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Forking in Finite Models

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Abstract We study properties of forking in the classes of all finite models of a complete theory in a finite variable logic. We also study model constructions under the assumption that forking is trivial.

Originally this paper was written at the end of the 1990s but then forgotten. However, recently Cameron Hill seems to have applied the methods developed here to give a new efficient algorithm for inverting the L^2-invariant (see Hill [1]; such an algorithm was first found in Otto [3]). Thus I have decided to give this paper a second chance.

We assume that the reader is familiar with Hyttinen [2] and the usual tricks of stability theory. We assume that τ is a finite similarity type, k is greater than or equal to the arity of τ and in any case at least 2, $\mathscr{L} = L^k$, T is a complete $\mathscr{L}$-theory, and $\mathbf{K} = \{\mathscr{A} \mid \mathscr{A} \models T \text{ is a finite } \tau\text{-model}\}$. In addition, we assume that $\{\mathscr{A}/\cong \mid \mathscr{A} \in \mathbf{K}\}$ is infinite and that the property K2 (i.e., amalgamation over subsets of $\mathbf{K}$-models) from [2] is satisfied. K2 is a strong assumption, but one has to start from somewhere; otherwise we could do everything in a more general setup. But by working out only this special case, we avoid complicated assumptions.

Now the properties K1, K3, and L1 from [2] are satisfied and so, since $\mathscr{L} = L^k$, the number of $\mathscr{L}$-formulas modulo equivalence in $\mathbf{K}$ is finite. We let $\mathbf{M}$ be the "monster model" proved to exist in [2]. The monster model is a limit model, $\mathbf{M} = (\mathscr{A}_i)_{i<\omega}$, but again since $\mathscr{L} = L^k$, we may replace $\mathbf{M}$ by $\bigcup_{i<\omega} \mathscr{A}_i$. By a set we mean a finite subset of $\mathbf{M}$ and write A, B, and so on for these, and A^m means the set of sequences of elements of A of length m. By a sequence we mean a finite sequence of elements of $\mathbf{M}$ and write a, b, and so on for these, and $a \in A$ means $a \in A^{\mathrm{length}(a)}$. We write $\models \varphi(a)$ for $\mathbf{M} \models \varphi(a)$, and we write x, y, and so on for the sequences of variables. As in [2], we allow the use of dummy variables, which do not belong to V $(= \{v_0, \dots, v_{k-1}\}$ = the set of variables which are allowed to

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appear in $\mathcal{L}$-formulas; see [2]). So for $\varphi = \varphi(x)$, $\models \varphi(a)$ makes sense also in the case when the length of a (= the length of x) is greater than k. By $p \vdash q$ we mean that every a which realizes p, realizes also q. By $S_n(A)$ we mean $S_n(A, \mathbf{M})$. Several authors use the notation $S_n^k(A)$ to indicate that only k-variable n-types are under consideration, but we do not do this here because we only ever consider k-variable types. By $R^n(p, \omega)$ we mean $R^n_{\mathcal{L}}(p, A, \omega)$, where A is any set such that p is over A. This is well defined since $\mathcal{L}$ is finite (modulo equivalence in $\mathbf{K}$), and, as pointed out in [2], if K2 is satisfied, then $R^n_{\Delta}(p, A, \omega)$ does not depend on the choice of A. We write $t(a, A)$ for $t_{\mathbf{M}}(a, A)$. Finally, we write $R^n(a, A, \omega)$ for $R^n(t(a, A), \omega)$.

In [2] we showed that if $\mathbf{K}$ is stable, then the ranks $R^n(a, A, \omega)$ exist. As pointed out in [2], ranks allow us to define forking. We do this in the first section, where we also prove the usual properties of forking. However, the dimension theory that follows is of limited value if we cannot construct $\mathbf{K}$-models, and as far as the author of this paper knows, there are no methods of constructing models to a quite arbitrary class of finite models. In the second section, we show that there is a method if forking is trivial (and the class is as above).

In this paper, we calculate several numbers that are elements of $\mathbb{N}$ and functions $\mathbb{N}^n \to \mathbb{N}$. We want to point out that we do not try to find the best possible values, we just show the existence by calculating some value and proving that it satisfies the requirements. Because there are many of these numbers, an index of symbols can be found at the end of this paper.

Of course, everything we do is based on the ideas from Shelah [4]. Basic stability-theoretic definitions are essentially the same; however, due to the fact that addition and multiplication do not behave the same way on the (finite) cardinals relevant here as they do on the (infinite) cardinals relevant in [4], some splitting of concepts happens. The proofs are sometimes verbatim copies of those in stability theory (see, e.g., the proof of Lemma 1.15), and sometimes additional work is needed (see, e.g., the proof of Theorem 2.3 and the discussion just before it). Also, one of the main differences between [4] and what we do here is that we have very little space to work with. That is, knowing that something is finite is usually not good enough; some kind of "uniform" upper bound is needed and similarly for "large enough" (see, e.g., Lemma 1.17). And, of course, the cardinals we are working with do not have nice closure properties (again, see the discussion immediately before Theorem 2.3).

Throughout this paper, we assume that $\mathbf{K}$ is stable.

1 Independence

Definition 1.1

(i) Let a be a sequence of length k. We write $a \downarrow_B C$ if $R^k(a, B \cup C, \omega) = R^k(a, B, \omega)$, and we say that $t(a, B \cup C)$ *does not fork* over B.

(ii) We write $A \downarrow_B C$ if for all $a \in A^k$, $a \downarrow_B C$.

Lemma 1.2 *Assume that $A \subseteq B \subseteq C \subseteq D$ and that a is of length k.*

(i) (Existence) If $a \downarrow_A B$, then there is b such that $t(b, B) = t(a, B)$ and $b \downarrow_A C$.

(ii) (Transitivity) We have that $a \downarrow_A C$ iff $a \downarrow_A B$ and $a \downarrow_B C$.

(iii) (Monotonicity) If $a \downarrow_A D$, then $a \downarrow_B C$.

(iv) We have $a \downarrow_A A$.

Proof Condition (i) follows from [2, Lemma 34], and (ii), (iii), and (iv) are trivial. $\square$

Let $I = (a_i)_{i<\alpha}$ and $J = (b_i)_{i<\alpha}$, $\alpha \le \omega$, be order-indiscernible sequences over A. We say that I and J are *isomorphic over* A if there is an $\mathcal{L}$-elementary function $f : A \cup \bigcup_{i<\alpha} a_i \to A \cup \bigcup_{i<\alpha} b_i$ such that $f \restriction A = \mathrm{id}_A$ and $f(a_i) = b_i$. We say that I is a *maximal order-indiscernible sequence over* A if $\alpha = \omega$ or for all a, $I \frown (a)$ is not order-indiscernible over A. By arity of I we mean length(a_0).

Lemma 1.3

(i) Let $I = (a_i)_{i<\alpha}$ and $J = (b_i)_{i<\alpha}$, $k \le \alpha \le \omega$, be order-indiscernible sequences over A. If there is an $\mathcal{L}$-elementary function $f : A \cup \bigcup_{i<k} a_i \to A \cup \bigcup_{i<k} b_i$ such that $f \restriction A = \mathrm{id}_A$ and for all $i < k$ $f(a_i) = b_i$, then I and J are isomorphic over A.

(ii) For all A, the number of maximal order-indiscernible sequences over A of fixed arity modulo isomorphism over A is finite.

(iii) There is a function $f'_I : \mathbb{N} \to \mathbb{N}$ such that for all A and a_i, $i < f'_I(|A|)$, the following holds. If $(a_i)_{i<f'_I(|A|)}$ is an order-indiscernible sequence over A of arity at most k, then there are a_i, $f'_I(|A|) \le i < \omega$, such that $(a_i)_{i<\omega}$ is order-indiscernible over A.

(iv) If $(a_i)_{i<n}$, $n \ge f'_I(|A|)$, is an order-indiscernible sequence over A of arity at most k, then it is indiscernible over A.

Proof Condition (i) follows immediately from the assumption that $\mathcal{L} = L^k$. Condition (ii) follows easily from (i) and K3 (if $(a_i)_{i<\alpha}$ and $(b_i)_{i<\beta}$ are maximal indiscernible sequences and $(a_i)_{i<k}$ and $(b_i)_{i<k}$ are isomorphic over A, then $\alpha = \beta$). Condition (iii) is immediate by (ii) (and K3). Condition (iv) follows from [2, Lemma 22, Theorem 25]. $\square$

Lemma 1.4 *There is $N'_{\mathrm{in}} \in \mathbb{N}$ such that for all indiscernible $I = (a_i)_{i<\alpha}$ of arity at most k, $\varphi(x, y)$ and a, either*

$$\left|\{i < \alpha \mid \models \varphi(a, a_i)\}\right| < N'_{\mathrm{in}}$$

or

$$\left|\{i < \alpha \mid \models \neg\varphi(a, a_i)\}\right| < N'_{\mathrm{in}}.$$

Proof The proof is immediate by Lemma 1.3 and the fact that **K** does not have the independence property. $\square$

Definition 1.5 Let $g : \mathbb{N} \to \mathbb{N}$ be such that $g(n) = \max\{f'_I(n), \mathbb{N}'_{\mathrm{in}} + 1\}$. We say that $t(a, B)$ *splits strongly over* $A \subseteq B$ if there are $I = (a_i)_{i<g(|A|)} \subseteq B$ indiscernible over A, $\varphi(x, y)$ and f such that $(\varphi(x, a_0), f) \in t(a, B)$ and $(\neg\varphi(x, a_1), f) \in t(a, B)$.

Lemma 1.6 *Assume that* length$(a) = k$. *If $t(a, B)$ splits strongly over $A \subseteq B$, then $a \not\downarrow_A B$.*

Proof Assume not. By Lemmas 1.3 and 1.2, for all $n \in \mathbb{N}$, we can find $(a_i)_{i<n}$, b, φ and f such that $t(b, A) = t(a, A)$, $(\varphi(x, a_0), f) \in t(b, A \cup a_0)$, $(\neg\varphi(x, a_i), f) \in t(b, A \cup a_i)$, $0 < i < n$, and $b \downarrow_A \bigcup_{i<n} a_i$. It is easy to see that this implies that $R^k(a, A, \omega) > R^k(a, A, \omega)$, which is a contradiction. $\square$

Lemma 1.7

(i) *There is* $N_{\mathrm{sp}} \in \mathbb{N}$ *such that for all* a *of length at most* k *and* A_i, $i < N_{\mathrm{sp}}$, *there is* $j < N_{\mathrm{sp}}$ *such that* $t(a, \bigcup_{i \le j} A_i)$ *does not split over* $\bigcup_{i<j} A_i$.

(ii) *Assume that* $N = N_{\mathrm{sp}} \cdot f_I'(|A_0|)$, A_i, $i \le N$, *and* B, a, *and* b *are such that*

  (a) *for all* $i < j \le N$, $A_i \subseteq A_j \subseteq B$,

  (b) $\mathrm{length}(a) = \mathrm{length}(b) = k$,

  (c) $t(a, A_N) = t(b, A_N)$,

  (d) $R^k(a, B) = R^k(b, B) = R^k(a, A_0)$,

  (e) *for all* $i < N$ *and* $c \in B$ *of length at most* k, *there is* $d \in A_{i+1}$ *such that* $t(d, A_i) = t(c, A_i)$.

  Then $t(a, B) = t(b, B)$.

(iii) *There is* $f : \mathbb{N} \to \mathbb{N}$ *such that for all* a *of length* k, *and* $A \subseteq B$, *if* $R^k(a, B) < R^k(a, A)$, *then there is* $C \subseteq B$ *such that* $A \subseteq C$, $|C| \le f(|A|)$, *and* $R^k(a, C) < R^k(a, A)$.

(iv) *There is* $\kappa(\mathbf{K}) \in \mathbb{N}$ *such that for all* a *of length* k *and* A_i, $i < \kappa(\mathbf{K})$, *there is* $j < \kappa(\mathbf{K})$ *such that* $a \downarrow_{\bigcup_{i<j} A_i} A_j$.

(v) *There is* $F_{fo} : \mathbb{N} \to \mathbb{N}$ *such that for all* $A \subseteq B$ *and* a *of length* k, *there is* $C \subseteq B$ *such that* $a \downarrow_C B$, $A \subseteq C$, *and* $|C| < F_{fo}(|A|)$. *We write* $\kappa^*(\mathbf{K}) = F_{fo}(0)$. *(Note that we can choose* F_{fo} *so that* $F_{fo}(n) = n + \kappa^*(\mathbf{K})$.*)*

(vi) *Let* $N_{\mathrm{in}} = \max\{f_I'(\kappa^*(\mathbf{K})), N_{\mathrm{in}}'\}$. *Then in (ii),* N *can be replaced by* $N_{\mathrm{sp}} \cdot N_{\mathrm{in}}$.

Proof (i) The proof is immediate since $\mathbf{K}$ is stable and $\mathcal{L}$ is finite.

(ii) Assume not. Choose $c \in B$ such that for some φ, $\models \varphi(a, c) \wedge \neg\varphi(b, c)$. Let $N' = f_I'(|A_0|)$. By the choice of N, we can find i such that $i + N' < N$ and $t(c, A_{i+N'})$ does not split over A_i. For all $j < N'$, choose $c_j \in A_{i+j+1}$ such that $t(c_j, A_{i+j}) = t(c, A_{i+j})$. Then it is easy to see that $I = (c_j)_{j<N'} \frown (c)$ is order-indiscernible over A_i. By Lemma 1.3(iv), I is indiscernible over A_0. But then it is easy to see that either $t(a, B)$ or $t(b, B)$ splits strongly over A, which contradicts Lemma 1.6.

(iii) For each pair $A \subseteq B$, choose sets A_i, $i < N$ as in (ii), $A_0 = A$. Then for this pair, we can let $f(|A|) = |A_N| + k$. Also it is easy to see that we can find for these an upper bound that depends only on A. Finally, the claim follows from the fact that the number of possible A (modulo isomorphism) in each cardinality is finite.

(iv) Clearly, it is enough to show that for all a and A, $R^k(a, A, \omega) < N_{\mathrm{sp}}$. For this it is enough to show that if $R^k(a, A, \omega) = n + 1$, then there are b and $B \supseteq A$ such that $t(b, A) = t(a, A)$, $t(b, B)$ splits over A, and $R^k(b, B, \omega) = n$. As in the proof of [2, Lemma 30(iv)], we can find p such that $t(a, A) \subseteq p$, p splits over A, and $R^k(p, \omega) = n$. By [2, Lemma 34], p can be extended into a complete type.

(v) Let $F_{fo}(n) = f^{\kappa(\mathbf{K})}(n)$, where f is from (iii) (and $f^{m+1} = f \circ f^m$, etc.).

(vi) Note that if A_i, $i < N_{\mathrm{sp}} \cdot N_{\mathrm{in}}$, a and b satisfy (a)–(e) in (ii), then all the assumptions hold if we replace A_0 by $A \subseteq A_0$ such that $|A| < \kappa^*(\mathbf{K})$ and $a \downarrow_A A_0$. Thus the claim follows. □

Definition 1.8

(i) Assume that $A \subseteq B$ and $a \downarrow_A B$. We say that $t(a, A)$ is *stationary inside* B if for all b, $t(b, A) = t(a, A)$ and $b \downarrow_A B$ implies $t(b, B) = t(a, B)$.

(ii) We say that $t(a, A)$ is *stationary* if for all $B \supseteq A$, b, and c, $t(b, A) = t(c, A) = t(a, A)$, $b \downarrow_A B$, and $c \downarrow_A B$ implies $t(b, B) = t(c, B)$.

(iii) Let $f : \mathbb{N} \to \mathbb{N}$ be any function such that for all $A, k \cdot |S_k(A)| < f(|A|)$. We define $F_{\mathrm{st}} : \mathbb{N} \to \mathbb{N}$ so that $F_{\mathrm{st}}(n) = f^N(n)$, where $N = N_{\mathrm{sp}} \cdot N_{\mathrm{in}}$.

(iv) We let f_I be the function such that for all $n \in \mathbb{N}$,

$$f_I(n) = \max\{f'_I(n), N_{\mathrm{in}}\}.$$

Lemma 1.9

(i) Assume that $A \subseteq B$. Then there is $C \subseteq B$ such that $A \subseteq C$, $|C| < F_{\mathrm{st}}(|A|)$ and for all a of length k, $a \downarrow_A C$ implies that $t(a, C)$ is stationary inside B.

(ii) For all A, there is $C \supseteq A$ such that $|C| < F_{\mathrm{st}}(|A|)$ and for all a of length k, $a \downarrow_A C$ implies that $t(a, C)$ is stationary. Furthermore, if a is given, then C can be chosen so that $a \downarrow_A C$.

Proof The proof is immediate by Lemmas 1.7(ii) and 1.2(i) and the homogeneity of $\mathbf{M}$. □

We say that $t(a, A)$ is *algebraic* if $|\{b \mid t(b, A) = t(a, A)\}| < \omega$.

Lemma 1.10

(i) There is $f_{\mathrm{alg}} : \mathbb{N} \to \mathbb{N}$ such that for all A and a of length k, if $|\{b \mid t(b, A) = t(a, A)\}| > f_{\mathrm{alg}}(|A|)$, then $|\{b \mid t(b, A) = t(a, A)\}| = \omega$.

(ii) If $t(a, A)$ is algebraic and a is of length k, then $a \downarrow_A B$ for all B.

(iii) If $t(a, A)$ is not algebraic but $t(a, B)$ is, then $a \not\downarrow_A B$.

Proof Condition (i) follows easily from Lemma 1.3 and Ramsey's theorem, and conditions (ii) and (iii) follow from the remark that $R^k(a, A, \omega) = 0$ iff $t(a, A)$ is algebraic. □

Lemma 1.11 *Assume that* $\mathrm{length}(a) = \mathrm{length}(b) = k$. *Then $a \downarrow_A b$ iff $b \downarrow_A a$.*

Proof We first prove the following claim.

Claim *If $t(a, A)$ is stationary and $a \downarrow_A b$, then $b \downarrow_A a$.*

Proof For a contradiction, suppose that $a \downarrow_A b$ but $b \not\downarrow_A a$. Choose a_i and b_i, $i < \omega$, so that $t(a_i, A) = t(a, A)$, $a_i \downarrow_A \bigcup_{j<i}(a_i \cup b_i)$, $t(b_j, A) = t(b, A)$, and $b_i \downarrow_A a_i \cup \bigcup_{j<i}(a_i \cup b_i)$. Then $b_j \not\downarrow_A a_i$ iff $j < i$. By Lemma 1.10, $(a_i \frown b_i)_{i<\omega}$ is infinite. So it is easy to get a contradiction with the fact that $\mathbf{K}$ does not have the order property. □

It suffices to prove Lemma 1.11 from left to right. By Lemma 1.9, we can find $B \supseteq A$ such that for all c of length k, if $c \downarrow_A B$, then $t(c, B)$ is stationary. Choose b' so that $t(b', A) = t(b, A)$ and $b' \downarrow_A B$. Choose a' so that $t(a' \frown b', A) = t(a \frown b, A)$ and $a' \downarrow_A B \cup b'$. Then

$$b' \downarrow_A B \wedge b' \downarrow_B a' \;\Rightarrow\; b' \downarrow_A B \cup a' \;\Rightarrow\; b' \downarrow_A a',$$

and so $b \downarrow_A a$. □

Theorem 1.12

(i) For all A, B, and C, there is A' such that $t(A', B) = t(A, B)$ and $C \downarrow_B A'$.

(ii) For all A, B, and a of length k, if $A \downarrow_B a$ and $t(a, B)$ is stationary, then $a \downarrow_B A$.

(iii) *For all A, B, and a of length k, if $A \downarrow_B a$, then $a \downarrow_B A$.*
(iv) *(Symmetry) For all A, B, and C, $A \downarrow_B C$ iff $C \downarrow_B A$.*

Proof (i) Choose sequences a_i, $i < n$, so that they are of length k and $\bigcup_{i<n} a_i = A$ (use, e.g., constant sequences). Choose sequences a'_i so that for all $i < n$, $t(\bigcup_{j\le i} a'_i, B) = t(\bigcup_{j\le i} a_i, B)$ and $a'_i \downarrow_{B\cup\bigcup_{j<i} a'_j} C$. Then by Lemmas 1.11 and 1.2, $A' = \bigcup_{i<n} a'_i$ is as desired.

(ii) If not, then (by stationarity) we can find $a' \in A^k$ such that $a \not\downarrow_B a'$, which contradicts Lemma 1.11.

(iii) Choose $C \supseteq B$ so that
  (a) for all c of length k, if $c \downarrow_B C$, then $t(c, C)$ is stationary,
  (b) $A \cup a \downarrow_B C$ and $A \downarrow_{B\cup a} C$.

By (the proof of) (i), this is possible. Then by Lemma 1.2 and the latter half of (b), $A \downarrow_C a$ and so by (ii), $a \downarrow_C A$. By Lemma 1.2 again, $a \downarrow_B A$.

(iv) This is immediate by (iii). □

The following lemma is an analogue for the finite character property in the traditional forking calculus.

Lemma 1.13 *If $A \subseteq B$, a is of length k, and $a \not\downarrow_A B$, then there is $b \in B^k$ such that $a \not\downarrow_A b$.*

Proof By Theorem 1.12, $B \not\downarrow_A a$. By the definition of $\downarrow$, there is $b \in B^k$ such that $b \not\downarrow_A a$. By Lemma 1.11, $a \not\downarrow_A b$. □

Corollary 1.14 *We can change our definition of $\kappa^*(\mathbf{K})$ in Lemma 1.7(v) by $\kappa^*(\mathbf{K}) = k \cdot \kappa(\mathbf{K})$.*

Proof The proof is immediate by Lemma 1.13. □

Lemma 1.15

(i) *Assume that for all $i < n$, $B_i \downarrow_A \bigcup_{j<i} B_j$. Then $(B_i)_{i<n}$ is A-independent, that is, for all $w \subseteq n$,*

$$\bigcup_{i\in w} B_i \downarrow_A \bigcup_{i\in n-w} B_i .$$

(ii) *For all n there is n' such that for all C, A, and B_i, $i < n'$, the following holds. If $(B_i)_{i<n'}$ is A-independent and $|C| < n$, then there is $i < n'$ such that $C \downarrow_A B_i$.*

Proof These are some of the usual consequences of a well-behaved independence notion. As an example we prove (i) by induction on $n \geq 1$. The case $n = 1$ is trivial. Suppose that the claim is proved for n and we prove it for $n + 1$. We assume that $n \notin w$; the other case is similar. Now by the assumption and monotonicity,

$$B_n \downarrow_{A\cup\bigcup_{i\in n-w} B_i} \bigcup_{i\in w} B_i .$$

By symmetry, $\bigcup_{i\in w} B_i \downarrow_{A\cup\bigcup_{i\in n-w} B_i} B_n$. But then by the induction assumption and transitivity, the claim follows. □

We finish this section with two additional observations.

Definition 1.16 Assume that $n \geq 2 \cdot N_{\text{in}}$ and that $I = (a_i)_{i<n}$ is an indiscernible sequence of arity at most k. We let $\mathrm{Av}(I, A)$ be the set of those pairs $(\varphi(x, a), f)$ such that $a \in A$ and

$$\left|\left\{i < n \;\middle|\; \big(\varphi(x, a), f\big) \in t(a_i, A)\right\}\right| \geq N_{\text{in}}.$$

Lemma 1.17 *Assume that $I = (a_i)_{i<n}$ is an indiscernible sequence of arity at most k and that $n \geq 2 \cdot N_{\text{in}}$.*

(i) *For all A, $\mathrm{Av}(I, I \cup A)$ is $\mathbf{K}$-consistent. Furthermore, there is $f' : \mathbb{N} \to \mathbb{N}$ such that if $n > f'(|A|)$, then there is $i < n$ such that $t(a_i, A) = \mathrm{Av}(I, A)$.*

(ii) *For all A, $\mathrm{Av}(I, I \cup A)$ does not fork over $\bigcup_{i<2\cdot N_{\text{in}}} a_i$ and $\mathrm{Av}(I, I)$ is stationary.*

(iii) *There is a function $f : \mathbb{N} \to \mathbb{N}$ such that the following holds for every indiscernible sequence $I = (a_i)_{i<n}$ over A of arity at most k. Assume that $n \geq 2 \cdot N_{\text{in}}$, $J = (b_i)_{i<n'}$, and $K = (c_i)_{i<n''}$ are indiscernible sequences over A, $n' > n$, $n'' > f(n')$ and for all $i < n$, $c_i = b_i = a_i$. Then there is $i < n''$ such that $t(c_i, A \cup J) = \mathrm{Av}(J, A \cup J)$.*

Proof (i) This is immediate by Lemmas 1.3 and 1.4.

(ii) Assume not. Then, by (i) (and by extending and re-enumerating I, if necessary), we can find a of length k and $j \geq 2 \cdot N_{\text{in}}$, so that if we write $B = \bigcup_{i<j} a_i$, then
  (a) $a \downarrow_B \cup I$,
  (b) $j + N_{\text{in}} < n$,
  (c) $t(a_j, a \cup B) \neq \mathrm{Av}(I, a \cup B)$,
  (d) for all $j < i < n$, $t(a_i, a \cup B) = \mathrm{Av}(I, a \cup B)$.
But by (c) and (d), $t(a, \cup I)$ splits strongly over B. By Lemma 1.6, this contradicts (a).

(iii) Notice first that by (i) and (ii), $(c_i)_{n \leq i < n''}$ is $(A \cup I)$-independent. We let $f(n') = (k \cdot n')^k \cdot \kappa^*(\mathbf{K}) + n'$. Then we can find $w \subseteq n''$ such that $n \subseteq w$, $|w| \leq f'(n')$, and for all $a \in \cup J$, $a \downarrow_{A \cup \bigcup_{i \in w} c_i} A \cup K$. By (i) and (ii) above, if $i \in n'' - w$, then $t(c_i, B \cup J) = \mathrm{Av}(J, B \cup J)$. □

The second additional observation is a weak compactness theorem. We say that a k-type p over A is *complete* if for all φ, $a \in A$ and f, either $(\varphi(x, a), f) \in p$ or $(\neg\varphi(x, a), f) \in p$. Notice that ($\mathbf{K}$-)consistency is not included in our notion of completeness.

Lemma 1.18 *Let $m = F_{\text{st}}(k \cdot \kappa(\mathbf{K})) + k$, and let p be a complete k-type over A. Assume that for all $B \subseteq A$, if $|B| \leq m$, then $p \restriction B = \{(\varphi(x, a), f) \in p \mid a \in B\}$ is $\mathbf{K}$-consistent. Then p is $\mathbf{K}$-consistent.*

Proof Choose a maximal sequence of k-sequences a_i and sets A_i, $i \leq n$, such that

(i) $A_0 = \emptyset$,
(ii) for all $i \leq n$, $t(a_i, A_i) = p \restriction A_i$,
(iii) for all $i < n$, $a_{i+1} \not\downarrow_{A_i} A_{i+1}$, $A_i \subseteq A_{i+1}$, and $|A_{i+1} - A_i| \leq k$.

Clearly this is possible, and $|A_n| < k \cdot \kappa(\mathbf{K})$. Then choose $B \subseteq A$ such that $A_n \subseteq B$, $|B| \leq F_{\text{st}}(k \cdot \kappa(\mathbf{K}))$, and for all b, $b \downarrow_{A_n} B$ implies that $t(b, B)$ is stationary inside A. By Lemma 1.9, this is possible. Then choose a so that $t(a, B) = p \restriction B$ and $a \downarrow_B A$. We claim that $t(a, A) = p$. Assume not. Choose $C \subseteq A$ such that $B \subseteq C$,

$t(a, C) \neq p \restriction C$, and $|C| \leq m$. Let a' be such that $t(a', C) = p \restriction C$. By the choice of B and a, $a' \not\downarrow_B C$. Especially $a' \not\downarrow_{A_n} C$. By Lemma 1.13, this contradicts the choice of A_n. □

2 *n*-prime K-models

In this section, we study the possibilities of using our independence relation to construct nice finite models which are a little bit saturated.

Definition 2.1

(i) We say that A is *m-saturated* if for all $B \subseteq A$ of power m and $p \in S_1(B)$, there is $a \in A$ such that a realizes p.

(ii) We say that B is *m-atomic over A* if for all $a \in B$ of length k, there is $C \subseteq A$ of power less than m such that $t(a, C) \vdash t(a, A)$, that is, $t(a, C)$ m-isolates $t(a, A)$.

(iii) We say that forking is *trivial* if for all a, B, C, and D, the following holds. If $a \not\downarrow_D B \cup C$ and $B \downarrow_D C$, then $a \not\downarrow_D B$ or $a \not\downarrow_D C$.

Remark 2.2 Note that if A is k-saturated, then A is a **K**-model. Note also that if $k = 2$, then forking is trivial (always). Finally, if a is of length at most k and A is $F_{\mathrm{st}}(\kappa^*(T))$-saturated, then $t(a, A)$ is stationary. In fact, if A is $(\kappa^*(\mathbf{K}) + 2 \cdot k \cdot N_{\mathrm{sp}} + k \cdot N_{\mathrm{in}})$-saturated, then $t(a, A)$ is stationary (see the proof of Lemma 1.7).

The basic idea behind the proof of the following theorem is the same as in an easy proof of the fact that ω-stable theories have saturated models in every (infinite) cardinality κ (including singular cardinals). We recall this argument in order to make it easier for the reader to follow the proof of Theorem 2.3 below.

For all $i < \omega$, let A_i be as follows: $A_0 = \emptyset$ and $A_{i+1} = A_i \cup \bigcup_{j<\kappa} b^i_j$, where the sequence $(b^i_j)_{j<\kappa}$ is such that it is independent over A_i and every strong type over A_i is realized by κ many b_i (algebraic types only once). Now $M = \bigcup_{i<\omega} A_i$ is saturated: For this let $A \subseteq M$ be of power less than κ, and let b be arbitrary. By ω-stability we can find $n < \omega$ such that $b \downarrow_{A_n} M$. Also since $|A| < \kappa$ and since $(b^i_j)_{j<\kappa}$ is independent over A_i, there is $X \subseteq \kappa$ of power less than κ such that $b^i_j \not\downarrow_{A_i} A$ iff $j \in X$. Thus there is $j < \kappa$ such that $\mathrm{stp}(b^i_j, A_i) = \mathrm{stp}(b, A_i)$ and $b^i_j \downarrow_{A_i} A$. By stationarity of strong types, even $t(b^i_j, A_i \cup A) = t(b, A_i \cup A)$.

The reason why our argument below is much more complicated than the argument above, is that the n above need not exist (finite cardinals are not regular limit ordinals), we do not have strong types, and addition and multiplication on finite cardinals are much more complicated functions than what they are on infinite cardinals. So what we do is that at each step, we do not realize all the types but only suitably isolated ones; this together with the triviality assumption reduces the existence of n to an existence of a large enough "gap." Instead of stationarity of strong types, Lemma 1.9(ii) is used. Note also that it does not really matter what the various numbers we calculate are as long as they are large enough. We calculate them only to show that there are some numbers that actually are large enough (so one does not lose much by reading the proof so that one forgets the numbers and just believes that in the models there is enough space for the constructions).

Theorem 2.3 *Assume that forking is trivial. For all $m' \in \mathbb{N}$, there is an m'-saturated* $\mathbf{K}$*-model A.*

Proof Clearly, we may assume that $m' \geq k$, and so if A is m'-saturated, it is also a $\mathbf{K}$-model. Let $m = m' + \kappa(\mathbf{K}) \cdot \kappa^*(\mathbf{K})$, $N = N_{\mathrm{sp}} \cdot N_{\mathrm{in}}$, and $n^* = m \cdot (N+1)$. We construct A in layers:

(a) $A_0 = \emptyset$,
(b) $A_{i+1} = A_i \cup B_i$,
(c) $A = A_{n^*}$ ($\cup A_{n^*}$ to be precise, A_{n^*} is a set of k-sequences),
(d) B_i is a set of sequences b of length k such that
- B_i is A_i-independent;
- for every sequence $b \in B_i$, $t(b, A_i)$ is N_i-isolated (we will define the numbers N_i below);
- $B_i \downarrow_{A_i} A'_i$, where $A'_i \supseteq A_i$ is chosen so that for all b, if $b \downarrow_{A_i} A'_i$, then $t(b, A'_i)$ is stationary;
- for all k-sequences b, if $b \downarrow_{A_i} A'_i$, $t(b, A_i)$ is N_i-isolated and $b \notin A_i$, then there is $b' \in B_i$ such that $t(b', A'_i) = t(b, A'_i)$; furthermore, if $t(b, A'_i)$ is algebraic, then the number of such b' is 1 and otherwise the number is $2 \cdot f_{\mathrm{cl}}^{n^*}(N_{n^*})$ (see the next paragraph for the definition of the functions f_{cl}^{j}).

Our first goal is to determine the numbers N_i. We do this by induction on i. We let $N_0 = 1$. Assume that N_j, $j < i$, have been defined. We say that $C \subseteq A$ is *closed* if for all $i < n^*$ and $b \in C \cap B_i$, there is $B_b \subseteq C \cap A_i$ of power less than N_i such that $t(b, B_b) \vdash t(b, A_i)$. We let $f_{\mathrm{cl}}^i : \mathbb{N} \to \mathbb{N}$ be such that for all $C \subseteq A_i$, the closure of C is of power less than $f_{\mathrm{cl}}^i(|C|)$. Note that f_{cl}^i can be calculated from $\{N_j \mid j < i\}$. Let $f : \mathbb{N} \to \mathbb{N}$ be such that for all A, $k^2 \cdot |S(A)| < f(|A|)$. Note that for all n, $F_{\mathrm{st}}(n) \leq f^N(n)$. Then we let

(e) $N_{i+1} = (f)^{3 \cdot N}(f_{\mathrm{cl}}^i(m))$.

We recall from above that for all i, we have chosen $A'_i \supseteq A_i$ so that for all b, if $b \downarrow_{A_i} A'_i$, then $t(b, A'_i)$ is stationary. Furthermore, we require that $A'_i \subseteq A'_{i+1}$. We will use these in place of strong types.
Let $N^* = f_{\mathrm{cl}}^{n^*}(N_{n^*})$.

Notice the following.

Claim 1 *Assume that $C \subseteq A_i$ is of power $f^N(f_{\mathrm{cl}}^i(m'))$. Then for all b there is $b' \in B_i$ such that $t(b', C) = t(b, C)$.*

Proof The proof is immediate by Lemma 1.13 and the definition of f and N_i. □

We show that $A = A_{n^*}$ is m'-saturated. For this it is enough to prove the following. Assume that $B \subseteq A$ is of power m, a is a singleton, and for all $i \leq n^*$, $a \downarrow_{B \cap A_i} A_i$. Then there exists $b \in A$, such that $t(b, B) = t(a, B)$. Let $I \subseteq n^*$ be such that $i \in I$ iff $B \cap B_i \neq \emptyset$. By the pigeonhole principal, we can find $i^* < n^*$ such that for all j, if $i^* \leq j \leq i^* + N$, then $j \notin I$. Let $i' = i^* + N$, and let B^* be the least set such that it is closed and $B \cap A_{i^*} \subseteq B^* \subseteq A_{i^*}$. Then $|B^*| < f_{\mathrm{cl}}^{i^*}(m)$.

Now choose $C_0 \subseteq C_1 \subseteq C_2 \subseteq A_{i'}$ so that

(1) $B^* \subseteq C_0$,
(2) for all c, if $c \downarrow_{B^*} C_0$, then $t(c, C_0)$ is stationary (use the method from Lemma 1.7(ii)),

(3) for all c there is d such that $t(d, C_0) = t(c, C_0)$ and $t(d, C_1)$ isolates $t(d, A_{i'})$ (use the method from Lemma 1.7(ii)),
(4) C_2 is the least closed set such that $C_1 \subseteq C_2$,
(5) $|C_1| < N_{i'}$.

By Claim 1 and the construction, this is possible. Also $|C_2| < N^*$. Let $E \subseteq A$ be the least set such that it is closed and $B \subseteq E$. Then $B^* \subseteq E$ and $|E| < N_{n^*} \leq N^*$.

Claim 2 *We can choose C_2 (together with C_0 and C_1) so that $C_2 \downarrow_{B^*} E$.*

Proof We start with the following two subclaims. We say that D is *separate* from C over $C' \subseteq C \cap D$ if the following holds: for all $i < n^*$ and

$$d \in (D \cap C \cap B_i) - C',$$

$t(d, A_i)$ is algebraic. Note that if, for example, D is closed, then $t(d, A_i \cap D)$ is algebraic.

Subclaim 1

(i) Assume that D is closed and separate from C over C'. Then for all $i < n^*$, $D \downarrow_{D \cap A_i} A'_i \cup ((C \cap A_{i+1}) - C')$.
(ii) Assume that D, C, and C' are closed and that D is separate from C over C'. Then $D \downarrow_{C'} C$.

Proof (i): Since D is closed, we can see that

(*) for all $j < n^*$ and $d \in B_j \cap D$, $d \downarrow_{D \cap A_j} A'_j \cup \big(B_j - \{d\}\big)$.

So by transitivity, $D \downarrow_{D \cap A_{i+1}} A'_i \cup ((C \cap A_{i+1}) - C')$. So it is enough to prove that $D \cap B_i \downarrow_{D \cap A_i} A'_i \cup ((C \cap A_{i+1}) - C')$. By (*), it is enough to show that for all $d \in D \cap B_i$, $d \downarrow_{A'_i} ((C \cap B_i) - C') \cup D^i_d$, where $D^i_d = D \cap B_i - \{d\}$. If $t(d, A_i)$ is algebraic, this is clear. Otherwise, $d \notin (C \cap B_i) - C'$, that is, $d \notin ((C \cap B_i) - C') \cup D^i_d$, and so the claim follows from the definition of B_i.

(ii): By induction on i we show that $D \cap A_i \downarrow_{C' \cap A_i} C \cap A_i$. The case $i = 0$ is trivial. We prove the case $i + 1$. Because C' is closed, $C' \downarrow_{C' \cap A_i} (D \cup C) \cap A_i$. By this and the induction assumption,

(α) $C \cap A_i \downarrow_{C' \cap A_{i+1}} D \cap A_i$.

By (i), $C \cap A_{i+1} \downarrow_{C \cap A_i} D \cap A_i$, and so by ($\alpha$),

(β) $C \cap A_{i+1} \downarrow_{C' \cap A_{i+1}} D \cap A_i$.

By (i), $D \cap A_{i+1} \downarrow_{D \cap A_i} C \cap A_{i+1} - C'$, and so (ii) follows from this and (β). □

Assume that $C \subseteq A$ is closed. We say that $f : C \to A$ is *good* if it is $\mathcal{L}$-elementary, preserves the levels of the construction, and $f(C)$ is closed. Note that if $f : C \to A$ is good, then so is $f^{-1} : f(C) \to A$.

Subclaim 2 *Let $i < n^*$.*

(i) Assume that $C \subseteq A_i$ is closed and of power less than N^* and that $f : C \to A_i$ is good. Let c be such that $t(c, C)$ isolates $t(c, A_i)$. If $f(t(c, C)) = t(c', f(C))$, then $t(c', f(C))$ isolates $t(c', A_i)$.
(ii) Assume that $C \subseteq A$ and $D \subseteq A_{i+1}$ are sets, C and $D \cup C$ are closed, $C \cup D$ is of power less than $2 \cdot N^*$, and $f : C \to A$ is good. Then there is a good function $g : C \cup D \to A$ such that $f \subseteq g$. Furthermore, if E^* and E' are closed and such that $f(C)$ is separate from E^* over E' and

$|f(C) \cup E^*| + |D| < 2 \cdot N^*$, *then g can be chosen so that $g(C \cup D)$ is separate from E^* over E'.*

Proof We prove these simultaneously by induction on i. For the case $i = 0$, (i) is trivial and (ii) goes as in the case $i+1$. We prove the case $i+1$. Now (i) follows immediately from item (ii) of the induction assumption. So let C and D be as in (ii). By the induction assumption we may assume that $D \subseteq B_i$ and that for all $d \in D$, $d \notin C$. Choose D' so that $t(D', f(C)) = f(t(D, C))$ and $D' \downarrow_{f(C)} A'_i$. By Subclaim 1(i), $D \downarrow_{C \cap A_i} C$, and so $D' \downarrow_{f(C) \cap A_i} f(C) \cup A'_i$. Especially, D' is A'_i-independent. By the choice of A'_i, it is enough to find $D'' \subseteq B_i$ such that $t(D'', A'_i) = t(D', A'_i)$ and for all $d \in D''$, $d \notin f(C)$. By (i), for all $d \in D'$, $t(d, A_i)$ is sufficiently isolated. So for all $d' \in D'$, we can find $d \in B_i$ such that $t(d, A'_i) = t(d', A'_i)$. By the definition of B_i, the set of these is the wanted D'', if we can choose the sequences so that they do not belong to $f(C)$ (and are distinct from each other). If $t(d, A_i)$ is not algebraic, then this is clear. If $t(d, A_i)$ is algebraic, then it is enough to show that for all $c \in f(C) \cap B_i$, $t(d, A'_i) \neq t(c, A'_i)$. But this is clear, since otherwise $t(d, A'_i)$ is not stationary ($d \neq c$ for all $c \in f(C) \cap B_i$).

The furthermore part follows immediately from the construction. □

Now Claim 2 follows from Subclaim 2(ii) by an easy induction. Note that after we have moved C_2 so that $C_2 \downarrow_{B^*} E$, (3) above still holds by Subclaim 2(ii) ((1), (2), (4), and (5) are clear). □

Then we can choose b so that $t(b, C_0) = t(a, C_0)$ and $t(b, C_1)$ isolates $t(b, A_{i'})$. So we can choose b so that in addition $b \downarrow_{C_1} A'_{i'}$, $b \in B_{i'}$ and $b \notin E$ or $t(b, C_1)$ is algebraic. Then by Claim 2, $b \downarrow_{C_0} E$. So if $a \downarrow_{C_0} E$, $t(b, B) = t(a, B)$. So it is enough to show that $a \downarrow_{C_0} E$.

Assume not. Let $j < n^*$ be such that $a \downarrow_{C_0} A_j \cap E$ and $a \not\downarrow_{(A_j \cap E) \cup C_0} c$ for some $c \in E \cap B_j$ (since forking is trivial, such c exists). Also $j \geq i' + 1$, because $a \downarrow_{A_{i'+1} \cap B} A_{i'+1}$ and $A_{i'+1} \cap B = A_{i^*} \cap B \subseteq C_0$. Then $t(b, (A_j \cap E) \cup C_0) = t(a, (A_j \cap E) \cup C_0)$, but $t(b, (A_j \cap E) \cup C_0 \cup c) \neq t(a, (A_j \cap E) \cup C_0 \cup c)$. This contradicts the fact that E is closed, and this also concludes the proof of Theorem 2.3. □

By taking a closer look at the proof of Theorem 2.3, we can see that the following theorem also holds.

Theorem 2.4 *Assume that forking is trivial.*

(i) There are $g, h : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that the following holds. For all $m' \in \mathbb{N}$ and D, if (*) below holds, then there is an m'-saturated $\mathbf{K}$-model $A \supseteq D$ such that it is $h(m', |D_0|)$-atomic over D.

(*) $D = D_0 \cup D_1$ and D_1 is $g(m', |D_0|)$-saturated.

(ii) There are $g, h : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that the following holds. For all $m', m \in \mathbb{N}$ and D, if (**) below holds, then there is an m'-saturated $\mathbf{K}$-model $A \supseteq D$ such that it is $h(m', m)$-atomic over D.

(**) $D = \bigcup_{i \in I} D^i_0 \cup D_1$, D_1 is $g(m', m)$-saturated, $|I| \geq g(m', m)$, for all $i \in I$, $|D^i_0| < m$, and for all $i, j \in I$, $t(D^i_0, D_1) = t(D^j_0, D_1)$.

Proof (i) Just repeat the construction from the proof of Theorem 2.3 with the following changes. First of all, note that (by choosing g properly) we can assume that every type over D_1 is stationary. So if $C \subseteq D$ and $t(c, C)$ is stationary inside D, then $t(c, C)$ is stationary. We let $A_0 = D$. Then we choose $D_2 \subseteq D_1$ so that $D_0 \downarrow_{D_2} D_1$ and $t(D_0, D_2)$ is stationary inside D_1. Clearly the size of D_2 depends only on $|D_0|$. Then we define closedness of C as in the proof of Theorem 2.3, except that in addition we require that $D_0 \cup D_2 \subseteq C$ and $t(C, C \cap D_1)$ is stationary inside D_1. Note that

(*) if C is closed, then by triviality of forking, $C' \downarrow_{C \cap D_1} D_1$.

Then the maximal size of N_{n^*} depends only on m' and $|D_0|$. So we define g so that always $g(m', |C_0|) \geq 2 \cdot N_{n^*}$ and h is defined so that $h(m', |C_0|) \geq N_{n^*}$. By (*), it is easy to see that if C is closed, then $t(C, C \cap D) \vdash t(C, D)$ and so A is $h(m', |C_0|)$-atomic (in fact the value of $h(m', |C_0|)$ can be improved a lot).

(ii) Just repeat the construction from (i) with the following changes. As in (ii), we can find $D_2 \subseteq D_0$ so that for all $i \in I$, $D_0^i \downarrow_{D_2} D_1$ and $t(D_0^i, D_2)$ is stationary inside D_1. Then we define closedness of C as in the proof of Theorem 2.3, except that in addition we require that $D_2 \subseteq C$, $t(C, C \cap D_1)$ is stationary inside D_1, and if $D_0^i \cap C \neq \emptyset$, then $D_0^i \subseteq C$. Then the proof of (ii) works. □

Definition 2.5

(i) By F_{at} we mean the function h from Theorem 2.4(i), and by G_{sat} we mean the function g from Theorem 2.4(i).

(ii) We say that A is *m-constructible* over B if there is $(a_i)_{i<n}$ such that $A = B \cup \bigcup_{i<n} a_i$ and for all $i < n$, $t(a_i, B \cup \bigcup_{j<i} a_j)$ is m-isolated. In this case we say that $(a_i)_{i<n}$ is an m-construction over B.

(iii) We say that A is *m-primary* over B if A is $F_{\mathrm{is}}(m)$-constructible over B and m-saturated, where $F_{\mathrm{is}}(m) = F_{\mathrm{st}}(F_{fo}(m))$.

(iv) We say that A is *m-primitive* over B if for all m-saturated $C \supseteq B$, there is an $\mathcal{L}$-elementary embedding $f : A \cup B \to C$ such that $f \restriction B = \mathrm{id}_B$.

Observe that by Lemma 1.7(v) and Lemma 1.9(i), for all $A \subseteq B$ and a (of length k), there are b and $A \subseteq C \subseteq B$ such that $t(b, A) = t(a, A)$, $t(b, C)$ isolates $t(b, B)$ and $|C| < F_{\mathrm{is}}(|A|)$ (choose b so that the rank of $t(b, B)$ is minimal among those types over B that extend the type $t(a, A)$).

Corollary 2.6 *Assume that forking is trivial, that $m \geq k$, and that D is a set.*

(i) There is an m-primary $\mathbf{K}$-model over D.

(ii) m-constructible sets over D are m-primitive over D.

(iii) If $D = D_0 \cup D_1$ and D_1 is $G_{\mathrm{sat}}(m, |D_0|)$-saturated, then m-constructible sets over D are $F_{\mathrm{at}}(m, |D_0|)$-atomic over D.

Proof Assertion (ii) is trivial; (i) follows immediately from (ii), the observation above, and Theorem 2.3; and (iii) follows from (ii) and Theorem 2.4(i). □

Remark 2.7 For all $A \neq \emptyset$,

$$\left|S_k(A)\right| < f\left(F_{\mathrm{st}}\left(\kappa^*(\mathbf{K})\right)\right) \cdot |A|^{\kappa^*(\mathbf{K})},$$

where f is from Definition 1.8(iii), that is, (any) function such that for all B, $k \cdot |S_k(B)| < f(|B|)$.

Proof There is no change from the classical proof (see the proof of [4, Lemma III.3.6]). □

So by going through the proof of Theorem 2.3, we can calculate a recursive upper bound for $\min\{|A| \mid A \in \mathbf{K}\}$ from N_{sp} and N_{in}.

We finish this paper by proving the usual property of primary models.

Definition 2.8 We write $A \rhd_B C$ if for all a of length k, $a \downarrow_B A$ implies $a \downarrow_B C$.

Lemma 2.9 *Assume that a is of length k, B is m-atomic over $A \cup a$, and A is m^*-saturated, where $m^* = F_{\mathrm{st}}(\kappa^*(\mathbf{K})) + m + k$. Then $a \rhd_A B$.*

Proof Assume not. Then we can choose $A' \subseteq A$ and b and c of length k such that

(i) $a \downarrow_{A'} A \cup b$ and $t(a, A')$ is stationary,
(ii) $t(c, A' \cup a) \vdash t(c, A \cup a)$,
(iii) $c \not\downarrow_{A \cup a} b$,
(iv) $|A'| < F_{\mathrm{st}}(\kappa^*(\mathbf{K})) + m$.

Then we can choose $b' \in (A \cup b)^k$ such that $c \not\downarrow_{A' \cup a} b'$. By (i), (iv) and since A is m^*-saturated, we can find $b^* \in A^k$ such that $t(b^*, A' \cup a) = t(b', A' \cup a)$. Clearly this contradicts (ii). $\square$

Corollary 2.10 *Assume that forking is trivial. Let a be of length k, let B be m-primary over $A \cup a$, and let A be m^*-saturated, where*

$$m^* = \max\bigl\{F_{\mathrm{st}}\bigl(\kappa^*(\mathbf{K})\bigr) + F_{\mathrm{at}}\bigl(F_{\mathrm{is}}(m), k\bigr) + k, G_{\mathrm{sat}}\bigl(F_{\mathrm{is}}(m), k\bigr)\bigr\}.$$

Then $a \rhd_A B$.

Proof The proof is immediate by Lemma 2.9 and Corollary 2.6. $\square$

Index of symbols

f_{alg} Lemma 1.10(i),
F_{at} Definition 2.5(i),
f^i_{cl} beginning of the proof of Theorem 2.3,
F_{fo} Lemma 1.7(v),
f_I Definition 1.8(iv),
f'_I Lemma 1.3(iii),
F_{is} Definition 2.5(iii),
F_{st} Definition 1.8(iii),
G_{sat} Definition 2.5(i),
$\kappa(\mathbf{K})$ Lemma 1.7(iv),
$\kappa^*(\mathbf{K})$ Lemma 1.7(v),
N_{in} Lemma 1.7(vi),
N'_I Lemma 1.4,
N_{sp} Lemma 1.7(i).

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