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Indiscernibles, EM-Types, and Ramsey Classes of Trees

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Abstract The author has previously shown that for a certain class of structures $\mathcal{I}$, $\mathcal{I}$-indexed indiscernible sets have the modeling property just in case the age of $\mathcal{I}$ is a Ramsey class. We expand this known class of structures from ordered structures in a finite relational language to ordered, locally finite structures which isolate quantifier-free types by way of quantifier-free formulas. This result is applied to give new proofs that certain classes of trees are Ramsey. To aid this project we develop the logic of EM-types.

1 Introduction

A generalized indiscernible set (which we will abbreviate as an *indiscernible*) is a set of tuples from a model $\mathcal{M}$, $(a_i : i \in \mathcal{I})$, indexed by a structure $\mathcal{I}$ in a homogeneous way: the complete type of a finite tuple of parameters $(a_{i_1}, \ldots, a_{i_n})$ in $\mathcal{M}$ is fully determined by the quantifier-free type of the indices $(i_1, \ldots, i_n)$ in $\mathcal{I}$. If $\mathcal{I}$ is known, we call the indiscernible an *$\mathcal{I}$-indexed indiscernible set.* Generalized indiscernible sets were originally developed in Shelah [17] and have been used in many places, for example, Baldwin and Shelah [2], Laskowski and Shelah [11], Džamonja and Shelah [3], Guingona [6], Kim and Kim [9], and Takeuchi and Tsuboi [21]. In [3], indiscernibles indexed by trees were studied, and a specific property was proved of them. One of the main goals of Scow [16] was to consider this specific property generalized from a tree to an arbitrary structure, $\mathcal{I}$, named the *modeling property (for $\mathcal{I}$-indexed indiscernibles)*, and relate this property to a combinatorial property of the age of $\mathcal{I}$. The appropriate notion turned out to be the one of *Ramsey class* (see Definition 3.6). A "dictionary" theorem was proved: if $\mathcal{I}$ is a structure in a finite relational language, linearly ordered by one of its relations, then the age of $\mathcal{I}$ is a Ramsey class just in case $\mathcal{I}$-indexed indiscernible sets have the modeling property (see Definition 3.1). In Theorem 3.12, we extend this dictionary to the case where $\mathcal{I}$ is locally finite, linearly ordered by one of its relations, and has a certain technical property,

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`qfi`: quantifier-free types realized in $\mathcal{I}$ are isolated by quantifier-free formulas. This generalizes the dictionary theorem to certain situations where we have an infinite language containing function symbols, in particular to the case where $\mathcal{I}$ is ordered and locally finite in a finite language. The *locally finite-linearly ordered-*`qfi` case encompasses two indexing structures $\mathcal{I}$ from the literature, $I_0 = (\omega^{<\omega}, \trianglelefteq, \wedge, <_{\text{lex}})$ and $I_s = (\omega^{<\omega}, \trianglelefteq, \wedge, <_{\text{lex}}, (P_n)_{n<\omega})$, where $\trianglelefteq, \wedge, <_{\text{lex}}, P_n$ are interpreted as the partial tree-order, the meet function in this order, the lexicographic order on sequences, and the nth level of the tree, respectively. It is known from Kim, Kim, and Scow [10] and [21] that both of these structures index indiscernibles with the modeling property. Corollaries 3.17 and 3.18 conclude that the ages of I_0, I_s, respectively, form Ramsey classes. The latter constitutes an alternative proof of a known result (see Nguyen Van Thé [15], Fouché [4]). It was brought to the author's attention that the former result appears in Leeb [12] and was later surveyed in Graham and Rothschild [5]. Subsequent proofs have been found by Solecki (unpublished) using ideas from [20] and also by Sokic ([19, Theorem 2]). It was conjectured that results might travel both ways through the dictionary theorem: known Ramsey classes would yield new structures to index indiscernibles; known results on indiscernibles would yield new Ramsey classes. The author hopes some new example of a Ramsey class may yet come to light as a result of the dictionary theorem.

In Section 2 we give the basic lemmas around `qfi` and further develop a notion of EM-type used in [10]. In the process, we give restatements of certain definitions from [16] in Definitions 2.1, 2.5, and 2.8 that drop reference to a linear order on $\mathcal{I}$. The technology of EM-types primarily addresses the question, "What uniform definable character of an initial, indexed set of parameters may be preserved in an indiscernible indexed by the same set?" In the technology developed in this section, there is no use of a linear order on the index structure, $\mathcal{I}$. Though indiscernibles indexed by unordered $\mathcal{I}$ do not exist in *all* structures M (see [16]), the technical lemmas of this section are still of some independent interest for studying unordered indiscernibles in a limited setting.

In Section 3 we prove the main theorem, Theorem 3.12, that in the more general case of *locally finite-linearly ordered-*`qfi`, I-indexed indiscernibles have the modeling property just in case $\text{age}(I)$ is a Ramsey class. From this theorem we deduce the partition result, Corollary 3.17, that $\text{age}(I_0)$ is a Ramsey class.

In Section 4 we provide an alternate proof of the result that I_0-indexed indiscernibles have the modeling property (from [21]) using only a result of [4], Theorem 3.12, and the technology of EM-types. The arguments in Theorem A.5 are finitary and can be adapted to a direct proof of Corollary 3.17, modulo a few applications of compactness.

1.1 Conventions Much of our model-theoretic notation is standard (see Hodges [7] and Marker [13] for references). For $t \in \{0, 1\}$, by φ^t we mean φ if $t = 0$, and $\neg\varphi$ if $t = 1$. For an L'-structure $\mathcal{I}$ and a sublanguage $L^* \subseteq L'$, by $\mathcal{I}|L^*$ we mean the reduct of $\mathcal{I}$ to L^*. By $\text{qftp}^{L'}(i_1, \ldots, i_n; \mathcal{I})$ we mean the complete quantifier-free L'-type of $(i_1, \ldots, i_n)$ in $\mathcal{I}$ (if L' is clear, it is omitted). The complete quantifier-free type of a substructure of $\mathcal{I}$ is the complete quantifier-free type of a tuple that enumerates some substructure of $\mathcal{I}$. By $\text{Diag}(\mathcal{N})$, we mean the atomic diagram of $\mathcal{N}$. By $\text{age}(\mathcal{I})$ we mean the class of all finitely generated substructures of $\mathcal{I}$ closed under isomorphisms. In this paper, a complete quantifier-free type is always a type in a finite list of variables.

For a tuple $\overline{a} = (a_1, \ldots, a_m)$ and a subsequence $\sigma = \langle i_1, \ldots, i_k \rangle$ of $\langle 1, \ldots, m \rangle$, by $\overline{a} \upharpoonright \sigma$ we mean $(a_{i_1}, \ldots, a_{i_k})$. For a subset $Y \subseteq I$ and a type $\Gamma(\{x_i : i \in I\})$, by $\Gamma|_{\{x_i : i \in Y\}}$ we mean the restriction of q to formulas containing variables in $\{x_i : i \in Y\}$. If a tuple $\overline{a}$ satisfies a type $\Gamma(\overline{x})$ in a structure $\mathcal{M}$, we write $\overline{a} \vDash_{\mathcal{M}} \Gamma$, where $\mathcal{M}$ is omitted if it is the monster model (see Convention 2.1).

We write $\overline{x}, \overline{a}, \overline{\imath}$ to denote finite tuples and α, β to denote ordinals. The underlying set of a structure $\mathcal{I}$ is given by the unscripted letter, I. For a sequence $\eta := \langle \eta_0, \ldots, \eta_{n-1} \rangle$, we denote the length by $\ell(\eta) = n$. Given a tuple $\overline{a} = (a_1, \ldots, a_n)$, by $(\overline{a})_i$ we mean a_i and by $\bigcup \overline{a}$ we mean $\{a_i : 1 \leq i \leq n\}$. We often abbreviate expressions $(a_{i_1}, \ldots, a_{i_n})$ by $\overline{a}_{\overline{\imath}}$.

2 Basic Notions

The definition for $\mathcal{I}$-indexed indiscernible sets was first presented in [17]. We set our notation in the following.

Definition 2.1 (Generalized indiscernible set) Fix an L'-structure $\mathcal{I}$ and an L-structure $\mathcal{M}$ for some languages L and L'. Let a_i be same-length tuples of parameters from M indexed by the underlying set I of $\mathcal{I}$.

(1) We say that $(a_i : i \in I)$ is an *$\mathcal{I}$-indexed indiscernible (set in $\mathcal{M}$)* if for all $n \geq 1$, for all sequences $i_1, \ldots, i_n, j_1, \ldots, j_n$ from I,

$$\mathrm{qftp}^{L'}(i_1, \ldots, i_n; \mathcal{I}) = \mathrm{qftp}^{L'}(j_1, \ldots, j_n; \mathcal{I}) \Rightarrow \mathrm{tp}^{L}(a_{i_1}, \ldots, a_{i_n}; \mathcal{M}) = \mathrm{tp}^{L}(a_{j_1}, \ldots, a_{j_n}; \mathcal{M}).$$

We omit $\mathcal{M}$ where it is clear from context.

(2) In the case where the L'-structure $\mathcal{I}$ is clear from context, we say that the $\mathcal{I}$-indexed indiscernible $(a_i : i \in I)$ is *L'-generalized indiscernible*.

(3) Given a sublanguage $L^* \subseteq L'$, we say that the L'-generalized indiscernible set $(a_i : i \in I)$ is *L^*-generalized indiscernible* if it is an $\mathcal{I}|L^*$-indexed indiscernible.

(4) A *generalized indiscernible (set)* is an I-indexed set $(a_i : i \in I)$ for some set I that is an $\mathcal{I}$-indexed indiscernible for some choice of structure $\mathcal{I}$ on I.

We will always assume that generalized indiscernible sets are *nontrivial*, that is, that whenever $i \neq j$, $a_i \neq a_j$.

Notation 2.2 For convenience, $\mathcal{I}$ as in Definition 2.1 is referred to as the *index model* and L' is the *index language*; $\mathcal{M}$ is referred to as the *target model* and L is the *target language*. In this paper, parameters $(a_i : i \in I)$ in $\mathcal{M}$ are always assumed to be tuples such that $\ell(a_i) = \ell(a_j)$ for all i, j, and without loss of generality we often assume that $\ell(a_i) = 1$.

Convention 2.1 For our purposes, there is no loss in generality to assume that we are working not just in a target model $\mathcal{M}$ but in a monster model $\mathbb{M}$ of $\mathrm{Th}(\mathcal{M})$. From now on we write $\vDash \varphi$ for $\vDash_{\mathbb{M}} \varphi$. We will reserve L for the language of this model. Parameters with no identified location come from $\mathbb{M}$.

We define certain technical restrictions on $\mathcal{I}$ that we make in this paper and follow with a proposition.

Definition 2.3

(1) Say that $\mathcal{I}$ has *quantifier-free types equivalent to quantifier-free formulas (`qteqf`)* if for every complete quantifier-free type $q(\overline{x})$ realized in $\mathcal{I}$, there is a quantifier-free formula $\theta(\overline{x})$ equivalent to q in $\mathcal{I}$, that is, such that $q(\mathcal{I}) = \theta(\mathcal{I})$.

(2) Say that $\mathcal{I}$ is `qfi` if, for any complete quantifier-free type $q(\overline{x})$ realized in $\mathcal{I}$, there is a quantifier-free formula $\theta_q(\overline{x})$ such that $\mathrm{Th}(\mathcal{I})_\forall \cup \theta_q(\overline{x}) \vdash q(\overline{x})$.

Observation 2.1 If $\mathcal{I}$ realizes finitely many quantifier-free n types, for each n, then it is clear that $\mathcal{I}$ is `qteqf`. For example, if $\mathcal{I}$ is a uniformly locally finite L'-structure where L' is a finite language, or more specifically, $\mathcal{I}$ is an L'-structure where L' is a finite relational language, then $\mathcal{I}$ is `qteqf`.

Proposition 1

(1) *$\mathcal{I}$ is `qfi` just in case it has `qteqf`.*

(2) *In the case that $\mathcal{I}$ is a structure in a finite language and is locally finite, then $\mathcal{I}$ is `qfi`.*

Proof (1) If $\mathcal{I}$ is `qfi`, clearly it has `qteqf` (note that $\theta_q \in q$). Suppose that $\mathcal{I}$ has `qteqf`. Fix a complete quantifier-free type $q(\overline{x})$ realized in $\mathcal{I}$, and say that it is equivalent to the quantifier-free θ_q in $\mathcal{I}$. Then, for all $\psi_\alpha \in q$, $\mathcal{I} \models \forall\overline{x}(\theta_q(\overline{x}) \to \psi_\alpha(\overline{x}))$. Thus, $\mathrm{Th}(\mathcal{I})_\forall \vdash \forall\overline{x}(\theta_q(\overline{x}) \to \psi_\alpha(\overline{x}))$ and so $\mathcal{I}$ is `qfi`.

(2) This surprisingly helpful observation is surely folklore, but we provide a proof for completeness. Fix n. We will show that $\mathcal{I}$ has `qteqf`. By assumption, every n-tuple from $\mathcal{I}$ generates a finite substructure of $\mathcal{I}$, and L' is finite. Thus we may enumerate the finite L'-structures up to isomorphism type as $(D_i)_{i<\omega}$ (ω is not important here). Let $\overline{d}_i$ be an enumeration of D_i; for each i, say that $|D_i| = N(i)$. For a particular i, let $\Phi_{\mathrm{gr}}^{\overline{d}_i}$ be a formula in variables $(x_1, \ldots, x_{N(i)})$, satisfied by $\overline{d}_i$ in $\mathcal{I}$, that describes the extensions of the relation symbols on $\overline{d}_i$, the graphs of the function symbols on $\overline{d}_i$, and any equalities or inequalities between constant symbols and the $(\overline{d}_i)_j$. Clearly such a formula exists as a finite conjunction of literals.

Now given a complete quantifier-free n-type realized in $\mathcal{I}$, $q(y_1, \ldots, y_n)$, there must be some $l < \omega$ and some $(x_{i_j} : j \le n)$ for $i_j \le N(i)$ such that $q(x_{i_1}, \ldots, x_{i_n}) \cup \{\Phi_{\mathrm{gr}}^{\overline{d}_l}\}$ is consistent. But then there are terms $\tau_k = \tau_k(x_{i_1}, \ldots, x_{i_n})$ such that

$$q(x_{i_1}, \ldots, x_{i_n}) \cup \{\Phi_{\mathrm{gr}}^{\overline{d}_l}\} \vdash (\tau_k = x_k)$$

for all $1 \le k \le N(i)$. Let $\sigma_k = \tau_k(x_{i_1}, \ldots, x_{i_n}; y_1, \ldots, y_n)$, and substitute σ_k for x_k in $\Phi_{\mathrm{gr}}^{\overline{d}_l}$ to obtain

$$\Phi_{\mathrm{gr}}^{\overline{d}_l}\left(x_1, \ldots, x_{N(i)}; \sigma_1(\overline{y}), \ldots, \sigma_{N(i)}(\overline{y})\right).$$

The latter is a quantifier-free formula equivalent to q in $\mathcal{I}$. By (1) we are done. $\square$

Remark 2.4 Note that for a locally finite structure $\mathcal{I}$, $\mathcal{I}$ is `qfi` just in case for every complete quantifier-free type $q(\overline{x})$ of a finite substructure of $\mathcal{I}$, there is a quantifier-free formula $\theta_q(\overline{x})$ such that $\mathrm{Th}(\mathcal{I})_\forall \cup \theta_q(\overline{x}) \vdash q(\overline{x})$.

The assumption made on index models $\mathcal{I}$ for an $\mathcal{I}$-indexed indiscernible in [17] is exactly that $\mathcal{I}$ has `qteqf` (equivalently, `qfi`). The statements of `qteqf` and `qfi` offer different perspectives on the same condition, and we will use the terms interchangeably.

We define what it means for a generalized indiscernible to inherit the local structure of a set of parameters. In this definition, the parameters and the indiscernible need not be indexed by the same structure, only by structures in the same language. In Ziegler [24] the following notion is named *lokal wie*. The same notion is referred to as *based on* in [10], [16], and Simon [18]. Below we promote a synthesis of the two names.

Definition 2.5 (Locally based on) Fix $\mathcal{I}$, $\mathcal{J}$ L'-structures and a sublanguage $L^* \subseteq L'$. Fix a set of parameters $\mathbf{I} := (a_i : i \in I)$.

(1) We say that the J-indexed set $(b_i : i \in J)$ is *L^*-locally based on the a_i (L^*-locally based on $\mathbf{I}$)* if for any finite set of L-formulas, Δ, and for any finite tuple $(t_1, \ldots, t_n)$ from J, there exists a tuple $(s_1, \ldots, s_n)$ from I such that
$$\mathrm{qftp}^{L^*}(\bar{t}; \mathcal{J}) = \mathrm{qftp}^{L^*}(\bar{s}; \mathcal{I}),$$
and
$$\mathrm{tp}^{\Delta}(\bar{b}_{\bar{t}}; \mathbb{M}) = \mathrm{tp}^{\Delta}(\bar{a}_{\bar{s}}; \mathbb{M}).$$
We abbreviate this condition by “the b_i are L^*-locally based on the a_i.”

(2) If the J-indexed set $(b_i : i \in J)$ is L'-locally based on the a_i, we omit mention of L'.

Observation 2.2 It is easy to see that the property of one indexed set being *locally based on* another is transitive. Fix L'-structures $\mathcal{I}, \mathcal{J}, \mathcal{J}'$, and fix parameters $(a_i : i \in I)$, $(b_j : j \in J)$, and $\mathbf{W} := (c_k : k \in J')$. Then, if $\mathbf{W}$ is locally based on the b_i, and the b_i are locally based on the a_i, we may conclude that $\mathbf{W}$ is locally based on the a_i. In fact, we may further conclude that $\mathrm{age}(\mathcal{J}') \subseteq \mathrm{age}(\mathcal{J}) \subseteq \mathrm{age}(\mathcal{I})$ by focusing attention on the complete quantifier-free types of substructures.

Definition 2.6 Fix languages $L^* \subseteq L'$. Given an L'-structure $\mathcal{I}$ and an I-indexed set $\mathbf{I} := (a_i : i \in I)$, define the *($L^*$-)EM-type of* $\mathbf{I}$ to be:
$$\begin{aligned}&\mathrm{EMtp}_{L^*}(\mathbf{I})(x_i : i \in I)\\ &\quad = \big\{\psi(x_{i_1}, \ldots, x_{i_n}) : \psi \text{ from } L, i_1, \ldots, i_n \text{ from } I,\\ &\qquad \text{and for any } (j_1, \ldots, j_n) \text{ from } I \text{ such that}\\ &\qquad \mathrm{qftp}^{L^*}(j_1, \ldots, j_n; \mathcal{I}) = \mathrm{qftp}^{L^*}(i_1, \ldots, i_n; \mathcal{I}), \vDash \psi(a_{j_1}, \ldots, a_{j_n})\big\}.\end{aligned}$$
If $L^* = L'$, we may omit mention of it.

Remark 2.7 The specific case of the above definition for $\mathcal{I}$ a linear order is called an “EM-type” in Tent and Ziegler [22]. This notation is not to be confused with $\mathrm{EM}(I, \Phi)$, which in Baldwin [1] and Shelah [17] refers to a certain kind of structure. The relevant similarity is that $\Phi(x_i : i \in I)$ is *proper* for $(\mathcal{I}, \mathrm{Th}(\mathbb{M}))$ in the sense of [1] and [17] if it is the set of formulas satisfied in $\mathbb{M}$ by an $\mathcal{I}$-indexed indiscernible. By Proposition 2(3), given an L'-structure $\mathcal{I}$, L'-EM-types indexed by I may always be extended to a set Φ proper for $(\mathcal{I}, \mathrm{Th}(\mathbb{M}))$, provided that $\mathcal{I}$-indexed indiscernible sets have the modeling property.

The following notation for the type of an indiscernible follows [13]. In the classical case of order indiscernibles, where the index structure is a linear order of the form

$(\mathbb{N}, <)$, there is a canonical orientation of the variables in any quantifier-free n-type (e.g., $q(x_1, \ldots, x_n)$, where $x_1 < \cdots < x_n$). Here we deal with an arbitrary structure $\mathcal{I}$ where there may not be such a canonical orientation, and so we define the type of an indiscernible to include all orientations of variables in all types. From this perspective, the use of canonical orientations of variables is something of an aesthetic device for special cases.

Definition 2.8 Given an $\mathcal{I}$-indexed indiscernible set $\mathbf{I} := (a_i : i \in I)$, define

(1) for any complete quantifier-free type $\eta(v_1, \ldots, v_n)$ realized in $\mathcal{I}$:

$$p^\eta(\mathbf{I}) = \big\{\psi(x_1, \ldots, x_n) : \psi \text{ from } L \text{ and there exists } i_1, \ldots, i_n \text{ from } I \text{ such that } (i_1, \ldots, i_n) \vDash_{\mathcal{I}} \eta \text{ and } \vDash \psi(a_{i_1}, \ldots, a_{i_n})\big\};$$

(2)

$$\mathrm{tp}(\mathbf{I}) := \big\langle p^\eta(\mathbf{I}) : n < \omega, \eta \text{ is a complete quantifier-free } n\text{-type realized in } \mathcal{I}\big\rangle.$$

Observation 2.3 Let $\eta(v_1, \ldots, v_n)$ be the complete quantifier-free type of a finite substructure of $\mathcal{I}$ in some enumeration. Suppose there is a permutation τ of $\{1, \ldots, n\}$ such that realizations of $\eta(v_1, \ldots, v_n)$, $\eta_\tau := \eta(v_{\tau(1)}, \ldots, v_{\tau(n)})$ are isomorphic as tuples. If $\mathbf{I}$ is an $\mathcal{I}$-indexed indiscernible set, then the following information will be contained in $\mathrm{tp}(\mathbf{I})$: $\psi(x_1, \ldots, x_n) \in p^\eta(\mathbf{I}) \Leftrightarrow \psi(x_1, \ldots, x_n) \in p^{\eta_\tau}(\mathbf{I})$.

Remark 2.9 The set $p^\eta(\mathbf{I})$ does not seem terribly useful for a set of parameters $\mathbf{I}$ if $\mathbf{I}$ is not generalized indiscernible, as $p^\eta(\mathbf{I})$ may not be a consistent type. The type $\mathrm{EMtp}_{L'}(\mathbf{I})$ is always a consistent type, though it may be trivial.

The following definitions are for Proposition 2.

Definition 2.10 Fix an L'-structure $\mathcal{I}$ and a language $\mathcal{L}$. We define $\mathbf{Ind}(\mathcal{I}, \mathcal{L})$ to be

$$\mathbf{Ind}(\mathcal{I}, \mathcal{L})(x_i : i \in I) := \big\{\varphi(x_{i_1}, \ldots, x_{i_n}) \to \varphi(x_{j_1}, \ldots, x_{j_n}) : n < \omega, \bar{\imath}, \bar{\jmath} \text{ from } I, \mathrm{qftp}^{L'}(\bar{\imath}; \mathcal{I}) = \mathrm{qftp}^{L'}(\bar{\jmath}; \mathcal{I}), \varphi(x_1, \ldots, x_n) \in \mathcal{L}\big\}.$$

Definition 2.11 Let $\Gamma(x_i : i \in I)$ be an L-type, and let $\mathcal{U} = (a_i : i \in I)$ be an I-indexed set of parameters in $\mathbb{M}$. We say that Γ *is finitely satisfiable in* $\mathcal{U}$ if for every finite $I_0 \subseteq I$, there is a $J_0 \subseteq I$, a bijection $f : I_0 \to J_0$, and an enumeration $\bar{\imath}$ of I_0 such that $\mathrm{qftp}^{L'}(\bar{\imath}; \mathcal{I}) = \mathrm{qftp}^{L'}(f(\bar{\imath}); \mathcal{I})$ and $(a_{f(i)} : i \in I_0) \vDash \Gamma|_{\{x_i : i \in I_0\}}$.

Observation 2.4 If $\mathcal{I}$ and $\mathcal{J}$ are L'-structures with the same age, then they realize the same complete quantifier-free types. Suppose that $\bar{\imath}$ from $\mathcal{I}$ realizes a complete quantifier-free type $\eta(v_1, \ldots, v_n)$. Since $\mathcal{I}$ and $\mathcal{J}$ have the same age, the substructure of $\mathcal{I}$ generated by $\bar{\imath}$ is isomorphic to some substructure of $\mathcal{J}$. An isomorphism taking one substructure to the other takes $\bar{\imath}$ to a tuple $\bar{\jmath}$ from $\mathcal{J}$ satisfying the same complete quantifier-free type.

In the next proposition we detail how two sets of parameters indexed by L'-structures may interact by way of EM-type, tp, and the property of being *locally based on*. These sets of parameters are indexed by sets I, J, and the parameters may or may not be indiscernible according to the intended structures $\mathcal{I}$, $\mathcal{J}$ on I, J. Table 1 illustrates the roles of the different bold-face letters.

Table 1 Indexed sets used in Proposition 2.

Indexing set	$\mathcal{I}/\mathcal{J}$-indexed indiscernible set	I/J-indexed set
I	$\mathbf{I} = (c_i)_i, \mathbf{W} = (d_i)_i$	$\mathbf{U} = (a_i)_i, \mathbf{V}$
J	$\mathbf{J} = (b_i)_i$	$\mathbf{T} = (e_i)_i$

Proposition 2 *Fix an L'-structure $\mathcal{I}$, any I-indexed set of parameters $\mathbf{U} = (a_i : i \in I)$ (possibly indiscernible), and an $\mathcal{I}$-indexed indiscernible set $\mathbf{I} = (c_i : i \in I)$. Let $\mathcal{J}$ be an L'-structure with the same age as $\mathcal{I}$, and let $\mathbf{J} := (b_i : i \in J)$ be any $\mathcal{J}$-indexed indiscernible set. Assume that $\mathcal{I} \subseteq \mathcal{J}$ is a substructure in items (3), (7), and (8).*

(1) *For any complete quantifier-free type η realized in $\mathcal{J}$, if $p^\eta(\mathbf{I}) \subseteq p^\eta(\mathbf{J})$, then $p^\eta(\mathbf{J}) \subseteq p^\eta(\mathbf{I})$.*
(2) *[Two sets of indiscernibles] $\mathbf{J}$ is locally based on the c_i just in case* $\mathrm{tp}(\mathbf{I}) = \mathrm{tp}(\mathbf{J})$.
(3) *[Two sets of parameters] A J-indexed set of parameters $\mathbf{T} = (e_i : i \in J)$ is locally based on the a_i just in case* $\mathrm{EMtp}_{L'}(\mathbf{T}) \supseteq \mathrm{EMtp}_{L'}(\mathbf{U})$.
(4) *For an I-indexed set of parameters $\mathbf{V}$, $\mathbf{V} \vDash \mathrm{EMtp}_{L'}(\mathbf{U})$ if and only if* $\mathrm{EMtp}_{L'}(\mathbf{V}) \supseteq \mathrm{EMtp}_{L'}(\mathbf{U})$.
(5) *For an $\mathcal{I}$-indexed indiscernible set $\mathbf{W} := (d_i : i \in I)$, $\mathrm{tp}(\mathbf{W}) = \mathrm{tp}(\mathbf{I})$ just in case $\mathbf{W} \vDash \mathrm{EMtp}_{L'}(\mathbf{I})$, just in case $\mathrm{EMtp}_{L'}(\mathbf{W}) = \mathrm{EMtp}_{L'}(\mathbf{I})$.*
(6) *If $\mathbf{Ind}(\mathcal{I}, L)$ is finitely satisfiable in $\mathbf{U}$, then there is an $\mathcal{I}$-indexed indiscernible $\mathbf{W} := (d_i : i \in I)$ locally based on the a_i.*
(7) *There is a J-indexed set of parameters $\mathbf{T} = (e_i : i \in J)$ such that* $\mathrm{EMtp}_{L'}(\mathbf{U}) \subseteq \mathrm{EMtp}_{L'}(\mathbf{T})$.
(8) *Suppose that $\mathbf{T}$ is any J-indexed set of parameters, and $L^* \subseteq L'$. If $\mathrm{EMtp}_{L'}(\mathbf{U}) \subseteq \mathrm{EMtp}_{L'}(\mathbf{T})$, then $\mathrm{EMtp}_{L^*}(\mathbf{U}) \subseteq \mathrm{EMtp}_{L^*}(\mathbf{T})$.*

Proof

(1) Suppose that $p^\eta(\mathbf{I}) \subseteq p^\eta(\mathbf{J})$. Let $\varphi(\overline{x}) \in p^\eta(\mathbf{J})$. Assume, for contradiction, that there is no tuple from I witnessing that $\varphi \in p^\eta(\mathbf{I})$. Then there is a tuple from I that witnesses that $(\neg\varphi) \in p^\eta(\mathbf{I})$, by Observation 2.4 and the fact that $\mathcal{I}$ and $\mathcal{J}$ have the same age. Since $\mathbf{J}$ is indiscernible and $\varphi(\overline{x}) \in p^\eta(\mathbf{J})$, in fact for all $\overline{\jmath}$ from J satisfying η, $\vDash \varphi(\overline{b}_{\overline{\jmath}})$, and so it is not possible that $(\neg\varphi) \in p^\eta(\mathbf{J})$, as our assumption would have us conclude.
(2) Suppose that $\mathbf{J}$ is locally based on the c_i. Fix a complete quantifier-free type $\eta(\overline{v})$ realized in $\mathcal{J}$. By (1), we need only show that $p^\eta(\mathbf{I}) \subseteq p^\eta(\mathbf{J})$ to show that $\mathrm{tp}(\mathbf{I}) = \mathrm{tp}(\mathbf{J})$. Suppose that some tuple from I witnesses that $\varphi(\overline{x}) \in p^\eta(\mathbf{I})$. Then by indiscernibility, every tuple $\overline{\imath}$ from I satisfying η is witness to $\vDash \varphi(\overline{c}_{\overline{\imath}})$. By the property of being *locally based on*, it would be impossible for a tuple $\overline{\jmath}$ from J satisfying $\eta(\overline{v})$ to have $\vDash \neg\varphi(\overline{b}_{\overline{\jmath}})$. Thus all tuples $\overline{\jmath}$ from $\mathcal{J}$ satisfying η (and there is at least one) witness that $\varphi \in p^\eta(\mathbf{J})$.

The other direction follows from the technique in (3) for representing Δ-types as formulas.
(3) Suppose that $\mathbf{T}$ is locally based on the a_i, and fix $\varphi(x_{i_1}, \dots, x_{i_n}) \in \mathrm{EMtp}_{L'}(\mathbf{U})$. Let $\overline{\imath} := (i_1, \dots, i_n)$. If $\varphi(x_{i_1}, \dots, x_{i_n}) \notin \mathrm{EMtp}_{L'}(\mathbf{T})$, then $\vDash \neg\varphi(\overline{e}_{\overline{\jmath}})$ for some $\overline{\jmath}$ from J satisfying the same quantifier-free type

as $\bar{\imath}$. By assumption, there exists $\bar{\imath}'$ from I satisfying the same quantifier-free type as $\bar{\jmath}$ and $\models \neg\varphi(\bar{a}_{\bar{\imath}'})$. But the condition $\varphi(x_{i_1}, \ldots, x_{i_n}) \in \mathrm{EMtp}_{L'}(\mathbf{U})$ implies that such an $\bar{\imath}'$ cannot exist.

Suppose that $\mathrm{EMtp}_{L'}(\mathbf{T}) \supseteq \mathrm{EMtp}_{L'}(\mathbf{U})$. Fix a finite $\Delta \subset L$ and any $\bar{e}_{\bar{\jmath}}$ from $\mathbf{T}$. Let $\bar{\jmath} := (j_1, \ldots, j_n)$. For contradiction, suppose that

$$\begin{aligned}&\text{no } \bar{\imath} \text{ exists in } I, \text{ with the same quantifier-free type as } \bar{\jmath}\\ &\text{and such that } \bar{a}_{\bar{\imath}} \equiv_\Delta \bar{e}_{\bar{\jmath}}.\end{aligned} \tag{1}$$

Let φ be the conjunction of positive and negative instances of formulas from Δ satisfied by $\bar{e}_{\bar{\jmath}}$. So $\models \varphi(\bar{e}_{\bar{\jmath}})$. By equation (1), for arbitrary $\bar{\imath} = (i_1, \ldots, i_n)$ from I with the same quantifier-free type as $\bar{\jmath}$, $\models \neg\varphi(\bar{a}_{\bar{\imath}})$. Thus $\neg\varphi(x_{i_1}, \ldots, x_{i_n}) \in \mathrm{EMtp}_{L'}(\mathbf{U}) \subseteq \mathrm{EMtp}_{L'}(\mathbf{T})$. But then since $\bar{\jmath}$ satisfies the same quantifier-free type as $\bar{\imath}$, $\models \neg\varphi(\bar{e}_{\bar{\jmath}})$, which is a contradiction.

(4) This is clear.

(5) This follows because the indiscernibility assumption conflates the "there exists" condition in $\mathrm{tp}(\mathbf{I})$ with the "for all" condition in $\mathrm{EMtp}_{L'}(\mathbf{I})$. We use (2) and (3) to conclude that $\mathrm{tp}(\mathbf{W}) = \mathrm{tp}(\mathbf{I}) \Leftrightarrow \mathrm{EMtp}_{L'}(\mathbf{W}) \supseteq \mathrm{EMtp}_{L'}(\mathbf{I})$. However, the first condition is symmetric and $\mathbf{W}, \mathbf{I}$ are both $\mathcal{I}$-indexed indiscernible sets, so we may substitute $\mathrm{EMtp}_{L'}(\mathbf{W}) = \mathrm{EMtp}_{L'}(\mathbf{I})$ for the second condition. To obtain the equivalence with $\mathbf{W} \models \mathrm{EMtp}_{L'}(\mathbf{I})$, use (4).

(6) First observe that if $\Gamma(x_i : i \in I)$ is finitely satisfiable in $\mathbf{U}$, then $\Gamma \cup \mathrm{EMtp}_{L'}(\mathbf{U})$ is satisfiable. So there exists $\mathbf{W}$ satisfying $\mathbf{Ind}(\mathcal{I}, L) \cup \mathrm{EMtp}_{L'}(\mathbf{U})$. Thus $\mathbf{W}$ is generalized indiscernible and $\mathbf{W} \models \mathrm{EMtp}_{L'}(\mathbf{U})$. By (3) and (4), $\mathbf{W}$ is locally based on the a_i.

(7) We obtain $\mathbf{T} = (e_i : i \in J)$ as a realization of the type

$$\begin{aligned}&\Gamma(x_j : j \in J)\\ &\quad = \big\{\varphi(x_{j_1}, \ldots, x_{j_n}) : \bar{\jmath} \text{ from } J \text{ such that for some } \bar{\imath} \text{ from } I \text{ with}\\ &\qquad \mathrm{qftp}^{L'}(\bar{\jmath}; \mathcal{J}) = \mathrm{qftp}^{L'}(\bar{\imath}; \mathcal{I}), \varphi(x_{i_1}, \ldots, x_{i_n}) \in \mathrm{EMtp}_{L'}(\mathbf{U})\big\}.\end{aligned}$$

But this type is clearly finitely satisfiable in $\mathbf{U}$, as $\mathcal{I}$ and $\mathcal{J}$ have the same age.

(8) This is clear, as a union of quantifier-free L'-types is equivalent to each quantifier-free L^*-type. □

For an L'-structure $\mathcal{I}$, if $\mathcal{I}$-indexed indiscernibles have the modeling property, we may find $\mathcal{J}$-indexed indiscernibles locally based on an $\mathcal{I}$-indexed set of parameters, for any L'-structure $\mathcal{J}$ with $\mathrm{age}(\mathcal{J}) \subseteq \mathrm{age}(\mathcal{I})$, as is observed in [24] and [16] (equivalently, if every complete quantifier-free type realized in $\mathcal{J}$ is realized in $\mathcal{I}$). We prove a weaker result below, for clarity. The term "stretching" is well-known terminology in the linear order case (see Hodges [7] and Baldwin [1]).

Definition 2.12 Fix L'-structures $\mathcal{I}$ and $\mathcal{J}$ such that $\mathrm{age}(\mathcal{J}) = \mathrm{age}(\mathcal{I})$. Given an $\mathcal{I}$-indexed indiscernible $\mathbf{I} = (a_i : i \in I)$, we say that a $\mathcal{J}$-indexed indiscernible $\mathbf{J} = (b_i : i \in J)$ is a *stretching of* $\mathbf{I}$ *onto* $\mathcal{J}$ if $\mathrm{tp}(\mathbf{I}) = \mathrm{tp}(\mathbf{J})$.

The lemma below is only a slight generalization of [17, Chapter VII, Lemma 2.2] in that the `qteqf` hypothesis is not needed.

Lemma 2.13 *For any L'-structures $\mathcal{I}$ and $\mathcal{J}$ such that* $\mathrm{age}(\mathcal{J}) = \mathrm{age}(\mathcal{I})$ *and $\mathcal{I}$-indexed indiscernible* $\mathbf{I} = (a_i : i \in I)$*, there is a stretching of* $\mathbf{I}$ *onto $\mathcal{J}$.*

Proof Fix $\mathbf{I} = (a_i : i \in I)$, $\mathcal{I}$, $\mathcal{J}$ as above. Define Γ to be the type

$$\Gamma(x_s : s \in J) := \big\{\varphi(x_{s_1}, \dots, x_{s_n}) : (s_1, \dots, s_n) \text{ from } J, \eta(v_1, \dots, v_n) \text{ is a complete quantifier-free type in } \mathcal{I}, \operatorname{qftp}(s_1, \dots, s_n; \mathcal{J}) = \eta, \text{ and } \varphi(x_1, \dots, x_n) \in p^{\eta}(\mathbf{I})\big\}.$$

Claim 2.14 *Any realization* $\mathbf{J} = (b_i : i \in J)$ *of* Γ *will be a stretching of* $\mathbf{I}$ *onto* $\mathcal{J}$.

Proof Let $\mathbf{J} \models \Gamma$. By Observation 2.4, $\mathcal{I}$ and $\mathcal{J}$ realize the same complete quantifier-free types. By Proposition 2(1), to see that $\operatorname{tp}(\mathbf{I}) = \operatorname{tp}(\mathbf{J})$ holds we need only show that $p^{\eta}(\mathbf{I}) \subseteq p^{\eta}(\mathbf{J})$ for an arbitrary complete quantifier-free type η realized in $\mathcal{J}$. Note that any formula $\varphi(\overline{x})$ in $p^{\eta}(\mathbf{I})$ will automatically be in $p^{\eta}(\mathbf{J})$, by definition of Γ. A realization of Γ is automatically $\mathcal{J}$-indexed indiscernible by the fact that $\operatorname{tp}(\mathbf{I}) = \operatorname{tp}(\mathbf{J})$ and that $\mathcal{I}, \mathcal{J}$ realize the same complete quantifier-free types. □

To see that Γ is finitely satisfiable in $\mathbb{M}$, take a finite subset $\Gamma_0 \subset \Gamma$. Let $\{j_k : k \leq N\}$ list all the members of J mentioned in any formula in Γ_0. Let B be the substructure of J generated by $\{j_k : k \leq N\}$. By assumption, there is a substructure A of $\mathcal{I}$ isomorphic to B, by some isomorphism $f : B \to A$. Then $(f(j_k))_{k \leq N}$ has the same complete quantifier-free type as $(j_k)_{k \leq n}$ and the tuple $(a_{f(j_k)} : k \leq N)$ works to satisfy $\Gamma_0(x_{j_0}, \dots, x_{j_N})$, by generalized indiscernibility of $\mathbf{I}$. □

3 Modeling Property and Ramsey Classes

In applications one looks for $\mathcal{I}$-indexed indiscernibles to have the *modeling property*, meaning that $\mathcal{I}$-indexed indiscernible sets can be produced in the monster model of any theory so as to inherit the local structure of an initial I-indexed set of parameters.

Definition 3.1 (Modeling property) Fix an L'-structure $\mathcal{I}$. We say that *$\mathcal{I}$-indexed indiscernibles have the modeling property* if given any parameters $(a_i : i \in I)$ in the monster model of some theory, $\mathbb{M}$, there exists an $\mathcal{I}$-indexed indiscernible $(b_i : i \in I)$ in $\mathbb{M}$ locally based on the a_i.

We repeat definitions for Ramsey classes given in Kechris, Pestov, and Todorcevic [8] and Nešetřil [14].

Definition 3.2 Define an *A-substructure of C* to be a substructure $A' \subseteq C$ isomorphic to A where we do not reference a particular enumeration of A'.

We refer to the set of A-substructures of C as $\binom{C}{A}$.

Remark 3.3 We may think of an A-substructure of C as the range of an embedding $e : A \to C$. If A has no nontrivial automorphisms, then A-substructures may be identified with embeddings of A in C.

Definition 3.4 For an integer $k > 0$, by a *k-coloring of* $\binom{C}{A}$ we mean a function $f : \binom{C}{A} \to \eta$, where η is some set of size k (typically $\eta := \{0, \dots, k-1\}$).

Definition 3.5 Fix a class U of L'-structures, for some language L'. Let A, B, C be structures in U, and let k be some positive integer.

(1) By
$$C \to (B)^A_k$$
we mean that for all k-colorings f of $\binom{C}{A}$, there is a $B' \subseteq C$, where B' is L'-isomorphic to B and the restricted map, $f \restriction \binom{B'}{A}$, is constant.

(2) If, for a particular coloring $f : \binom{C}{A} \to k$ we have a $B' \subseteq C$ such that $f \restriction \binom{B'}{A}$ is constant, we say that B' *is homogeneous for this coloring (homogeneous for f)*.

Definition 3.6 Let $\mathcal{U}$ be a class of finite L'-structures, for some language L'. The class $\mathcal{U}$ is a *Ramsey class* if for any $A, B \in \mathcal{U}$ and positive integer k, there is a C in $\mathcal{U}$ such that $C \to (B)^A_k$.

Remark 3.7 In the case where L' contains a linear ordering, coloring substructures $A \subseteq C$ is equivalent to coloring embeddings of A into C. It is observed in [14] that if we color embeddings, we can never find homogeneous $B \subseteq C$ containing A, if A has a nontrivial automorphism and we color the embedded copies of A in C different colors. If $\mathcal{I}$-indexed indiscernibles have the modeling property, then because of the case of $\mathcal{M}$ a linear order, there cannot exist a finite substructure $A \subset \mathcal{I}$ with a nontrivial automorphism (see Observation 2.3). The example of this phenomenon with $\mathcal{I}$ an unordered symmetric graph is worked out in [16].

We want some additional notation for the function symbols case. For the rest of this section we work with index structures $\mathcal{I}$ that are linearly ordered by some relation, $<$. By *increasing* we will always mean $<^{\mathcal{I}}$-increasing.

Definition 3.8 For $\mathcal{I}$ locally finite and linearly ordered by $<$, define $\overline{\mathrm{cl}}(\cdot)$ on I to take finite tuples $\overline{a}$ in increasing enumeration in $\mathcal{I}$ to the smallest substructure of $\mathcal{I}$ containing $\overline{a}$, also listed in increasing enumeration.

Remark 3.9 In Definition 3.8, $\overline{\mathrm{cl}}(\overline{a})$ is a finite, increasing tuple in $\mathcal{I}$.

Observation 3.1 Let $\mathcal{I}$ be as in Definition 3.8. For a finite subset $A \subseteq I$, let $C(A) := \bigcup \overline{\mathrm{cl}}(\overline{a})$, where $\overline{a}$ lists A in increasing order. Then $C(\cdot)$ defines a closure property on finite subsets $A, B \subseteq I$; that is, $A \subseteq C(A)$, $C(C(A)) = C(A)$, and if $A \subseteq B$, then $C(A) \subseteq C(B)$.

Remark 3.10 Our use of $\overline{\mathrm{cl}}(\cdot)$ in the next theorem and also in Corollary A.3 is quite similar to the technique of the strong-subtree envelopes in Todorcevic [23, Section 6.2].

The next theorem uses some additional notation.

Definition 3.11 Fix a structure $\mathcal{I}$ linearly ordered by a relation $<$. Fix a finite tuple $\overline{b}$ from I and a finite subset $A \subseteq I$.

(1) By $p_{\overline{b}}(\overline{x})$ we mean the complete quantifier-free type of $\overline{b}$ in $\mathcal{I}$.
(2) By $p_A(\overline{x})$ we mean $p_{\overline{a}}(\overline{x})$, where $\overline{a}$ is A listed in increasing enumeration.
(3) We say that $\overline{b}$ is an *increasing copy of* A if the substructure B of $\mathcal{I}$ on $\bigcup \overline{b}$ is isomorphic to A.
(4) Fix a finite tuple $\overline{i}$ from A (i.e., $\bigcup \overline{i} \subseteq A$), and let $\overline{a}$ list A in $<^{\mathcal{I}}$-increasing order. We say that $\overline{i}$ *isolates* τ *in* A if $\overline{a} \restriction \tau = \overline{i}$.

We give the main theorem.

Theorem 3.12 *Suppose that $\mathcal{I}$ is a* $\mathtt{qfi}$*, locally finite structure in a language L' with a relation $<$ linearly ordering I. Then $\mathcal{I}$-indexed indiscernible sets have the modeling property just in case* $\mathrm{age}(\mathcal{I})$ *is a Ramsey class.*

Proof $\Leftarrow$: Here we use the locally finite and ordered hypotheses. Suppose that $\mathrm{age}(\mathcal{I})$ is a Ramsey class. Fix an initial set of parameters $\mathbf{I} := (a_i : i \in I)$ in $\mathbb{M}$. We wish to find an $\mathcal{I}$-indexed indiscernible $\mathbf{J} := (b_j : j \in \mathcal{I})$ locally based on the a_i. By Proposition 2(6), it suffices to show that $\mathbf{Ind}(\mathcal{I}, L)$ is finitely satisfiable in $\mathbf{I}$.

Let η be a complete quantifier-free n-type realized by some tuple $\overline{\imath}$ in $\mathcal{I}$. Let A be the substructure generated by $\overline{\imath}$ in $\mathcal{I}$ (say A has size N). There is some sequence τ so that $\overline{\imath}$ isolates τ in A. Fix this τ and call it σ_η. If $\overline{\jmath} \vDash_{\mathcal{I}} \eta$, $\bigcup\overline{\mathrm{cl}}(\overline{\jmath})$ is isomorphic to A by the homomorphism induced by $\overline{\jmath} \mapsto \overline{\imath}$. If $\overline{b}$ is an increasing copy of A, then $\overline{b} \restriction \sigma_\eta \vDash_{\mathcal{I}} \eta$ and $\overline{\mathrm{cl}}(\overline{b} \restriction \sigma_\eta) = \overline{b}$. Note that for realizations $\overline{\jmath} \vDash_{\mathcal{I}} \eta$, $\overline{\mathrm{cl}}(\overline{\jmath}) \restriction \sigma_\eta = \overline{\jmath}$; thus for $\overline{\jmath}, \overline{\jmath}' \vDash_{\mathcal{I}} \eta$, $\overline{\mathrm{cl}}(\overline{\jmath}) = \overline{\mathrm{cl}}(\overline{\jmath}') \Rightarrow \overline{\jmath} = \overline{\jmath}'$. So we have shown that σ_η sets up a correspondence

$$\overline{\jmath} \mapsto \overline{\mathrm{cl}}(\overline{\jmath}) \tag{2}$$

between realizations of η in $\mathcal{I}$ and copies of A in $\mathcal{I}$.

Now let $\Gamma_0 \subseteq \Gamma$ be a finite subset. Γ_0 mentions only finitely many formulas $\{\varphi_1, \ldots, \varphi_l\} =: \Delta$. We may assume that the variables occurring in Γ_0 are $x_{p_1}, \ldots, x_{p_r}$ for some increasing tuple $\overline{p}$ in $\mathcal{I}$. Let $B := \bigcup\overline{\mathrm{cl}}(p_1, \ldots, p_r)$, and let $\overline{p}$ isolate the sequence τ_B in B. Let $\eta_1, \ldots, \eta_s$ be the complete quantifier-free types realized in the set $\{p_1, \ldots, p_r\}$. It suffices to find a copy B' of B in $\mathcal{I}$ such that

$$\text{for all } 1 \le t \le s,\ \text{for all realizations } \overline{\jmath}, \overline{\jmath}' \text{ of } \eta_t \text{ in } B', \overline{a}_{\overline{\jmath}} \equiv_\Delta \overline{a}_{\overline{\jmath}'} \tag{3}$$

since then $\overline{b}' \restriction \tau_B \vDash \Gamma_0$, for $\overline{b}'$ the increasing enumeration of B'.

The argument in [16, Claim 4.16] shows that we only need to accomplish equation (3) for one η_t, as the rest follows by induction. So fix a complete quantifier-free n-type η_t realized in $\mathcal{I}$. For some choice of $\overline{\imath} \vDash_{\mathcal{I}} \eta_t$, let $\bigcup\overline{\mathrm{cl}}(\overline{\imath}) =: E$. Linearly order the finitely many (Δ, n)-types, and suppose that there are K of them, for some finite K. Define a K-coloring on all copies E' of E in $\mathcal{I}$: E' gets the kth color if its increasing enumeration $\overline{e}'$ has the property that $\overline{e}' \restriction \sigma_{\eta_t} =: \overline{\jmath}$ indexes $\overline{a}_{\overline{\jmath}}$ with the kth Δ-type. By the assumption of a Ramsey class, there is a copy B_t of B in $\mathcal{I}$ that is homogeneous for this coloring. Since all copies E' of E in B_t get the same color, by definition of the coloring, there is a (Δ, n)-type $\pi(\overline{x})$, and all $\overline{\jmath} \vDash_{\mathcal{I}} \eta$ such that $\overline{\jmath} = \overline{e}' \restriction \sigma_{\eta_t}$ for $\overline{e}'$ the increasing enumeration of some $E' \cong E$ in B_t are such that $\overline{a}_{\overline{\jmath}} \vDash \pi$. But every realization of η in B_t is such a $\overline{\jmath}$ by equation (2) and the fact that $\bigcup\overline{\mathrm{cl}}(\cdot)$ acts as a closure relation under which B_t is closed. $\square$

Proof $\Rightarrow$: Let $\mathcal{K} := \mathrm{age}(\mathcal{I})$. Suppose that $\mathcal{I}$-indexed indiscernible sets have the modeling property. We want to show that $\mathrm{age}(\mathcal{I})$ is a Ramsey class. We adapt the well-known technique of compactness in partition results to our context.

Claim 3.13 *Let $\mathcal{I}$ be* $\mathtt{qfi}$*, locally finite, and linearly ordered by one of its relations. If for all $k < \omega$ and $A, B \in \mathcal{K}$: $I \to (B)^A_k$, then $\mathcal{K}$ is a Ramsey class.*

Proof Let $T := \mathrm{Th}(\mathcal{I})$, $k, A, B, \mathcal{I}$ as above, and suppose that A, B have cardinality n, N, respectively. Let $L^+ := L' \cup \{P_0, \ldots, P_{k-1}\}$, and consider the following L^+-theory S. For the complete quantifier-free types p_D for finite substructures

$D \subseteq \mathcal{I}$, substitute a formula equivalent modulo $T_\forall$, using the `qfi` hypothesis:

$$\begin{aligned} S := T_\forall &\cup \mathrm{Diag}(\mathcal{I}) \cup \Big\{\forall \overline{x}\Big(p_A(\overline{x}) \to \bigvee_{i<k} P_i(\overline{x})\Big)\Big\} \\ &\cup \big\{\neg\exists\overline{x}\big(P_i(\overline{x}) \wedge P_j(\overline{x})\big) : i \neq j < k\big\} \\ &\cup \Big\{\neg\exists\overline{x}\Big(p_B(\overline{x}) \\ &\wedge \bigvee_{s<k}\Big(\bigwedge_{1\le i_1<\cdots<i_n\le N}\big(p_A(x_{i_1},\ldots,x_{i_n}) \to P_s(x_{i_1},\ldots,x_{i_n})\big)\Big)\Big)\Big\}. \end{aligned}$$

If we assume that no C exists in $\mathcal{K}$ such that $C \to (B)^A_k$, then S is finitely satisfiable, by taking finitely generated substructures of $\mathcal{I}$ and a bad coloring on such a substructure in order to interpret the new predicates, P_i. Note that the formulas equivalent to complete quantifier-free types in $\mathcal{I}$ are equivalent to the same types in models of $T_\forall$ (in particular, in substructures of $\mathcal{I}$). By compactness, S is satisfied by some structure $\mathcal{J}$ whose restriction to the constants in $\mathrm{Diag}(\mathcal{I})$ is a structure $\mathcal{I}^*$ whose L'-reduct is isomorphic to $\mathcal{I}$ by some map $f : I^* \to I$. There is a coloring by the $P_i^{\mathcal{J}}$ of the A-substructures of $\mathcal{J}$ for which there is no copy of B in $\mathcal{J}$ homogeneous for this coloring. If we restrict this coloring to $\binom{\mathcal{I}^*}{A}$, there is still no homogeneous copy of B. By standard methods of reducts and expansions, the map f yields a k-coloring of the A-substructures of $\mathcal{I}$ for which there is no homogeneous copy of B. □

Now fix $\mathcal{I}$ as in the statement of the theorem. The proof continues as in [16]; we repeat a shortened proof here for completeness. At this point the `qfi` hypothesis is no longer needed.

Claim 3.14 *Fix $A, B \in \mathcal{K}$ and $k < \omega$. Then $I \to (B)^A_k$.*

Proof Fix a k-coloring of the A-substructures of $\mathcal{I}$, $g : \binom{\mathcal{I}}{A} \to \{1,\ldots,k\}$. Since $\mathcal{I}$ is linearly ordered, we can understand g as being defined on n-tuples $\overline{a} \models_{\mathcal{I}} p_A$. We need to find $B' \subseteq I$ isomorphic to B, homogeneous for this coloring.

Let A have size n. Fix a language $L = \{R_1,\ldots,R_k\}$ with k n-ary relations, and construct an L-structure $\mathcal{M}$ as follows:

(1) $|\mathcal{M}| = I$;
(2) the relation R_s, $1 \le s \le k$, is interpreted as follows:
For $i_1,\ldots,i_n$ from $|\mathcal{M}|$,

$$R_s^{\mathcal{M}}(i_1,\ldots,i_n) \Leftrightarrow$$

(a) $\overline{\imath} \models_{\mathcal{I}} p_A$, and
(b) $g((i_1,\ldots,i_n)) = s$.

Let $(a_i : i \in I)$ be the I-indexed set in $\mathcal{M}$ such that $a_i = i$. We work in a monster model $\mathbb{M}$ of $\mathrm{Th}(\mathcal{M})$. By assumption, we can find an L'-generalized indiscernible $(b_j : j \in I)$ in $\mathbb{M}$ locally based on the a_i. Since $\mathcal{K} = \mathrm{age}(\mathcal{I})$, we may find a copy of B in $\mathcal{I}$, D'. By assumption, D' is a finite structure. Enumerate D' in $<^{D'}$-increasing order as $(j_k : k \le N)$. By the modeling property, for $\Delta := L$, there is some $i_1,\ldots,i_N$ such that

$$\begin{aligned} \mathrm{qftp}^{L'}(i_1,\ldots,i_N;\mathcal{I}) &= \mathrm{qftp}^{L'}(j_1,\ldots,j_N;\mathcal{I}), \qquad \text{and} \\ \mathrm{tp}^{\Delta}(b_{j_1},\ldots,b_{j_N};\mathcal{M}_1) &= \mathrm{tp}^{\Delta}(a_{i_1},\ldots,a_{i_N};\mathcal{M}). \end{aligned} \tag{4}$$

Subclaim 3.15 *$D := (i_k : k \le N) \subseteq I$ is a copy of B in I that is homogeneous for the coloring, g.*

Proof $D \cong D'$, as $\mathrm{qftp}^{L'}(\overline{\imath}) = \mathrm{qftp}^{L'}(\overline{\jmath})$ and D, D' are structures. So D is a copy of B, and it remains to show that D is homogeneous for the coloring, g. The b_i are generalized indiscernible, so there is some choice of l_0 so that for any increasing copy $\overline{c}'$ of A in D', $\models R_{l_0}(\overline{c}')$. We show that all copies of A in D are colored l_0 under g.

Let $\overline{c}$ be any increasing copy of A in D. There is some sequence σ so that $\overline{c}$ isolates σ in $\overline{\imath}$. By the first part of equation (4), for $\overline{c}' := \overline{\jmath} \restriction \sigma$, $\overline{c}'$ is an increasing copy of A. Thus $\models R_{l_0}(\overline{c}')$. By the second part of equation (4), $\models R_{l_0}(\overline{c})$, that is, $g(\overline{c}) = l_0$. □

□

□

3.1 Applications We make use of L_i-generalized indiscernible sets for $i = s, 1, 2$, where the languages L_i are defined as follows.

Definition 3.16

(1) We fix languages

$$L_s = \{\trianglelefteq, \wedge, <_{\mathrm{lex}}, (P_n)_{n<\omega}\}, \qquad L_1 = \{\trianglelefteq, \wedge, <_{\mathrm{lex}}, <_{\mathrm{len}}\},$$
$$L_0 = \{\trianglelefteq, \wedge, <_{\mathrm{lex}}\}.$$

(2) We let I_s, I_1, I_0 be the intended interpretations of L_s, L_1, L_0, respectively, on $\omega^{<\omega}$: $\trianglelefteq$ is interpreted as the partial tree-order; $\wedge$ as the meet function in this order; $<_{\mathrm{lex}}$ as the lexicographic ordering on sequences extending the partial tree-order; P_n to hold of η just in case $\ell(\eta) = n$; $\eta <_{\mathrm{len}} \nu$ to hold just in case $\ell(\eta) < \ell(\nu)$.

Corollary 3.17 $\mathrm{age}(I_0)$ *is a Ramsey class.*

Proof I_0-indexed indiscernible sets have the modeling property by a result from [21]. For completeness, an alternate proof of this result is given as Theorem A.5.

It remains to verify the conditions of Theorem 3.12. Since I_0 is locally finite in a finite language, I_0 is `qfi` by Proposition 1. Thus by Theorem 3.12, $\mathrm{age}(I_0)$ is a Ramsey class. □

Corollary 3.18 ([4]) *We have that* $\mathrm{age}(I_s)$ *is a Ramsey class.*

Proof In [10] and [21], it was concluded that I_s-indexed indiscernible sets have the modeling property, relying on a key result from [17].[1] It remains to verify the conditions in Theorem 3.12.

Note that $I_0 = I_s \restriction \{\trianglelefteq, \wedge, <_{\mathrm{lex}}\}$. In Corollary 3.17 we argue that I_0 is `qfi` by way of Remark 2.4. Let T_s be the theory of I_s, and let T_0 be the theory of I_0.

Thus, for any complete quantifier-free (L_0, m)-type of a substructure of I_0, p, there exists an (L_0, m)-formula θ_p such that

$$(T_0)_\forall \cup \{\theta_p(\overline{x})\} \vdash p(\overline{x}). \tag{5}$$

For any complete quantifier-free (L_s, m)-type $q(\overline{x})$ realized in I_s, there is some p_0 so that $p_0 = q \restriction L_0$. Thus, for some choice of $t_l \in \{0, 1\}$ for $l < \omega$:

$$p_0(\overline{x}) \cup \{P_l(x_i)^{t_l} : i < m, l < \omega\} \vdash q(\overline{x}). \tag{6}$$

Using equation (5), we have

$$(T_s)_\forall \cup \left\{\theta_{p_0}(\overline{x})\right\} \cup \left\{P_l(x_i)^{t_l} : i < m, l < \omega\right\} \vdash q(\overline{x}). \tag{7}$$

We use the fact that, for all $i \neq k < \omega$,

$$(T_s)_\forall \vdash \left(\forall y \neg\left(P_i(y) \wedge P_k(y)\right)\right), \tag{8}$$

and that any complete quantifier-free type q realized in I_s contains at least one $P_k(x_j)$ for every $j < m$ (though in other models of T_s this may not be the case). Thus there exist $i_0, \ldots, i_{m-1} < \omega$ such that,

$$(T_s)_\forall \cup \left\{\theta_{p_0}(\overline{x}) \wedge \left(\bigwedge_{j<m} P_{i_j}(x_j)\right)\right\} \vdash q(\overline{x}). \tag{9}$$

Thus we have shown that I_s is `qfi`. By Theorem 3.12, age(I_s) is a Ramsey class. □

We give an additional remark in connection with [21, Example 17]. Here the authors provide the example of $I_t := I_0 \restriction \{\trianglelefteq, <_{\text{lex}}\}$ and show that I_t-indexed indiscernibles do not have the modeling property. We observe that this fact is also a corollary of Theorem 3.12. Let $L_t := \{\trianglelefteq, <_{\text{lex}}\}$.

Corollary 3.19 ([21, Example 17]) *I_t-indexed indiscernibles do not have the modeling property.*

Proof Let $K_t := \text{age}(I_t)$. By Theorem 3.12, I_t-indexed indiscernibles have the modeling property just in case K_t is a Ramsey class, by a quick verification of the conditions. By [14, Theorem 4.2(i)] and the presence of a linear ordering, if K_t is a Ramsey class, then K_t has the amalgamation property. However, an example analyzed in [21, Example 17] provides the counterexample to amalgamation. Let A be the finite structure given by $a_0 \trianglelefteq a_1, a_2, a_3$ and $a_0 <_{\text{lex}} a_1 <_{\text{lex}} a_2 <_{\text{lex}} a_3$. Let B_i be the structures below, where a diagonal edge between nodes denotes that the bottom node is $\trianglelefteq$-related to the top node, the absence of an edge between nodes denotes no $\trianglelefteq$-relation, and $<_{\text{lex}}$ both refines $\trianglelefteq$ and obeys the rule that $x <_{\text{lex}} y$ if x is to the left of y on the page. Then A L_t-embeds into B_1, B_2 by $a_i \mapsto b_i, c_i$ (see Figure 1).

Suppose there exists some amalgam C for (A, B_1, B_2). By a small abuse of notation, we use the labels "b_i, c_i," $0 \leq i \leq 4$, to refer to the *images* of these points in C. First, observe that b_4, c_4 in C must be $\trianglelefteq$-comparable (by inspection of K_t), as both points are $\trianglelefteq$-predecessors of the same point, $b_2 (= c_2)$. If $b_4 \trianglelefteq c_4$, then $b_4 \trianglelefteq c_4 \trianglelefteq c_3 = b_3$, contradicting the data in B_1. If $c_4 \trianglelefteq b_4$,

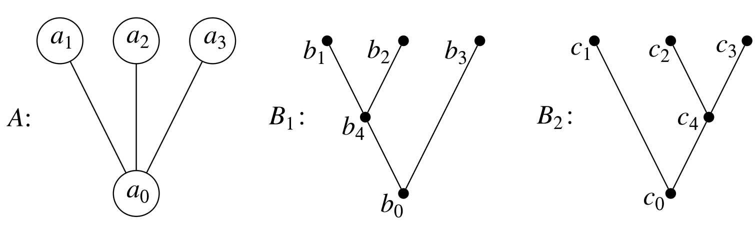

Figure 1 B_1 and B_2 have no L_t-amalgam over A.

then $c_4 \trianglelefteq b_4 \trianglelefteq b_1 = c_1$, contradicting the data in B_2. Thus, no such amalgam exists. $\square$

Appendix

As an application of EM-types, we give an alternate proof that I_0-indexed indiscernible sets have the modeling property. This proof eschews [17, Appendix 2.6] in favor of Lemma A.2 below, whose statement is taken from Nguyen [15], where the original result is attributed to [4].

First we clarify the notion of height we are using.

Definition A.1 Fix a finite tree T partially ordered by $\trianglelefteq$, and let $\nu \in T$.

(1) We say that $\mathrm{ht}(\nu) = |\{\eta : \eta \trianglelefteq \nu, \eta \neq \nu\}|$.
(2) We say that $\mathrm{ht}(T) = \max\{\mathrm{ht}(\nu) : \nu \in T\}$.

Lemma A.2 ([15, Section 2.2, Lemma 2]) *Fix $m \in \omega$, and let $\mathcal{K}_u^m$ be the class of all finite L_t-substructures of $\omega^{\leq m}$ of height m, all of whose maximal nodes have height m.*[2] *Then $\mathcal{K}_u^m$ is a Ramsey class.*

Corollary A.3 *$\mathcal{K}_s$ is a Ramsey class.*

Proof The idea is simple, but we fill in the steps. Fix D_t in $\mathcal{K}_u^m$. We may interpret the $(P_n)_n$ naturally on D_t so that for $\eta \in D_t$, $\mathrm{ht}(\eta) = n \leftrightarrow P_n(\eta)$, and we may interpret the meet function $\wedge$ on D_t in the usual way, as it is definable from $\trianglelefteq$. In this way we obtain a natural L_s-expansion of D_t, which we call $\mathtt{exp}(D_t)$. In fact any L_t-embedding $f : A_t \to B_t$ for $A_t, B_t \in \mathcal{K}_u^m$ naturally induces an L_s-embedding $\bar{f} : \mathtt{exp}(A_t) \to \mathtt{exp}(B_t)$.

Fix $D \in \mathcal{K}_s$ such that n is maximal so that $P_n^D \neq \emptyset$, and let $n \leq m$. We define an L_t-structure from D uniquely up to L_t-isomorphism. Let k be least so that the L_s-substructure $E_m \subseteq I_s$ on the set $k^{\leq m}$ contains a copy of D, and fix one such copy $D' \subseteq E_m$. Suppose that D' has i-many $\trianglelefteq$-maximal elements, and choose a size-i subset Y of k^m that $\trianglelefteq$-majorizes these maximal elements. Let $\mathtt{fill}_m(D')$ be the L_t-reduct in E_m on the set $\{\eta \in k^{\leq m} : (\exists x \in Y)\eta \trianglelefteq x\}$. Then $\mathtt{fill}_m(D') \in \mathcal{K}_u^m$. There is a first-order L_t-formula $\Psi = \Psi_D$ that carves out D', that is, $\Psi(\mathtt{fill}_m(D')) = D'$. For an L_t-structure $D_t \cong_{L_t} \mathtt{fill}_m(D')$, let $\mathcal{S}(D_t)$ be defined as the L_s-substructure of $\mathtt{exp}(D_t)$ defined on the set $\Psi(D_t)$. Then $\mathcal{S}(D_t) \cong_{L_s} D'$.

Fix A, B in $\mathcal{K}_s$ and $k \in \omega$. Let m be maximal so that P_m^B is nonempty. By Lemma A.2, we may choose $C_t \in \mathcal{K}_u^m$ so that

$$C_t \to \left(\mathtt{fill}_m(B)\right)_k^{\mathtt{fill}_m(A)}. \tag{10}$$

Let $C := \mathtt{exp}(C_t)$.

Claim A.4 $C \to (B)_k^A$.

Proof Fix a coloring $c : \binom{C}{A} \to k$. We convert c into a coloring $c' : \binom{C_t}{\mathtt{fill}_m(A)} \to k$ as follows. Given A_t a copy of $\mathtt{fill}_m(A)$ in C_t, let $c'(A_t) := c(\mathcal{S}(A_t))$ (by the above, $\mathcal{S}(A_t) \cong_{L_s} A$). By equation (10), there is a copy B_t of $\mathtt{fill}_m(B)$ in C_t homogeneous for this coloring. Then $\mathcal{S}(B_t)$ is a copy of B in C that is homogeneous for c, as every copy of A in $\mathcal{S}(B_t)$ extends to a copy of A_t in B_t. $\square$

$\square$

The use of EM-types and Corollary A.3 allows us to finitize the proof of Theorem A.5 below, up to some applications of compactness. All the other techniques and ideas below are not new, and may be seen in [17] and [10] as well as the original argument in [21].

Theorem A.5 ([21, Theorem 16]) *$\mathcal{I}_0$-indexed indiscernible sets have the modeling property.*

Proof In the following, numbers (n) refer to items from Proposition 2. Let $\mathbf{I} := (a_i : i \in \omega^{<\omega})$ be a set of parameters in a monster model $\mathbb{M}$ of some theory. We must show that there is an I_0-indexed indiscernible set L_0-locally based on the a_i.

Step 1. By Corollary A.3 and Theorem 3.12, there is an I_s-indexed indiscernible $\mathbf{T} := (d_i : i \in \omega^{<\omega})$ that is L_s-locally based on the a_i. By (3), $\mathrm{EMtp}_{L_s}(\mathbf{T}) \supseteq \mathrm{EMtp}_{L_s}(\mathbf{I})$, so by (8),

$$\mathrm{EMtp}_{L_0}(\mathbf{T}) \supseteq \mathrm{EMtp}_{L_0}(\mathbf{I}). \tag{11}$$

Step 2. We aim to find an I_1-indexed indiscernible $\mathbf{U} := (e_i : i \in \omega^{<\omega})$ that is L_1-locally based on $\mathbf{T}$. By (6), $\mathbf{U}$ may be obtained by the following claim.

Claim A.6 *$\mathbf{Ind}(I_1, L)$ is finitely satisfiable in $\mathbf{T}$.*

Proof Let $F_1 \subset \mathbf{Ind}(I_1, L)$ be some finite subset. There is some n so that all variables occurring in F_1 are indexed by nodes in $\omega^{<n}$. There is some finite set $\Delta \subset \mathcal{L}$ such that all formulas occurring in F_1 are from Δ. Let $(\mu^i(x_0, \ldots, x_{m-1}) : i < N)$ enumerate the quantifier-free L_1-types of size-m substructures of $\omega^{<n}$, where we may assume that Δ is a set of L-formulas in m variables. Because expansions of μ^i to complete quantifier-free L_s-types may allow $P_k(x_i)$ and $P_k(x_j)$ for $i \neq j$, we do some coding. For any function $f : m \to m$ and $(j_0, \ldots, j_{m-1}) =: \overline{\jmath} \in \omega^m$, define

$$\mu^i_{f,\overline{\jmath}} := \mu^i \cup \left\{P_{j_{f(0)}}(x_0), \ldots, P_{j_{f(m-1)}}(x_{m-1})\right\}. \tag{12}$$

By L_s-indiscernibility, we know that for any increasing tuple $\overline{\jmath} \in \omega^m$ and $f : m \to m$, if $\mu^i_{f,\overline{\jmath}}$ is realized in I_s, then there is a complete type p in $\mathbb{M}$ such that for any $\overline{l} \vDash_{\mathcal{I}_s} \mu^i_{f,\overline{\jmath}}$, $\mathrm{tp}(\overline{d}_{\overline{l}}; \mathbb{M}) = p$. Enumerate the (Δ, m)-types in $\mathbb{M}$ as $(\delta_i : 1 \leq i < K)$ for some $K \in \omega$, and fix $\delta_0 := \emptyset$. Let $N' := N \cdot m^m$. Fix an enumeration $((f_\beta, \mu_\beta) : \beta < N')$ of functions $f : m \to m$ and types $\mu = \mu^i$, for $i < N$. Let

$$f : [\aleph_0]^m \to K^{N'}$$

map an m-tuple $\overline{\jmath} \mapsto \alpha$, for $\alpha < K^{N'}$ if

1. $(s_\beta)_{\beta < N'}$ is the αth sequence from $K^{N'}$, and
2. for all $\beta < N'$, if there exists $\overline{l}$ from I_s satisfying $\mu^\beta_{f_\beta,\overline{\jmath}}$, then $\mathrm{tp}^\Delta(\overline{d}_{\overline{l}}; \mathbb{M}) = \delta_{s_\beta}$; otherwise, $s_\beta = 0$.

By Ramsey's theorem, there is an infinite subset of $\aleph_0$ that is homogeneous for this coloring. The L_1-subtree of I_1 obtained by restricting to the levels in this infinite set indexes a subset of $\mathbf{T} = (d_i : i < \omega^{<\omega})$, a finite subset of which will satisfy F_1. □

By (3), $\mathrm{EMtp}_{L_1}(\mathbf{U}) \supseteq \mathrm{EMtp}_{L_1}(\mathbf{T})$. Thus,

$$\mathrm{EMtp}_{L_0}(\mathbf{U}) \supseteq_{\text{by (8).}} \mathrm{EMtp}_{L_0}(\mathbf{T}) \supseteq_{\text{by equation (11)}} \mathrm{EMtp}_{L_0}(\mathbf{I}). \tag{13}$$

Step 3. If we show that $\mathbf{Ind}(I_0, L)$ is finitely satisfiable in $\mathbf{U}$, then by (6), there is an I_0-indexed indiscernible $\mathbf{J} := (b_i : i \in \omega^{<\omega})$ locally based on the e_i. By equation (13) and item (3), the e_i are L_0-locally based on the a_i, so by Observation 2.2, we are done. It remains to show the following.

Claim A.7 *$\mathbf{Ind}(I_0, L)$ is finitely satisfiable in $\mathbf{U}$.*

Proof A finite subset $F_0 \subset \mathbf{Ind}(I_0, L)$ contains only variables indexed by nodes in $\omega^{\leq n}$ for some n. To satisfy F_0 in $\mathbf{U}$, it suffices to show that the type of an L_0-generalized indiscernible k-branching tree of height n is satisfiable in $\mathbf{U}$.

We follow [3] to show that there is an L_0-embedding of $\sigma : k^{\leq n} \to \omega^{<\omega}$ such that for all $i <_{\text{lex}} j$, we have $\sigma(i) <_{\text{len}} \sigma(j)$. We define $l_m < \omega, h_m : k^{\leq m} \to \omega^{<\omega}$ by induction on m:

$$\begin{aligned} h_i\big(\langle\rangle\big) &= \langle\rangle, \quad \text{for all } i < \omega, \\ l_m &= \max\big\{\ell\big(h_m(\eta)\big) + 1 : \eta \in k^{\leq m}\big\}, \\ h_{m+1}\big(\langle t\rangle^\frown \nu\big) &= \langle t\rangle^\frown \underbrace{\langle 0, \ldots, 0\rangle}_{(t+1)\cdot l_m}{}^\frown h_m(\nu). \end{aligned} \tag{14}$$

Define $\sigma := h_n$. The range of $k^{\leq n}$ under σ is an L_1-subtree $W \subset I_1$, sometimes called a *skew subtree*. We know that $\mathbf{U}$ is already L_1-generalized indiscernible. Since the L_0-type of a tuple in W determines its L_1-type in I_1, $(e_{\sigma(i)} : i \in k^{\leq n})$ is L_0-generalized indiscernible. □

□

Notes

1. By Theorem 3.12, Corollary A.3 presents an alternate route to proof.

2. The latter condition is not entirely explicit in the statement, but appears in the proof and is intended by the author.

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