Notre Dame Journal of Formal Logic
Volume 55, Number 4, 2014

Classical Negation and Game-Theoretical Semantics

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Abstract Typical applications of Hintikka's game-theoretical semantics (GTS) give rise to semantic attributes—truth, falsity—expressible in the Σ^1_1-fragment of second-order logic. Actually a much more general notion of semantic attribute is motivated by strategic considerations. When identifying such a generalization, the notion of classical negation plays a crucial role. We study two languages, L_1 and L_2, in both of which two negation signs are available: $\rightharpoondown$ and $\sim$. The latter is the usual GTS negation which transposes the players' roles, while the former will be interpreted via the notion of *mode*. Logic L_1 extends independence-friendly (IF) logic; $\rightharpoondown$ behaves as classical negation in L_1. Logic L_2 extends L_1, and it is shown to capture the Σ^2_1-fragment of third-order logic. Consequently the classical negation remains inexpressible in L_2.

1 Introduction

In game-theoretical semantics (GTS), which Hintikka originally formulated in [7], satisfaction conditions for formulas of first-order logic are formulated by associating a two-player zero-sum game $G(\varphi, \mathcal{M}, \gamma)$ between players I and II with every formula φ, relevant model $\mathcal{M}$, and assignment γ and defining φ to be satisfied in $\mathcal{M}$ by γ if there exists a winning strategy, w.s. for short, for player II in game $G(\varphi, \mathcal{M}, \gamma)$. The details of the definition of these *semantic games* can be phrased in a variety of ways, for example as follows.

We restrict attention to *regular* formulas, that is, formulas in which no two nested quantifiers carry the same variable and in which no variable appears both free and bound (for this terminology, see Caicedo, Dechesne, and Janssen [2]). Here and henceforth, if $\mathcal{M}$ is a model, M is its domain. We denote the empty assignment by λ. Positions in semantic games are quadruples $(\psi, \mathcal{M}, \epsilon, \rho)$, where ψ is a subformula of φ; ϵ is an assignment; and ρ is a bijective function of type $\{\mathbb{V}, \mathbb{F}\} \to \{I, II\}$, termed a *role distribution*. Here $\mathbb{V}$ (verifier) and $\mathbb{F}$ (falsifier) are two *roles*. Henceforth

Received June 8, 2012; accepted January 10, 2013
2010 Mathematics Subject Classification: Primary 03B60, 03C80; Secondary 03B15
Keywords: game-theoretical semantics, higher-order logic, independence-friendly logic, negation
 10.1215/00294527-2798709

we write ρ_0 for the role distribution satisfying $\rho_0(\mathbb{V}) = \mathit{II}$ and $\rho_0(\mathbb{F}) = I$. The initial position of game $G(\varphi, \mathcal{M}, \gamma)$ is $(\varphi, \mathcal{M}, \gamma, \rho_0)$. Suppose, then, that position $(\psi, \mathcal{M}, \epsilon, \rho)$ has been reached. If $\circ \in \{\vee, \wedge\}$ and $\psi = (\theta_1 \circ \theta_2)$, the next position is $(\theta_i, \mathcal{M}, \lambda, \rho)$. The term θ_i is chosen by player $\rho(\mathbb{V})$ if $\circ = \vee$ and by player $\rho(\mathbb{F})$ if $\circ = \wedge$. If $\mathrm{Q} \in \{\exists, \forall\}$ and $\psi = \mathrm{Q}x\theta$, the next position is $(\theta, \mathcal{M}, x \mapsto b, \rho)$, where $b \in M$ is chosen by $\rho(\mathbb{V})$ if $\mathrm{Q} = \exists$ and by $\rho(\mathbb{F})$ if $\mathrm{Q} = \forall$. If $\psi = {\sim}\theta$, the next position is $(\theta, \mathcal{M}, \lambda, \rho^*)$, where ρ^* is the transposition of ρ, that is, a map satisfying $\rho^*(\mathbb{V}) = \rho(\mathbb{F})$ and $\rho^*(\mathbb{F}) = \rho(\mathbb{V})$. If ψ is atomic, there are no further moves. If the sequence of positions from the initial position to the current position is $(\psi_0, \mathcal{M}, \epsilon_0, \rho_0), \ldots, (\psi_n, \mathcal{M}, \epsilon_n, \rho_n)$, this sequence induces an assignment γ via the components $\epsilon_i \neq \lambda$. If ψ_n is atomic and $\mathcal{M}, \gamma \models \psi_n$, then $\rho(\mathbb{V})$ wins, else $\rho(\mathbb{F})$ wins.[1] A *strategy* of player $j \in \{I, \mathit{II}\}$ is a tuple of *strategy functions*, one for each operator token for which the player must make a move. A strategy function yields a move for the corresponding operator token, depending on the adversary's earlier moves. It can be shown, if the axiom of choice is assumed, that a first-order formula φ is satisfied in a structure $(\mathcal{M}, \gamma)$ according to the standard Tarskian semantics if and only if there exists a w.s. for player *II* in $G(\varphi, \mathcal{M}, \gamma)$ (see Hodges [14, p. 94]).

GTS is useful in connection with certain extensions of first-order logic. While strategy functions of a given player in first-order semantic games may take as arguments any earlier moves of the adversary, in *independence-friendly logic* (IF logic, here denoted L_{IF}) there is a syntactic mechanism available for indicating that for specified operator tokens only some of the adversary's earlier moves are available as arguments (for L_{IF}, see Hintikka [8]). The game rules for IF-logical games are the same as those for first-order games; the difference lies at the level of strategies. For example, if $\mathcal{M}$ is a model the domain of which consists of numbers 1 and 2 and $\varphi_0 := \forall x(\exists y/\{\forall x\})\, x = y$, there are 4 plays in game $G(\varphi_0, \mathcal{M})$, namely, $(1, 1)$, $(1, 2)$, $(2, 1)$, and $(2, 2)$. The strategy function of player *II* for $(\exists y/\{\forall x\})$ must not take for its argument the value player *I* has chosen for variable x; it must be a constant. Since choosing neither 1 nor 2 as a value for y leads to a win for both possible values of x, there is no w.s. for *II* in game $G({\sim}\varphi_0, \mathcal{M})$, that is, φ_0 is not true in $\mathcal{M}$. On the other hand, neither is ${\sim}\varphi_0$ true in $\mathcal{M}$. In order for there to exist a w.s. for *II* in this game, it should be possible for *II*, in the role of falsifier, to choose a value a of x so that for all values b of y that *I* can choose, we have $a \neq b$. Since neither φ nor ${\sim}\varphi$ is true in $\mathcal{M}$, the negation $\sim$ interpreted via the idea of switching roles does *not* capture classical negation $\neg$: $\neg\psi$ is true precisely in those situations in which ψ is not true. We will refer to $\sim$ as *dual negation* and $\neg$ as *classical negation*. We denote first-order logic formulated using $\sim$ as its negation symbol by L_{FO}, while first-order logic with $\neg$ as its negation symbol is denoted by **FO**.

On various occasions Hintikka [8], [9], [10] has claimed that there cannot be a game rule for classical negation. Since satisfaction conditions formulated in terms of semantic games, as defined by Hintikka, are always expressible in the Σ^1_1-fragment of second-order logic (**SO**), it is indeed impossible that semantic games hence formulated could incorporate a rule for classical negation. (Σ^1_1 is not closed under complementation.) We wish to show, however, that a game-theoretical analysis of classical negation is possible. This is achieved by enriching the structure of game positions by an additional component—to be termed a *mode*—and suitably interpreting the effect of a mode on the strategy level. This approach for capturing classical negation has its limits; it applies only to certain languages (including L_{IF}). These limits will be identified. Related earlier research is discussed in Section 9.

In higher-order logics, we write $\equiv$ for logical equivalence and we use $\neg$ to denote negation. We write $L \leq L'$ to indicate that logic L can be translated into logic L'; that is, for every $\varphi \in L$ there is $\psi_\varphi \in L'$ such that $\mathcal{M}, \gamma \models \varphi$ if and only if $\mathcal{M}, \gamma \models \psi_\varphi$ for all suitable models $\mathcal{M}$ and assignments γ. Writing $L < L'$ means that $L \leq L'$ but $L' \nleq L$, while $L = L'$ means that $L \leq L'$ and $L' \leq L$. We denote the empty tuple by $()$. When it serves clarity, we use angle brackets to mark a tuple: $\langle a, b, c\rangle$ means (a, b, c). If X is a set, we write $|X|$ for its cardinality. If $\mathcal{M}$ is a model of vocabulary τ with $P \in \tau$, then $P^{\mathcal{M}}$ stands for the interpretation of P in $\mathcal{M}$. We abbreviate $({\sim}\theta_1 \vee \theta_2)$ by $(\theta_1 \rightarrow \theta_2)$ and $((\theta_1 \rightarrow \theta_2) \wedge (\theta_2 \rightarrow \theta_1))$ by $(\theta_1 \leftrightarrow \theta_2)$. Further, we write $(\theta_1 \supset \theta_2)$ for $(\neg\theta_1 \vee \theta_2)$ and $(\theta_1 \subset\!\supset \theta_2)$ for $((\theta_1 \supset \theta_2) \wedge (\theta_2 \supset \theta_1))$. If f is a function of type $A \rightarrow B$ and $C \subseteq A$, we write $f|_C$ for the restriction of f to the set C. For later use, we define the notion of game as follows.

Definition 1.1 (Game, play) A *game* between players I and II is a quintuple (X, S, p_0, c, u), the components of which are as follows. X is a set and $p_0 \in X$. The elements of X are *positions*; p_0 is the *initial position*. Further, S is function of type $X \rightarrow \mathcal{P}(X)$. If $p \in X$, the set $S(p)$ is said to consist of the *possible successors* of p. If the set $S(p)$ is empty, the position p is *terminal*. The component c, called a *player function*, is defined on a subset of all nonterminal positions; its codomain is the set $\{I, II\}$. If c is defined on p, the elements of $S(p)$ are considered to be resulting from different moves available to player $c(p)$ at p. A *play* is any sequence $(p_0, \ldots, p_n)$ such that p_n is terminal and $p_{i+1} \in S(p_i)$ for all $0 \leq i < n$. Any nonempty initial segment of a play is a *partial play*. Finally, u is a *utility function* assigning to each play exactly one of the tuples (*win*, *loss*) or (*loss*, *win*). If $u(\pi) = (\textit{win}, \textit{loss})$, we say that I wins and II loses the play π, and if $u(\pi) = (\textit{loss}, \textit{win})$ we say that I loses and II wins π.

2 First-Order Semantic Games Relativized to Modes

The existence of a w.s. for a given player is a strategic property of a game, but there are other properties of interest. Insofar as it makes sense to reason in terms of the class of strategies of a given player in the first place, we may pose questions pertaining to *all* those strategies, as well as to the existence of a strategy with such-and-such features. In particular, we may turn attention to the following property: for all strategies g of player I in $G(\varphi, \mathcal{M}, \gamma)$, there is a sequence of moves $\vec{b}$ of player II such that II wins the play determined by g and $\vec{b}$. We proceed to study a greater variety of strategic properties than has been commonplace in connection with semantic games.

We will utilize semantic games whose positions are quintuples (rather than quadruples as in first-order semantic games), consisting not only of a formula, a model, an assignment, and a role distribution, but also a *mode*. We consider two modes, to be labeled as $+$ and $-$. On the play level the modes have a very modest role, but at the strategic level they are of importance. The situation may be compared with the case of L_{IF}, where the independence indications have no effect whatsoever at the play level, but they impose a constraint on strategies available to a given player. The modes will be used for game-theoretically interpreting a unary connective denoted by $\rightharpoondown$ and considered as a negation symbol, to be termed *mode negation*. This connective, hence interpreted, will capture classical negation in connection with certain logics.

Let us write L_0 for the fragment of L_{FO} consisting of formulas of the form $P_1 \cdots P_n\, \chi$, where $1 \le n < \omega$ and χ is a quantifier-free formula of a relational vocabulary and every P_i is one of the symbols $\sim$, $\forall x_i$, and $\exists x_i$. The string $P_1 \cdots P_n$ is the *prefix* and χ the *matrix* of the formula $P_1 \cdots P_n\, \chi$. Note that if P_i is a quantifier, the variable it carries is x_i. This syntactic restriction could be relaxed, but we stay with it for simplicity. So $\forall x_1 {\sim} \exists x_3 R(x_1, x_3)$ is a formula, but $\forall x_1 {\sim} \exists x_2 R(x_1, x_2)$ is not. By definition a (winning) strategy of player *II* in the semantic game $G(\varphi, \mathcal{M}, \gamma, +)$ with $\varphi \in L_0$ is simply any (winning) strategy for *II* in semantic game $G(\varphi, \mathcal{M}, \gamma)$, and a (winning) strategy of player *I* in the semantic game $G(\varphi, \mathcal{M}, \gamma, -)$ is any (winning) strategy for *I* in semantic game $G(\varphi, \mathcal{M}, \gamma)$. By contrast, a (winning) strategy of *II* in $G(\varphi, \mathcal{M}, \gamma, -)$ is any *functional* F which for every strategy f of *I* in semantic game $G(\varphi, \mathcal{M}, \gamma)$ yields a sequence of moves $F(f)$ by *II* complying with the game rules of $G(\varphi, \mathcal{M}, \gamma)$ such that f and $F(f)$ together determine a play of that game (won by *II*); and a (winning) strategy of *I* in $G(\varphi, \mathcal{M}, \gamma, +)$ is any functional F which for every strategy f of *II* in $G(\varphi, \mathcal{M}, \gamma)$ yields a sequence of moves $F(f)$ by *I* complying with the rules of $G(\varphi, \mathcal{M}, \gamma)$ such that f and $F(f)$ together determine a play of that game (won by *I*). For L_0, we define semantic games *relativized to a role distribution* as follows: $G(\varphi, \mathcal{M}, \gamma, \rho_0, \star)$ equals $G(\varphi, \mathcal{M}, \gamma, \star)$, and $G(\varphi, \mathcal{M}, \gamma, \rho_0^*, \star)$ equals $G({\sim}\varphi, \mathcal{M}, \gamma, \star)$.

Mode-relative semantic games for L_0 are mode-invariant; the proof uses the well-known property of determinacy of standard semantic games for L_0. Mode invariance fails for more general languages to be studied in this paper.

Fact 2.1 (Mode invariance for L_0) For any L_0-formula φ, model $\mathcal{M}$, assignment γ, and player $j \in \{I, II\}$, there is a w.s. for j in game $G(\varphi, \mathcal{M}, \gamma, +)$ if and only if there is a w.s. for j in game $G(\varphi, \mathcal{M}, \gamma, -)$.

Proof We begin with the case $j := II$. Suppose that there is a w.s., call it g, for *II* in $G(\varphi, \mathcal{M}, \gamma, +)$, but still there is no w.s. for *II* in $G(\varphi, \mathcal{M}, \gamma, -)$. Hence there is a strategy of player *I* in $G(\varphi, \mathcal{M}, \gamma)$, call it f, such that for any sequence of moves by *II*, it is *I* who wins the resulting play. So the play obtained when *II* applies g and *I* applies f is won by both players, which is impossible. Conversely, suppose that there is a w.s., call it F, for *II* in $G(\varphi, \mathcal{M}, \gamma, -)$. Suppose for contradiction that there is no w.s. for player *II* in $G(\varphi, \mathcal{M}, \gamma, +)$. By determinacy of semantic games for L_0, there is a w.s., call it h, for *I* in $G(\varphi, \mathcal{M}, \gamma)$. Hence in $G(\varphi, \mathcal{M}, \gamma)$ the play determined by the strategy h and the sequence of moves $F(h)$ is won by both players, which again is impossible. We may reason similarly if $j := I$. $\square$

3 Languages L_1 and L_2

We begin to investigate to which extent and how we can game-theoretically capture classical negation in connection with certain extensions of L_0. In addition to L_0 with $\sim$ as its negation sign, we define two further languages: L_1 and L_2. Their formulas will have the general form $P_1 \cdots P_n\, \chi$, where $0 \le n < \omega$ and χ is a quantifier-free L_0-formula, every P_i being one of the symbols $\sim$, $\neg$, $(\forall x_i / W_i)$, and $(\exists x_i / W_i)$, given that W_i stands for a set of quantifiers $\mathsf{Q}_k x_k$ with $1 \le k < i$ and $\mathsf{Q}_k \in \{\forall, \exists\}$. The expressions $(\forall x_i / W_i)$ and $(\exists x_i / W_i)$ are called *quantifiers*. If clarity so demands, they may be termed *slashed quantifiers* in contradistinction to the expressions $\forall x_i$ and $\exists x_i$, which may then be referred to as *plain quantifiers*. Note that the expressions W_i stand for sets of plain quantifiers, not slashed quantifiers. If $(\mathsf{Q}x_i/W_i)$ is a slashed quantifier, $/W_i$ is its *independence indication*. If the set W_i is empty, we adopt the convention of writing $\exists x_i$ for $(\exists x_i/W_i)$ and $\forall x_i$ for $(\forall x_i/W_i)$.

The *polarity* of expression P_i in the prefix $P_1 \cdots P_n$ is *positive* if the total number of negation signs in the string $P_1 \cdots P_{i-1}$ is even (occurrences of both $\rightharpoondown$ and $\sim$ counted), otherwise it is *negative*. The quantifiers appearing in prefixes of formulas of the logics L_1 and L_2 are required to meet the following conditions.

- L_2: Suppose $P_i = (\mathsf{Q}_i x_i/W_i)$ and $P_j = (\mathsf{Q}_j x_j/W_j)$ and $\mathsf{Q}_i x_i \in W_j$ with $i < j$. If P_i and P_j have the same polarity and P_i is an existential (universal) quantifier, then P_j is a universal (existential) quantifier. If, again, P_i and P_j have different polarities and P_i is an existential (universal) quantifier, then also P_j is an existential (universal) quantifier.
- L_1: We have the condition for L_2 with the following additional requirement: if $P_i = (\mathsf{Q}_i x_i/W_i)$ and $P_j = (\mathsf{Q}_j x_j/W_j)$ and $\mathsf{Q}_i x_i \in W_j$ with $i < j$, then between P_i and P_j no $\rightharpoondown$-sign occurs in the prefix.

For example, if χ and θ are quantifier-free, then

$$\rightharpoondown \forall x_2 \exists x_3 \forall x_4 \big(\exists x_5/\{\forall x_2\}\big) \rightharpoondown \forall x_7 \exists x_8 \forall x_9 \big(\exists x_{10}/\{\forall x_7\}\big)\, \chi$$

is a formula of L_1 (but not of L_0), and

$$\rightharpoondown \forall x_2 \exists x_3 \forall x_4 \big(\exists x_5/\{\forall x_2\}\big) \rightharpoondown \forall x_7 \exists x_8 \forall x_9 \big(\exists x_{10}/\{\forall x_7, \exists x_3, \exists x_5\}\big)\, \theta$$

is a formula of L_2 (but not of L_1). Directly by definition, the sets of formulas of the logics introduced thus far are related as follows: $L_0 \subsetneq L_1 \subsetneq L_2$.

If $P_1 \cdots P_n\, \chi$ is an L_2-formula, any string $P_i \cdots P_n\, \chi$ with $1 \leq i \leq n+1$ is its *subformula*. If $P_i \cdots P_n\, \chi$ is a subformula, the set $\mathrm{Free}_1(\psi)$ of its *free atomic variables* consists of those variables x_j appearing in χ for which there is no quantifier $(\mathsf{Q}_j x_j/W_j)$ in the string $P_i \cdots P_n$. The set $\mathrm{Free}_2(\psi)$ of its *free independence variables* consists of variables x_j such that there is a quantifier $(\mathsf{Q}_k x_k/W_k)$ in the string $P_i \cdots P_n$ and a quantifier $\mathsf{Q}_j x_j$ in W_k with $j < i$. For example, if $\psi := (\exists x_7/\{\forall x_3\}) R(x_3, x_5, x_7)$, then $\mathrm{Free}_1(\psi) = \{x_3, x_5\}$ and $\mathrm{Free}_2(\psi) = \{x_3\}$. Note that $\mathrm{Free}_2(\varphi) = \varnothing$ for all $\varphi \in L_2$. Subformulas containing free independence variables are *not* L_2-formulas.

The fragment of L_1 without $\rightharpoondown$ coincides, syntactically, with the fragment L_{IF}^{pr} of L_{IF} consisting of formulas with a prefix of slashed quantifiers followed by a quantifier-free matrix formula. By Theorem 7.2, the negation $\rightharpoondown$ actually behaves as classical negation in L_1. In fact, then, formulas $\rightharpoondown\theta$ with $\theta \in L_{\mathrm{IF}}^{pr}$ belong to what Hintikka [8] has called extended IF logic. The whole logic L_1 consists of formulas in prenex form of the fully extended IF logic (to be denoted L_{FeIF}) discussed in [10], whereas L_2 goes even beyond L_{FeIF}.

4 Semantic Games Generalized

We associate a semantics with L_2-formulas via a two-fold procedure. First, we define correlated semantic games; then we explain how the semantic attributes of interest are defined with reference to these games by using what we call metagames. Both semantic games and metagames are games in the sense of Definition 1.1. We define these games so that it will be absolutely clear what their corresponding components are (set X of positions, successor function S, initial position p_0, player function c, utility function u), although we do not explicitly phrase the definitions as definitions of these five components.

Define a function $(\cdot)' : \{+,-\} \to \{+,-\}$ by setting $+' = -$ and $-' = +$. A *simple assignment* of variable x_i to a value ξ, denoted $x \mapsto \xi$, is the map $\{(x_i, \xi)\}$. Recall that λ stands for the empty assignment. For every L_2-formula $\varphi = P_1 \cdots P_n\, \chi$, suitable model $\mathcal{M}$, assignment $\gamma : \mathrm{Free}_1(\varphi) \to M$, and mode $\star \in \{+,-\}$, we associate a *semantic game* $G(\varphi, \mathcal{M}, \gamma, \star)$ defined as follows.

1. The initial position is $(\varphi, \mathcal{M}, \gamma, \rho_0, \star)$.
2. If $(\sim\psi, \mathcal{M}, \epsilon, \rho, \star)$ is a position, so is $(\psi, \mathcal{M}, \lambda, \rho^*, \star)$.
3. If $(\rightharpoondown\psi, \mathcal{M}, \epsilon, \rho, \star)$ is a position, so is $(\psi, \mathcal{M}, \lambda, \rho^*, \star')$.
4. If $((\mathsf{Q}x_i / W_i)\psi, \mathcal{M}, \epsilon, \rho, \star)$ is a position with $\mathsf{Q} \in \{\forall, \exists\}$, and $b \in M$, then $(\psi, \mathcal{M}, x_i \mapsto b, \rho, \star)$ is a position. If $\mathsf{Q} = \forall$, player $\rho(\mathbb{F})$ chooses one such position, else it is player $\rho(\mathbb{V})$ who chooses one such position.
5. If $(\psi, \mathcal{M}, \epsilon, \rho, \star)$ is a position, ψ equals the matrix χ, and the play which led to this position is $\pi = ((\psi_0, \mathcal{M}, \epsilon_0, \rho_0, \star_0), \ldots, (\psi_n, \mathcal{M}, \epsilon_n, \rho_n, \star_n))$, then player $\rho(\mathbb{V})$ wins the play π if $\mathcal{M}, \epsilon_0, \ldots, \epsilon_n \models \psi$, else $\rho(\mathbb{F})$ wins.

For simplicity we let plays terminate with the matrix formula, although it may not be atomic. Given that the ϵ_i are pairwise distinct variable assignments, the notation $\mathcal{M}, \epsilon_0, \ldots, \epsilon_n \models \psi$ means that $\mathcal{M}, \delta \models \psi$ in the usual sense of first-order logic, with $\delta = \bigcup_{0 \le i \le n} \epsilon_i$. Note that in the above game rules, $\epsilon_0 = \gamma$. Viewed as sets, each ϵ_i with $i > 0$ is either empty or a singleton.

Semantic games can be relativized to a role distribution by stipulating that $G(\varphi, \mathcal{M}, \gamma, \rho_0, \star)$ equals $G(\varphi, \mathcal{M}, \gamma, \star)$ and $G(\varphi, \mathcal{M}, \gamma, \rho_0^*, \star)$ equals $G(\sim\varphi, \mathcal{M}, \gamma, \star)$. If φ is a sentence, $G(\varphi, \mathcal{M}, \star)$ equals $G(\varphi, \mathcal{M}, \lambda, \star)$. Note that independence indications play no role in the game rules; they will become operative at the strategy level. While the dual negation $\sim$ acts exclusively on the role distribution (transposing it), the mode negation $\rightharpoondown$ acts *also* on the mode (changing it). The impact of the modes will become manifest at the strategic level.

Recall that in Section 2 it was explained, for formulas φ of L_0, what counts as a winning strategy of a given player in a semantic game $G(\varphi, \mathcal{M}, \gamma, \star)$. If $\varphi \in L_0$, write $\mathcal{M}, \gamma \models \varphi$ to indicate that there is a w.s. for player *II* in game $G(\varphi, \mathcal{M}, \gamma)$. We may observe that when applied to L_0-formulas, the connective $\rightharpoondown$ captures the classical negation.

Fact 4.1 Let $\varphi \in L_0$. For all suitable structures $(\mathcal{M}, \gamma)$, there is a w.s. for player *II* in game $G(\rightharpoondown\varphi, \mathcal{M}, \gamma, +)$ if and only if $\mathcal{M}, \gamma \not\models \varphi$.

Proof $\mathcal{M}, \gamma \not\models \varphi$ if and only if there is no w.s. for player *II* in $G(\varphi, \mathcal{M}, \gamma)$ if and only if for every strategy of *II* in $G(\varphi, \mathcal{M}, \gamma, \rho_0, +)$ there is a sequence of moves of *I* such that *I* wins the resulting play if and only if (†) for every strategy of *I* in $G(\varphi, \mathcal{M}, \gamma, \rho_0^*, +) = G(\varphi, \mathcal{M}, \gamma, \rho_0^*)$ there is a sequence of moves of *II* such that *II* wins the resulting play if and only if there is a w.s. for *II* in $G(\varphi, \mathcal{M}, \gamma, \rho_0^*, -)$ if and only if there is a w.s. for *II* in $G(\rightharpoondown\varphi, \mathcal{M}, \gamma, \rho_0, +)$. The equivalence (†) holds because in L_0, any strategy for *II* in $G(\theta, \mathcal{N}, \delta)$ is a strategy for *I* in $G(\sim\theta, \mathcal{N}, \delta)$, and vice versa. □

5 Metagames

The strategic impact of a mode change with formulas discussed thus far (formulas of L_0 prefixed by $\rightharpoondown$) has been that of effecting a switch between the attributes "there is a strategy for player *II* such that for all sequences of moves by player *I* ..." and "for all strategies of player *I* there is a sequence of moves by player *II* ...." It is to be expected, then, that nested occurrences of $\rightharpoondown$ give rise to rather complicated strategic properties. For instance, insofar as $\rightharpoondown$ is to capture classical negation, on model $\mathcal{M}$ the formula $\rightharpoondown \forall x_2 \exists x_3 \rightharpoondown \forall x_4 \exists x_5 R(x_2, x_3, x_4, x_5)$ should state that for any function $f_{x_3} : M \rightarrow M$ there is a value a of x_2 and a function $g_{x_5} : M \rightarrow M$ such that for any value b of x_4, we have: $\langle a, f(a), b, g(b)\rangle \in R^{\mathcal{M}}$. The function g_{x_5} may in principle depend on the function f_{x_3}. Indeed, the sentence should state the existence of functionals $F_{x_2} : M^M \rightarrow M$ and $F_{x_5} : M^M \rightarrow M^M$ such that for any function $f_{x_3} : M \rightarrow M$ and for any value b of x_4, we have $\langle F_{x_2}(f_{x_3}), f_{x_3}(F_{x_2}(f_{x_3})), b, F_{x_5}(f_{x_3})(b)\rangle \in R^{\mathcal{M}}$. How should the relevant attributes be specified in connection with logics L_1 and L_2? We opt for explicating this by using *metagames*. In terms of such metagames $\Gamma(\varphi, \mathcal{M}, \gamma, \star)$, we may conveniently discuss strategic properties of the *object games*—the plain semantic games $G(\varphi, \mathcal{M}, \gamma, \star)$ with $\varphi \in L_i$ with $i := 0, 1, 2$.

For L_0-formulas such metagames are extremely simple: in metagame $\Gamma(\varphi, \mathcal{M}, \gamma, +)$, first player *II* picks out, for each quantifier in the prefix for which it is his or her turn to move, what would be a corresponding strategy function f_i in semantic game $G(\varphi, \mathcal{M}, \gamma)$. Then player *I* picks out, for each quantifier in the prefix for which it would be her or his turn to move in semantic game $G(\varphi, \mathcal{M}, \gamma)$, an element a_j of M. Metagame $\Gamma(\varphi, \mathcal{M}, \gamma, -)$ is played similarly, the difference being that it is player *I* who picks out what would be strategy functions for him in semantic game $G(\varphi, \mathcal{M}, \gamma)$, whereafter player *II* selects what would be moves for her in semantic game $G(\varphi, \mathcal{M}, \gamma)$. In either case, if the choices are $\vec{f}$ and $\vec{a}$ and the play of the semantic game $G(\varphi, \mathcal{M}, \gamma)$ determined by $\vec{f}$ and $\vec{a}$ is won by player *II*, then player *II* wins the play $(\vec{f}, \vec{a})$ of the metagame $\Gamma(\varphi, \mathcal{M}, \gamma, \star)$, else player *I* wins the play. There is no *need* to resort to the idea of metagame when explicating the notions of strategy and winning strategy for mode-relative L_0-games; indeed these notions were introduced for L_0 already in Section 2. However, in order to formulate the semantics of L_2 we need a generalization, and such a generalization is conveniently devised at the level of metagames. In the general case of L_2-formulas, plays of the metagame are structured as follows:

(tuple of functions ; tuple of moves), (tuple of functions ; tuple of moves), ..., (tuple of functions ; tuple of moves).

There is a finite number of *rounds* (tuple of functions, tuple of moves), in each of which first a finite number of functions are chosen, one by one, by one of the players (these we call *function moves*)—whereafter a finite number of elements from the domain are chosen, one by one, by the other player (*element moves*). The players alternate in making moves in the sense that one of the players makes function moves in the rounds with odd order position, and the other player in rounds with even order position. Once metagames and the corresponding notions of strategy are defined, we will declare that "strategy of player j in a semantic game $G(\varphi, \mathcal{M}, \gamma, \star)$" *means* "strategy of player j in a metagame $\Gamma(\varphi, \mathcal{M}, \gamma, \star)$."

We need some auxiliary notions in order to define the general concept of metagame. If $S_1 \cdots S_r$ is a string, any string $S_i \cdots S_k$ with $1 \leq i \leq k \leq r$ is its *substring*. The *length* of a string $S_1 \cdots S_r$ equals r. Any string not containing the symbol $\rightharpoondown$ is *$\rightharpoondown$-free*. A substring $S_i \cdots S_k$ of a string $S_1 \cdots S_r$ is *maximally $\rightharpoondown$-free* if it is $\rightharpoondown$-free and there is no $\rightharpoondown$-free substring of $S_1 \cdots S_r$ of length greater than $(k-i)+1$ having $S_i \cdots S_k$ as its substring. A quantifier P_l in the prefix $P_1 \cdots P_n$ has *existential force* if the quantifier P_l is existential and its polarity is positive in the prefix $P_1 \cdots P_n$, or P_l is universal and its polarity is negative in the prefix $P_1 \cdots P_n$. The notion of a quantifier having *universal force* can be defined dually. Observe that the notions of existential and universal force are purely syntactic. (This will no longer be the case for the notions of the weak and strong force to be defined in Section 6.) There are two types of objects that the players may choose in the course of a metagame: elements of the domain and what we will call *local strategy functions*. It will depend on the position which types of objects can be chosen. Some further notions are needed to facilitate formulating the game rules for metagames.

Definition 5.1 (Barrier, local visibility) Let $\varphi := P_1 \cdots P_n\, \chi$ be an L_2-formula. Suppose that P_i is a quantifier. If the prefix $P_1 \cdots P_i$ is not $\rightharpoondown$-free, the *backward barrier* of the quantifier P_i is the operator P_j with $j < i$ such that $P_j = \rightharpoondown$ and the string $P_{j+1} \cdots P_i$ is $\rightharpoondown$-free. Similarly, if the suffix $P_i \cdots P_n$ is not $\rightharpoondown$-free, the *forward barrier* of P_i is the operator P_j with $j > i$ such that $P_j = \rightharpoondown$ and the string $P_i \cdots P_{j-1}$ is $\rightharpoondown$-free. That is, the backward (forward) barrier of P_i is the $\rightharpoondown$-sign closest to P_i on the left (right) if one exists. The operators *close* to P_i are those in the string $P_k \cdots P_l$ with $1 \leq k \leq i \leq l \leq n$, given that ($P_{k-1}$ is the backward barrier of P_i or if none exists $k = 1$) and (P_{l+1} is the forward barrier of P_i or if none exists $l = n$). If P_j is the backward barrier of P_i, the operators in the string $P_1 \cdots P_{j-1}$ are *far from* P_i. If $P_i = (\mathsf{Q}x_i / W_i)$ has existential force in φ, a quantifier P_k is said to be *locally visible* for P_i provided that the following conditions are satisfied: $k < i$; P_k is close to P_i; P_k has universal force in φ; and $P_k = (\mathsf{Q}' x_k / W_k)$ with $\mathsf{Q}' x_k \notin W_i$. Derivatively, if P_i is a quantifier with existential force in φ, $\pi = (p_0, \ldots, p_{m-1})$ is a partial play of a semantic game correlated with φ, and the subformula component of the position p_{i-1} is $P_i \cdots P_n\, \chi$, the *assignment locally visible* at π equals $\gamma \cup \delta$, where γ is the assignment in p_0 and δ is the union of the simple assignments $x_j \mapsto b_j$ in positions p_j such that quantifier P_j is locally visible for P_i. These simple assignments result from having assigned a value to quantifier P_j at position p_{j-1}. The notions of locally visible quantifier (assignment) can be defined dually for quantifiers with universal force.

Definition 5.2 (Local strategy function, local strategy) Consider a semantic game $G(\varphi, \mathcal{M}, \gamma, \star)$ with $\varphi := O_1 \cdots O_n\, \chi$. If $O_i = (\mathsf{Q}x_i / W_i)$ and O_i has existential force in φ, a *local strategy function* for O_i is a function f_i satisfying the following: whenever $\pi = (p_0, \ldots, p_{i-1})$ is a partial play of game $G(\varphi, \mathcal{M}, \gamma, \star)$ and δ is the assignment locally visible at π, the function f_i is defined on the map δ and for some $b \in M$ we have $f_i(\delta) = x_i \mapsto b$. That is, the local strategy function f_i tells player *II* which element $b \in M$ to assign to the variable x_i; the choice is allowed to depend on the assignment locally visible at π and on nothing else. If $O_j \cdots O_k$ is a maximally $\rightharpoondown$-free string in of the prefix $O_1 \cdots O_n$, a *local strategy* for player *II* is a sequence of local strategy functions, one for each O_i with $j \leq i \leq k$ having existential force in φ. The notions of local strategy function and local strategy for player *I* can be defined dually.

For any local strategy function f there are fixed variables $x_{i_1}, \ldots, x_{i_{n+1}}$ such that f takes assignments $\{(x_{i_1}, a_{i_1}), \ldots, (x_{i_n}, a_{i_n})\}$ as arguments and yields a simple assignment $\{(x_{i_{n+1}}, a_{i_{n+1}})\}$ as its value. Without danger of confusion we may, then, view local strategy functions as functions mapping tuples of elements $(a_{i_1}, \ldots, a_{i_n})$ to elements $a_{i_{n+1}}$.

In semantic games for L_0 and even for L_{IF} all strategies are local strategies. Local strategies encode strategic reasonings restricted to parts of plays during which the negation $\neg$ is not encountered. We will now define the notion of metagame, to be used in explicating the notions of truth and falsity of L_2-sentences. For every formula $\varphi = P_1 \cdots P_n\, \chi$ of L_2, suitable structure $(\mathcal{M}, \gamma)$, and mode $\star \in \{+, -\}$, we associate a *metagame* $\Gamma(\varphi, \mathcal{M}, \gamma, \star)$ as follows.

1. The initial position is $(\varphi, \mathcal{M}, \gamma, \rho_0, \star)$.
2. If $(\neg P_i \cdots P_n\, \chi, \mathcal{M}, \epsilon, \rho, \star)$ is a position, the next position is $(P_i \cdots P_n\, \chi, \mathcal{M}, \lambda, \rho^*, \star')$.

3a. If $(P_i \cdots P_n\, \chi, \mathcal{M}, \epsilon, \rho, \star)$ is a position and $P_i \cdots P_k$ with $i \leq k \leq n$ is a maximally $\neg$-free substring of $P_i \cdots P_n$ containing exactly r dual negation symbols, the next position is

$$\big(\langle(P_{i_1}, \ldots, P_{i_s}),\, (P_{j_1}, \ldots, P_{j_t})\rangle\, P_{k+1} \cdots P_n\, \chi, \mathcal{M}, \lambda, \rho_r, \star\big),$$

where the P_{i_y} (resp., the P_{j_z}) are those quantifiers in the string $P_i \cdots P_k$ that have existential (universal) force in φ and $\rho_r = \rho$ if r is even, whereas $\rho_r = \rho^*$ if r is odd.

3b. If $(\langle(), ()\rangle\, P_{k+1} \cdots P_n\, \chi, \mathcal{M}, \epsilon, \rho, \star)$ is a position, the next position is $(P_{k+1} \cdots P_n\, \chi, \mathcal{M}, \lambda, \rho, \star)$.

4a. If $(\langle(P_{i_r}, \ldots, P_{i_s}),\, (P_{j_1}, \ldots, P_{j_t})\rangle \neg P_{k+1} \cdots P_n\, \chi, \mathcal{M}, \epsilon, \rho, +)$ is a position and $r \leq s$, player *II* chooses a local strategy function f_{i_r} for P_{i_r}. The next position is $(\langle(P_{i_{r+1}}, \ldots, P_{i_s}),\, (P_{j_1}, \ldots, P_{j_t})\rangle \neg P_{k+1} \cdots P_n\, \chi, \mathcal{M}, x_{i_r} \mapsto f_{i_r}, \rho, +)$.

4b. If $(\langle(), (P_{j_r}, \ldots, P_{j_t})\rangle \neg P_{k+1} \cdots P_n\, \chi, \mathcal{M}, \epsilon, \rho, +)$ is a position and $r \leq t$, player *I* picks out an element $b_{i_r} \in M$. The next position is $(\langle(), (P_{j_{r+1}}, \ldots, P_{j_t})\rangle \neg P_{k+1} \cdots P_n\, \chi, \mathcal{M}, x_{i_r} \mapsto b_{i_r}, \rho, +)$.

4c. If $(\langle(P_{i_1}, \ldots, P_{i_s}),\, (P_{j_r}, \ldots, P_{j_t})\rangle \neg P_{k+1} \cdots P_n\, \chi, \mathcal{M}, \epsilon, \rho, -)$ is a position and $r \leq t$, player *I* chooses a local strategy function f_{i_r} for P_{i_r}. The next position is $(\langle(P_{i_1}, \ldots, P_{i_s}),\, (P_{j_{r+1}}, \ldots, P_{j_t})\rangle \neg P_{k+1} \cdots P_n\, \chi, \mathcal{M}, x_{i_r} \mapsto f_{i_r}, \rho, -)$.

4d. If $(\langle(P_{i_r}, \ldots, P_{i_s}), ()\rangle \neg P_{k+1} \cdots P_n\, \chi, \mathcal{M}, \epsilon, \rho, -)$ is a position and $r \leq s$, player *II* picks out an element $b_{i_r} \in M$. The next position is $(\langle(P_{i_{r+1}}, \ldots, P_{i_s}), ()\rangle \neg P_{k+1} \cdots P_n\, \chi, \mathcal{M}, x_{i_r} \mapsto b_{i_r}, \rho, -)$.

5. If $(\psi, \mathcal{M}, \epsilon, \rho, \star)$ is a position, ψ equals the matrix χ, and the play that led to this position is $\pi = ((\psi_0, \mathcal{M}, \epsilon_0, \rho_0, \star_0), \ldots, (\psi_n, \mathcal{M}, \epsilon_n, \rho_n, \star_n))$, then player $\rho(\mathbb{V})$ wins the play π if $\mathcal{M}, \epsilon_0, \ldots, \epsilon_n \models \psi$, else $\rho(\mathbb{F})$ wins.

Metagames relativized to a role distribution are defined by stipulating that $\Gamma(\varphi, \mathcal{M}, \gamma, \rho_0, \star)$ equals $\Gamma(\varphi, \mathcal{M}, \gamma, \star)$ and $\Gamma(\varphi, \mathcal{M}, \gamma, \rho_0^*, \star)$ equals $\Gamma(\sim\varphi, \mathcal{M}, \gamma, \star)$. Further, $\Gamma(\varphi, \mathcal{M}, \star)$ equals $\Gamma(\varphi, \mathcal{M}, \lambda, \star)$. Note that in item (3a) either $k = n$ or else $P_{k+1} = \neg$. Note also that in noninitial metagame positions there may appear simple assignments ϵ_i whose values are functions. In item (5), the notation $\mathcal{M}, \epsilon_0, \ldots, \epsilon_n \models \psi$ means $\mathcal{M}, \delta \models \psi$, where $\delta : \mathrm{Free}_1(\psi) \to M$ is the assignment defined as follows: if $\epsilon_j = (x_{i_j} \mapsto b_{i_j})$, then $\delta(x_{i_j}) = b_{i_j}$, while if $\epsilon_j = (x_{i_j} \mapsto f_{i_j})$ and η is the assignment locally visible at the partial play $((\psi_0, \mathcal{M}, \epsilon_0, \rho_0, \star_0), \ldots, (\psi_{j-1}, \mathcal{M}, \epsilon_{j-1}, \rho_{j-1}, \star_{j-1}))$, then $\delta(x_{i_j}) = f_{i_j}(\eta)$.

Remark Rule (3a) serves to group quantifiers of the string $P_i \cdots P_{k-1}$ into two subgroups according to whether they are of existential or universal force, by introducing a pair of lists of quantifiers: $\langle (P_{i_1}, \ldots, P_{i_s}), (P_{j_1}, \ldots, P_{j_t}) \rangle$. Rules (4a)–(4d) explicate how such pairs of lists are processed, one quantifier at a time, until both lists are empty. Thereafter rule (3b) applies and we are back with a string of symbols without pairs of list indicators. Expressions such as $\langle (P_{i_r}, \ldots, P_{i_s}), (P_{j_1}, \ldots, P_{j_t}) \rangle \rightharpoondown P_{k+1} \cdots P_n \, \chi$ are not formulas according to the syntax of L_2. We employ such auxiliary expressions in order to facilitate the formulation of the game rules. Observe that no special rule is needed for dual negation in metagames, since rule (3a) is so formulated that it takes into account the possible occurrences of dual negations when forming the two lists of quantifiers and when determining the role distribution ρ_r. Finally note that rules (4a)–(4d) are formulated in terms of players and not in terms of their roles. That is, for example, according to rule (4a), it is player *II* who chooses a local strategy function for P_{i_r}, irrespective of the role she or he happens to occupy in the corresponding object game when making a move for the quantifier P_{i_r}.

6 Strategic Properties of Object Games

If φ is a sentence of logic L_0, a strategy $\vec{f}$ is winning for player *II* in semantic game $G(\varphi, \mathcal{M})$ if and only if making the initial moves $\vec{f}$ in metagame $\Gamma(\varphi, \mathcal{M}, +)$ yields a win to *II* against any sequence of elements thereafter chosen by player *I*. Similarly, $\vec{g}$ is a w.s. for *I* in $G(\varphi, \mathcal{M})$ if and only if the initial moves $\vec{g}$ in $\Gamma(\varphi, \mathcal{M}, -)$ yield a win to *I* against any sequence of elements chosen by *II*. This observation will guide our generalizations: for truth, we turn attention to the positive mode and player *II*, for falsity to the negative mode and player *I*.

Generally the existence of a w.s. for player *II* in metagame $\Gamma(\varphi, \mathcal{M}, \gamma, +)$ corresponds to a certain rather complicated strategic property of the object game $G(\varphi, \mathcal{M}, \gamma, +)$. In connection with L_1 the property takes the form

> (there is a local strategy f_1 chosen by player *II* such that for any tuple $\vec{x}_1$ of elements chosen by *I*) (there is a local strategy f_2 chosen by player *I* such that for any tuple of elements $\vec{x}_2$ chosen by *II*) ....

Logic L_2 introduces a further complication: those *functionals* that are utilized as strategy functions of player *II* in the metagame may not be allowed to take for arguments certain earlier moves by player *I*.

Consider a semantic game $G(\varphi, \mathcal{M}, \gamma, \star)$, where $\varphi = P_1 \cdots P_n \, \chi$ is an L_2-formula. Suppose that P_r is a quantifier with existential force in the prefix $P_1 \cdots P_n$. The existential force of P_r is said to be *weak* in the semantic game $G(\varphi, \mathcal{M}, \gamma, \star)$ if the initial mode $\star$ equals $+$ and the number of $\rightharpoondown$-signs in the substring $P_1 \cdots P_{r-1}$ is odd, or the initial mode $\star$ equals $-$ and the number of $\rightharpoondown$-signs in the substring $P_1 \cdots P_{r-1}$ is even. Its existential force is *strong* otherwise, that is, if the initial mode $\star$ equals $+$ and the number of $\rightharpoondown$-signs in the substring $P_1 \cdots P_{r-1}$ is even, or the initial mode $\star$ equals $-$ and the number of $\rightharpoondown$-signs in the substring $P_1 \cdots P_{r-1}$ is odd. Dually, a quantifier P_r is of strong (weak) universal force in $G(\varphi, \mathcal{M}, \gamma, \star)$ if it is of universal force and either the initial mode is positive and P_r is preceded by an odd (even) number of $\rightharpoondown$-signs, or else the initial mode is negative and P_r is preceded by an even (odd) number of $\rightharpoondown$-signs. It should be noted that the notions of weak and strong force are not purely syntactic; they are relative not only to a prefix but also to the initial mode. In metagames players make function moves for quantifiers which are in the corresponding object game of strong force and element moves for quantifiers which are of weak force in the corresponding object game. We introduce a couple of auxiliary notions and then formulate the definition of strategy function applicable in connection with metagames.

Definition 6.1 (Visibility) Let $\varphi := P_1 \cdots P_n \, \chi$ be an L_2-formula. Suppose first that P_i is a quantifier of existential force, and let $P_j \cdots P_l$ be the string of operators close to P_i $(j \leq i \leq l)$. A quantifier P_k is said to be *visible* for P_i if it satisfies the following conditions: P_k is far from P_i (whence $k < j$); P_k has universal force in φ; and $x_k \notin W_i$. If P_i has strong force, no other quantifiers are visible for P_i. If, again, P_i has weak force, also all those P_k are visible for P_i that are close to P_i and have universal force relative to φ. Derivatively, if P_r is a quantifier with existential force in φ, $\pi = (p_0, \ldots, p_{m-1})$ is a partial play of a metagame correlated with φ and the expression $\langle (P_r, \ldots, P_{i_s}), (P_{j_1}, \ldots, P_{j_t}) \rangle \psi$ appears in the position p_{m-1}, the *assignment visible* at π equals $\gamma \cup \delta$, where γ is the assignment in p_0 and δ is the union of those simple assignments $x_j \mapsto \xi_j$ appearing in positions p_j such that quantifier P_j is visible for P_r. If P_j has strong universal force, the object ξ_j is a local strategy function, whereas if P_j has weak universal force, ξ_j is an element of the domain. The notions of visible quantifier and visible assignment can be defined dually for quantifiers with universal force.

The notion of *locally* visible assignment—defined for semantic games—must not be confused with the notion of visible assignment—defined for metagames. Also the notions of locally visible quantifier and visible quantifier must be kept apart. In particular if P_i has strong existential force, none of the quantifiers locally visible for P_i are visible for P_i. Note also that if P_i has weak existential force, *all* quantifiers close to it are visible for P_i—including those that lie in its syntactic scope. In accordance with the metagame rules, the adversary will have associated a local strategy function with all those quantifiers before an element is assigned to P_i.

Definition 6.2 (Strategy in a metagame) Let $\varphi = P_1 \cdots P_n \, \chi$ be an L_2-formula. If P_i is a quantifier of existential force, a *strategy function* of player *II* for P_i in metagame $\Gamma(\varphi, \mathcal{M}, \gamma, \star)$ is a function F_i satisfying the following. Whenever $\pi = (p_0, \ldots, p_{i-1})$ is a partial play of $\Gamma(\varphi, \mathcal{M}, \gamma, \star)$ and the assignment visible at π is δ, the function F_{i_r} is defined on δ and we have the following:

- if $p_{i-1} = (\langle (P_{i_r}, \ldots, P_{i_s}),\ (P_{j_1}, \ldots, P_{j_t}) \rangle \psi, \mathcal{M}, \epsilon, \rho, +)$ and P_{i_r} has strong force, then F_{i_r} tells player *II* which local strategy function to pick; that is, there is a local strategy function f_{i_r} for P_{i_r} such that $F_{i_r}(\delta) = f_{i_r}$;
- if $(\langle (P_{i_r}, \ldots, P_{i_s}),\ () \rangle \psi, \mathcal{M}, \epsilon, \rho, -)$ and P_{i_r} has weak force, then F_{i_r} tells player *II* which element $b \in M$ to pick, that is, $F_{i_r}(\delta) \in M$.

A strategy F for player *II* is a sequence of strategy functions, one strategy function F_i for each quantifier P_i of existential force in the prefix. We say that player *II* has *used* strategy F in a play $(p_0, \ldots, p_m)$ of game $\Gamma(\varphi, \mathcal{M}, \gamma, \star)$ if the following two conditions are met for all $i \leq m$:

- if $p_{i-1} = (\langle (P_{i_r}, \ldots, P_{i_s}),\ (\langle P_{j_1}, \ldots, P_{j_t}) \rangle \psi, \mathcal{M}, \epsilon, \rho, +)$, and P_{i_r} has strong force, and δ_{i-1} is the assignment visible to player *II* at $(p_0, \ldots, p_{i-1})$, then $p_i = (\langle (P_{i_{r+1}}, \ldots, P_{i_s}),\ (P_{j_1}, \ldots, P_{j_t}) \rangle \psi, \mathcal{M}, x_{i_r} \mapsto F_{i_r}(\delta_{i-1}), \rho, +)$;

- if $p_{i-1} = (\langle (P_{i_r}, \ldots, P_{i_s}), ()\rangle \psi, \mathcal{M}, \epsilon, \rho, -)$, and P_{i_r} has weak force, and δ_{i-1} is the assignment visible to player II at $(p_0, \ldots, p_{i-1})$, then $p_i = (\langle (P_{i_{r+1}}, \ldots, P_{i_s}), ()\rangle \psi, \mathcal{M}, x_{i_r} \mapsto F_{i_r}(\delta_{i-1}), \rho, -)$.

The notions of strategy and using a strategy can be defined dually for player I. A strategy is *winning* for player j if for all sequences of moves by his or her adversary, it yields moves so that the resulting play is won by player j.

Similarly to the case of local strategy functions, in the case of strategy functions we may ignore the fact that their arguments are assignments $\{(x_{i_1}, \xi_{i_1}), \ldots, (x_{i_n}, \xi_{i_n})\}$ and their values are simple assignments $\{(x_{i_{n+1}}, \xi_{i_{n+1}})\}$; we may simply treat them as functions taking tuples $(\xi_{i_1}, \ldots, \xi_{i_n})$ as their arguments and yielding the object $\xi_{i_{n+1}}$ as their value.

The effect of independence indications $/W_i$ that results from the definitions of strategy and local strategy may be summarized as follows. If $P_1 \cdots P_n\, \chi$ is a formula and $P_i = (\mathsf{Q}_i x_i / W_i)$ appears in the maximally $\rightharpoondown$-free string $P_h \cdots P_k$, let (V_i, U_i) be a partition of W_i such that quantifiers in V_i occur in the string $P_1 \cdots P_{h-1}$ (i.e., are far from P_i), while those in U_i occur in the string $P_h \cdots P_k$ (i.e., are close to P_i). The set V_i regulates *strategy functions* for P_i in metagames but does not affect the play level, while the set U_i regulates its *local strategy functions* and therefore affects the play level in metagames. If P_i is of strong existential force, the quantifiers in U_i determine the type of the local strategy functions for P_i. If, again, P_i is of weak existential force, the quantifiers in U_i play no role whatsoever. It is obvious they cannot have a role at the play level, since the move corresponding to P_i is an element of the domain if P_i is of weak force. However, we do not wish quantifiers in U_i to have a role at the strategy level either. That is, if P_i has weak existential force, P_j is a quantifier of universal force close to P_i (i.e., $h \leq j \leq k$), and $\mathsf{Q}_j x_j \in W_i$, we wish the indicated independence vis-à-vis $\mathsf{Q}_j x_j$ to be vacuous. We take this to be motivated by the behavior of L_{IF}. Locally we are interested—in any given maximally $\rightharpoondown$-free string—in strategies of one player only, in analogy with the case of L_{IF}, in connection with which, considering the truth (falsity) of a sentence, only independence indications of quantifiers with existential (universal) force matter, that is, quantifiers of only one type are interpreted in terms of strategies and restrictions imposed on them. As to the set V_i, it determines which earlier moves by the adversary are allowed as arguments of the strategy function for P_i. If P_i has weak existential force, among those earlier moves there are automatically in particular all moves that the adversary has made for the quantifiers of strong universal force earlier in the current maximally $\rightharpoondown$-free string, among which there may well be quantifiers close to P_i *in the syntactic scope* of P_i.

In order to have available the notion of (winning) strategy also in object games, we stipulate the following.

Definition 6.3 (Strategy in an object game) Let φ be an L_2-formula, and let $j \in \{I, II\}$. We say that a tuple of functions F is a *(winning) strategy* for player j in an object game $G(\varphi, \mathcal{M}, \gamma, \star)$ if F is a (winning) strategy for j in the metagame $\Gamma(\varphi, \mathcal{M}, \gamma, \star)$.

The semantic attributes we are interested in are defined as follows.

Definition 6.4 (Satisfaction, satisfaction equivalence) If $\varphi \in L_2$, $j \in \{I, II\}$, and $\star \in \{+, -\}$, we write $\mathcal{M}, \gamma \models^{\star}_{j} \varphi$, if there exists a w.s. for player j in game

$G(\varphi, \mathcal{M}, \gamma, \star)$. If $\mathcal{M}, \gamma \models^+_{II} \varphi$, we say that formula φ is *satisfied* in model $\mathcal{M}$ by assignment γ. If φ is a sentence satisfied in $\mathcal{M}$ by λ, we say that it is *true* in $\mathcal{M}$, symbolically $\mathcal{M} \models^+_{II} \varphi$. If $\mathcal{M}, \gamma \models^-_{I} \varphi$, we speak of *dissatisfaction* and *falsity*. Formulas $\varphi(x_1, \ldots, x_n)$ and $\psi(x_1, \ldots, x_n)$ are *satisfaction-equivalent* if for all suitable structures $(\mathcal{M}, \gamma)$, we have $\mathcal{M}, \gamma \models^+_{II} \varphi$ if and only if $\mathcal{M}, \gamma \models^+_{II} \psi$. If φ and ψ are sentences, we speak of *truth equivalence*. Formulas $\varphi(x_1, \ldots, x_n)$ and $\psi(x_1, \ldots, x_n)$ are *strongly equivalent* if for all suitable $(\mathcal{M}, \gamma)$, all $j \in \{I, II\}$, and all $\star \in \{+, -\}$, we have $\mathcal{M}, \gamma \models^\star_j \varphi$ if and only if $\mathcal{M}, \gamma \models^\star_j \psi$.

In Sections 7 and 8, we will study the expressivity of the logics L_1 and L_2. The notation $L \leq L'$ used when comparing the expressive powers of logics L and L' was introduced at the end of Section 1 supposing that the satisfaction relation of both logics is $\models$. When one of the logics is L_1 or L_2, it must be understood that the intended satisfaction relation for that logic is $\models^+_{II}$.

The following interconnections of the relations just defined are immediate.[2]

Fact 6.5 Let $\varphi \in L_2$ be arbitrary. Then

(a) $\mathcal{M}, \gamma \models^+_{I} \varphi$ if and only if $\mathcal{M}, \gamma \models^-_{II} {\sim}\varphi$ if and only if $\mathcal{M}, \gamma \models^+_{II} \neg\varphi$,
(b) $\mathcal{M}, \gamma \models^-_{I} \varphi$ if and only if $\mathcal{M}, \gamma \models^+_{II} {\sim}\varphi$ if and only if $\mathcal{M}, \gamma \models^-_{II} \neg\varphi$.

We say that a formula φ is *bivalent* if either $\mathcal{M}, \gamma \models^+_{II} \varphi$ or $\mathcal{M}, \gamma \models^-_{I} \varphi$, for all suitable structures $(\mathcal{M}, \gamma)$.

Observation 6.6 Not all negated formulas $\neg\,\varphi$ are bivalent. For example, let $\psi_0 := \forall x (\exists y/\{\forall x\})\, x = y$, and let $\mathcal{M}_0$ be a model with at least two elements. Then ψ_0 is neither true nor false: neither $\mathcal{M}_0 \not\models^+_{II} \psi_0$ nor $\mathcal{M}_0 \not\models^-_{I} \psi_0$. Yet $\neg\neg\psi_0$ is strongly equivalent to ψ_0, so $\neg\neg\,\psi_0$ is a formula prefixed by $\neg$ which is not bivalent. In Theorem 7.2 we will see that for all L_1-sentences φ and models $\mathcal{M}$, we have that $\neg\varphi$ is true in $\mathcal{M}$ if and only if φ is not true in $\mathcal{M}$. By what was just observed, the nontruth of $\neg\varphi$ does *not* in general imply its falsity.

Whenever $P_1 \cdots P_n\, \chi$ is an L_2-formula, let $P_1^+ \cdots P_n^+$ be the string defined as follows: $P_i^+ = P_i$ if P_i is not a quantifier; $P_i^+ = \mathsf{Q}_i x_i$ if $P_i = (\mathsf{Q}_i x_i/W_i)$ is a quantifier of weak universal force; $P_i^+ = (\mathsf{Q}_i x_i/U_i')$ if $P_i = (\mathsf{Q}_i x_i/V_i \cup U_i)$ is a quantifier of strong universal force and $U_i' = \{\mathsf{Q} : \mathsf{Q} \in U_i$ and Q is of existential force$\}$; $P_i^+ = (\mathsf{Q}_i x_i/V_i')$ if $P_i = (\mathsf{Q}_i x_i/V_i \cup U_i)$ is of weak existential force and $V_i' = \{\mathsf{Q} : \mathsf{Q} \in V_i$ and Q is of universal force$\}$; and $P_i^+ = (\mathsf{Q}_i x_i/W_i')$ if $P_i = (\mathsf{Q}_i x_i/W_i)$ is of strong existential force and $W_i' = \{\mathsf{Q} : \mathsf{Q} \in W_i$ and Q is of universal force$\}$; here the partition (V_i, U_i) of W_i is defined as explained above. We define the string $P_1^- \cdots P_n^-$ in a dual fashion. The following fact is a direct consequence of the semantics.

Fact 6.7 A formula $P_1 \cdots P_n\, \chi$ of L_2 is satisfied (resp., dissatisfied) in $\mathcal{M}$ by γ if and only if $P_1^+ \cdots P_n^+\, \chi$ is satisfied (resp., $P_1^- \cdots P_n^-\, \chi$ is dissatisfied) in $\mathcal{M}$ by γ.

If we write L_1' for the fragment of logic L_1 whose formulas do not use the connective $\neg$, then $L_1' = \Sigma^1_1$. For, L_1' coincides not only syntactically but also semantically with L^{pr}_{IF}, and by well-known results this fragment coincides with Σ^1_1 (see [8]). Before proceeding to further expressivity issues, let us consider some examples. Below, when no confusion is likely, we allow using in the syntax variables such as x, y, z, and so on, requiring, however, that the formulas be regular in the sense mentioned in the beginning of this paper.

Example 6.8 (Logic L_1) We claim that the L_1-sentence

$$\forall t \rightharpoondown \forall x \forall y \big(\exists z/\{\forall y\}\big)\big(\exists v/\{\forall x\}\big) \rightharpoondown \big([x = y \leftrightarrow z = v] \ \rightarrow \ t = z\big),$$

call it φ, is true in a model $\mathcal{M}$ if and only if the domain M is finite. Its truth condition can be expressed by the Π^1_1-sentence $\forall t\, \forall \mathsf{f}\, \forall \mathsf{g}\, \exists x\, \exists y\, ([x = y \subset\supset \mathsf{f}(x) = \mathsf{g}(y)] \supset t = \mathsf{f}(x))$. What this sentence states is that if $t \in M$, $f \in M^M$ and $g \in M^M$ are arbitrary, either $f \neq g$ (in which case there are $a, b \in M$ so that $a = b$ and $f(a) \neq g(b)$) or $f = g$ but f is not injective (and so there are $a, b \in M$ such that $a \neq b$ and $f(a) = g(b)$) or else $f = g$ and f is injective and there is $a \in M$ so that $t = f(a)$. That is, the Π^1_1-sentence states that every injective function of type $M \to M$ is surjective, that is, that the domain M is finite. Let us check directly in terms of the semantics of L_1 that φ expresses the finiteness of the domain.

First suppose that M is finite. Define a strategy $(F_{\forall x}, F_{\forall y})$ for player *II* in metagame $\Gamma(\varphi, \mathcal{M}, +)$ as follows. If a, f, and g are the choices of player *I* for $\forall t$, $(\exists z/\{\forall y\})$, respectively, $(\exists v/\{\forall x\})$, let $F_{\forall x}(a, f, g) = F_{\forall y}(a, f, g) = b$ if $f \neq g$, where b satisfies $f(b) \neq g(b)$; further let $F_{\forall x}(a, f, g) = b'$ and $F_{\forall y}(a, f, g) = b''$ if $f = g$ but f is not injective, where b', b'' with $b' \neq b''$ satisfy $f(b') = g(b'')$; finally, if $f = g$ and f is injective, choose the value of $F_{\forall y}(a, f, g)$ in an arbitrary fashion, and let $F_{\forall x}(a, f, g) = e$ for the uniquely determined e such that $f(e) = a$. (Such a value e exists because M is finite.) Clearly the strategy $(F_{\forall x}, F_{\forall y})$ is winning for *II*. Note that the values of the strategy functions $F_{\forall x}$ and $F_{\forall y}$ depend on function choices of player *I*, and if $f = g$ and f is injective, the value also depends on the element choice of player *I*. Second, suppose $\mathcal{M} \models^+_{II} \varphi$, and let $(F_{\forall x}, F_{\forall y})$ be a w.s. for *II* in metagame $\Gamma(\varphi, \mathcal{M}, +)$. We show M to be finite. Let $f : M \to M$ be injective; let $a \in M$ be arbitrary. We must show that there is $b \in M$ such that $a = f(b)$. Consider any play of the metagame where the choices of player *I* are, in the order in which they are made, a, f, and f. Letting $b := F_{\forall x}(a, f, f)$, we have $a = f(b)$, since $(F_{\forall x}, F_{\forall y})$ is a winning strategy.

Example 6.9 (Logic L_2) Let $\psi(A, B, C)$ be the L_2-sentence

$$\begin{aligned}&\forall t \rightharpoondown \forall x \forall x' \big(\exists y/\{\forall x'\}\big)\big(\exists y'/\{\forall x\}\big) \rightharpoondown \big(\exists v/\{\forall t, \exists y'\}\big)\big(\exists v'/\{\forall t, \exists y\}\big)\\ &\quad \big(\big[A(t) \wedge B(y) \wedge B(y')\big] \to \big[x = x' \wedge A(x) \wedge C(v) \wedge C(v')\\ &\quad \wedge \big([y \neq y' \wedge v \neq v'] \vee [x = t \wedge y = y' \wedge v = v']\big)\big]\big).\end{aligned}$$

We claim that whenever $\mathcal{M}$ is a (finite or infinite) model with each of the sets $A^{\mathcal{M}}$, $B^{\mathcal{M}}$, $C^{\mathcal{M}}$ nonempty, we have $\mathcal{M} \models^+_{II} \psi(A, B, C)$ if and only if $|B^{\mathcal{M}}|^{|A^{\mathcal{M}}|} \leq |C^{\mathcal{M}}|$.

Left to right. Let $(F_{\forall x}, F_{\forall x'}, F_{\exists v}, F_{\exists v'})$ be a w.s. for player *II* in metagame $\Gamma(\psi(A, B, C), \mathcal{M}, +)$. Thus, there are functionals S and S' of type $M^M \to C^{\mathcal{M}}$ with $S = F_{\exists v}$ and $S' = F_{\exists v'}$ such that for all maps f and f' of type $M \to B^{\mathcal{M}}$, either there is an element $b = F_{\forall x} = F_{\forall x'}$ in $A^{\mathcal{M}}$ such that $f(b) \neq f'(b)$ and $S(f) \neq S'(f')$, or else for all $c \in A^{\mathcal{M}}$ we have $f(c) = f'(c)$ and $S(f) = S'(f')$. Fix an element $d \in C^{\mathcal{M}}$, and for all $g : A^{\mathcal{M}} \to B^{\mathcal{M}}$, let h_g be the map of type $M \to B^{\mathcal{M}}$ satisfying $h_g(a) = g(a)$ if $a \in A^{\mathcal{M}}$, and $h_g(a) = d$ otherwise. Define maps T and T' of type $(B^{\mathcal{M}})^{A^{\mathcal{M}}} \to C^{\mathcal{M}}$ by setting $T(g) = S(h_g)$ and $T'(g) = S'(h_g)$ for all maps $g : A^{\mathcal{M}} \to B^{\mathcal{M}}$. By the properties of S and S', $T = T'$ and this function is injective.

Right to left. Suppose that there is an injective function $T : (B^{\mathcal{M}})^{A^{\mathcal{M}}} \to C^{\mathcal{M}}$. Define a strategy $(F_{\forall x}, F_{\forall x'}, F_{\exists v}, F_{\exists v'})$ for player *II* in metagame $\Gamma(\psi(A, B, C), \mathcal{M}, +)$ as follows. Suppose that $a \in M$, $f \in M^M$, and $f' \in M^M$ are the choices of player *I* for $\forall t$, $(\exists y/\{\forall x'\})$, respectively, $(\exists y'/\{\forall x\})$. Let $g = f|_{A^{\mathcal{M}}}$ and $g' = f'|_{A^{\mathcal{M}}}$. We may suppose that the images of these maps are contained in $B^{\mathcal{M}}$, for if they are not, there is $c \in M$ with $(g \cup g')(c) \notin B^{\mathcal{M}}$ such that by choosing c as the common value of x and x' player *II* wins the resulting play, no matter which moves are made for $\forall t$, $\exists v$, and $\exists v'$. Likewise we may suppose $a \in A^{\mathcal{M}}$. Now, we put $F_{\exists v} = F_{\exists v'} = S$, where $S : M^{B^{\mathcal{M}}} \to M$ is a functional defined as follows: $S(h) = T(h|_{A^{\mathcal{M}}})$, for all $h \in M^{B^{\mathcal{M}}}$. Further, (1) if $g = g'$, we set $F_{\forall x} = F_{\forall x'} = a$, while (2) if $g \neq g'$, we put $F_{\forall x} = F_{\forall x'} = b$, where b is a fixed element satisfying $g(b) \neq g'(b)$. Let us check that player *II* wins the resulting play in both cases. Consider case (1) first. The assignment δ induced by these moves satisfies $\delta(t) = \delta(x) = \delta(x') = a$; and $\delta(v) = S(f) = T(g) = T(g') = S(f') = \delta(v')$; and $\delta(y) = f(a) = g(a) = g'(a) = \delta(y')$ (because $a \in A^{\mathcal{M}}$). Hence in particular $\mathcal{M}, \delta \models C(v) \wedge C(v') \wedge x = t \wedge y = y' \wedge v = v'$. In case (2), the induced assignment δ satisfies the following: $\delta(x) = \delta(x') = b$; and $\delta(v) = S(f) = T(g) \neq T(g') = S(f')$ (since T is injective); and $\delta(y) = f(b) = g(b) \neq g'(b) = \delta(y')$ (because $b \in A^{\mathcal{M}}$). Hence in particular $\mathcal{M}, \delta \models C(v) \wedge C(v') \wedge A(x) \wedge y \neq y' \wedge v \neq v'$.

Below we will use the following fact.

Fact 6.10 Every formula φ of L_2 is strongly equivalent to a formula of L_2 whose prefix contains no $\sim$-sign.

Proof Let us first agree on the following notation. Suppose that W is a set of plain quantifiers. If $\mathrm{Q}x \in W$, write $W^d_{\mathrm{Q}x}$ for the result of replacing $\mathrm{Q}x$ in W by its dual; otherwise let $W^d_{\mathrm{Q}x} = W$. (The dual of $\exists x$ is $\forall x$, and vice versa.) If $(\mathrm{Q}'x'/W)$ is a slashed quantifier, $(\mathrm{Q}'x'/W)^d_{\mathrm{Q}x}$ denotes $(\mathrm{Q}'x'/W^d_{\mathrm{Q}x})$. Given a prefix $P_1 \cdots P_n$ of an L_2-formula, define a sequence of prefixes $\vec{P}_i$ recursively as follows. Let $\vec{P}_0 := P_1 \cdots P_n$. If $\vec{P}_i := O_1 \cdots O_k \sim O_{k+1} \cdots O_m$ and $O_1 \cdots O_k$ is $\sim$-free, define $\vec{P}_{i+1}$ as follows depending on the symbol O_{k+1}.

- If $O_{k+1} = \sim$, then $\vec{P}_{i+1} := O_1 \cdots O_k O_{k+2} \cdots O_m$.
- If $O_{k+1} = \neg$, then $\vec{P}_{i+1} := O_1 \cdots O_k \neg \sim O_{k+2} \cdots O_m$.
- If $O_{k+1} = (\exists x/W)$, then $\vec{P}_{i+1} := O_1 \cdots O_k (\forall x/W) \sim O^\star_{k+2} \cdots O^\star_m$, where $O^\star_j = O_j$ if O_j is one of the two negation symbols, whereas $O^\star_j = {O_j}^d_{\exists x}$ if O_j is a quantifier ($k + 2 \leq j \leq m$). Similarly, if $O_{k+1} = (\forall x/W)$, then $\vec{P}_{i+1} := O_1 \cdots O_k (\exists x/W) \sim O^\star_{k+2} \cdots O^\star_m$, where $O^\star_j = {O_j}^d_{\forall x}$ if O_j if O_j is a quantifier, else $O^\star_j = O_j$.

These rules apply until a prefix $O_1 \cdots O_r O_{r+1}$ is reached (in at most n steps) such that $O_1 \cdots O_r$ is $\sim$-free. For example, the rules transform $\sim \forall x (\exists y/\{\forall x\}) R(x, y)$ into $\exists x \sim (\exists y/\{\exists x\}) R(x, y)$, which is transformed into $\exists x (\forall y/\{\exists x\}) \sim R(x, y)$. It is not difficult to verify that $\vec{P}_i \chi$ and $\vec{P}_{i+1} \chi$ are strongly equivalent for all quantifier-free formulas χ. $\square$

7 L_1 and Second-Order Logic

We prove that our semantics interprets $\rightharpoondown$ in L_1 as classical negation, and we establish a strict upper bound to the expressivity of L_1 as a fragment of **SO**. In the present section we will restrict attention to L_1-formulas whose prefix is $\sim$-free; by Fact 6.10 we may do so without loss of generality. We will use known results about the Henkin quantifier logic L^* (see Krynicki and Mostowski [19]).

A *Henkin quantifier* $H\vec{x}\vec{y}$ is an expression of the form

$$\begin{pmatrix} \forall x_1^1 & \dots & \forall x_{n_1}^1 & \exists y_1 \\ \vdots & \ddots & \vdots & \vdots \\ \forall x_1^k & \dots & \forall x_{n_k}^k & \exists y_k \end{pmatrix},$$

where $k, n_i \geq 1$ for all $i := 1, \dots, k$ and the x_j^i and the y_l are (object-language) variables. Given a relational vocabulary τ, the syntax of the logic L^* is generated by the following grammar:

$$R(x_1, \dots, x_m) \mid x_1 = x_2 \mid \neg\psi \mid (\psi \vee \psi) \mid (\psi \wedge \psi) \mid \forall x \psi \mid \exists x \psi \mid H\vec{x}\vec{y}\,\psi,$$

where $R \in \tau$ is m-ary, x and the x_i are variables, $\vec{y}$ is a k-tuple of variables, and $\vec{x}$ is a $\sum_{i=1}^{k} n_i$-tuple of variables with $n_1, \dots, n_k \geq 1$. We require that L^*-formulas be regular: one and the same variable never occurs in nested quantifiers (in particular not in nested Henkin quantifiers). The semantics of L^* is obtained by adding to the recursive clauses defining the satisfaction relation of **FO** the following clause: $\mathcal{M}, \gamma \models H\vec{x}\vec{y}\psi$ if and only if there are functions $f_i : M^{n_i} \to M$ (with $1 \leq i \leq k$) such that for all $a_j^i \in M$ (with $1 \leq i \leq k$, $1 \leq j \leq n_i$) we have $\mathcal{M}, \delta_{\vec{f},\vec{a}} \models \psi$, where $\delta_{\vec{f},\vec{a}}$ is the extension of γ satisfying $\delta_{\vec{f},\vec{a}}(x_j^i) = a_j^i$ and $\delta_{\vec{f},\vec{a}}(y_i) = f_i(a_1^i, \dots, a_{n_i}^i)$. Note that by this semantics, $\mathcal{M}, \gamma \models \neg H\vec{x}\vec{y}\neg\psi$ means that for all functions $f_i : M^{n_i} \to M$ (with $1 \leq i \leq k$) there are elements $a_j^i \in M$ (with $1 \leq i \leq k$, $1 \leq j \leq n_i$) such that $\mathcal{M}, \delta_{\vec{f},\vec{a}} \models \psi$. Write L_+^* for the fragment of L^* consisting of formulas of the form $H\vec{x}\vec{y}\psi$, where ψ is quantifier-free. Write L_-^* for the fragment of L^* consisting of negations of L_+^*-formulas. Enderton [4] and Walkoe [22] showed that $\Sigma_1^1 = L_+^*$, whence $\Pi_1^1 = L_-^*$. Enderton [4] proved that $L^* \leq \Delta_2^1$, while M. Mostowski [20] showed that the converse does not hold.

We proceed to prove that logic L_1 can be translated into L^*: for every $\varphi \in L_1$ there is $\psi_\varphi \in L^*$ such that $\mathcal{M}, \gamma \models_{II}^{+} \varphi$ if and only if $\mathcal{M}, \gamma \models \psi_\varphi$ for all suitable structures $(\mathcal{M}, \gamma)$. Let us first agree on some notation. Suppose that $\vec{P}_{\vec{z},\vec{v}}$ is a substring of the prefix of an L_1-formula containing only quantifiers, the existential quantifiers of $\vec{P}_{\vec{z},\vec{v}}$ being $(\exists z_1/Z_1), \dots, (\exists z_m/Z_m)$ and its universal quantifiers $\forall v_1, \dots, \forall v_n$, with $\vec{z} = z_1 \cdots z_m$ and $\vec{v} = v_1 \cdots v_n$. For each $1 \leq l \leq m$, let U_l be the set of variables v_j such that the quantifier $\forall v_j$ precedes $(\exists z_l/Z_l)$ in the string $\vec{P}_{\vec{z},\vec{v}}$ but does not belong to the set Z_l. There may well be variables v_j with $v_j \in U_l \cap U_{l'}$ for distinct l, l'. For all $1 \leq l \leq m$, let $V_l := \{v_j^l : v_j \in U_l\}$. Hence by syntactic criteria the sets of variables V_l and $V_{l'}$ are disjoint whenever l, l' are distinct. For all $1 \leq j \leq n$, let $I_j := \{l : v_j \in U_l\}$. Write $\theta[\vec{P}_{\vec{z},\vec{v}}] := \bigwedge_{1 \leq j \leq n, l \in I_j} v_j = v_j^l$. Let $H[\vec{P}_{\vec{z},\vec{v}}]$ be the Henkin quantifier $H\vec{\alpha}\vec{z}$, where $\vec{\alpha} = \alpha_1^1, \dots, \alpha_{n_m}^m$ is a tuple of metavariables with $\alpha_l^l, \dots, \alpha_{n_l}^l$ standing for the variables in the set V_l ordered according to increasing subscripts. If $\chi \in L_1$ is quantifier-free, let $\chi^{\neg}$ be the result of replacing all occurrences of $\sim$ in χ by $\neg$. With each L_1-formula φ, we associate L^*-formulas φ^+ and φ^- to be termed the positive and negative translation of φ into L^*, respectively,

- $\chi^+ = \chi^{\neg}$ and $\chi^- = \neg\chi^{\neg}$ if χ is quantifier-free;
- if $\vec{P}_{\vec{z},\vec{v}}\,\vec{O}\chi$ is an L_1-formula such that $\vec{P}_{\vec{z},\vec{v}}$ is a string of quantifiers and $\vec{O}$ is either empty or begins with $\rightharpoondown$, then
  - ⋆ $(\vec{P}_{\vec{z},\vec{v}}\,\vec{O}\chi)^+ = H[\vec{P}_{\vec{z},\vec{v}}]\,\forall v_1 \cdots \forall v_n(\theta[\vec{P}_{\vec{z},\vec{v}}] \supset (\vec{O}\chi)^+)$,
  - ⋆ $(\vec{P}_{\vec{z},\vec{v}}\,\vec{O}\chi)^- = \neg H[\vec{P}_{\vec{z},\vec{v}}]\neg\exists v_1 \cdots \exists v_n(\theta[\vec{P}_{\vec{z},\vec{v}}] \wedge (\vec{O}\chi)^-)$;
- $(\rightharpoondown\varphi)^+ = \varphi^-$ and $(\rightharpoondown\varphi)^- = \varphi^+$.

Lemma 7.1 *For all $\varphi \in L_1$ and all structures $(\mathcal{M}, \gamma)$, we have $\mathcal{M}, \gamma \models^+_{II} \varphi$ if and only if $\mathcal{M}, \gamma \models \varphi^+$ and $\mathcal{M}, \gamma \models^-_{II} \sim\varphi$ if and only if $\mathcal{M}, \gamma \models \varphi^-$. In particular, then, $L_1 \leq L^*$.*

Proof We prove the claim by induction on the number of $\rightharpoondown$-signs in the prefix of an L_1-formula. For the base case of 0 $\rightharpoondown$-signs, suppose that $\chi \in L_1$ is quantifier-free. Note that syntactically $\chi \in L_{\text{FO}}$ and $\chi^{\neg} \in$ **FO** and $\chi^{\neg} \in L^*$. By the interrelations of the semantics of the logics L_1, L_{FO}, **FO**, and L^*, we have $\mathcal{M}, \gamma \models^+_{II} \chi$ if and only if χ is satisfied in $(\mathcal{M}, \gamma)$ according to the semantics of L_{FO} if and only if $\chi^{\neg}$ is satisfied in $(\mathcal{M}, \gamma)$ according to the semantics of **FO** if and only if $\chi^+ = \chi^{\neg}$ is satisfied in $(\mathcal{M}, \gamma)$ according to the semantics of L^*. Further, $\mathcal{M}, \gamma \models^-_{II} \sim\chi$ if and only if (Fact 2.1) $\mathcal{M}, \gamma \models^+_{II} \sim\chi$ if and only if $\sim\chi$ is satisfied in $(\mathcal{M}, \gamma)$ according to the semantics of L_{FO} if and only if $\chi^- = \neg\chi^{\neg}$ is satisfied in $(\mathcal{M}, \gamma)$ according to the semantics of L^*. Suppose, then, that the claim holds for formulas with n $\rightharpoondown$-signs in the prefix. Let φ be a formula with $n + 1$ $\rightharpoondown$-signs. We distinguish two cases.

(1) Suppose $\varphi = \rightharpoondown \psi$. We have $\mathcal{M}, \gamma \models^+_{II} \rightharpoondown \psi$ if and only if there is a w.s. for player II in metagame $\Gamma(\psi, \mathcal{M}, \gamma, \rho^*_0, -)$ if and only if $\mathcal{M}, \gamma \models^-_{II} \sim \psi$ if and only if (inductive hypothesis) $\mathcal{M}, \gamma \models \psi^-$, where $\psi^- = (\rightharpoondown \psi)^+$. Further, $\mathcal{M}, \gamma \models^-_{II} \sim\rightharpoondown \psi$ if and only if there is a w.s. for player II in metagame $\Gamma(\rightharpoondown \psi, \mathcal{M}, \gamma, \rho^*_0, -)$ if and only if $\mathcal{M}, \gamma \models^+_{II} \psi$ if and only if (ind. hyp.) $\mathcal{M}, \gamma \models \psi^+$, where $\psi^+ = (\rightharpoondown\psi)^-$.

(2) Suppose $\varphi = \vec{P}_{\vec{z},\vec{v}}\,\vec{O}\chi$, where $\vec{P}_{\vec{z},\vec{v}}$ is nonempty and $\vec{O}$ is either empty or begins with $\rightharpoondown$. Consider first the positive translation. Assume $\mathcal{M}, \gamma \models^+_{II} \vec{P}_{\vec{z},\vec{v}}\,\vec{O}\chi$. Thus, there is a w.s. for II in metagame $\Gamma(\vec{P}_{\vec{z},\vec{v}}\,\vec{O}\chi, \mathcal{M}, \gamma, +)$ which yields for every existential quantifier $(\exists z_i / W_i)$ in $\vec{P}_{\vec{z},\vec{v}}$ a local strategy function f_i such that against any sequence of moves $\vec{a}$ by I for the universal quantifiers in $\vec{P}_{\vec{z},\vec{v}}$, we have $\mathcal{M}, \delta \models^+_{II} \vec{O}\chi$, where δ is the extension of the assignment γ determined by $\vec{f}$ and $\vec{a}$. By (1) we have $\mathcal{M}, \delta \models (\vec{O}\chi)^+$. Given how the Henkin quantifier $H[\vec{P}_{\vec{z},\vec{v}}]$ and the formula $\theta[\vec{P}_{\vec{z},\vec{v}}]$ are defined, the local strategy function for $(\exists z_i / W_i)$ witnesses the existential quantifier $\exists z_i$ ($1 \leq i \leq m$) in $H[\vec{P}_{\vec{z},\vec{v}}]$ so that $\mathcal{M}, \gamma \models H[\vec{P}_{\vec{z},\vec{v}}]\,\forall v_1 \cdots \forall v_n(\theta[\vec{P}_{\vec{z},\vec{v}}] \supset (\vec{O}\chi)^+)$. Conversely, if $\mathcal{M}, \gamma \models H[\vec{P}_{\vec{z},\vec{v}}]\,\forall x_1 \cdots \forall v_n(\theta[\vec{P}_{\vec{z},\vec{v}}] \supset (\vec{O}\chi)^+)$, let $f_1, \ldots, f_m$ be the witnesses of $\exists z_1, \ldots, \exists z_m$ in $H[\vec{P}_{\vec{z},\vec{v}}]$ yielding for all values $a_j \in M$ of v_j with $1 \leq j \leq n$ values of $z_1, \ldots, z_m$ such that the assignment $\delta_{\vec{f},\vec{a}}$ extending γ hence obtained satisfies $\mathcal{M}, \delta_{\vec{f},\vec{a}} \models (\vec{O}\chi)^+$. By (1) there is a w.s., call it $F_{\vec{f},\vec{a}}$, for II in $\Gamma(\vec{O}\chi, \mathcal{M}, \delta_{\vec{f},\vec{a}}, +)$.

Define a strategy F' in $\Gamma(\vec{P}_{\vec{z},\vec{v}}\,\vec{O}\chi, \mathcal{M}, \gamma, +)$ as follows: choose for each quantifier $(\exists z_i/W_i)$ in the string $\vec{P}_{\vec{z},\vec{v}}$ the function f_i, and for any tuple of values $(a_1, \dots, a_n)$ chosen for the universal quantifiers $\forall v_1, \dots, \forall v_n$ in the string $\vec{P}_{\vec{z},\vec{v}}$, from the position $(\vec{O}\chi, \mathcal{M}, \delta_{\vec{f},\vec{a}}, \rho_0, +)$ onwards use the strategy $F_{\vec{f},\vec{a}}$. Clearly F' is winning for *II*.

We must still check the claim concerning the negative translation. Assuming $\mathcal{M}, \gamma \models^-_{II} \sim \vec{P}_{\vec{z},\vec{v}}\,\vec{O}\chi$, there is a w.s., call it F, for player *II* in metagame $\Gamma(\vec{P}_{\vec{z},\vec{v}}\,\vec{O}\chi, \mathcal{M}, \gamma, \rho_0^*, -)$. Against any sequence $\vec{f}$ of local strategy functions chosen by player *I* for the existential quantifiers in $\vec{P}_{\vec{z},\vec{v}}$, the strategy F determines a value a_i for each universal quantifier $\forall v_i$ in $\vec{P}_{\vec{z},\vec{v}}$ such that $\mathcal{M}, \delta \models^-_{II} \sim \vec{O}\chi$, where δ is the extension of the assignment γ determined by F and $\vec{f}$. Thus, by (1) we have $\mathcal{M}, \delta \models (\vec{O}\chi)^-$. Given how the Henkin quantifier $H[\vec{P}_{\vec{z},\vec{v}}]$ and the formula $\theta[\vec{P}_{\vec{z},\vec{v}}]$ are defined, we have $\mathcal{M}, \gamma \models \neg H[\vec{P}_{\vec{z},\vec{v}}]\neg\exists v_1 \cdots \exists v_n(\theta[\vec{P}_{\vec{z},\vec{v}}] \wedge (\vec{O}\chi)^-)$, since whenever $f_1, \dots, f_n$ are functions chosen for the existential quantifiers in $H[\vec{P}_{\vec{z},\vec{v}}]$ and $a_1, \dots, a_n$ are the values assigned by the strategy F to the universal quantifiers $\forall v_1, \dots, \forall v_n$ in $\vec{P}_{\vec{z},\vec{v}}$, each existential quantifier $\exists v_i$ is witnessed by the value a_i and each universal quantifier $\forall \alpha^r_k$ in $H[\vec{P}_{\vec{z},\vec{v}}]$ is witnessed by the value a_j if v^l_j is the variable for which the metavariable α^r_k stands. Conversely, if $\mathcal{M}, \gamma \models \neg H[\vec{P}_{\vec{z},\vec{v}}]\neg\exists v_1 \cdots \exists v_n(\theta[\vec{P}_{\vec{z},\vec{v}}] \wedge (\vec{O}\chi)^-)$, let F be a map assigning to any tuple $(f_1, \dots, f_m)$ of functions corresponding to $\exists x_1, \dots, \exists x_m$ in $H[\vec{P}_{\vec{z},\vec{v}}]$ values of the variables $\alpha^1_1, \dots, \alpha^m_{n_m}, v_1, \dots, v_n$ so that the resulting assignment $\delta_{F,\vec{f}}$ satisfies $\mathcal{M}, \delta_{F,\vec{f}} \models (\theta[\vec{P}_{\vec{z},\vec{v}}] \wedge (\vec{O}\chi)^-)$. In particular, then, by item (1) we have $\mathcal{M}, \delta_{F,\vec{f}} \models^-_{II} \sim \vec{O}\chi$; that is, there is a w.s., call it $F'_{\vec{f}}$, for player *II* in metagame $\Gamma(\vec{O}\chi, \mathcal{M}, \delta_{F,\vec{f}}, \rho_0^*, -)$. Clearly the strategy which consists of letting the map F determine the moves for the universal quantifiers in the string $\vec{P}_{\vec{z},\vec{v}}$ against the adversary's moves $\vec{f}$ for the existential quantifiers therein, and thereafter using the strategy $F'_{\vec{f}}$, is a w.s. for player *II* in metagame $\Gamma(\vec{P}_{\vec{z},\vec{v}}\,\vec{O}\chi, \mathcal{M}, \gamma, \rho_0^*, -)$. Hence, $\mathcal{M}, \gamma \models^-_{II} \sim\vec{P}_{\vec{z},\vec{v}}\vec{O}\chi$. □

We are in a position to see that in L_1 the connective $\rightharpoondown$ captures classical negation. Further, we see that L_1 is strictly less expressive than the Δ^1_2-fragment of **SO**. Note that the positive translation φ^+ of an L_1-formula φ is syntactically in Σ^1_{k+1} if the number of $\rightharpoondown$-signs in the prefix of φ equals k.

Theorem 7.2 *Let $\varphi \in L_1$.*

(a) For all suitable models $\mathcal{M}$ and assignments γ, we have $\mathcal{M}, \gamma \models^+_{II} \rightharpoondown\varphi$ if and only if $\mathcal{M}, \gamma \not\models^+_{II} \varphi$.

(b) $\Sigma^1_1 \cup \Pi^1_1 \leq L_1 < \Delta^1_2$.

Proof We begin with item (a). First we prove the following claim: $\varphi^- \equiv \neg\varphi^+$ for all $\varphi \in L_1$. If φ is quantifier-free, this is immediate: $\varphi^- = \neg\varphi^\neg = \neg\varphi^+$. Suppose inductively that the claim $\psi^- \equiv \neg\psi^+$ holds for all L_1-formulas ψ with n $\rightharpoondown$-signs in the prefix. Let φ be a formula with $n + 1$ $\rightharpoondown$-signs. (1) If $\varphi = \rightharpoondown\psi$, then $\varphi^- = (\rightharpoondown\psi)^- = \psi^+$. By the inductive hypothesis, $\psi^+ \equiv \neg\psi^-$. And

$\neg\psi^- = \neg(\neg\!\!\!\rightharpoondown \psi)^+ = \neg\varphi^+$. (2) If $\varphi = \vec{P}_{\vec{z},\vec{v}}\,\vec{O}\chi$, then $\varphi^- = (\vec{P}_{\vec{z},\vec{v}}\,\vec{O}\chi)^- = \neg H[\vec{P}_{\vec{z},\vec{v}}]\neg\exists v_1\cdots\exists v_n(\theta[\vec{P}_{\vec{z},\vec{v}}]\wedge(\vec{O}\chi)^-)$, where $H[\vec{P}_{\vec{z},\vec{v}}]$ and $\theta[\vec{P}_{\vec{z},\vec{v}}]$ are as defined in connection with the translations $(\cdot)^+$ and $(\cdot)^-$ before the statement of Lemma 7.1. The formula $\neg H[\vec{P}_{\vec{z},\vec{v}}]\neg\exists v_1\cdots\exists v_n(\theta[\vec{P}_{\vec{z},\vec{v}}]\wedge(\vec{O}\chi)^-)$ is, by the inductive hypothesis, equivalent to the formula $\neg H[\vec{P}_{\vec{z},\vec{v}}]\neg\exists v_1\cdots\exists v_n(\theta[\vec{P}_{\vec{z},\vec{v}}]\wedge\neg(\vec{O}\chi)^+)$, which again is equivalent to $\neg H[\vec{P}_{\vec{z},\vec{v}}]\forall v_1\cdots\forall v_n(\theta[\vec{P}_{\vec{z},\vec{v}}]\supset(\vec{O}\chi)^+) = \neg(\vec{P}_{\vec{z},\vec{v}}\,\vec{O}\chi)^+$. Having proven the claim, we have for an arbitrary L_1-formula φ and any structure $(\mathcal{M},\gamma)$: $\mathcal{M},\gamma\models^+_{II}\neg\!\!\!\rightharpoondown\varphi$ if and only if (Lemma 7.1) $\mathcal{M},\gamma\models(\neg\!\!\!\rightharpoondown\varphi)^+$ if and only if $\mathcal{M},\gamma\models\varphi^-$ if and only if (above claim) $\mathcal{M},\gamma\models\neg\varphi^+$ if and only if $\mathcal{M},\gamma\not\models\varphi^+$ if and only if (Lemma 7.1) $\mathcal{M},\gamma\not\models^+_{II}\varphi$.

Let us, then, move on to item (b). To see that $\Sigma^1_1\leq L_1$, by the result of Enderton and Walkoe it suffices to show that $L^*_+\leq L_1$. Given an L^*_+-formula $\varphi := H\vec{x}\vec{y}\chi$ with $\vec{x} = x^1_1\cdots x^k_{n_k}$ and $\vec{y} = y_1\cdots y_k$, let ψ_φ be the L_1-formula

$$\forall x^1_1\cdots\forall x^k_{n_k}(\exists y_1/W_1)\cdots(\exists y_k/W_k)\,\chi,$$

where $W_i = \{\forall x^1_1,\ldots,\forall x^k_{n_k}\}\setminus\{\forall x^i_1,\ldots,\forall x^i_{n_i}\}$ for all $1\leq i\leq k$. Clearly we have $\mathcal{M},\gamma\models\varphi$ if and only if $\mathcal{M},\gamma\models^+_{II}\psi_\varphi$ for all suitable structures $(\mathcal{M},\gamma)$, and we may conclude that $\Sigma^1_1\leq L_1$. It follows that also $\Pi^1_1\leq L_1$. For, suppose that ζ is a Π^1_1-formula. Since $\Sigma^1_1 = L^*_+\leq L_1$, there is a formula ξ of L_1 such that for all suitable structures $(\mathcal{M},\gamma)$ we have $\mathcal{M},\gamma\models\zeta$ if and only if $\mathcal{M},\gamma\not\models^+_{II}\xi$ if and only if (by item (a)) $\mathcal{M},\gamma\models^+_{II}\neg\!\!\!\rightharpoondown\xi$. Since $\neg\!\!\!\rightharpoondown\xi\in L_1$, we may infer that $\Pi^1_1\leq L_1$. Finally, since $L^* < \Delta^1_2$ by the results of Enderton and M. Mostowski, and $L_1\leq L^*$ by Lemma 7.1, it follows that $L_1 < \Delta^1_2$. □

8 The Place of L_2 in Type Hierarchy

8.1 The negation $\neg\!\!\!\rightharpoondown$ in L_2 While in L_1 our semantic framework assigns to $\neg\!\!\!\rightharpoondown$ the meaning of classical negation, this is not so in the larger context of L_2.

Theorem 8.1 *Let $\varphi\in L_2$. If $\mathcal{M},\gamma\models^+_{II}\neg\!\!\!\rightharpoondown\varphi$, then $\mathcal{M},\gamma\not\models^+_{II}\varphi$. However, the converse does not hold in general.*

Proof For the positive claim, note that if we had both $\mathcal{M},\gamma\models^+_{II}\neg\!\!\!\rightharpoondown\varphi$ and $\mathcal{M},\gamma\models^+_{II}\varphi$, by Fact 6.5 we would have $\mathcal{M},\gamma\models^-_{II}\sim\varphi$ and $\mathcal{M},\gamma\models^-_{I}\sim\varphi$, which is impossible: there cannot exist a w.s. for both players in metagame $\Gamma(\sim\varphi,\mathcal{M},\gamma,-)$. Consider, then, the following L_2-sentence, to be called ψ: $\exists x\neg\!\!\!\rightharpoondown(\exists y/\{\exists x\})\,x = y$. Let $\mathcal{M}$ be a model of the empty vocabulary with $|M|\geq 2$. If we show that there is no w.s. for II in either of the games $\Gamma(\psi,\mathcal{M},+)$ or $\Gamma(\sim\psi,\mathcal{M},-)$, the negative claim follows. A strategy for II in metagame $\Gamma(\psi,\mathcal{M},+)$ is an element $a\in M$ chosen for $\exists x$. If I chooses the same element a for y, player I in the role of *verifier* wins the resulting play. Thus, there is no w.s. for player II in $\Gamma(\psi,\mathcal{M},+)$. As to metagame $\Gamma(\sim\psi,\mathcal{M},-)$, a strategy for II in this game is an element $a\in M$ chosen for $\exists y$. Let b be an element of M distinct from a. The play in which I in the role of *verifier* chooses b for $\exists x$ and in which II using her or his strategy picks out a for $\exists y$ terminates, while I has the role of *falsifier* and is consequently won by I. We conclude that there is no w.s. for player II in $\Gamma(\sim\psi,\mathcal{M},-)$ either. □

By Theorem 8.1, we cannot express classical negation of each L_2-formula φ in L_2 by the formula $\neg\!\!\!\rightharpoondown\varphi$. This of course does not yet prove that the classical negation of φ could not have a translation into L_2; it only shows that the simple syntactic operation of prefixing φ by $\neg$ does not provide one. However, below we see that for its expressive power L_2 equals Σ^2_1. Given that $\Sigma^2_1 \neq \Pi^2_1$, we may then infer that the classical negation of an arbitrary L_2-formula φ is not expressible in L_2 in the first place.

8.2 Fragment Σ^2_1 of third-order logic When a model $\mathcal{M}$ is clear from the context, the elements of the domain M are termed *first-order objects*. If $n \geq 0$, let $\mathcal{F}_n$ be the set of all functions of type $M^n \to M$; its elements are *n-ary second-order objects*. Let $\mathcal{F} := \bigcup_{n<\omega}\mathcal{F}_n$. If $n, m, k \geq 0$, let $\mathfrak{F}_{n,m,k}$ be the set of all functions of type $\mathcal{F}^n \times M^m \to \mathcal{F}_k$; its elements are (n, m, k)-ary *third-order objects*. We write $\mathfrak{F} := \bigcup_{n,m,k<\omega}\mathfrak{F}_{n,m,k}$. First-order objects are a degenerate case of second-order objects, and second-order objects a degenerate case of third-order objects. In third-order logic we have, in addition to individual variables, also n-ary second-order and (n, m, k)-ary third-order function variables with n-ary second-order, respectively, (n, m, k)-ary third-order objects as values, for all $n, m, k \geq 0$. Given a set V of first-, second-, and third-order variables, the sets of first- and second-order *terms* are defined recursively as follows.

- First-order variables are first-order terms, and second-order variables are second-order terms.
- If s is an n-ary second-order term and $\mathsf{t}_1, \ldots, \mathsf{t}_n$ are first-order terms, then $\mathsf{s}(\mathsf{t}_1, \ldots, \mathsf{t}_n)$ is a first-order term.
- If F is an (n, m, k)-ary third-order variable, $\mathsf{s}_1, \ldots, \mathsf{s}_n$ are second-order terms, and $\mathsf{t}_1, \ldots, \mathsf{t}_m$ are first-order terms, then $\mathsf{F}(\mathsf{s}_1, \ldots, \mathsf{s}_n, \mathsf{t}_1, \ldots, \mathsf{t}_m)$ is a k-ary second-order term.

If τ is a relational vocabulary, V_i is a set of variables of order i (with $i := 1, 2, 3$), and $V = V_1 \cup V_2 \cup V_3$, then the syntax of third-order logic of vocabulary τ over V, denoted $\mathbf{TO}[\tau, V]$, is given by the following grammar:

$$\varphi \ ::= \ \mathbf{a} \mid \neg\mathbf{a} \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid \forall \mathsf{x}\varphi \mid \exists \mathsf{x}\varphi \mid \forall \mathsf{f}\varphi \mid \exists \mathsf{f}\varphi \mid \forall \mathsf{F}\varphi \mid \exists \mathsf{F}\varphi,$$

where $\mathbf{a}$ is either a string $R(\mathsf{t}_1, \ldots, \mathsf{t}_n)$ for some positive integer n, n-ary relation symbol $R \in \tau$, and first-order terms $\mathsf{t}_1, \ldots, \mathsf{t}_n$ over V, or else $\mathbf{a}$ is a string $\mathsf{t} = \mathsf{t}'$ for first-order terms t, t' over V; $\mathsf{x} \in V_1$; $\mathsf{f} \in V_2$; and $\mathsf{F} \in V_3$. If ξ is a variable of order i, then $\forall\xi$ and $\exists\xi$ are *quantifiers of order i*.

Given functions $\alpha : V_1 \to M$ and $\beta : V_2 \to \mathcal{F}$ and $\gamma : V_3 \to \mathfrak{F}$ such that $\beta(\mathsf{f}_i) \in \mathcal{F}_n$ if f_i is n-ary and $\gamma(\mathsf{F}_i) \in \mathfrak{F}_{n,m,k}$ if F_i is (n, m, k)-ary, the function $\alpha \cup \beta \cup \gamma$ is an *assignment* in $\mathcal{M}$. If δ is an assignment, ζ is an $\vec{a}$-ary object of order i, and ξ is an $\vec{a}$-ary variable of order i, we write $\delta(\xi/\zeta)$ for the assignment that agrees with δ except that $\delta(\xi/\zeta)(\xi) = \zeta$. The definition of the *satisfaction relation* $\mathcal{M}, \delta \models \varphi$ uses the notion of *value* u^δ of a term u under assignment δ, specified for first- and second-order terms as follows:

- $\mathsf{x}^\delta = \delta(\mathsf{x})$ and $\mathsf{f}^\delta = \delta(\mathsf{f})$,
- $\mathsf{s}(\mathsf{t}_1, \ldots, \mathsf{t}_n)^\delta = \mathsf{s}^\delta(\mathsf{t}^\delta_1, \ldots, \mathsf{t}^\delta_n)$,
- $\mathsf{F}(\mathsf{s}_1, \ldots, \mathsf{s}_n, \mathsf{t}_1, \ldots, \mathsf{t}_m)^\delta = \delta(\mathsf{F})(\mathsf{s}^\delta_1, \ldots, \mathsf{s}^\delta_n, \mathsf{t}^\delta_1, \ldots, \mathsf{t}^\delta_n)$.

In particular, we define $\mathcal{M}, \delta \models R(\mathsf{t}_1, \ldots, \mathsf{t}_n)$ if and only if $\langle \mathsf{t}^\delta_1, \ldots, \mathsf{t}^\delta_n \rangle \in R^{\mathcal{M}}$; $\mathcal{M}, \delta \models \exists \mathsf{f}\varphi$ if and only if $\mathcal{M}, \delta(\mathsf{f}/f) \models \varphi$ for some suitable $f \in \mathcal{F}$; and $\mathcal{M}, \delta \models \exists \mathsf{F}\varphi$ if and only if $\mathcal{M}, \delta(\mathsf{F}/F) \models \varphi$ for some suitable $F \in \mathfrak{F}$.

We define $\Sigma^2_1[\tau, V]$ as the fragment of $\mathbf{TO}[\tau, V]$ whose formulas are of the form $\exists \mathsf{F}_1 \cdots \exists \mathsf{F}_n \varphi$, where φ contains no third-order quantifiers. It is not difficult to see that Σ^2_1 is closed under disjunction and conjunction as well as first- and second-order existential and universal quantification. Now, Σ^2_1-formulas can be proven to admit the following *Skolem form*.

Fact 8.2 Every Σ^2_1-formula φ is equivalent to a formula ψ_φ of the form $\exists \mathsf{F}_1 \cdots \exists \mathsf{F}_n \forall \mathsf{f}_1 \cdots \forall \mathsf{f}_m \forall \mathsf{x}_1 \cdots \forall \mathsf{x}_k\, \theta$, where θ is quantifier-free, and φ and ψ_φ have the same free variables.

A straightforward way of obtaining the Skolem form is as follows. First write the formula in negation normal form. Then eliminate in the resulting formula all occurrences of first- and second-order existential quantifiers, replacing the variables each such quantifier binds by a suitable term $\mathsf{F}(\mathsf{f}_1, \ldots, \mathsf{f}_n, \mathsf{x}_1, \ldots, \mathsf{x}_m)$; the third-order variable F has the arity (n, m, k), where $k = 0$ if the term replaces a first-order variable, while if it replaces a second-order variable, k equals the arity of that variable. Finally, the resulting formula is prefixed by the relevant third-order existential quantifiers. The following specific assumptions can actually be made on formulas in Skolem form.

Fact 8.3 For every $\varphi \in \Sigma^2_1$ there is a logically equivalent $\psi \in \Sigma^2_1$ such that the following hold.

(a) For every third-order variable F in ψ, there are pairwise distinct second-order variables $\mathsf{f}_1, \ldots, \mathsf{f}_m$ and pairwise distinct first-order variables $\mathsf{x}_1, \ldots, \mathsf{x}_m$, $\mathsf{y}_1, \ldots, \mathsf{y}_k$ such that in all its occurrences, F appears in the term $\mathsf{F}(\mathsf{f}_1, \ldots, \mathsf{f}_m,$ $\mathsf{x}_1, \ldots, \mathsf{x}_n)(\mathsf{y}_1, \ldots, \mathsf{y}_k)$ with these same variables in the same order.

(b) For every second-order variable f in ψ which does not appear as an argument of a third-order variable, there are fixed first-order variables $\mathsf{y}_1, \ldots, \mathsf{y}_k$ such that in all its occurrences, f appears in the term $\mathsf{f}(\mathsf{y}_1, \ldots, \mathsf{y}_k)$ with these same variables in the same order.

(c) All third-order variables in ψ are of arity $(n, m, 0)$ for some $n, m \geq 0$.

Proof By Fact 8.2 any Σ^2_1-formula may be assumed to be of the form

$$\exists \mathsf{F}_1 \cdots \exists \mathsf{F}_p \forall \mathsf{f}_1 \cdots \forall \mathsf{f}_q \forall \mathsf{x}_1 \cdots \forall \mathsf{x}_r\, \theta,$$

with θ quantifier-free. For (a), we indicate suitable transformation rules which turn formulas in Skolem form into equivalent formulas in Skolem form. We will use $(\psi_1 \rightsquigarrow \psi_2)$ as an abbreviation of $(\mathrm{neg}(\psi_1) \vee \psi_2)$, where $\mathrm{neg}(\psi_1)$ is a negation normal form of ψ_1; recall that in third-order formulas the negation symbol $\neg$ may only appear in front of atomic formulas.

Claim (a.1) *We may assume that third-order variables* F *appear only in expressions* $\mathsf{F}(\mathsf{f}_1, \ldots, \mathsf{f}_m, \mathsf{x}_1, \ldots, \mathsf{x}_n)(\mathsf{x}_{n+1}, \ldots, \mathsf{x}_{n+k})$*, where the* f_i *are second-order and the* x_j *first-order variables. Namely, we may replace* $\forall \vec{\mathsf{g}}\, \forall \vec{\mathsf{y}}\, \theta[\mathsf{F}(\mathsf{s}_1, \ldots, \mathsf{s}_m,$ $\mathsf{t}_1, \ldots, \mathsf{t}_n)(\mathsf{t}_{n+1}, \ldots, \mathsf{t}_{n+k})]$ *by the formula*

$$\forall \vec{\mathsf{g}}\, \forall \mathsf{f}_1 \cdots \forall \mathsf{f}_m\, \forall \vec{\mathsf{y}}\, \forall \mathsf{x}_1 \cdots \forall \mathsf{x}_{n+k}$$
$$\big(\chi \rightsquigarrow \theta\big[\mathsf{F}(\mathsf{f}_1, \ldots, \mathsf{f}_n, \mathsf{x}_1, \ldots, \mathsf{x}_k)(\mathsf{x}_{n+1}, \ldots, \mathsf{x}_{n+k})\big]\big),$$

where χ *equals* $(\bigwedge_{1 \leq i \leq m} \forall \vec{z_i}\, [\mathsf{f}_i(\vec{z_i}) = \mathsf{s}_i(\vec{z_i})] \wedge \bigwedge_{1 \leq j \leq n+k} \mathsf{x}_j = \mathsf{t}_j)$*. In this formula there are still positive occurrences of first-order universal quantifiers in the* *antecedent of the implication (and therefore negative occurrences in the implication itself). However, a formula of the form* $\forall\vec{\mathsf{h}}\forall\vec{\mathsf{v}}\,(\forall \mathsf{x}[\mathsf{f}(\mathsf{x}) = \mathsf{s}(\mathsf{x})] \rightsquigarrow \psi)$ *is equivalent to the formula* $\exists \mathsf{F}\forall\vec{\mathsf{h}}\forall\vec{\mathsf{v}}\forall \mathsf{u}\,(\mathsf{u} = \mathsf{F}(\vec{\mathsf{h}},\vec{\mathsf{v}}) \supset [\mathsf{f}(\mathsf{u}) \neq \mathsf{s}(\mathsf{u}) \vee \psi])$*, where* u *is a fresh first-order variable. Similar equivalences can be used when the antecedent involves a conjunction of formulas* $\forall\vec{\mathsf{x}}_\mathsf{i}[\mathsf{f}_i(\vec{\mathsf{x}}_\mathsf{i}) = \mathsf{s}_i(\vec{\mathsf{x}}_\mathsf{i})]$*, the* $\vec{x}_i$ *being tuples of any length.*

Claim (a.2) *We may assume that if an expression of the form* $\mathsf{F}(\mathsf{f}_1,\ldots,\mathsf{f}_m,\mathsf{x}_1,\ldots,\mathsf{x}_n)(\mathsf{x}_{n+1},\ldots,\mathsf{x}_{n+k})$ *appears in* θ*, then the* f_i *are pairwise distinct second-order variables and the* x_j *are pairwise distinct first-order variables. As to second-order variables, if syntactically* $\mathsf{f}_i = \mathsf{f}_j$ *with* $i < j$*, then let* h *be a fresh second-order variable of the same arity as* f_i*. We may, first, replace* $\forall\vec{\mathsf{g}}\ \forall\vec{\mathsf{y}}\ \theta[\mathsf{F}(\mathsf{f}_1,\ldots,\mathsf{f}_m,\vec{\mathsf{x}_1})(\vec{\mathsf{x}_2})]$ *by the formula*

$$\forall\vec{\mathsf{g}}\ \forall\vec{\mathsf{y}}\ (\forall\vec{z}\ \big[\mathsf{h}(\vec{z}) = \mathsf{f}_i(\vec{z})\big] \rightsquigarrow \theta\big[\mathsf{F}(\mathsf{f}_1,\ldots,\mathsf{f}_{j-1},\mathsf{h},\mathsf{f}_{j+1},\ldots,\mathsf{f}_m,\vec{\mathsf{x}_1})(\vec{\mathsf{x}_2})\big].$$

The negative occurrences of the first-order universal quantifier can be eliminated as explained in Claim (a.1). We can get rid of repetitions of first-order variables in the expression $\mathsf{F}(\mathsf{f}_1,\ldots,\mathsf{f}_m,\mathsf{x}_1,\ldots,\mathsf{x}_n)(\mathsf{x}_{n+1},\ldots,\mathsf{x}_{n+k})$ *similarly, but without the need to introduce third-order existential quantifiers.*

Claim (a.3) *We may assume that in any two first-order terms in* θ *that contain a given third-order variable* F*, the same string of second-order variables and the same strings of first-order variables appear. For, if* $(\vec{\mathsf{f}},\vec{\mathsf{x}},\vec{\mathsf{y}})$ *and* $(\vec{\mathsf{f}'},\vec{\mathsf{x}'},\vec{\mathsf{y}'})$ *are distinct strings with* $\vec{\mathsf{f}} = \mathsf{f}_1\cdots\mathsf{f}_m$*,* $\vec{\mathsf{f}'} = \mathsf{f}'_1\cdots\mathsf{f}'_\mathsf{m}$*,* $\vec{\mathsf{x}} = \mathsf{x}_1\cdots\mathsf{x}_n$*,* $\vec{\mathsf{x}'} = \mathsf{x}'_1\cdots\mathsf{x}'_\mathsf{n}$*,* $\vec{\mathsf{y}} = \mathsf{y}_1\cdots\mathsf{y}_k$*,* $\vec{\mathsf{y}'} = \mathsf{y}'_1\cdots\mathsf{y}'_\mathsf{k}$*, we may first introduce a new third-order variable* F' *and replace* $\forall\vec{\mathsf{g}}\ \forall\vec{\mathsf{z}}\ \ \theta[\mathsf{F}(\vec{\mathsf{f}},\vec{\mathsf{x}})(\vec{\mathsf{y}}),\ \mathsf{F}(\vec{\mathsf{f}'},\vec{\mathsf{x}'})(\vec{\mathsf{y}'})]$ *by*

$$\exists\mathsf{F}'\ \forall\vec{\mathsf{g}}\ \forall\vec{\mathsf{h}}\ \forall\vec{\mathsf{h}'}\ \forall\vec{\mathsf{z}}\ \forall\vec{\mathsf{u}}\ \forall\vec{\mathsf{u}'}\ \forall\vec{\mathsf{v}}\ \forall\vec{\mathsf{v}'}\ \big(\theta\big[\mathsf{F}(\vec{\mathsf{f}},\vec{\mathsf{x}})(\vec{\mathsf{y}}),\ \mathsf{F}'(\vec{\mathsf{f}'},\vec{\mathsf{x}'})(\vec{\mathsf{y}'})\big]\ \wedge\ \chi\big),$$

where χ *equals* $[(\bigwedge_{1\le i\le m}\forall\vec{w}_i\ [\mathsf{h}_i(\vec{w}_i) = \mathsf{h}'_\mathsf{i}(\vec{w}_i)] \wedge \bigwedge_{1\le i\le n}\mathsf{u}_i = \mathsf{u}'_\mathsf{i} \wedge \bigwedge_{1\le i\le k}\mathsf{v}_i = \mathsf{v}'_\mathsf{i}) \rightsquigarrow \mathsf{F}(\vec{\mathsf{h}},\vec{\mathsf{u}})(\vec{\mathsf{v}}) = \mathsf{F}'(\vec{\mathsf{h}'},\vec{\mathsf{u}'})(\vec{\mathsf{v}'})]$ *and the arity of* h_i *equals the arity of* f_i*, the arity of* h'_i *equals the arity of* f'_i*, and the variables* h_i*,* h'_i*,* u_j*,* u'_j*,* v_l*,* v'_l *are fresh. Then we may again get rid of negative occurrences of first-order universal quantifiers in the implication as in connection with Claim (a.1).*

We have just proven item (a). Using similar equivalences we can prove item (b). For item (c), if F' is an $(m, n+k, 0)$-ary variable, formulas $\exists\mathsf{F}\forall\vec{\mathsf{f}}\,\vec{\mathsf{x}}\,\vec{\mathsf{y}}\,\theta[\mathsf{F}(\vec{\mathsf{f}},\vec{\mathsf{x}})(\vec{\mathsf{y}})]$ and $\exists\mathsf{F}'\forall\vec{\mathsf{f}}\,\vec{\mathsf{x}}\,\vec{\mathsf{y}}\,\theta[\mathsf{F}'(\vec{\mathsf{f}},\vec{\mathsf{x}},\vec{\mathsf{y}})]$ are equivalent. Namely, if F is a witness of F, then F' defined as follows is a witness of F': $F'(\vec{f},\vec{a},\vec{b}) = F(\vec{f},\vec{a})(\vec{b})$ for all suitable tuples $\vec{f}$, $\vec{a}$, and $\vec{b}$. Conversely, if F' is a witness of F', for fixed $\vec{h}$, $\vec{c}$ define first $g_{\vec{h},\vec{c}}$ as the k-ary second-order object satisfying $g_{\vec{h},\vec{c}}(\vec{b}) = F'(\vec{h},\vec{c},\vec{b})$ for all suitable $\vec{b}$; then define F as follows: $F(\vec{f},\vec{a}) = g_{\vec{f},\vec{a}}$ for all suitable $\vec{f}$, $\vec{a}$. □

8.3 From L_2 to Σ^2_1 We prove that logic L_2 can be translated into Σ^2_1.

Fact 8.4 There is a translation of L_2 into Σ^2_1.

Proof Let $\psi := P_1\cdots P_n\,\chi$ be an L_2-formula. By Fact 6.10 we may suppose without loss of generality that the prefix contains no $\sim$-sign. Further, since we are interested in the *satisfaction conditions* of ψ, by Fact 6.7 we may assume that

$P_1 \cdots P_n = P_1^+ \cdots P_n^+$. Write k for the number of $\neg$-signs in the prefix. If $k = 0$, ψ is a formula of L_{IF} and therefore trivially equivalent to a Σ_1^2-formula. If $k > 0$, there are strings $\vec{O}_i$ of quantifiers (with $0 \leq i \leq k+1$) such that $P_1 \cdots P_n = \vec{O}_1 \neg \vec{O}_2 \cdots \neg \vec{O}_k \neg \vec{O}_{k+1}$. Since any L_2-formula $\neg\neg\,\psi$ is strongly equivalent to the formula ψ, we may assume that the strings $\vec{O}_2, \ldots, \vec{O}_k$ are nonempty. By contrast, the strings $\vec{O}_1$ and $\vec{O}_{k+1}$ may be empty.

- If i is odd, write $\mathbf{A}_i$ for the set of universal quantifiers of the string $\vec{O}_i$. Let $\forall x_{s_1}, \ldots, \forall x_{s_K}$ be the list of all elements of $\bigcup_i \mathbf{A}_i$. In the relevant game, player *I* chooses an element of the domain for each $\forall x_{s_j}$.
- If i is even, write $\mathbf{B}_i$ for the set of existential quantifiers of the string $\vec{O}_i$. Player *I* chooses a local strategy function for each of these quantifiers. For every $(\exists x_j / W_j) \in \mathbf{B}_i$, introduce a term $\mathsf{f}_j(x_{i_1}, \ldots, x_{i_{a_j}})$, where f_j is an a_j-ary second-order variable and $x_{i_1}, \ldots, x_{i_{a_j}}$ are those first-order variables that are bound by a universal quantifier preceding the quantifier $(\exists x_j / W_j)$ in the string $\vec{O}_i$ but not belonging to the set W_j. Let $\mathsf{f}_{j_1}, \ldots, \mathsf{f}_{j_N}$ be the list of all second-order function variables hence introduced by quantifiers $(\exists x_j / W_j) \in \bigcup_i \mathbf{B}_i$.
- If i is even, write $\mathbf{C}_i$ for the set of universal quantifiers of the string $\vec{O}_i$. Player *II* chooses an element of the domain for each of these quantifiers. For every $(\forall x_j / W_j) \in \mathbf{C}_i$, introduce a term $\mathsf{F}_j(\mathsf{f}_{i_1}, \ldots, \mathsf{f}_{i_{n_j}}, x_{r_1}, \ldots, x_{r_{m_j}})$ satisfying the following conditions. First, $\mathsf{f}_{i_1}, \ldots, \mathsf{f}_{i_{n_j}}$ are the second-order function variables introduced for those existential quantifiers that (a) appear in a string $\vec{O}_k$ with $k \leq i$ for an even number k, and (b) do not belong to the set W_j. Note that among them there are automatically all existential quantifiers of the string $\vec{O}_i$, also those coming syntactically after $(\forall x_j / W_j)$. Second, $x_{r_1}, \ldots, x_{r_{m_j}}$ are the first-order variables bound by a universal quantifier which (a) appears in a string $\vec{O}_k$ with $k < i$ for an odd number k, but (b) does not belong to W_j. Let $\mathsf{F}_{t_1}, \ldots, \mathsf{F}_{t_H}$ be the list of all third-order function variables thus introduced by quantifiers in $\bigcup_i \mathbf{C}_i$.
- If i is odd, write $\mathbf{D}_i$ for the set of existential quantifiers of the string $\vec{O}_i$. Player *II* chooses a local strategy function for each of these quantifiers. For every $(\exists x_j / W_j) \in \mathbf{D}_i$, introduce a term $\mathsf{F}_j(\mathsf{f}_{i_1}, \ldots, \mathsf{f}_{i_{n_j}}, x_{r_1}, \ldots, x_{r_{m_j}})$ satisfying the following. First, $\mathsf{f}_{i_1}, \ldots, \mathsf{f}_{i_{n_j}}$ are the second-order function variables introduced for those existential quantifiers that (a) appear in a string $\vec{O}_k$ with $k < i$ for an even number k, and (b) do not belong to the set W_i. Second, $x_{r_1}, \ldots, x_{r_{m_j}}$ are the first-order variables that are bound by a universal quantifier which (a) appears in a string $\vec{O}_k$ with $k < i$ for an odd number k, but (b) does not belong to the set W_j. Let $\mathsf{F}_{k_1}, \ldots, \mathsf{F}_{k_M}$ be the list of all third-order function variables thus introduced by quantifiers in $\bigcup_i \mathbf{D}_i$.

Let χ^* be the result of replacing in χ every variable x_j such that either $(\forall x_j / W_j) \in \bigcup_i \mathbf{C}_i$ or $(\exists x_j / W_j) \in \bigcup_i \mathbf{D}_i$ by the corresponding term $\mathsf{F}_j(\mathsf{f}_{i_1}, \ldots, \mathsf{f}_{i_{n_j}}, \mathsf{x}_{r_1}, \ldots, \mathsf{x}_{r_{m_j}})$. Let χ^+ be the result of first replacing in χ^* every variable x_j such that $(\exists x_j / W_j) \in \bigcup_i \mathbf{B}_i$ by the term $\mathsf{f}_j(\mathsf{x}_{i_1}, \ldots, \mathsf{x}_{i_{a_j}})$ and then replacing in the resulting string all occurrences of $\sim$ by $\neg$. Let φ_ψ be the Σ_1^2-formula

$\exists \mathsf{F}_{t_1} \cdots \exists \mathsf{F}_{t_H} \exists \mathsf{F}_{k_1} \cdots \exists \mathsf{F}_{k_M} \forall \mathsf{f}_{j_1} \cdots \forall \mathsf{f}_{j_N} \forall \mathsf{x}_{s_1} \cdots \forall \mathsf{x}_{s_K} \pm \chi^*$, where $\pm\chi^* = \chi^*$ if the number k is even, and $\pm\chi^* = \neg\chi^*$ if k is odd. Now, given a model $\mathcal{M}$ and an assignment γ over the free variables of ψ, we clearly have that $(F_{t_1}, \ldots, F_{t_H}, F_{k_1}, \ldots, F_{k_M})$ is a w.s. for *II* in metagame $\Gamma(\mathcal{M}, \psi, \gamma, +)$ if and only if $\mathcal{M}, \delta_\gamma \models \forall \mathsf{f}_{j_1} \cdots \forall \mathsf{f}_{j_N} \forall \mathsf{x}_{s_1} \cdots, \forall \mathsf{x}_{s_K} \pm \chi^*$, where δ_γ is an extension of γ satisfying $\delta_\gamma(\mathsf{F}_i) = F_i$ for all $i \in \{t_1, \ldots, t_H, k_1, \ldots, k_M\}$. □

8.4 From Σ^2_1 to L_2 We prove that conversely, Σ^2_1 is translatable into L_2.

Theorem 8.5 *There is a translation of Σ^2_1 into L_2.*

Proof Let $\psi \in \Sigma^2_1$ be arbitrary. By Facts 8.2 and 8.3, ψ may be assumed to be of the form $\exists \mathsf{F}_1 \cdots \exists \mathsf{F}_n \forall \mathsf{f}_1 \cdots \forall \mathsf{f}_m \forall \mathsf{x}_1 \cdots \forall \mathsf{x}_k\, \theta$, where the first-order terms satisfy the conditions (a), (b), and (c) laid down in Fact 8.3, and θ is quantifier-free. Let φ_ψ be the L_2-formula

$$\forall x_1 \cdots \forall x_k \rightharpoondown \forall y_1 \cdots \forall y_k (\exists z_1/W_1) \cdots (\exists z_m/W_m) \rightharpoondown (\exists t_1/V_1) \cdots (\exists t_n/V_n)\ (\chi \wedge \theta^*),$$

where the sets W_i and V_l and the formulas χ and θ^* are as follows. First, for every second-order function variable f_i in θ, if $\mathsf{x}_{j_1}, \ldots, \mathsf{x}_{j_{r_i}}$ are the pairwise distinct first-order variables such that all those occurrences of f_i in θ that do not occur as arguments of a third-order variable appear in the expression $\mathsf{f}_i(\mathsf{x}_{j_1}, \ldots, \mathsf{x}_{j_{r_i}})$, then let $\mathbf{X}_i := \{\mathsf{x}_{j_1}, \ldots, \mathsf{x}_{j_{r_i}}\}$. For every third-order function variable F_i in θ, if $\mathsf{f}_{j_1}, \ldots, \mathsf{f}_{j_{s_i}}, \mathsf{x}_{j'_1}, \ldots, \mathsf{x}_{j'_{s'_i}}$ are the pairwise distinct variables such that all occurrences of F_i in θ appear in the expression $\mathsf{F}_i(\mathsf{f}_{j_1}, \ldots, \mathsf{f}_{j_{s_i}}, \mathsf{x}_{j'_1}, \ldots, \mathsf{x}_{j'_{s'_i}})$, then let $\mathbf{Y}_i := \{\mathsf{f}_{j_1}, \ldots, \mathsf{f}_{j_{s_i}}\}$ and $\mathbf{Z}_i := \{\mathsf{x}_{j'_1}, \ldots, \mathsf{x}_{j'_{s'_i}}\}$. For all $1 \leq i \leq m$ and $1 \leq l \leq n$, we set

- $W_i := \{\forall y_j : 1 \leq j \leq k \text{ and } \mathsf{x}_j \notin \mathbf{X}_i\}$;
- $V_l := \{\exists z_j : 1 \leq j \leq m \text{ and } \mathsf{f}_j \notin \mathbf{Y}_l\} \cup \{\forall x_j : 1 \leq j \leq k \text{ and } \mathsf{x}_j \notin \mathbf{Z}_l\}$;
- $\chi := \bigwedge_{1 \leq i \leq k} x_i = y_i$;
- θ^* is the result of first replacing in θ, for all $1 \leq i \leq m$ and $1 \leq l \leq n$, every occurrence of the term $\mathsf{f}_i(\mathsf{x}_{j_1}, \ldots, \mathsf{x}_{j_{r_i}})$ by the variable z_i and every occurrence of the term $\mathsf{F}_l(\mathsf{f}_{j_1}, \ldots, \mathsf{f}_{j_{s_l}}, \mathsf{x}_{j'_1}, \ldots, \mathsf{x}_{j'_{s'_l}})$ by the variable t_l, and then replacing in the resulting string all occurrences of $\neg$ by $\sim$.

Note that $(\chi \wedge \theta^*)$ is a quantifier-free formula which does not contain second- or third-order variables. Indeed, it is a quantifier-free L_{FO}-formula. Consequently φ_ψ is an L_2-formula. Let, then, $(\mathcal{M}, \gamma)$ be any suitable structure. If δ_γ is an extension of γ providing witnesses for the existential quantifiers $\exists \mathsf{F}_1, \ldots, \exists \mathsf{F}_n$ so that $\mathcal{M}, \delta_\gamma \models \forall \mathsf{f}_1 \cdots \forall \mathsf{f}_m \forall \mathsf{x}_1 \cdots \forall \mathsf{x}_k\, \theta$, then clearly the sequence $(F_{\forall y_1}, \ldots, F_{\forall y_k}, F_{\exists t_1}, \ldots, F_{\exists t_n})$ defined as follows is a w.s. for *II* in metagame $\Gamma(\mathcal{M}, \varphi_\psi, \gamma, +)$: $F_{\forall y_j}$ assigns to y_j the same value that player *I* has earlier assigned to x_j, and $F_{\exists t_i} = \delta_\gamma(\mathsf{F}_i)$. Conversely, if $F_{\exists t_1}, \ldots, F_{\exists t_n}$ are the strategy functions for the quantifiers $(\exists t_1/V_1), \ldots, (\exists t_n/V_n)$ belonging to a w.s. for *II* in $\Gamma(\mathcal{M}, \varphi_\psi, \gamma, +)$, then clearly $\mathcal{M}, \delta_\gamma \models \forall \mathsf{f}_1 \ldots \forall \mathsf{f}_m \forall \mathsf{x}_1 \ldots \forall \mathsf{x}_k\, \theta$, given that δ_γ is the extension of γ satisfying $\delta_\gamma(\mathsf{F}_i) = F_{\exists t_i}$. □

Corollary 8.6

(a) $L_2 = \Sigma^2_1$;

(b) classical negation is not expressible in L_2.

Proof Item (a) is immediate from Fact 8.4 and Theorem 8.5. Item (b) follows by the fact that Σ^2_1 is not closed under complementation. □

Extending L_{FO} by allowing independence indications as in L_{IF} yields a logic with the expressive power of Σ^1_1. Observe that by Corollary 8.6, liberalizing the slashing conventions of L_1 so as to obtain L_2 yields an even more dramatic increase in expressive power: from a logic less expressive than Δ^1_2, a logic is obtained that is more expressive than the full second-order logic.

9 Conclusion

Let us say that a semantic game is of *order* 1, if all model-related entities with which the players operate are individuals. Given a fragment X of higher-order logic, we can ask whether a logic L_X can be found whose semantic games are of order 1 and which has the same expressive power as X. It is not entirely trivial to find out whether for a given X such a logic L_X exists. It took for instance some effort to find out that the fragment Σ^2_1 of **TO** is of this kind.

We provided a game-theoretical interpretation to the connective $\neg\!\!\rightharpoondown$, which yielded to $\neg\!\!\rightharpoondown$ the meaning of classical negation relative to logic L_1. This was achieved by enriching positions of semantic games by an additional component—a mode. The import of $\neg\!\!\rightharpoondown$ on the play level is actually the same as that of $\sim$. However, through effecting a mode change, it has important repercussions at the strategy level. As mentioned in Section 1, Hintikka has repeatedly claimed that classical negation does not admit a game-theoretic interpretation. Our semantic games show that this claim does not categorically hold if relatively small modifications in game rules are allowed. Earlier a way of capturing classical negation using 3-player strategic games was found by Figueira, Gorín, and Grimson [6] (see Section 9.3 below). Hintikka could not object to our formulation of semantic games on the basis that we interpret $\neg\!\!\rightharpoondown$ as acting on the strategy level. For, this is precisely how Hintikka himself has interpreted the independence indications in L_{IF}: they regulate the players' strategies but have no bearing at the play level. By contrast, the games used by Figueira, Gorín and Grimson differ from Hintikka's semantic games in several respects.

9.1 Hintikka's subgame semantics The only way in which Hintikka has considered it possible to deal with classical negation is via the following strategy-level rule with a limited range of application: there is a w.s. for *II* in game $G(\neg\varphi, \mathcal{M}, \gamma)$ if and only if there is no w.s. for *II* in $G(\varphi, \mathcal{M}, \gamma)$. He has usually limited attention to the case where φ is a sentence, but in L_{FeIF} (see [10]) he allows any formulas subject to the condition that if $(\mathsf{Q}x/W)$ is a quantifier in the syntactic scope of an occurrence of $\neg$, then all quantifiers referred to via the set W are likewise in the syntactic scope of this occurrence of $\neg$. Thus, formulas like $\forall x \neg(\exists y/\{\forall x\})R(x, y)$ are excluded. Apart from denoting the classical negation by $\neg$ rather than by $\neg\!\!\rightharpoondown$, the L_{FeIF}-formulas in prenex form are exactly the formulas of L_1. Hintikka rules out the possibility of finding an interpretation to formulas not complying with the mentioned syntactic constraint—such as those in $L_2 \setminus L_1$. In order to conceptualize the use of the strategy-level rule for $\neg$ in L_{FeIF}, Hintikka [9], [10] resorts to the idea of semantic games with *subgames* (see Carlson and Hintikka [3], Hintikka and Kulas [11]). Hintikka or his associates have not developed this notion in full formal detail, but the basic idea is clear. In such games, the continuation of a given play may require having played an entire smaller "game" (a subgame), meaning that the players have selected corresponding strategy functions. Hence there is, for example, a w.s. for *II* in game $G(\neg\forall x\exists y\neg\exists z R(x,y,z),\mathcal{M})$ if and only if *II* lacks a w.s. in subgame $G(\forall x\exists y\neg\exists z R(x,y,z),\mathcal{M})$. There would be a w.s. for *II* in the subgame if and only if there was a map $f : M \to M$ such that for all $a \in M$, player *II* lacks a w.s. in subgame $G(\exists z R(x,y,z),\mathcal{M},x\mapsto a,y\mapsto f(a))$, that is, $\langle a, f(a), b\rangle \notin R^{\mathcal{M}}$ for all $b \in M$. It is a part of the subgame semantics idea that the moves of player j in later subgames are allowed to depend on the strategy functions chosen by j's adversary in earlier subgames. We see that a w.s. for *II* in the sample game would be a pair of functionals (F_x, F_z) such that if $f : M \to M$ is a function determining a value of y depending on a value of x, then $F_x(f)$ is a value of x, and $F_z(f)$ is a value of z, and $\langle F_x(f), f(F_x(f)), F_z(f)\rangle \in R^{\mathcal{M}}$. Using the notion of mode, we were able to formulate semantic games for L_2 which on the play level involve sequences of moves of the same kind as in plain first-order semantic games. What Hintikka achieves with subgames, we achieve with metagames, used for singling out the sorts of strategy-level attributes that we are interested in. Hintikka's subgame semantics for L_1 agrees with the semantics we have assigned to this logic. Incidentally, we saw in Theorem 7.2 that $L_1 < \Delta^1_2$. This contradicts Hintikka's argument in [10] to the effect that L_{FeIF} has the expressive power of full second-order logic. This may be contrasted with the case of *team logic* (see Väänänen [21]), the result of extending *dependence logic* with classical negation, which in fact has the expressive power of **SO** (see Kontinen and Nurmi [17]).

9.2 Flattening and Hodges's compositional semantics Hodges [12], [13] formulated a compositional semantics to L_{IF} by defining the relations $\mathcal{M} \models^t_X \varphi$ (X is a *trump* for φ in $\mathcal{M}$) and $\mathcal{M} \models^{\mathrm{cot}}_X \varphi$ (X is a *cotrump* for φ in $\mathcal{M}$) recursively on the structure of L_{IF}-subformulas φ.[3] Here X is a *set* of assignments, and φ may have arbitrary free atomic or independence variables. Dual negation is interpreted via the clause $\mathcal{M} \models^t_X {\sim}\varphi$ if and only if $\mathcal{M} \models^{\mathrm{cot}}_X \varphi$. Hodges's semantics captures the game-theoretic semantics for L_{IF}: for every formula $\varphi \in L_{\mathrm{IF}}$ (by definition without free independence variables), there is a w.s. for player *II* in $G(\varphi,\mathcal{M},\gamma)$ if and only if $\mathcal{M} \models^t_{\{\gamma\}} \varphi$. Hodges considered also an extended language having available the *flattening operator* $\downarrow$ with the following semantics:

- $\mathcal{M} \models^t_X {\downarrow}\varphi$ if and only if X is nonempty and for every $\gamma \in X$ we have $\mathcal{M} \models^t_{\{\gamma\}} \varphi$;
- $\mathcal{M} \models^{\mathrm{cot}}_X {\downarrow}\varphi$ if and only if X is nonempty and for every $\gamma \in X$ we have $\mathcal{M} \not\models^t_{\{\gamma\}} \varphi$.

The idea behind the semantics of $\downarrow$ is this. Independence indications impose uniform choices to be made relative to a multitude of possible variable assignments. Thus, the semantic effect of independence indications is void when the evaluation is relative to a single assignment. If ψ is a subformula with free independence variables, let the "flattening of ψ" be the formula ψ^f *without* free independence variables, obtained from ψ by removing from its independence indications all quantifiers $\mathsf{Q}x$ with x free. Whenever X_0 is a *singleton*, we have $\mathcal{M} \models^t_{X_0} \varphi$ if and only if $\mathcal{M} \models^t_{X_0} \varphi^f$, and $\mathcal{M} \models^{\text{cot}}_{X_0} \varphi$ if and only if $\mathcal{M} \models^{\text{cot}}_{X_0} \varphi^f$. Therefore the formulas $\downarrow\varphi$ and $\downarrow\varphi^f$ have the same trumps and the same cotrumps, and the semantic effect of applying the flattening operator $\downarrow$ is that the formula to which it is applied will be treated as if it had no free independence variables. Hodges proposed to define $\neg$ by stipulating that $\neg\varphi$ means $\sim\downarrow\varphi$.

Regarding "intuitions" on classical negation in the IF-logical setting, Hodges actually agrees with Hintikka, who blocks at the outset syntactic contexts like $\forall x\neg(\exists y/W)$ with $\forall x \in W$. Hodges allows such contexts, but his semantics takes care that independence indications crossing a classical negation sign are vacuous. For example, $\mathcal{M} \models^t \forall x\neg(\forall y/\{\forall x\})R(x, y)$ if and only if for all $a \in M$ we have $\mathcal{M} \models^t_{\{x\mapsto a\}} \sim\downarrow(\forall y/\{\forall x\})R(x, y)$, that is, $\mathcal{M} \models^{\text{cot}}_{\{x\mapsto a\}} \downarrow(\forall y/\{\forall x\})R(x, y)$, that is, $\mathcal{M} \not\models^t_{\{x\mapsto a\}} (\forall y/\{\forall x\})R(x, y)$, that is, $\mathcal{M} \not\models^t_{\{x\mapsto a\}} \forall yR(x, y)$, that is, $\mathcal{M} \models^t_{\{x\mapsto a\}} \neg\forall yR(x, y)$. Thus, $\forall x\neg(\forall y/\{\forall x\})R(x, y)$ and $\forall x\neg\forall yR(x, y)$ are truth-equivalent on Hodges's semantics. This is to be contrasted with our treatment of L_2 in which the independence indications crossing a $\rightharpoondown$-sign have a perfectly nonvacuous interpretation in the metagame setting. On our semantics $\forall x\rightharpoondown(\forall y/\{\forall x\})R(x, y)$ is actually truth-equivalent to $\exists y\forall x\rightharpoondown R(x, y)$.

9.3 Capturing classical negation game-theoretically Hodges [12] did not attempt to phrase the semantics of $\downarrow$ game-theoretically, and neither did he comment on the expressive power of the language obtained by having $\downarrow$ available in the IF-like logic he formulated. Both of these endeavors are undertaken by Figueira, Gorín, and Grimson [5], [6]. Write $L(\downarrow)$ for the set of formulas obtained from L_{IF}-formulas by allowing arbitrary occurrences of $\downarrow$ in the prefix. Observe that consequently formulas of $L(\downarrow)$ are in prenex form, they are regular, and they contain no free independence variables. It follows from [6, Theorem 2] that $L(\downarrow)$ has the same expressive power as the logic $\text{SL}(\downarrow)$ discussed by the mentioned authors, if we restrict attention to $\text{SL}(\downarrow)$-formulas whose all free variables appear in atomic subformulas. In [5] Figueira, Gorín, and Grimson associate with all $L(\downarrow)$-formulas and all models a 3-player game in strategic form; their original motivation for defining such games was to find a semantic analysis of IF-logical formulas avoiding certain problems related to renaming of variables (see Janssen [16]). If the players are *I*, *II*, and *III*, then in each *turn* both players *I* and *II* select functions reminiscent of local strategies in our metagames, whereafter player *III* carries out a certain sort of evaluation. With k nested $\downarrow$-signs, there are k turns in the correlated game. The authors show that for $L(\downarrow)$-formulas, Hodges's compositional semantics and their game semantics coincide. In [6], the authors prove, utilizing their game-theoretically formulated semantics, that $L(\downarrow) \leq \Delta^1_2$. We may note that actually $L_1 = L(\downarrow)$: for every $\varphi \in L_1$ there is $\psi_\varphi \in L(\downarrow)$ such that $\mathcal{M}, \gamma \models^+_{II} \varphi$ if and only if $\mathcal{M} \models^t_{\{\gamma\}} \psi_\varphi$; and for every $\psi \in L(\downarrow)$ there is $\varphi_\psi \in L_1$ such that $\mathcal{M} \models^t_{\{\gamma\}} \psi$ if and only if $\mathcal{M}, \gamma \models^+_{II} \varphi_\psi$. That $L_1 \leq L(\downarrow)$ follows because in L_1 the connective $\rightharpoondown$ has the force of classical negation, and when applied to a formula without free independence variables, the combination $\sim\downarrow$ likewise has the force of classical negation. By the semantics of $\sim$ and by what is observed above about flattening, any $L(\downarrow)$-formula ψ is satisfaction-equivalent to an $L(\downarrow)$-formula ψ^* in which independence indications do not cross a $\downarrow$-sign and in which $\sim$ may only occur immediately before $\downarrow$ and in the matrix formula. Replacing in ψ^* first all occurrences of $\sim\downarrow$ by $\rightharpoondown$ and then eliminating in the resulting string all occurrences of $\downarrow$, we obtain an L_1-formula satisfaction-equivalent to ψ. Therefore $L(\downarrow) \leq L_1$. This allows us to conclude, in view of Theorem 7.2, that actually $L(\downarrow) < \Delta^1_2$.

The game-based semantics of [5] is interesting as a game-theoretical analysis of Hodges's compositional semantics and more generally in relation to problems due to renaming of variables of IF-logical formulas. The games employed are relatively far removed from Hintikka's GTS: they involve more than two players, and the evaluation procedures carried out by player *III* are rather complicated. Our metagame analysis has its own complications, but the games remain 2-player games. In the context of L_1 they provide a game-theoretical analysis of classical negation and offer a way of reformulating Hintikka's subgame notion, which precisely turns on the idea that the strategy functions a player uses in later subgames may take as arguments the adversary's strategy functions in earlier subgames. When moving beyond L_1, the connective $\rightharpoondown$ no longer corresponds to classical negation. So, we have found a game-theoretical conceptualization which makes sense in the context of the whole of L_2 but captures classical negation only in L_1. Future research will help to assess the general interest of the ideas used to define logic L_2.

9.4 Negation as a model-theoretic operation Burgess [1] showed (in the equivalent context of Henkin quantifier sentences) that for any mutually incompatible L_{IF}-sentences φ and ψ there is a sentence θ such that φ and θ are truth-equivalent and so are ψ and $\sim\theta$. Kontinen and Väänänen [18] generalized this result to arbitrary formulas of dependence logic. As Burgess noted, it follows that the dual negation fails to correspond to a semantic operation on classes of models: $\sim\theta$ and $\sim\theta'$ may even be incompatible while θ and θ' are truth-equivalent. Because $\sim$ may be eliminated from prefixes of L_1-formulas, in this logic we need only model-theoretically well-behaved connectives. However, in the context of L_2 the connective $\rightharpoondown$ no longer expresses a semantic operation. If, for example, $\varphi := \exists x \rightharpoondown (\exists y/\{\exists x\}) R(x, y)$ and $\psi := \exists x \rightharpoondown \exists y R(x, y)$, then φ and ψ are truth-equivalent (the independence indication only restricts strategies of player *I*), though the sentences $\rightharpoondown\varphi$ and $\rightharpoondown\psi$ are not.

If we wanted to extend our framework so as to capture stronger fragments of higher-order logic, we might consider introducing a hierarchy of modes in the games and a hierarchy of negations $\rightharpoondown_n$ in the syntax. The negation $\rightharpoondown_n$ would be interpreted via "metagames of degree n," in which depending on the mode one of the players would select a tuple of "local strategy functions of degree n," the adversary responding by a tuple of local strategy functions of degree $n-1$. Local strategy functions of degree n would be strategy functions *simpliciter* in metagames of degree $n-1$. We conjecture that with n negations, we would capture the Σ^{n+1}_1-fragment of $(n+2)$th-order logic and therefore obtain in particular the full expressivity of $(n+1)$th-order logic; here only one of the negations (i.e., $\rightharpoondown_n$) would be model-theoretically ill-behaved. Developing this generalization is, however, left for another occasion.

Notes

1. When positions are formulated as above, the component ϵ_i keeps track of the local change, if any, in the overall variable assignment: after the initial position, ϵ_i is always

empty or a singleton. In this way we avoid encoding in a position more information than needed.

2. Unlike in L_0 (and even in L_{IF}), in L_1 and L_2 there being a w.s. for I in $G(\varphi, \mathcal{M}, \gamma, +)$ is *not* equivalent to there being a w.s. for II in $G({\sim}\varphi, \mathcal{M}, \gamma, +)$.

3. Actually Hodges formulated the semantics to slash logic (see Hodges [15]), in which quantifiers are of the form $(\mathsf{Q}x/V)$, where V a set of variables (not quantifiers) and variables in V may be "bound" by quantifiers of either force. His semantics can easily be adapted to L_{IF}. It suffices to redefine the equivalence relation among assignments that is used to formulate the semantic clause for slashed existential quantifiers.

References

[1] Burgess, J. P., "A remark on Henkin sentences and their contraries," *Notre Dame Journal of Formal Logic*, vol. 44 (2003), pp. 185–88. Zbl 1071.03023. MR 2130790. DOI 10.1305/ndjfl/1091030856. 496

[2] Caicedo, X., F. Dechesne, and T. M. V. Janssen, "Equivalence and quantifier rules for logic with imperfect information," *Logic Journal of the IGPL*, vol. 17 (2009), pp. 91–129. Zbl 1160.03007. MR 2472294. DOI 10.1093/jigpal/jzn030. 469

[3] Carlson, L. and J. Hintikka, "Conditionals, generic quantifiers, and other applications of subgames" pp. 179–214 in *Game-Theoretical Semantics*, edited by E. Saarinen, vol. 5 of *Synthese Language Library*, Reidel, Dordrecht, Netherlands, 1978. 494

[4] Enderton, H. B., "Finite partially-ordered quantifiers," *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, vol. 16 (1970), pp. 393–97. Zbl 0193.29405. MR 0284317. 484

[5] Figueira, S., D. Gorín, and R. Grimson, "On the formal semantics of IF-like logics," *Journal of Computer and System Sciences*, vol. 76 (2010), pp. 333–46. Zbl 1197.03028. MR 2664354. DOI 10.1016/j.jcss.2009.10.006. 495, 496

[6] Figueira, S., D. Gorín, and R. Grimson, "On the expressive power of IF-logic with classical negation," pp. 135–45 in *Logic, Language, Information and Computation*, edited by L. Beklemishev and R. de Queiroz, Springer, Berlin, 2011. 493, 495

[7] Hintikka, J., "Language-games for quantifiers" pp. 46–72 in *Studies in Logical Theory*, edited by N. Rescher, Basil Blackwell, Oxford, 1968. 469

[8] Hintikka, J., *The Principles of Mathematics Revisited*, with an appendix by G. Sandu, Cambridge Univ. Press, Cambridge, 1996. MR 1410063. DOI 10.1017/CBO9780511624919. 470, 473, 481

[9] Hintikka, J., "Negation in logic and in natural language," *Linguistics and Philosophy*, vol. 25 (2002), pp. 585–600. 470, 493

[10] Hintikka, J., "Truth, negation and other basic notions of logic" pp. 195–219 in *The Age of Alternative Logics: Assessing Philosophy of Logic and Mathematics Today*, edited by J. van Benthem, et al., Springer, Berlin, 2006. 470, 473, 493, 494

[11] Hintikka, J., and J. Kulas, *The Game of Language: Studies in Game-Theoretical Semantics and Its Applications*, vol. 22 of *Synthese Language Library*, D. Reidel, Dordrecht, Netherlands, 1983. MR 0729683. DOI 10.1007/978-94-010-9847-2. 494

[12] Hodges, W., "Compositional semantics for a language of imperfect information," *Logic Journal of the IGPL*, vol. 5 (1997), pp. 539–63. Zbl 0945.03034. MR 1465612. DOI 10.1093/jigpal/5.4.539. 494, 495

[13] Hodges, W., "Some strange quantifiers," pp. 51–65 in *Structures in Logic and Computer Science*, edited by J. Mycielski, G. Rozenberg, and A. Salomaa, vol. 1261

of *Lecture Notes in Computer Science*, Springer, Berlin, 1997. MR 1638352. DOI 10.1007/3-540-63246-8_4. 494

[14] Hodges, W., "Elementary predicate logic" pp. 1–129 in *Handbook of Philosophical Logic, Vol. 1*, edited by D. M. Gabbay and F. Guenthner, Kluwer, Dordrecht, Netherlands, 2001. MR 1884623. 470

[15] Hodges, W., "Logics of imperfect information: Why sets of assignments?" pp. 117–33 in *Interactive Logic*, edited by J. van Benthem, B. Löwe, and D. Gabbay, vol. 1 of *Texts in Logic and Games*, Amsterdam Univ. Press, Amsterdam, 2007. MR 2508176. 497

[16] Janssen, T. M. V., "Independent choices and the interpretation of IF logic: Logic and games," *Journal of Logic, Language and Information*, vol. 11 (2002), pp. 367–87. Zbl 1003.03025. MR 1909620. DOI 10.1023/A:1015542413718. 495

[17] Kontinen, J., and V. Nurmi, "Team logic and second-order logic," *Fundamenta Informaticae*, vol. 106 (2011), pp. 259–72. Zbl 1250.03048. MR 2810064. 494

[18] Kontinen, J., and J. Väänänen, "A remark on negation in dependence logic," *Notre Dame Journal of Formal Logic*, vol. 52 (2011), pp. 55–65. Zbl 1216.03048. MR 2747162. DOI 10.1215/00294527-2010-036. 496

[19] Krynicki, M., "Henkin quantifiers" pp. 193–262 in *Quantifiers: Logics, Models and Computation, Vol. 1*, edited by M. Krynicki, M. Mostowski, and L. Szczerba, Kluwer, Dordrecht, Netherlands, 1995. 484

[20] Mostowski, M., "Arithmetic with the Henkin Quantifier and its Generalizations," pp. 1–25 in *Séminaire du Laboratoire Logique, Algorithmique et Informatique Clermontois, Vol. 2*, edited by F. Gaillard and D. Richard, Institut universitaire de technologie de Clermont-Ferrand, Aubière, 1991. 484

[21] Väänänen, J., *Dependence Logic: A New Approach to Independence Friendly Logic*, vol. 70 of *London Mathematical Society Student Texts*, Cambridge Univ. Press, Cambridge, 2007. MR 2351449. DOI 10.1017/CBO9780511611193. 494

[22] Walkoe, W., "Finite partially-ordered quantification," *Journal of Symbolic Logic*, vol. 35 (1970), pp. 535–55. Zbl 0219.02008. MR 0278920. 484

Acknowledgment

I wish to express my gratitude to the anonymous referees whose comments and criticism helped to improve the paper.

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