are positive and uniform, then σ is equivalent to a sentence in $\mathcal{L}^c_{\omega_1,\omega}(\mathtt{Uni})$.

In this paper all models will be countable, and we let $\mathfrak{M}$ and $\mathfrak{N}$ (and their variants) be L-structures with underlying sets M and N, respectively. We say that a map $\alpha : M \to N$ is a *strong homomorphism* if for any j-ary relation $R \in \mathrm{L} - \{=\}$,

$$(\forall m_1, \ldots, m_j \in M) \quad \mathfrak{M} \models R(m_1, \ldots, m_j) \Leftrightarrow \mathfrak{N} \models R\big(\alpha(m_1), \ldots, \alpha(m_j)\big)$$

and for any j-ary function f,

$$(\forall m_1, \ldots, m_j \in M) \quad \mathfrak{N} \models \alpha\big(f(m_1, \ldots, m_j)\big) = f\big(\alpha(m_1), \ldots, \alpha(m_j)\big)$$

(we will consider constants as 0-ary functions). Note that α is a strong homomorphism exactly when it preserves all formulas in $\mathtt{Pos}$.

We will assume that we are working in a background model Set of Zermelo–Frankel set theory. However, all of our statements about specific σ are $\Sigma^1_2(\sigma)$, and so they hold of σ in Set if and only if they hold of σ in $\mathsf{L}[\sigma]$. But, as $\mathsf{L}[\sigma]$ always satisfies the axiom of choice, we can assume without loss of generality that Set does as well.

For any definitions or results not in this paper, we refer the reader to such standard texts as Barwise [1] for infinitary logic, Hodges [2] for model theory, and Jech [3] for set theory.

2 Preserving and Reflecting Formulas

In this section we study equivalence relations whose quotients preserve and reflect formulas we care about.

Definition 2.1 If $\alpha : \mathfrak{M} \to \mathfrak{N}$ is a strong homomorphism, let $\mathrm{Pres}(\alpha)$ be the collection of formulas $\varphi \in \mathcal{L}_{\omega_1,\omega}(\mathrm{L})$ such that

$$(\forall m_1, \ldots, m_j) \quad \mathfrak{M} \models \varphi(m_1, \ldots, m_j) \Rightarrow \mathfrak{N} \models \varphi\big(\alpha(m_1), \ldots, \alpha(m_j)\big),$$

that is, the collection of formulas which are *preserved* by α. Also, let $\mathrm{Refl}(\alpha)$ be the collection of formulas $\varphi \in \mathcal{L}_{\omega_1,\omega}(\mathrm{L})$ such that

$$(\forall m_1, \ldots, m_j) \quad \mathfrak{N} \models \varphi\big(\alpha(m_1), \ldots, \alpha(m_j)\big) \Rightarrow \mathfrak{M} \models \varphi(m_1, \ldots, m_j),$$

that is, the collection of formulas which are *reflected* by α.

Lemma 2.2 *Suppose that $\alpha : \mathfrak{M} \to \mathfrak{N}$ is a surjective strong homomorphism. Then*

- $\mathcal{L}^c_{\omega_1,\omega}(\mathrm{Pres}(\alpha)) = \mathrm{Pres}(\alpha)$,
- $\mathcal{L}^c_{\omega_1,\omega}(\mathrm{Refl}(\alpha)) = \mathrm{Refl}(\alpha)$,

that is, $\mathrm{Pres}(\alpha)$ *and* $\mathrm{Refl}(\alpha)$ *are both closed under* $\bigwedge, \bigvee, \exists, \forall$.

Proof For any strong homomorphism it is immediate that both $\mathrm{Pres}(\alpha)$ and $\mathrm{Refl}(\alpha)$ are closed under $\bigwedge$ and $\bigvee$. It is also immediate that $\mathrm{Pres}(\alpha)$ is closed under $\exists$ and $\mathrm{Refl}(\alpha)$ is closed under $\forall$.

That $\mathrm{Pres}(\alpha)$ is closed under $\forall$ and that $\mathrm{Refl}(\alpha)$ is closed under $\exists$ follow from the surjectivity of α. $\square$

The following is then immediate.

Corollary 2.3 *If $\alpha : \mathfrak{M} \to \mathfrak{N}$ is a surjective strong homomorphism, then both $\mathcal{L}^c_{\omega_1,\omega}(\mathtt{Pos}) \subseteq \mathrm{Pres}(\alpha)$ and $\mathcal{L}^c_{\omega_1,\omega}(\mathtt{Neg}) \subseteq \mathrm{Refl}(\alpha)$.*

Every surjective map $\alpha : M \to N$ induces an equivalence relation $\equiv_\alpha$ on M given by $a \equiv_\alpha b$ if and only if $\alpha(a) = \alpha(b)$. Further, if α is a strong homomorphism, then $N \cong M/{\equiv_\alpha}$. Given an equivalence relation $\equiv$ on M, and $a \in M$, we define $[a]_\equiv := \{b \in M : b \equiv a\}$.

Definition 2.4 An equivalence relation $\equiv$ on M is said to *respect* L if there is a (necessarily unique) L-structure with underlying set $M/{\equiv}$ such that the quotient map $e_\equiv : M \to M/{\equiv}$ is a strong homomorphism.

As it turns out, on any structure there is a unique maximal equivalence relation which respects the language. This equivalence relation will play a significant role in what follows.

Definition 2.5 Let

$$\begin{aligned} &\asymp (y_0, y_1) \\ &\quad := \bigwedge\big\{(\forall x_1, \ldots, x_j) Q(y_0, x_1, \ldots, x_j) \leftrightarrow Q(y_1, x_1, \ldots, x_j) : Q \in \mathtt{Rel}\big\}. \end{aligned}$$

We will write $\asymp (y_0, y_1)$ as $y_0 \asymp y_1$.

It is immediate from the definition that $\asymp^{\mathfrak{M}}$ is always an equivalence relation on $\mathfrak{M}$. We will abbreviate the quotient map $e_{\asymp^{\mathfrak{M}}} : M \to M/\asymp^{\mathfrak{M}}$ by $e_{\mathfrak{M}}$. We now show several important properties which always hold of $\asymp^{\mathfrak{M}}$ for any L-structure $\mathfrak{M}$.

Proposition 2.6

(1) $\asymp^{\mathfrak{M}}$ *respects* L.

Suppose that $\equiv$ *is an equivalence relation on* M *which respects* L.

(2) $\equiv \subseteq \asymp^{\mathfrak{M}}$.

(3) If $\equiv$ *is definable by a formula in* $\mathcal{L}^{c}_{\omega_1,\omega}(\mathtt{Rel})$*, then* $\equiv^{\mathfrak{M}} = \asymp^{\mathfrak{M}}$.

Proof (1):

We will define an L-structure $\mathfrak{N}$ with underlying set $M/\asymp^{\mathfrak{M}}$ such that $e_{\mathfrak{M}} : \mathfrak{M} \to \mathfrak{N}$ is a strong homomorphism.

For any $a_i^*, a_1, \ldots, a_j \in M$ with $a_i^* \asymp^{\mathfrak{M}} a_i$ and any $Q \in \mathtt{Rel}$ we have

$$\begin{aligned} \mathfrak{M} \models Q(&a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_j) \\ &\leftrightarrow Q(a_1, \ldots, a_{i-1}, a_i^*, a_{i+1}, \ldots, a_j). \end{aligned} \tag{1}$$

Hence, by repeated use of (1), we have that whenever $b_1, \ldots, b_j \in M$ with $\bigwedge_{i \leq j} a_i \asymp^{\mathfrak{M}} b_i$, that $\mathfrak{M} \models Q(a_1, \ldots, a_j) \leftrightarrow Q(b_1, \ldots, b_j)$. Therefore for any j-ary relation R, whether or not $\mathfrak{M} \models R(a_1, \ldots, a_j)$ holds depends only on the $\asymp$-equivalence classes of $a_1, \ldots, a_j$, and so it is consistent to define $\mathfrak{N} \models R([a_1]_{\asymp}, \ldots, [a_j]_{\asymp})$ if and only if $\mathfrak{M} \models R(a_1, \ldots, a_j)$. It is then clear that the map $e_{\mathfrak{M}}$ preserves and reflects $R(x_1, \ldots, x_j)$.

Now suppose that f is any j-ary function symbol in L, $a_1, \ldots, a_j, b_1, \ldots, b_j \in M$ and $\bigwedge_{i \leq j} a_i \asymp^{\mathfrak{M}} b_i$. For any $Q \in \mathtt{Rel}$, let $Q'(x_1, \ldots, x_k, y_1, \ldots, y_j)$ be the formula $Q(x_1, \ldots, x_k, f(y_1, \ldots, y_j))$. By the argument of the previous paragraph, if $c_1, \ldots, c_k \in M$, we have $\mathfrak{M} \models Q'(c_1, \ldots, c_k, a_1, \ldots, a_j) \leftrightarrow Q'(c_1, \ldots, c_k, b_1, \ldots, b_j)$. Hence, as Q was arbitrary, $f^{\mathfrak{M}}(a_1, \ldots, a_j) \asymp^{\mathfrak{M}} f^{\mathfrak{M}}(b_1, \ldots, b_j)$. This means that the $\asymp$-equivalence class of $f^{\mathfrak{M}}(a_1, \ldots, a_j)$ depends only on the $\asymp$-equivalence classes of $a_1, \ldots, a_j$. Hence it is consistent to define $\mathfrak{N} \models f([a_1]_{\asymp}, \ldots, [a_j]_{\asymp}) = [f(a_1, \ldots, a_j)]_{\asymp}$, which we do. It is then clear that $\mathfrak{N}$ is an L-structure and $e_{\mathfrak{M}} : \mathfrak{M} \to \mathfrak{N}$ is a strong homomorphism.

(2):

Suppose that $a_1, \ldots, a_j, b, c \in M$, $b \equiv c$, and $Q \in \mathtt{Rel}$. We then have the following equivalences: $\mathfrak{M} \models Q(b, a_1, \ldots, a_j)$ if and only if $\mathfrak{M}/\equiv \models Q(e_{\equiv}(b), e_{\equiv}(a_1), \ldots, e_{\equiv}(a_j))$ if and only if $\mathfrak{M}/\equiv \models Q(e_{\equiv}(c), e_{\equiv}(a_1), \ldots, e_{\equiv}(a_j))$ if and only if $\mathfrak{M} \models Q(c, a_1, \ldots, a_j)$. Hence $\mathfrak{M} \models (\forall x_1, \ldots, x_j) Q(b, x_1, \ldots, x_j) \leftrightarrow Q(c, x_1, \ldots, x_j)$. But, as Q was arbitrary, this implies $b \asymp^{\mathfrak{M}} c$.

(3):

By (2) we have $\equiv \subseteq \asymp^{\mathfrak{M}}$. Suppose that $x \equiv y$ is definable by a formula $\psi(x, y) \in \mathcal{L}^{c}_{\omega_1,\omega}(\mathtt{Rel})$ and that $a, b \in M$ with $a \not\equiv b$. As $\mathcal{L}^{c}_{\omega_1,\omega}(\mathtt{Rel})$ is closed (up to equivalence) under negation, we have that $\neg\psi(x, y)$ is equivalent to a formula in $\mathcal{L}^{c}_{\omega_1,\omega}(\mathtt{Rel})$. But $e_{\mathfrak{M}}$ preserves all formulas of $\mathcal{L}^{c}_{\omega_1,\omega}(\mathtt{Rel})$ and so $\mathfrak{M}/\asymp^{\mathfrak{M}} \models \neg\psi(e_{\mathfrak{M}}(a), e_{\mathfrak{M}}(b))$. However this implies $e_{\mathfrak{M}}(a) \not\asymp^{\mathfrak{M}/\asymp^{\mathfrak{M}}} e_{\mathfrak{M}}(b)$ (as $\mathfrak{M} \models (\forall x)\psi(x, x)$, and so $\mathfrak{M}/\asymp^{\mathfrak{M}} \models (\forall x)\psi(x, x)$). Because $e_{\mathfrak{M}}$ reflects all formulas of $\mathcal{L}^{c}_{\omega_1,\omega}(\mathtt{Rel})$, including $\asymp$, we have $a \not\asymp^{\mathfrak{M}} b$. In particular, this implies $\asymp^{\mathfrak{M}} \subseteq \equiv$ and hence $\asymp^{\mathfrak{M}} = \equiv$. □

Because $\asymp$ respects L the following notion is well defined.

Definition 2.7 The *core* of $\mathfrak{M}$, denoted $\mathbf{C}(\mathfrak{M})$, is the unique L-structure with underlying set $M/\asymp^{\mathfrak{M}}$ such that $e_{\mathfrak{M}} : M \to M/\asymp^{\mathfrak{M}}$ is a strong homomorphism.

It is immediate that $\mathbf{C}(\mathbf{C}(\mathfrak{M})) \cong \mathbf{C}(\mathfrak{M})$, and so we say that $\mathfrak{M}$ is a core if $\mathbf{C}(\mathfrak{M}) \cong \mathfrak{M}$. In particular, $\mathfrak{M}$ is a core if and only if $\asymp^{\mathfrak{M}}$ is $=^{\mathfrak{M}}$. The following is a quintessential example of a core.

Example 2.8 Suppose that $\leq \in \mathrm{L}$ is a binary relation and that $\mathfrak{M} \models$ "$\leq$ is a partial order". Then $\mathfrak{M}$ is a core. To see this, observe that the formula "$x \leq y \wedge y \leq x$" is an equivalence relation definable in $\mathcal{L}^c_{\omega_1,\omega}(\mathtt{Rel})$ and hence must be equivalent (over $\mathfrak{M}$) to $\asymp$. But as $\leq$ is a partial order, $x \leq y \wedge y \leq x$ implies that $x = y$.

The following is an easy corollary of Corollary 2.3 and Proposition 2.6.

Corollary 2.9 *If* $\sigma \in \mathcal{L}^c_{\omega_1,\omega}(\mathtt{Pos})$ *and* $\mathfrak{M} \models \sigma$, *then* $\mathbf{C}(\mathfrak{M}) \models \sigma$.

3 Properties of Cores

In this section we discuss what can be said about a sentence just knowing that it is satisfied by a core.

Proposition 3.1 *Suppose that there is a* j*-ary function symbol* $g \in \mathrm{L}$ *with* $j > 0$, $\sigma \in \mathcal{L}^c_{\omega_1,\omega}(\mathtt{Uni} \cup \mathtt{Neg})$ *and that there is a core* $\mathfrak{M}$ *such that* $\mathfrak{M} \models \sigma$. *Then there is a perfect set of countable* L*-structures all of which satisfy* σ.

Proof Suppose that $\mathbf{C}(\mathfrak{N}) = \mathfrak{M}$ and that $e_{\mathfrak{N}} : \mathfrak{N} \to \mathbf{C}(\mathfrak{N})$ reflects all formulas in $\mathtt{Uni}$. Then by Lemma 2.2 and Corollary 2.3 $e_{\mathfrak{N}}$ reflects all formulas in $\mathcal{L}^c_{\omega_1,\omega}(\mathtt{Uni} \cup \mathtt{Neg})$. In particular, $e_{\mathfrak{N}}$ reflects σ and as $\mathbf{C}(\mathfrak{N}) = \mathfrak{M}$, we have $\mathfrak{N} \models \sigma$ as well.

It therefore suffices to construct, for each $S \subseteq \mathbb{N} - \{0\}$, a countable model $\mathfrak{M}_S$ such that $\mathbf{C}(\mathfrak{M}_S) = \mathfrak{M}$, $e_{\mathfrak{M}_S}$ reflects all formulas in $\mathtt{Uni}$, and if $S_0 \neq S_1$, then $\mathfrak{M}_{S_0} \ncong \mathfrak{M}_{S_1}$. We will define $\mathfrak{M}_S$ in three stages.

Stage 1:

Let $A_S = \bigcup_{n \in S} \{n\} \times n$. The underlying set of $\mathfrak{M}_S$ is $M_S = M \times A_S$.

Stage 2:

For any j-ary relation $R \in \mathrm{L}$ and any $\langle m_1, n_1, a_1\rangle, \dots, \langle m_j, n_j, a_j\rangle \in M_S$,

$$\mathfrak{M}_S \models R\big(\langle m_1, n_1, a_1\rangle, \dots, \langle m_j, n_j, a_j\rangle\big) \Leftrightarrow \mathfrak{M} \models R(m_1, \dots, m_j).$$

Stage 3:

For any j-ary function $f \in \mathrm{L}$ and any $\langle m_1, n_1, a_1\rangle, \dots, \langle m_j, n_j, a_j\rangle \in M_S$,

$$\mathfrak{M}_S \models f\big(\langle m_1, n_1, a_1\rangle, \dots, \langle m_j, n_j, a_j\rangle\big) = \langle m^*, n^*, a^*\rangle$$

if and only if

- $\mathfrak{M} \models f(m_1, \dots, m_j) = m^*$,
- $n^* = \min\{n_1, \dots, n_j\}$,
- $a^* = 0$.

Let $(m_1, n_1, a_1) \equiv (m_2, n_2, a_2)$ if and only if $m_1 = m_2$. It is then immediate that $\equiv$ is an equivalence relation which respects L and hence $\equiv \subseteq \asymp^{\mathfrak{M}_S}$. Further, as $\mathfrak{M} \cong \mathfrak{M}_S/\equiv$ and $\mathfrak{M}$ is a core, $\equiv$ must be the maximal equivalence relation which respects L. Hence by Proposition 2.6 we have $\equiv = \asymp^{\mathfrak{M}_S}$ and $\mathbf{C}(\mathfrak{M}_S) \cong \mathfrak{M}$.

If $t(x_1, \dots, x_n)$ is an arbitrary term containing at least one function symbol, then

$$\mathfrak{M}_S \models t\big(\langle m_1, n_1, a_1\rangle, \dots, \langle m_j, n_j, a_j\rangle\big) = \big\langle t^{\mathfrak{M}}(m_1, \dots, m_j), \min\{n_1, \dots, n_j\}, 0\big\rangle.$$

So, if $\mathfrak{M} \models t_0(m_1, \ldots, m_j) = t_1(m_1, \ldots, m_j)$, with $t_0(x_1, \ldots, x_j) = t_1(x_1, \ldots, x_j)$ a uniform equation, then for all $\langle n_1, a_1\rangle, \ldots, \langle n_j, a_j\rangle \in A_S$ we have

$$\begin{aligned}\mathfrak{M}_S \models t_0\big(\langle m_1, n_1, a_1\rangle, \ldots, \langle m_j, n_j, a_j\rangle\big) \\ = t_1\big(\langle m_1, n_1, a_1\rangle, \ldots, \langle m_j, n_j, a_j\rangle\big) \end{aligned} \tag{2}$$

and hence $e_{\mathfrak{M}_S}$ reflects $t_0(x_1, \ldots, x_j) = t_1(x_1, \ldots, x_j)$. Notice that (2) hinges on the fact that each n_i occurs on each side of the equality.

Now suppose that $S_0, S_1 \subseteq \mathbb{N} - \{0\}$ but that $S_0 \neq S_1$. Let $h(x) = g(x, x, \ldots, x)$. For any L-structure $\mathfrak{N}$ let $W(\mathfrak{N}) = \{|(h^{\mathfrak{N}})^{-1}(a) \cap [b]_{\asymp^{\mathfrak{N}}}| : a, b \in N\}$, that is, the possible sizes of the inverse images (under h) of an element in an $\asymp^{\mathfrak{N}}$-equivalence class. It is immediate from Definition 2.5 that whenever $\mathfrak{N}_0 \cong \mathfrak{N}_1$, $W(\mathfrak{N}_0) = W(\mathfrak{N}_1)$. But it is also immediate from the construction that $W(\mathfrak{M}_S) = S \cup \{0\}$. So $W(\mathfrak{M}_{S_0}) \neq W(\mathfrak{M}_{S_1})$ and hence $\mathfrak{M}_{S_0} \not\cong \mathfrak{M}_{S_1}$ and we are done. □

Proposition 3.2 *Suppose that $\sigma \in \mathcal{L}^c_{\omega_1,\omega}(\mathtt{Func} \cup \mathtt{Neg})$ and that there is an infinite core $\mathfrak{M}$ such that $\mathfrak{M} \models \sigma$. Then there is a perfect set of* L*-structures all of which satisfy σ.*

Proof It suffices to construct, for each $S \subseteq \mathbb{N} - \{0\}$, a model $\mathfrak{M}_S$ such that $\mathbf{C}(\mathfrak{M}_S) = \mathfrak{M}$, $e_{\mathfrak{M}_S}$ reflects all formulas of Func, and where $S_0 \neq S_1$ implies that $\mathfrak{M}_{S_0} \not\cong \mathfrak{M}_{S_1}$. We will define $\mathfrak{M}_S$ in three stages.

Stage 1:

Let $i : M \to S$ be a surjective map (which must exist as M is infinite). Then the underlying set of $\mathfrak{M}_S$ is $M_S = \bigcup_{m \in M}\{m\} \times i(m)$.

Stage 2:

For any j-ary relation $R \in$ L and any $\langle m_1, a_1\rangle, \ldots, \langle m_j, a_j\rangle \in M_S$,

$$\mathfrak{M}_S \models R\big(\langle m_1, a_1\rangle, \ldots, \langle m_j, a_j\rangle\big) \Leftrightarrow \mathfrak{M} \models R(m_1, \ldots, m_j).$$

Stage 3:

For any j-ary function $f \in$ L and any $\langle m_1, a_1\rangle, \ldots, \langle m_j, a_j\rangle \in M_S$ we let

$$\mathfrak{M}_S \models f\big(\langle m_1, a_1\rangle, \ldots, \langle m_j, a_j\rangle\big) = \langle m^*, a^*\rangle$$

exactly when

- $\mathfrak{M} \models f(m_1, \ldots, m_j) = m^*$,
- $a^* = 0$.

Let $(m_1, a_1) \equiv (m_2, a_2)$ if and only if $m_1 = m_2$. It is easily checked that $\equiv$ is an equivalence relation which respects L and hence is contained in $\asymp^{\mathfrak{M}_S}$. Further, as $\mathfrak{M} \cong \mathfrak{M}_S/\equiv$ and $\mathfrak{M}$ is a core, $\equiv$ must be the maximal equivalence relation which respects L. So by Proposition 2.6 we have $\equiv = \asymp^{\mathfrak{M}_S}$ and $\mathbf{C}(\mathfrak{M}_S) \cong \mathfrak{M}$. It is also immediate that $e_{\mathfrak{M}_S}$ reflects all formulas in Func. Hence, by Lemma 2.2 and Corollary 2.3, $e_{\mathfrak{M}_S}$ reflects all formulas of $\mathcal{L}^c_{\omega_1,\omega}(\mathtt{Func} \cup \mathtt{Neg})$. In particular, $e_{\mathfrak{M}_S}$ reflects σ, and so $\mathfrak{M}_S \models \sigma$ (as $\mathfrak{M} \models \sigma$).

Finally, let $E(\mathfrak{N}) = \{|[a]_{\asymp^{\mathfrak{N}}}| : a \in N\}$. As $\asymp$ is definable, $E(\mathfrak{N})$ is preserved by isomorphism. But by construction $E(\mathfrak{M}_S) = S$. So if $S_0 \neq S_1$, then $E(\mathfrak{M}_{S_0}) \neq E(\mathfrak{M}_{S_1})$ and $\mathfrak{M}_{S_0} \not\cong \mathfrak{M}_{S_1}$. □

Example 3.3 Suppose that $\leq \in$ L, $\sigma \in \mathcal{L}^c_{\omega_1,\omega}(\mathtt{Func} \cup \mathtt{Neg})$ and that there is an infinite $\mathfrak{M}$ such that

- $\mathfrak{M} \models \sigma$,

- $\mathfrak{M} \models \leq$ is a partial order.

Then σ has a perfect set of models. This follows from Proposition 3.2 and Example 2.8.

In Propositions 3.1 and 3.2 we have shown that given a core $\mathfrak{M}$, if either M is infinite, or if there is a function symbol of arity greater than zero in the language, then we can "blow up" $\mathfrak{M}$ to a perfect set of models all of whose cores are $\mathfrak{M}$ and all of which satisfy some of the same sentences as $\mathfrak{M}$. Next we show that if neither of these conditions is satisfied, that is, there are no functions of arity greater than zero, and if M is finite, then any model with core $\mathfrak{M}$ must have a simple description.

For any first-order theory $T \subseteq \mathcal{L}_{\omega,\omega}(\mathrm{L})$, let $S(T)$ be the collection of complete types over T. Let $L_1(T)$ be the smallest fragment of $\mathcal{L}_{\omega_1,\omega}(\mathrm{L})$ containing $\mathcal{L}_{\omega,\omega}(\mathrm{L}) \cup \{\bigwedge_{\varphi \in p} \varphi(\mathbf{x}) : p \in S(T)\}$. For a model $\mathfrak{M}$, let $\mathrm{Th}_0(\mathfrak{M})$ be the complete first-order theory of $\mathfrak{M}$ in L, and let $\mathrm{Th}_1(\mathfrak{M})$ be the complete theory of $\mathfrak{M}$ in $L_1(\mathrm{Th}_0(\mathfrak{M}))$.

Definition 3.4 We say that $\mathfrak{M}$ has the *Martin property* if $S(\mathrm{Th}_0(\mathfrak{M}))$ is countable and $\mathrm{Th}_1(\mathfrak{M})$ is $\aleph_0$-categorical.

In particular, if $\mathfrak{M}$ has the Martin property, then it has quantifier rank at most $\omega + \omega$. *Martin's conjecture* for a first-order theory T says that either T has a perfect set of countable models or else every model of T has the Martin property.

It is worth mentioning that Martin's conjecture does not hold if we replace "first-order theory" with "sentence of $\mathcal{L}_{\omega_1,\omega}(\mathrm{L})$." For example, if $\mathfrak{M}$ has high quantifier rank (such as if $\mathfrak{M}$ is a well-ordering of type $\beta \gg \omega$) and $\sigma_{\mathfrak{M}}$ is a Scott sentence of $\mathfrak{M}$, then $\sigma_{\mathfrak{M}}$ is $\aleph_0$-categorical even though $\mathfrak{M}$ in general will not have the Martin property.

The reason why Martin's conjecture fails in this case is that we are able to encode a great deal of complexity in the sentence $\sigma_{\mathfrak{M}}$, complexity which is lost when we drop down to the first-order theory. A better generalization of Martin's conjecture for $\sigma \in \mathcal{L}_{\omega_1,\omega}(\mathrm{L})$ would be something along the lines of "either σ has a perfect set of model, or the quantifier rank of any model of σ is at most $\beta+\omega+\omega$ where β is the quantifier rank of σ." Of course if σ satisfies the condition of Martin's conjecture, then it also satisfies this condition. We will not dwell more on this topic now. We mention it simply to prepare the reader for Corollary 4.2 in which we show that in fact if we replace "first-order theory" with "sentence of $\mathcal{L}^c_{\omega_1,\omega}(\mathtt{Uni})$," then Martin's conjecture will hold.

Proposition 3.5 *Suppose that* L *has no function symbols of arity greater than zero and that* $\mathbf{C}(\mathfrak{M})$ *is finite. Then* $\mathfrak{M}$ *has the Martin property.*

Proof Before we begin the proof, it is worth taking a moment to describe what such a model $\mathfrak{M}$ will look like. Let $\mathrm{L}_R \subseteq \mathrm{L}$ be the collection of all non-equality relations in L. Such a model $\mathfrak{M}$ will have one $\asymp$-equivalence class for each element of $\mathbf{C}(\mathfrak{M})$. Further, $\mathfrak{M}|_{\mathrm{L}_R}$ will be completely determined by the number of elements in each equivalence class along with the structure of $\mathbf{C}(\mathfrak{M})|_{\mathrm{L}_R}$. The model $\mathfrak{M}$ will also determine which constants are equal to which others as well as which constants are $\asymp$-equivalent to each other. The last piece in the description of $\mathfrak{M}$ is then the number of elements in each $\asymp$-equivalence class which are not equal to any constant. In particular, any model $\mathfrak{N}$ with $\mathbf{C}(\mathfrak{N}) = \mathbf{C}(\mathfrak{M})$ and which agrees with $\mathfrak{M}$ on the above will be isomorphic to $\mathfrak{M}$. We now make this precise.

Let $\mathbf{C}(\mathfrak{M}) = \{a_1, \ldots, a_j\}$. As $\mathbf{C}(\mathfrak{M})$ is finite, there is a finite $\mathrm{L}_0 \subseteq \mathrm{L}_R$ such that any automorphism of $\mathbf{C}(\mathfrak{M})|_{\mathrm{L}_0}$ is also an automorphism of $\mathbf{C}(\mathfrak{M})|_{\mathrm{L}_R}$. For each finite $L^* \subseteq \mathrm{L}_R$, let $\Psi_{L^*}(x_1, \ldots, x_j)$ be the (first-order formula which is the) conjunction of the complete L^*-type of $\langle a_1, \ldots, a_j\rangle$ in $\mathbf{C}(\mathfrak{M})$. Let $\asymp_{L^*}$ be the conjunction of $\asymp \cap \mathcal{L}_{\omega,\omega}(L^*)$. Let T_{Rel} consist of the following, for each finite L^* with $\mathrm{L}_0 \subseteq L^* \subseteq \mathrm{L}_R$:

- $(\forall x_1, \ldots, x_j)\ \Psi_{\mathrm{L}_0}(x_1, \ldots, x_j) \leftrightarrow \Psi_{L^*}(x_1, \ldots, x_j)$;
- $(\forall x_1, \ldots, x_j, x)\ \Psi_{\mathrm{L}_0}(x_1, \ldots, x_j) \rightarrow \bigvee_{i \leq j} x \asymp_{\mathrm{L}_0} x_i$;
- $(\forall x_0, x_1)\ x_0 \asymp_{\mathrm{L}_0} x_1 \leftrightarrow x_0 \asymp_{L^*} x_1$;
- $(\exists x_1, \ldots, x_j)\ \Psi_{\mathrm{L}_0}(x_1, \ldots, x_j)$.

We then also have $T_{\mathrm{Rel}} \models (\forall x_0, x_1)\ x_0 \asymp_{\mathrm{L}_0} x_1 \leftrightarrow x_0 \asymp x_1$ and that $\mathrm{Th}_0(\mathfrak{M}) \models T_{\mathrm{Rel}}$. In fact $\mathrm{Th}_0(\mathfrak{M})$, along with the statement that there exists exactly j-elements, determines $\mathbf{C}(\mathfrak{M})|_{\mathrm{L}_R}$ up to isomorphism.

Let $\{c_i : i \in \kappa\} = \mathrm{L}_c$ be the collection of function symbols of arity 0 (i.e., constant symbols) in L. Let $\mathrm{L}' = \mathrm{L} \cup \{d_1, \ldots, d_j\}$, where the d_i's are new constants. Define T_{Con} to consist of the following, where c, c' range over L_c:

- $c = c'$ if $\mathfrak{M} \models c = c'$ and $c \neq c'$ if $\mathfrak{M} \models c \neq c'$;
- $\Psi_{\mathrm{L}_0}(d_1, \ldots, d_j)$;
- $c \asymp_{\mathrm{L}_0} d_i$ if $\mathbf{C}(\mathfrak{M}) \models c = a_i$;
- $d_i = c_k$ if $\mathbf{C}(\mathfrak{M}) \models c_k = a_i$ and k is the least such that this holds.

The purpose of the new constants $d_1, \ldots, d_j$ is to allow us to explicitly talk about each $\asymp^{\mathfrak{M}}$-equivalence class.

For $i \leq j$, let $n_i = |e_{\mathfrak{M}}^{-1}(a_i)|$. Let T_{Size} consist of the following.

- If n_i is finite, then $(\exists^{n_i} x) x \asymp_{\mathrm{L}_0} d_i$ and $\neg(\exists^{n_i+1} x) x \asymp_{\mathrm{L}_0} d_i$.
- If $n_i = \omega$, then for all $n \in \omega$, $(\exists^n x) x \asymp_{\mathrm{L}_0} d_i$.

Let $T = T_{\mathrm{Rel}} \cup T_{\mathrm{Con}} \cup T_{\mathrm{Size}}$. It is easily seen that every model of $\mathrm{Th}_0(\mathfrak{M})$ has an expansion to an L'-structure which satisfies T and further, up to isomorphism, that this expansion is unique (as all we are doing is adding a new constant to $\asymp$-equivalence classes which may not have one). Let $\mathfrak{M}'$ be such an expansion of $\mathfrak{M}$.

Now suppose that $L^* \subseteq \mathrm{L}'$ has only finitely many constants. It is then easily checked that for any $\mathfrak{N} \models T$, $\mathfrak{N}|_{L^*} \cong \mathfrak{M}|_{L^*}$. In particular, this implies that T is a complete theory.

Let $p(x) := \{x \neq c : c \in \mathrm{L}_c\}$, and let $p_i(x) := p(x) \cup \{x \asymp_{\mathrm{L}_0} d_i, x \neq d_i\}$. It is easy to see that if $p_i(x)$ is consistent over T, then $\bigwedge_{\varphi \in p_i} \varphi(x)$ is equivalent over T to a complete type. Further, it is also immediate that every complete 1-type in $S(T)$ is equivalent over T to one of $\bigwedge_{\varphi \in p_i} \varphi(x)$, $x = c_k$ or $x = d_i$ (where $c_k \in \mathrm{L}_c$ and $i \leq j$). In addition, for every sequence of complete 1-types $t_1(x_1), \ldots, t_k(x_k) \in S(T)$, the statement $\bigwedge_{i \leq k} t_i(x_i) \wedge \bigwedge_{i \neq j} x_i \neq x_j$ is a complete type. Hence every complete type over T is of this form and $|S(T)| \leq \omega$.

However, as T is an expansion of $\mathrm{Th}_0(\mathfrak{M})$, we also have $|S(\mathrm{Th}_0(\mathfrak{M}))| \leq \omega$, and for every complete type $r(x_1, \ldots, x_k) \in S(\mathrm{Th}_0(\mathfrak{M}))$ there are complete 1-types $s_1, \ldots, s_k$ such that $\mathrm{Th}_0(\mathfrak{M}) \models (\forall x_1, \ldots, x_l)\ r(x_1, \ldots, x_l) \leftrightarrow \bigwedge_{i \leq l} s_i(x_i)$. Hence every model of $\mathrm{Th}_0(\mathfrak{M})$ is determined up to isomorphism by how many realizations there are of each 1-type. In particular, this implies that every complete theory in $L_1(\mathrm{Th}_0(\mathfrak{M}))$ is $\aleph_0$-categorical and that $\mathfrak{M}$ has the Martin property. $\square$

It is worth mentioning that if there are only finitely many constants in L, then $p(x)$ is equivalent to a first-order formula, and hence if $\mathbf{C}(\mathfrak{M})$ is finite, then T is $\aleph_0$-categorical. But, as every model of $\mathrm{Th}_0(\mathfrak{M})$ has an expansion to a model of T, $\mathrm{Th}_0(\mathfrak{M})$ is also $\aleph_0$-categorical.

4 Main Theorems

We are now ready to prove our main theorems.

Theorem 4.1 (Vaught's conjecture for $\mathcal{L}^c_{\omega_1,\omega}(\mathtt{Uni})$) *If $\sigma \in \mathcal{L}^c_{\omega_1,\omega}(\mathtt{Uni})$, then either σ has a perfect set of countable models or σ has only countably many countable models.*

Proof If σ has no countable models we are trivially done, so let us assume that σ has at least one countable model (i.e., is consistent). We then have three cases to consider.

Case 1: L contains a function of arity greater than zero.

By assumption, there is a model $\mathfrak{M} \models \sigma$, and by Corollary 2.3, $\mathbf{C}(\mathfrak{M}) \models \sigma$ as well. But then by Proposition 3.1, σ has a perfect set of models.

Case 2: There is an $\mathfrak{M} \models \sigma$ with $\mathbf{C}(\mathfrak{M})$ infinite.

By Corollary 2.3 $\mathbf{C}(\mathfrak{M}) \models \sigma$, and so by Proposition 3.2, σ has a perfect set of models.

Case 3: For every $\mathfrak{M} \models \sigma$, $\mathbf{C}(\mathfrak{M})$ is finite.

In this case, by Proposition 3.5, every model has the Martin property and hence has quantifier rank at most $\omega + \omega$. But then by results of Morley [5], σ satisfies Vaught's conjecture. □

In particular, Theorem 4.1 implies that Vaught's conjecture holds for all sentences of $\mathcal{L}_{\omega_1,\omega}(\mathrm{L})$ which do not have equality as a subformula (i.e., are equivalent to a formula in $\mathcal{L}^c_{\omega_1,\omega}(\mathtt{Rel})$).

Further, as an immediate corollary we have the following.

Corollary 4.2 (Martin's conjecture for $\sigma \in \mathcal{L}^c_{\omega_1,\omega}(\mathtt{Uni})$) *For any $\sigma \in \mathcal{L}^c_{\omega_1,\omega}(\mathtt{Uni})$, either σ has a perfect set of models or every model of σ has the Martin property.*

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