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A General Concept of Being a Part of a Whole

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Abstract The transitivity of the relation of part to whole is often questioned. But it is among the most basic principles of mereology. In this paper we present a general solution to the problem of transitivity of parthood which may be satisfactory for both its advocates and its opponents.

We will show that even without the transitivity of parthood one can define—basic in mereology—the notion of being a mereological sum of some objects. We formulate several proposals of general approaches to the concept of being a part of a whole, none of which contains any existential assumptions. By adding the transitivity of parthood we obtain an axiomatization of "existentially neutral" (or "nonexistential") mereology.[1]

1 Philosophical Introduction and Preliminaries

Mereology arose as a theory of collective sets (or mereological sums). It was formulated by the Polish logician Stanisław Leśniewski [2]. Collective sets are certain wholes composed of parts. In general, the concept of a collective set can be defined with the help of the relation of part to whole, and mereology may therefore be considered as a theory of "the relation of part to whole" (from the Greek: *meros*—part).

In everyday speech, the expression "part" is usually understood as having the sense of the expressions "fragment," "piece," "bit," and so forth. Under each such interpretation, the relation of part to whole has two properties.

(a) No object is its own part.
(b) There are no two objects such that the first could be part of the second and the second part of the first.

Thanks to (a), we have no difficulty in interpreting the phrase "two objects" in (b). One can see that it concerns "two different" objects. The sentences (a) and (b) state,

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respectively, that the relation of part to whole is *irreflexive* and *antisymmetric*:

$$\nexists_x x \text{ is part of } x, \tag{irr$_{\mathrm{PP}}$}$$

$$\nexists_{x,y}(x \neq y \wedge x \text{ is part of } y \wedge y \text{ is part of } x). \tag{antis$_{\mathrm{PP}}$}$$

The conjunction of (irr$_{\mathrm{PP}}$) and (antis$_{\mathrm{PP}}$) is logically equivalent to the sentence below:

$$\nexists_{x,y}(x \text{ is part of } y \wedge y \text{ is part of } x). \tag{as$_{\mathrm{PP}}$}$$

The sentence (as$_{\mathrm{PP}}$) states that the relation of part to whole is *asymmetric*. It is a known result that a relation is asymmetric if and only if it is irreflexive and antisymmetric.

In general, the relation of part to whole is *acyclic* in the following sense:

$$\nexists_{x,y_1,\dots,y_n}(x \text{ is part of } y_1 \wedge y_1 \text{ is part of } y_2 \wedge \cdots \wedge y_n \text{ is part of } x). \tag{ac$_{\mathrm{PP}}$}$$

Notice that from (ac$_{\mathrm{PP}}$) we obtain (as$_{\mathrm{PP}}$) and consequently also (irr$_{\mathrm{PP}}$) and (antis$_{\mathrm{PP}}$).

In the literature concerning mereology, the phrase "proper part" is often used instead of the expression "part" we have so far been using. The practice has become established of using "part" in a new sense, in which it embraces any given object together with its parts in the everyday sense of that word. Each part of an object distinct from the object itself is called a *proper part*. We shall also be using this terminology. With this new meaning, the expression "part" meets a condition contrary to (irr$_{\mathrm{PP}}$), for from the definition it follows directly that *every object is its own* (*improper*) *part*. We obtain moreover that *no object is its own proper part*. If, however, one understands the phrase "two objects" in the sense of "two different objects," then "part" understood in this way is antisymmetric.

Leśniewski [2] accepted that the relation of a proper part to whole is asymmetric, (as$_{\mathrm{PP}}$), and transitive, that is, that any (proper) part of a (proper) part of a given object is also its (proper) part. Formally:

$$\begin{aligned}\forall_{x,y,z}(&x \text{ is a proper part of } y \wedge y \text{ is a proper part of } z\\ &\Longrightarrow x \text{ is a proper part of } z).\end{aligned} \tag{t$_{\mathrm{PP}}$}$$

Notice that (irr$_{\mathrm{PP}}$) and (t$_{\mathrm{PP}}$) entail (ac$_{\mathrm{PP}}$) and consequently also (as$_{\mathrm{PP}}$). Of course, by (t$_{\mathrm{PP}}$), we obtain as well that the relation of part to whole is reflexive, antisymmetric, and transitive:

$$\forall_{x,y,z}(x \text{ is part of } y \wedge y \text{ is part of } z \Longrightarrow x \text{ is part of } z). \tag{t$_{\mathrm{P}}$}$$

In support of the property (t$_{\mathrm{PP}}$) the following example was given: my left arm is a proper part of my body, which entails that my left hand is also a proper part of my body. Rescher [6, p. 10] shows however that in the general case, the transitivity of the relation of part to whole is essentially problematic. He provides the following counterexample: a nucleus is a proper part of a cell, a cell is a proper part of an organ, but this nucleus is not a proper part of this organ.

Yet Simons [7, pp. 107–8] shows that the concept of part with transitivity corresponds to the spatiotemporal inclusion and in that sense it is true that a nucleus is a proper part of an organ. Simons states that the fact that the word "part" has an additional meaning does not undermine the mereological concept of part, because it is not being claimed that the mereological concept includes all the meanings of the word "part" but rather those that are fundamental and of greatest importance. He says that the transitivity of the relation of being part of causes no special difficulties when we refer to spatiotemporal relations, including those between events.

Various interesting opinions concerning the problem of the transitivity of parthood are presented, among others, in Johansson [1], Lyons [3], Rescher [6], Varzi [9], Vieu [10].

In the sequel, let U be any nonempty (distributive) set of objects (*universe of discourse*). In general, for any binary relation R in U, we shall agree that instead of $\langle x, y\rangle \in R$ (resp., $\langle x, y\rangle \notin R$) we will write $x\ R\ y$ for short (resp., $x \not R\ y$). We will also write $x\ R\ y\ R\ z$ instead of $x\ R\ y\ \wedge\ y\ R\ z$. The converse of R is the relation $\breve{R} := \{\langle x, y\rangle \in U \times U : y\ R\ x\}$. Moreover, for any subset X of U, let $R|_X$ be the restriction of the relation R to the set X; that is, $R|_X := R \cap (X \times X)$.

We say that R is, respectively, *reflexive*, *irreflexive*, *symmetric*, *antisymmetric*, *asymmetric*, *transitive*, and *acyclic* if and only if R satisfies, respectively,

$$\forall_{x\in U}\, x\ R\ x, \tag{r_R}$$

$$\nexists_{x\in U}\, x\ R\ x, \tag{irr_R}$$

$$\forall_{x,y\in U}(x\ R\ y \Longrightarrow y\ R\ x), \tag{s_R}$$

$$\nexists_{x,y\in U}(x \neq y\ \wedge\ x\ R\ y\ R\ x), \tag{anti_R}$$

$$\nexists_{x,y\in U}\, x\ R\ y\ R\ x, \tag{as_R}$$

$$\forall_{x,y,z\in U}(x\ R\ y\ R\ z \Longrightarrow x\ R\ z), \tag{t_R}$$

$$\forall_{n>0}\nexists_{x\in U}\, x\ R^n\ x, \tag{ac_R}$$

where $R^1 = R$ and for any $n > 0$, $R^{n+1} = R^n \circ R$, that is, for any $x, y \in U$: $x\ R^{n+1}\ y$ iff $\exists_{z\in U}(x\ R^n\ z \wedge z\ R\ y)$.

We put $\mathrm{Id}_U := \{\langle x, x\rangle : x \in U\}$. Obviously, by suitable definitions we obtain the following.

Lemma 1.1 *For any binary relation R in U,*

1. *if R is irreflexive and transitive, then R is acyclic;*
2. *if R is acyclic, then R is asymmetric;*
3. *R is asymmetric if and only if R is irreflexive and antisymmetric;*
4. *$R \cup \mathrm{Id}_U$ is reflexive;*
5. *R is irreflexive if and only if $R = (R \cup \mathrm{Id}_U) \setminus \mathrm{Id}_U = R \setminus \mathrm{Id}_U$;*
6. *R is asymmetric if and only if R is irreflexive and $R \cup \mathrm{Id}_U$ is antisymmetric;*
7. *if R is asymmetric, then $R = (R \cup \mathrm{Id}_U) \setminus (\breve{R} \cup \mathrm{Id}_U)$, where $\breve{R} \cup \mathrm{Id}_U = \widetilde{R \cup \mathrm{Id}_U}$;*
8. *if R is transitive, then $R \cup \mathrm{Id}_U$ is transitive.*

2 The First Two Axioms

Let $\sqsubset$ be the binary relation of being a proper part of holding between objects from U:

$$\sqsubset := \big\{\langle x, y\rangle \in U \times U : x \text{ is a proper part of } y\big\}.$$

We will consider some class of frames of the form $\mathcal{U} = \langle U, \sqsubset\rangle$, where U is a nonempty set and $\sqsubset$ is a binary relation on U, which satisfies some conditions. Of course, the relation $\sqsubset$ is to represent the relation of being a proper part in the set U.[2]

In this paper we will not be assuming that the relation $\sqsubset$ is transitive. Therefore we must assume the following (see Lemma 1.1).

(A1) **The first axiom**: the relation $\sqsubset$ is acyclic.

From this axiom we obtain the conditions $(\mathrm{ac}_\sqsubset)$, $(\mathrm{as}_\sqsubset)$, $(\mathrm{anti}_\sqsubset)$, and $(\mathrm{irr}_\sqsubset)$.

For any $x, y \in U$, a *path from x to y* is any finite sequence $(u_0, u_1, \dots, u_n)$, such that $n > 0$, $x = u_0$, $y = u_n$, and $u_i \sqsubset u_{i+1}$, for $i = 0, \dots, n-1$. Moreover, for any $x \in U$ and $S \in 2^U$ a path from x to S is any path from x to some member of S.

For any frame $\mathcal{U}$ which satisfies (A1) we have the following.

Fact 2.1 If $(u_0, u_1, \dots, u_n)$ is a path from x to y, then $x \neq y$. Consequently, $\mathcal{U}$ is cycle-free; that is, there is no path from a point to itself.

We assume only that the relation $\sqsubset$ is *locally transitive* in the following sense:

$$\forall_{x,y \in U}\big(x \sqsubset y \Longrightarrow \text{ for any path } (u_0, u_1, \dots, u_n) \text{ from } x \text{ to } y,\\ \sqsubset \text{ is transitive on the set } \{u_0, u_1, \dots, u_n\}\big). \tag{lt$_\sqsubset$}$$

(A2) **The second axiom**: the relation $\sqsubset$ is locally transitive.

By definitions and (A2) we obtain the following.

Lemma 2.2 *For any x, y such that $x \sqsubset y$, for any path $(u_0, u_1, \dots, u_n)$ from x to y, and for all $i, j \in \{0, \dots, n\}$, if $i < j$, then $u_i \sqsubset u_j$.*

3 Maximal Closed Transitive Sets

We assume that a frame $\mathcal{U} = \langle U, \sqsubset\rangle$ satisfies (A1) and (A2).

We say that a subset X of U is *closed with respect to* $\sqsubset$ if and only if for all $x, y \in X$, every path from x to y is included in X.[3]

We shall define the family $\mathbb{CT}_{\mathcal{U}}$ of subsets of U which are closed with respect to $\sqsubset$ and in which the relation $\sqsubset$ is transitive:

$$\mathbb{CT}_{\mathcal{U}} := \{X \in 2^U : X \text{ is closed with respect to } \sqsubset \text{ and } \sqsubset|_X \text{ is transitive}\}.$$

Let $\sqsubseteq$ be the binary relation of being part of holding between objects from U, that is, $\sqsubseteq := \sqsubset \cup \mathrm{Id}_U$; for any $x, y \in U$:

$$x \sqsubseteq y \stackrel{\text{df}}{\Longleftrightarrow} x \sqsubset y \vee x = y.$$

By the definition, $\sqsubseteq$ is reflexive. From asymmetry of $\sqsubset$ we obtain antisymmetry of $\sqsubseteq$.

In the paper we use set-theoretical notation. In this way we avoid using too many connectives and quantifiers. So for any $x \in U$ we put

$$\begin{aligned} \mathsf{PP}(x) &:= \{y \in U : y \sqsubset x\}, & \qquad \widetilde{\mathsf{PP}}(x) &:= \{y \in U : x \sqsubset y\},\\ \mathsf{P}(x) &:= \{y \in U : y \sqsubseteq x\}, & \check{\mathsf{P}}(x) &:= \{y \in U : x \sqsubseteq y\}. \end{aligned}$$

Lemma 3.1 *For any $x \in U$, the sets $\mathsf{P}(x)$ and $\check{\mathsf{P}}(x)$ belong to $\mathbb{CT}_{\mathcal{U}}$.*

Proof Let x be an arbitrary member of U. First, we show that $\sqsubset|_{\mathsf{P}(x)}$ is transitive. Let $y, z, u \in \mathsf{P}(x)$ and $y \sqsubset z \sqsubset u$. Then, by (irr$_\sqsubset$), (as$_\sqsubset$), and the assumption, the points y, z, and u are pairwise different and $y \neq x \neq z$. If $u = x$ (i.e., $y \sqsubset z \sqsubset x$), then $y \sqsubset x$, by the first assumption. If $u \neq x$ (i.e., $y \sqsubset z \sqsubset u \sqsubset x$), then $\sqsubset$ is transitive on the set $\{y, z, u, x\}$, because $y \sqsubset x$ and $\sqsubset$ is locally transitive. Hence $y \sqsubset u$.

Second, we show that $\mathsf{P}(x)$ is closed with respect to $\sqsubset$. Let $y, z \in \mathsf{P}(x)$ and $(u_0, u_1, \dots, u_m)$ be any path from y to z. Then, by (ac$_\sqsubset$), $y \neq z$. Since $y \sqsubset u_1 \sqsubset \cdots \sqsubset u_{m-1} \sqsubset z \sqsubseteq x$, so also $y \neq x$, by (ac$_\sqsubset$), (as$_\sqsubset$), or (irr$_\sqsubset$). Thus, either $(y, u_1, \dots, u_{m-1}, z, x)$ or $(y, u_1, \dots, u_{m-1}, x)$ is a path from y to x. Because $\sqsubset$ is locally transitive, so $\sqsubset$ is transitive on the set $\{y, u_1, \dots, u_{m-1}, z, x\}$. Hence, by Lemma 2.2, this set is included in $\mathsf{P}(x)$.

Similarly, we can prove that for any $x \in U$, the set $\breve{\mathsf{P}}(x)$ belongs to $\mathbb{CT}_{\mathcal{U}}$. □

In the standard way we may prove the following lemma.

Lemma 3.2 *Let C be any chain in $\mathbb{CT}_{\mathcal{U}}$ (i.e., totally ordered subset of $\mathbb{CT}_{\mathcal{U}}$ with respect to the relation of inclusion). Then $\bigcup C \in \mathbb{CT}_{\mathcal{U}}$. In other words, every chain in $\langle \mathbb{CT}_{\mathcal{U}}, \subseteq \rangle$ has an upper bound.*

Let $\mathbb{MCT}_{\mathcal{U}}$ be a subfamily of the family $\mathbb{CT}_{\mathcal{U}}$ composed of those sets which are maximal with respect to the relation of inclusion, that is,

$$\mathbb{MCT}_{\mathcal{U}} := \{X \in \mathbb{CT}_{\mathcal{U}} : \nexists_{Y \in \mathbb{CT}_{\mathcal{U}}} X \subsetneq Y\}.$$

From Lemma 3.2 and the Kuratowski–Zorn lemma we obtain the following.

Theorem 3.3 *Every set from $\mathbb{CT}_{\mathcal{U}}$ is included in some set from $\mathbb{MCT}_{\mathcal{U}}$.*

From Lemma 3.1 and Theorem 3.3 we obtain the following.

Corollary 3.4

1. *For any $x \in U$ there is $M \in \mathbb{MCT}_{\mathcal{U}}$ such that $\mathsf{P}(x) \subseteq M$.*
2. *For any $x \in U$ there is $M \in \mathbb{MCT}_{\mathcal{U}}$ such that $\breve{\mathsf{P}}(x) \subseteq M$.*
3. *For all $x, y \in U$, if $y \sqsubset x$, then there is $M \in \mathbb{MCT}_{\mathcal{U}}$ such that $x, y \in M$.*
4. *We have $\emptyset \notin \mathbb{MCT}_{\mathcal{U}} \neq \emptyset$.*

From (A1) and (A2), for each set M from $\mathbb{MCT}_{\mathcal{U}}$, the pair $\langle M, \sqsubset|_M \rangle$ is a strictly partially ordered set; that is, the relation $\sqsubset|_M$ is transitive and irreflexive (asymmetric). Moreover, M is closed with respect to $\sqsubset$.

4 The Third Axiom

For an arbitrary subset X of U we put

$$\max_{\sqsubset}(X) := \big\{x \in X : \widetilde{\mathsf{PP}}(x) \cap X = \emptyset\big\},$$
$$\min_{\sqsubset}(X) := \big\{x \in X : \mathsf{PP}(x) \cap X = \emptyset\big\}.$$

The elements of $\max_{\sqsubset}(X)$ (resp., $\min_{\sqsubset}(X)$) will be called *maximal elements* (resp., *minimal elements*) in the set X.

(A3) **The third axiom**: the family $\mathbb{MCT}_{\mathcal{U}}$ meets the following condition.
For any $M, M' \in \mathbb{MCT}_{\mathcal{U}}$, if $M \neq M'$, then either $M \cap M' \subseteq \max_{\sqsubset}(M)$ or $M \cap M' \subseteq \max_{\sqsubset}(M')$.

Remark 4.1 If we suppose that in an irreflexive frame $\mathcal{U} = \langle U, \sqsubset \rangle$ the relation $\sqsubset$ is transitive, then $\mathbb{MCT}_{\mathcal{U}} = \{U\}$, so $\mathcal{U}$ automatically fulfills the axioms (A1)–(A3) (as well as the stronger version of the third axiom given in Remark 4.13).

In the next section we will give some auxiliary facts. In Section 4.2 we will give an equivalent version of the third axiom (see Theorem 4.12). Moreover, in Section 4.3, we will give a stronger version ($A3_s$) of the third axiom. This stronger version may be more intuitive than axiom (A3). However (A3) is strong enough to prove all facts (from Section 4.1) that will be used in the rest of the paper. Finally, in Remark 4.15 we informally describe axiom (A3).

4.1 Some auxiliary facts We will use the following auxiliary definitions and facts, where we consider any frame $\mathcal{U} = \langle U, \sqsubset\rangle$ satisfying the first three axioms.

Let $\bigcirc$, $\between$, and $\wr$ be, respectively, the *overlapping relation*, the *proper overlapping relation*, and the *disjointness relation*, which hold between objects from U:

$$\begin{aligned}
x \bigcirc y &\overset{\text{df}}{\iff} \exists_{z\in U}(z \sqsubseteq x \wedge z \sqsubseteq y),\\
x \between y &\overset{\text{df}}{\iff} x \neq y \wedge x \not\sqsubset y \wedge y \not\sqsubset x \wedge \exists_{z\in U}(z \sqsubset x \wedge z \sqsubset y),\\
x \wr y &\overset{\text{df}}{\iff} \neg\, x \bigcirc y.
\end{aligned}$$

By definition, the relations $\bigcirc$, $\between$, and $\wr$ are symmetric; $\bigcirc$ is reflexive; $\between$ and $\wr$ are irreflexive. Moreover, by definition, for any $x, y \in U$ we have

$$\begin{aligned}
x \bigcirc y &\iff x = y \vee x \sqsubset y \vee y \sqsubset x \vee \exists_{z\in U}(z \sqsubset x \wedge z \sqsubset y)\\
&\iff x = y \vee x \sqsubset y \vee y \sqsubset x \vee x \between y.
\end{aligned} \tag{1}$$

So $\sqsubset \subseteq \sqsubseteq \subseteq \bigcirc$ and $\between \subseteq \bigcirc$.

Notice that by ($\mathrm{as}_\sqsubset$), ($\mathrm{irr}_\sqsubset$), and definitions for any $x, y \in U$ we obtain

$$x \not\sqsubseteq y \iff x \wr y \vee x \between y \vee y \sqsubset x. \tag{2}$$

Now let M be an arbitrary set from $\mathbb{MCT}_{\mathcal{U}}$. For any $x, y \in U$ we put

$$\begin{aligned}
x \bigcirc^M y &\overset{\text{df}}{\iff} x, y \in M \wedge \exists_{z\in M}(z \sqsubseteq x \wedge z \sqsubseteq y),\\
x \between^M y &\overset{\text{df}}{\iff} x, y \in M \wedge x \neq y \wedge x \not\sqsubset y \wedge y \not\sqsubset x \wedge \exists_{z\in M}(z \sqsubset x \wedge z \sqsubset y),\\
x \wr^M y &\overset{\text{df}}{\iff} x, y \in M \wedge \neg\, x \bigcirc^M y.
\end{aligned}$$

By definition, the relations $\bigcirc^M$, $\between^M$, and $\wr^M$ are symmetric and we have the following.

Lemma 4.2 *For any M from $\mathbb{MCT}_{\mathcal{U}}$,*

1. $\sqsubset|_M \subseteq \sqsubseteq|_M \subseteq \bigcirc^M \subseteq \bigcirc|_M$, $\between^M \subseteq \between|_M \subseteq \bigcirc|_M$, $\between^M \subseteq \bigcirc^X$ *and* $\wr|_M \subseteq \wr^M$,
2. $\bigcirc|_M \subseteq \bigcirc^M$ *iff* $\between|_M \subseteq \between^M$ *iff* $\wr^M \subseteq \wr|_M$,[4]
3. *all members of the set* $\min_\sqsubset(M)$ *are pairwise in the relation* $\wr^M$,
4. $\forall_{x,y,z\in M}(z \bigcirc^M y \wedge z \sqsubseteq x \Longrightarrow y \bigcirc^M x)$.

Proof *Ad* 4. From the transitivity of $\sqsubset|_M$ we have the transitivity of $\sqsubseteq|_M$. □

Lemma 4.3 *We have $\forall_{x,y\in U}(y \sqsubset x \Longrightarrow \exists^1_{M\in\mathbb{MCT}_{\mathcal{U}}} x, y \in M)$. Thus, for any $x, y \in U$ such that $y \sqsubset x$ we can put*[5]

$$\mathrm{M}^x_y := (\iota M) x, y \in M \in \mathbb{MCT}_{\mathcal{U}}.$$

Proof Let $y \sqsubset x$. Then, by Corollary 3.4, for some $M \in \mathbb{MCT}_{\mathcal{U}}$ we have $x, y \in \mathsf{P}(x) \subseteq M$. Let $M' \in \mathbb{MCT}_{\mathcal{U}}$ and $x, y \in M'$. We show that $M' = M$. Indeed, if $M' \neq M$, then by (A3) either $\{x, y\} \subseteq \max_\sqsubset(M)$ or $\{x, y\} \subseteq \max_\sqsubset(M')$, a contradiction. □

Lemma 4.4 *We have $\forall_{M\in\mathbb{MCT}_{\mathcal{U}}}\forall_{x,y\in M}(y \sqsubset x \Longrightarrow \mathsf{P}(x) \cup \check{\mathsf{P}}(y) \subseteq M = \mathrm{M}^x_y)$.*

Proof Let $x, y \in M \in \mathbb{MCT}_{\mathcal{U}}$ and $y \sqsubset x$. First, suppose that $z \sqsubset x$. Then, by Corollary 3.4, for some $M' \in \mathbb{MCT}_{\mathcal{U}}$: $z, x \in \mathsf{P}(x) \subseteq M'$. So also $y \in M'$. Thus, by Lemma 4.3, $M' = M$; that is, $z \in M$. Moreover, since $x, y \in M$, so $M = \mathrm{M}^x_y$.

Second, suppose that $y \sqsubset z$. Then, by Corollary 3.4, for some $M'' \in \mathbb{MCT}_{\mathcal{U}}$: $y, z \in \mathsf{P}(z) \subseteq M''$. We show that $M'' = M$; that is, $z \in M$. Indeed, if $M \neq M''$, then by (A3) either $y \in \max_\sqsubset(M)$ or $y \in \max_\sqsubset(M'')$, a contradiction. □

By Lemmas 4.3 and 4.4 we obtain the following.

Corollary 4.5 *We have* $\forall_{x\in U}(\mathsf{PP}(x) \neq \emptyset \Longrightarrow \exists^1_{M\in\mathbb{MCT}_{\mathcal{U}}}\mathsf{P}(x) \subseteq M)$. *Thus, for any* $x \in U$ *such that* $\mathsf{PP}(x) \neq \emptyset$ *we can put*

$$\mathrm{M}^x := (\iota M)\mathsf{P}(x) \subseteq M \in \mathbb{MCT}_{\mathcal{U}}.$$

Moreover, for any $y \in \mathsf{PP}(x)$ *we have* $\mathrm{M}^x_y = \mathrm{M}^x$.

Corollary 4.6 *We have* $\forall_{x,y\in U}(y \sqsubset x \Rightarrow \exists^1_{M\in\mathbb{MCT}_{\mathcal{U}}}(\mathsf{P}(x) \cup \breve{\mathsf{P}}(y) \subseteq M = \mathrm{M}^x_y = \mathrm{M}^x))$.

By Lemmas 4.3 and 4.4, and Corollaries 4.5 and 4.6 we obtain the next corollary.

Corollary 4.7 *We have* $\forall_{M\in\in\mathbb{MCT}_{\mathcal{U}}}\forall_{x\in M\smallsetminus\min_{\sqsubset}(M)} M = \mathrm{M}^x$.

Corollary 4.8 *We have the following.*

1. $\forall_{M\in\mathbb{MCT}_{\mathcal{U}}}\forall_{x\in M}(\mathsf{PP}(x) \cap M \neq \emptyset \Longrightarrow \mathsf{PP}(x) \subseteq M)$,
2. $\forall_{M\in\mathbb{MCT}_{\mathcal{U}}}\forall_{x\in M}(\mathsf{PP}(x) \nsubseteq M \Longrightarrow x \in \min_{\sqsubset}(M))$,
3. $\forall_{M\in\mathbb{MCT}_{\mathcal{U}}}\forall_{x\in M}(\breve{\mathsf{PP}}(x) \cap M \neq \emptyset \Longrightarrow \breve{\mathsf{PP}}(x) \subseteq M)$,
4. $\forall_{M\in\mathbb{MCT}_{\mathcal{U}}}\forall_{x\in M}(\breve{\mathsf{PP}}(x) \nsubseteq M \Longrightarrow x \in \max_{\sqsubset}(M))$.

Corollary 4.9 *If* $M \cap M' \subseteq \max_{\sqsubset}(M)$, *then* $(M \cap M')\smallsetminus\min_{\sqsubset}(M') \subseteq \min_{\sqsubset}(M)$.

Proof Let (a) $M \cap M' \subseteq \max_{\sqsubset}(M)$, (b) $x \in M$, (c) $x \in M'$, (d) $x \notin \min_{\sqsubset}(M')$. Then, by (c) and (d), for some $y \in M'$ we have $y \sqsubset x$. Therefore, by (a)–(c), $y \notin M$. So $x \in \min_{\sqsubset}(M)$, by Corollary 4.8(2). $\square$

Lemma 4.10 *We have* $\forall_{M\in\mathbb{MCT}_{\mathcal{U}}}\forall_{x,y\in M}(x \wr^M y \wedge x \bigcirc y \Longrightarrow x, y \in \min_{\sqsubset}(M))$.

Proof Let $x \wr^M y$ and $x \bigcirc y$. Then for some $z \in U$: $z \sqsubseteq x$, $z \sqsubseteq y$, and $z \notin M$. Hence $x \neq z \neq y$, that is, $z \sqsubset x$ and $z \sqsubset y$. Therefore $x, y \in \min_{\sqsubset}(M)$, by Corollary 4.8(2). $\square$

By Corollaries 3.4 and 4.8(1) we obtain the following.

Lemma 4.11 *We have* $\forall_{x,y\in U}(x \between y \Longrightarrow \mathrm{M}^x = \mathrm{M}^y \wedge x \between^{\mathrm{M}^x} y)$.

Proof Let $x \between y$. Then for some u: $u \sqsubset x$ and $u \sqsubset y$. Hence $u \in \mathrm{M}^x$ and $u \in \mathrm{M}^y$. We show that $\mathrm{M}^x = \mathrm{M}^y$. Indeed, if $\mathrm{M}^x \neq \mathrm{M}^y$, then, by (A3), $u \in \max_{\sqsubset}(\mathrm{M}^x)$ or $u \in \max_{\sqsubset}(\mathrm{M}^y)$, a contradiction. In consequence $x \between^{\mathrm{M}^x} y$. $\square$

4.2 An equivalent version of (A3) We can give the following equivalent version of the third axiom.

Theorem 4.12 *Let a frame* $\mathcal{U} = \langle U, \sqsubset\rangle$ *satisfy the first three axioms. Then*

(A3′) for any $M, M' \in \mathbb{MCT}_{\mathcal{U}}$, *if* $M \neq M'$, *then either* $M \cap M' \subseteq \max_{\sqsubset}(M) \cap \min_{\sqsubset}(M')$ *or* $M \cap M' \subseteq \min_{\sqsubset}(M) \cap \max_{\sqsubset}(M')$.

Since $\min_{\sqsubset}(M) \cap \max_{\sqsubset}(M') \subseteq M \cap M'$ *and* $\max_{\sqsubset}(M) \cap \min_{\sqsubset}(M') \subseteq M \cap M'$, *so in (A3′) we have* $=$ *in the place of* $\subseteq$.

Moreover, (A3′) entails (A3). Thus (A3) and (A3′) are equivalent in the class of all structures satisfying (A1) and (A2).

Proof Let $\mathcal{U} = \langle U, \sqsubset\rangle$ satisfy the first three axioms, and let $M, M' \in \mathbb{MCT}_{\mathcal{U}}$ be such that $M \neq M'$. By (A3), $M \cap M' \subseteq \max_{\sqsubset}(M)$ or $M \cap M' \subseteq \max_{\sqsubset}(M')$. So we consider the following three alternative cases:

(i) $M \cap M' \subseteq \max_{\sqsubset}(M) \cap \max_{\sqsubset}(M')$,
(ii) $M \cap M' \subseteq \max_{\sqsubset}(M)$ and $M \cap M' \nsubseteq \max_{\sqsubset}(M')$,
(iii) $M \cap M' \nsubseteq \max_{\sqsubset}(M)$ and $M \cap M' \subseteq \max_{\sqsubset}(M')$.

Ad (i) Let $M \cap M' \subseteq \max_{\sqsubset}(M) \cap \max_{\sqsubset}(M')$. We prove that $M \cap M' \subseteq \min_{\sqsubset}(M)$ or $M \cap M' \subseteq \min_{\sqsubset}(M')$.

Suppose towards contradiction that $M \cap M' \nsubseteq \min_{\sqsubset}(M')$ and $M \cap M' \nsubseteq \min_{\sqsubset}(M)$. Then there is $x_0 \in M \cap M'$ such that $x_0 \notin \min_{\sqsubset}(M)$. Hence $M = \mathrm{M}^{x_0}$, by Corollary 4.7. Moreover, there is $y_0 \in M \cap M'$ such that $y_0 \notin \min_{\sqsubset}(M')$. Hence $M' = \mathrm{M}^{y_0}$. So we have $\mathrm{M}^{x_0} \neq \mathrm{M}^{y_0}$ and $x_0 \neq y_0$. Since $x_0, y_0 \in \max_{\sqsubset}(M)$, so $x_0 \not\sqsubset y_0$ and $y_0 \not\sqsubset x_0$. Moreover, $\neg\, x_0 \between y_0$, by Lemma 4.11. Hence $x_0 \wr y_0$, by (1). From Corollary 4.9 we have $x_0 \in \min_{\sqsubset}(M')$ and $y_0 \in \min_{\sqsubset}(M)$.

Now we prove that $\mathsf{P}(x_0) \cup \mathsf{P}(y_0) \in \mathbb{CT}_{\mathcal{U}}$. Then, by Theorem 3.3, we obtain $M'' \in \mathbb{MCT}_{\mathcal{U}}$ such that $\mathsf{P}(x_0) \cup \mathsf{P}(y_0) \subseteq M''$. So, by Corollary 4.5, $M = M'' = M'$ and we obtain a contradiction.

First, we prove that $\sqsubset|_{\mathsf{P}(x_0)\cup\mathsf{P}(y_0)}$ is transitive. Let $z, u, v \in \mathsf{P}(x_0) \cup \mathsf{P}(y_0)$ and $z \sqsubset u \sqsubset v$. By ($\mathrm{ac}_{\sqsubset}$), $v \neq z \neq u \neq v$ and $v \not\sqsubset z$. We show that for any $\alpha, \beta \in \{z, u, v\}$ the following cases are not possible: (a) $\alpha \in \mathsf{P}(x_0)$ and $\beta \in \mathsf{P}(y_0)$; (b) $\beta \in \mathsf{P}(x_0)$ and $\alpha \in \mathsf{P}(y_0)$. Hence we have that for any $\alpha, \beta \in \{z, u, v\}$, either $\alpha, \beta \in \mathsf{P}(x_0)$ or $\alpha, \beta \in \mathsf{P}(y_0)$. Thus $z \sqsubset v$, because the sets $\mathsf{P}(x_0)$ and $\mathsf{P}(y_0)$ belong to $\mathbb{CT}_{\mathcal{U}}$, by Lemma 3.1.

Suppose that there are $\alpha, \beta \in \{z, u, v\}$ such that $\alpha \in \mathsf{P}(x_0)$ and $\beta \in \mathsf{P}(y_0)$. Then $\alpha \neq \beta$, $\alpha \not\sqsubseteq y_0$, and $\beta \not\sqsubseteq y_0$, because $x_0 \wr y_0$. If $\alpha = z$ and $\beta = u$ (resp., $\beta = v$), then $x_0 \sqsupseteq z \sqsubset u \sqsubseteq y_0$ (resp., $x_0 \sqsupseteq z \sqsubset u \sqsubset v \sqsubseteq y_0$). Thus we have a path from z to y_0 and $z, y_0 \in \mathrm{M}^{x_0}$. So this path is included in M^{x_0}. Hence $z \sqsubset y_0$, so we obtain a contradiction: $x_0 \bigcirc y_0$. If $\alpha = u$ and $\beta = v$, then $x_0 \sqsupseteq u \sqsubset v \sqsubseteq y_0$ and we obtain a contradiction again: $x_0 \bigcirc y_0$. If $\alpha = u$ and $\beta = z$, then $y_0 \sqsupseteq z \sqsubset u \sqsubseteq x_0$ and we have a path from z to x_0 and $z, x_0 \in \mathrm{M}^{y_0}$. So this path is included in M^{y_0}. Hence $z \sqsubset x_0$, and we obtain a contradiction again: $x_0 \bigcirc y_0$. Analogously, we show that the rest of subcases from (a) and the cases (b) are also not possible.

Second, we prove that $\mathsf{P}(x_0) \cup \mathsf{P}(y_0)$ is closed with respect to $\sqsubset$. Let $z, u \in \mathsf{P}(x_0) \cup \mathsf{P}(y_0)$ and $(u_0, u_1, \dots, u_m)$ be any path from z to u, for some $m \geqslant 1$. Then, by ($\mathrm{ac}_{\sqsubset}$), $z \neq u$. If either $z, u \in \mathsf{P}(x_0)$ or $z, u \in \mathsf{P}(y_0)$, then this path is included, respectively, in $\mathsf{P}(x_0)$ or $\mathsf{P}(y_0)$, by Lemma 3.1.

Besides, we prove that the following two cases are not possible: $z \in \mathsf{P}(x_0)$ and $u \in \mathsf{P}(y_0)$; $z \in \mathsf{P}(y_0)$ and $u \in \mathsf{P}(x_0)$. Indeed, in the first case $x_0 \sqsupseteq z \sqsubset u_1 \sqsubset \dots \sqsubset u_m \sqsubseteq y_0$. So we have a path from z to y_0 and $z, y_0 \in \mathrm{M}^{x_0}$. Thus, this path is included in M^{x_0}. Hence $z \sqsubseteq y_0$. Therefore we obtain a contradiction: $x_0 \bigcirc y_0$. Analogously, we show that the second case is also not possible.

Ad (ii) Let $M \cap M' \subseteq \max_{\sqsubset}(M)$ and $M \cap M' \nsubseteq \max_{\sqsubset}(M')$. Hence there is $y_0 \in M \cap M'$ such that $y_0 \in \max_{\sqsubset}(M)$ and $y_0 \notin \max_{\sqsubset}(M')$. So there is $y_1 \in M'$ such that $y_0 \sqsubset y_1$.

We prove that $M \cap M' \subseteq \min_{\sqsubset}(M')$. Suppose towards contradiction that $M \cap M' \nsubseteq \min_{\sqsubset}(M')$. Then there is $x_0 \in M \cap M'$ such that $x_0 \in \max_{\sqsubset}(M)$ and

$x_0 \notin \min_{\sqsubset}(M')$. So there is $x_1 \in M'$ such that $x_1 \sqsubset x_0$ and $x_0 \in \min_{\sqsubset}(M)$, by Corollary 4.9.

Moreover, by Lemma 4.3, since $M \neq M'$, $x_0, y_0 \in M \cap M'$, $x_1, y_1 \in M'$, $x_1 \sqsubset x_0$, and $y_0 \sqsubset y_1$, so $x_0 \not\sqsubset y_0$, $y_0 \not\sqsubset x_0$, and $x_1, y_1 \notin M$.

We have $M' = \mathrm{M}^{x_0} = \mathrm{M}^{y_1}$, $\mathsf{P}(x_0) \subseteq M'$, $\mathsf{P}(y_1) \subseteq M'$, and $\breve{\mathsf{P}}(y_0) \subseteq M'$ (see Lemma 4.4, Corollary 4.5). So $\breve{\mathsf{PP}}(y_0) \subseteq M' \smallsetminus M$, $\breve{\mathsf{P}}(y_0) \cap (M \smallsetminus M') = \emptyset = \mathsf{P}(x_0) \cap (M \smallsetminus M')$, and $\mathsf{PP}(x_0) \subseteq M' \smallsetminus M$.

Now we consider some auxiliary claims.

Claim A There is a path from x_1 to $M \smallsetminus M'$.

Proof Suppose towards contradiction that for any $u \in M \smallsetminus M'$ there is no path from x_1 to u. We put

$$\mathrm{P} := \{u \in U : u \text{ is a member of some path from } x_1 \text{ to } M \cap M'\}.$$

Since $x_1 \sqsubset x_0$, so $x_1 \in \mathrm{P} \smallsetminus M$. We prove that $M \cup \mathrm{P} \in \mathbb{CT}_{\mathcal{U}}$.

First we show that $M \cup \mathrm{P}$ is closed with respect to $\sqsubset$. Let $x, y \in M \cup \mathrm{P}$ and $(u_0, u_1, \ldots, u_n)$ be any path from x to y, for some $n \geqslant 1$. (Let us denote this path by $[x \cdots y]$ for short.) We have $x \neq y$. We show that the path $[x \cdots y]$ is included in $M \cup \mathrm{P}$.

Notice that either $x \notin M$ or $y \neq x_1$. Otherwise we have a path $(x, \ldots, x_1, x_0)$ which is included in M. But since $x, x_0 \in M$ and $M \in \mathbb{CT}_{\mathcal{U}}$, we have a contradiction: $x_1 \in M$ and $x_1 \notin M$. Moreover, either $x \notin \mathrm{P}$ or $y \neq x_1$, because there is no path from x_1 to x_1. Below we consider all possible cases for x and y.

If $x, y \in M$, then the path $[x \cdots y]$ is included in M, since $M \in \mathbb{CT}_{\mathcal{U}}$.

If $x \in M$ and $y \in \mathrm{P} \smallsetminus M$, then $y \neq x_1$ and we have a path $(x_1, \ldots, y, \ldots, v)$ from x_1 to $M \cap M'$. So we have also a path $(x, \ldots, y, \ldots, v)$ which is included in M. Hence $[x \cdots y]$ is also included in M.

If $x = x_1$ and $y \in M$, then $y \in M \cap M'$, by the assumption towards contradiction. So we have a path $(x_1, u_1, \ldots, u_{n-1}, y)$ from x_1 to $M \cap M'$. Hence $[x \cdots y]$ is also included in P.

If $x \in \mathrm{P} \smallsetminus (M \cup \{x_1\})$ and $y \in M$, then we have a path $(x_1, \ldots, x, \ldots, v)$ from x_1 to $M \cap M'$. So we have also a path $(x_1, u_1, \ldots, x, \ldots, u_{n_1}, y)$ from x_1 to $M \cap M'$, by the assumption towards contradiction. Hence $[x \cdots y]$ is included in P.

If $x = x_1$ and $y \in \mathrm{P} \smallsetminus M$, then $y \neq x_1$ and we have a path $(x_1, w_1, \ldots, w_k, y, \ldots, v)$, where $v \in M \cap M'$. So we obtain also a path $(x_1, u_1, \ldots, u_{n-1}, y, \ldots, v)$ from x_1 to $M \cap M'$. Hence $[x \cdots y]$ is included in P.

If $x \in \mathrm{P} \smallsetminus (M \cup \{x_1\})$ and $y \in \mathrm{P} \smallsetminus (M \cup \{x_1\})$, then we have two paths $(x_1, \ldots, x, \ldots, v)$ and $(x_1, \ldots, y, \ldots, w)$, where $v, w \in M \cap M'$. So we have a path $(x_1, \ldots, x, u_1, \ldots, u_{n-1}, y, \ldots, w)$ from x_1 to $M \cap M'$. Thus $[x \cdots y]$ is included in P.

Second, we prove that the relation $\sqsubset|_{M \cup \mathrm{P}}$ is transitive. Let $x, y, z \in M \cup \mathrm{P}$ and $x \sqsubset y \sqsubset z$. By $(\mathrm{ac}_{\sqsubset})$, $z \neq x \neq y \neq z$ and $z \not\sqsubset x$. Moreover, we notice that the following cases are not possible:

(a) $x \in M$ and either $y \in \mathrm{P} \smallsetminus M$ or $z \in \mathrm{P} \smallsetminus M$,
(b) $y \in M$ and $z \in \mathrm{P} \smallsetminus M$,
(c) $x \in \mathrm{P} \smallsetminus M$ and $y, z \in M$.

Indeed, in the case when $x \in M$ and $z \in \mathrm{P} \smallsetminus M$, we have a path $(x_1, \ldots, z, \ldots, v)$ from x_1 to $M \cap M'$. So we create a path $(x, y, z, \ldots, v)$ which is included in M.

Thus, we obtain a contradiction. Similarly, we show that the rest of the subcases from (a) are not possible. In the case (b) we have a path $(x_1, \ldots, z, \ldots, w)$ from x_1 to $M \cap M'$. So we create a path $(y, z, \ldots, w)$ from y to $M \cap M'$, which is included in M. So we obtain a contradiction: $z \in M$. In the case (c) we have a path $(x_1, \ldots, x, \ldots, v)$ from x_1 to $M \cap M'$. So we create a path $(x_1, \ldots, x, y, z)$. By the assumption towards contradiction, $y, z \in M \cap M'$. Yet this is incompatible with the main assumption, which says that $M \cap M' \subseteq \max_{\sqsubset}(M)$.

Thus only the following three cases are possible. In the first case, if $x, y, z \in M$, then $x \sqsubset z$, since the relation $\sqsubset|_M$ is transitive. In the second, if $x, y \in \mathrm{P} \smallsetminus M$ and $z \in M$, then we have two paths $(x_1, \ldots, x, \ldots, v)$ and $(x_1, \ldots, y, \ldots, w)$ from x_1 to $M \cap M'$. Since $x_1 \in M'$ and $M' \in \mathbb{CT}_{\mathcal{U}}$, so these paths are included in M' and $x, y \in M'$. Notice that we also have a path $(x_1, \ldots, y, z)$. So $z \in M \cap M'$, by the assumption towards contradiction. Therefore $x \sqsubset z$, because the relation $\sqsubset|_{M'}$ is transitive. In the third, if $x, y, z \in \mathrm{P} \smallsetminus M$, then we have three paths $(x_1, \ldots, x, \ldots, v)$, $(x_1, \ldots, y, \ldots, w)$ and $(x_1, \ldots, z, \ldots, u)$ from x_1 to $M \cap M'$. Since $x_1 \in M'$ and $M' \in \mathbb{CT}_{\mathcal{U}}$, so these paths are included in M' and $x, y, z \in M'$. Hence $x \sqsubset z$.

Thus, we prove that $M \cup \mathrm{P} \in \mathbb{CT}_{\mathcal{U}}$. Hence, by Theorem 3.3, for some $M'' \in \mathbb{MCT}_{\mathcal{U}}$ we have $M \cup \mathrm{P} \subseteq M''$. So $M = M''$, because $M \in \mathbb{MCT}_{\mathcal{U}}$. Hence we obtain a contradiction: $\mathrm{P} \subseteq M$, so $x_1 \in M$. □

Claim B For all $v \in \check{\mathrm{P}}(y_0)$ and $u \in M \smallsetminus M'$ there is no path from v to u.

Proof Otherwise there is a path which has either the form $(y_0, v, \ldots, u)$, if $v \neq y_0$, or $(y_0, \ldots, u)$, if $v = y_0$. In both cases this path is included in M. So we obtain a contradiction: $y_0 \sqsubset u$. (In the first case we also have a contradiction: $v \in M$.) □

Claim C For any $u \in M \smallsetminus M'$ there is a path from u to $\widecheck{\mathrm{PP}}(y_0)$.

Proof Suppose that there is $u \in M \smallsetminus M'$ such that for any $v \in \widecheck{\mathrm{PP}}(y_0)$ there is no path from u to v. Hence there is no path from u to y_0. Then, by Claim B and Lemma 3.1, $\check{\mathrm{P}}(y_0) \cup \{u\} \in \mathbb{CT}_{\mathcal{U}}$. Thus, by Theorem 3.3, for some $M'' \in \mathbb{MCT}_{\mathcal{U}}$ we have $\check{\mathrm{P}}(y_0) \cup \{u\} \subseteq M''$. So $M' = M''$, by Corollary 4.5. Hence we obtain a contradiction: $u \in M'$. □

Now, by Claim A, there is a path $(x_1, \ldots, u)$ from x_1 to u, for some $u \in M \smallsetminus M'$. By Claim C, there is a path $(u, \ldots, v)$ from u to v, for some $v \in \widecheck{\mathrm{PP}}(y_0) \subseteq M'$. Hence we obtain a path $(x_1, \ldots, u, \ldots, v)$ from x_1 to v. Since $x_1, v \in M'$ and $M' \in \mathbb{MCT}_{\mathcal{U}}$, so this path is included in M'. Hence we have a contradiction: $u \in M'$ and $u \notin M'$.

Ad (iii) Let $M \cap M' \not\subseteq \max_{\sqsubset}(M)$ and $M \cap M' \subseteq \max_{\sqsubset}(M')$. Similarly, as in the case (ii) we can prove that $M \cap M' \subseteq \min_{\sqsubset}(M)$.

Thus, $M \cap M' \subseteq \max_{\sqsubset}(M) \cap \min_{\sqsubset}(M')$ or $M \cap M' \subseteq \min_{\sqsubset}(M) \cap \max_{\sqsubset}(M')$. □

4.3 A stronger version of (A3) In the following remark we give a stronger version of the third axiom. Next we present only additional remarks and an example.

Remark 4.13 Notice that the frame that is depicted in the diagram in Figure 1 satisfies axioms (A1)–(A3) (plus also two axioms (A4) and (A5) given below). Of

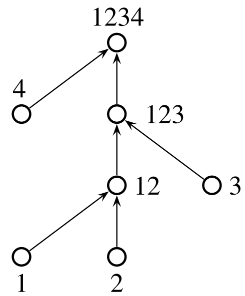

Figure 1

course, for this diagram the relation $\sqsubset$, represented by $\rightarrow$, is locally transitive and acyclic.

Remember that $\sqsubset$ is not transitive. Therefore we have $\mathbb{MCT}_{\mathcal{U}} = \{M_1, M_2, M_3\}$, where $M_1 = \{1234, 4, 123\}$, $M_2 = \{4, 123, 12, 3\}$, and $M_3 = \{4, 12, 3, 1, 2\}$; $M_1 \cap M_2 = \{4, 123\} = \min_{\sqsubset}(M_1) = \max_{\sqsubset}(M_2)$; $M_2 \cap M_3 = \{4, 12, 3\} = \min_{\sqsubset}(M_2) = \max_{\sqsubset}(M_3)$; $M_1 \cap M_3 = \{4\} = \min_{\sqsubset}(M_1) \cap \max_{\sqsubset}(M_3)$.

We can see that the frame does not fulfill the following condition, which is stronger than axiom (A3).

(A3$_s$) The family $\mathbb{MCT}_{\mathcal{U}}$ meets the following condition:
for any M and M' from $\mathbb{MCT}_{\mathcal{U}}$, if $M \neq M'$ and $M \cap M' \neq \emptyset$, then either $\max_{\sqsubset}(M) = M \cap M' = \min_{\sqsubset}(M')$ or $\min_{\sqsubset}(M) = M \cap M' = \max_{\sqsubset}(M')$.

Example 4.14 In connection with Rescher's counterexample (see the introduction, p. 360), it is possible to find an example of a set U, for which $\mathbb{MCT}_{\mathcal{U}} \neq \{U\}$ and the sets from $\mathbb{MCT}_{\mathcal{U}}$ intersect. For example, we put $U := O \cup C \cup N \cup A$, where O is a set of one-celled and many-celled organisms, C is the set of cells in organisms from O, N is the set of nuclei of cells from C, and A is the set of other parts of cells from C. We have $\mathbb{MCT}_{\mathcal{U}} = \{M_1, M_2\}$, where $M_1 := O \cup C$ and $M_2 := C \cup N \cup A$. Then the members of C are maximal elements in the set M_2 and minimal in the set M_1; $M_1 \cap M_2 = C = \min_{\sqsubset}(M_1) = \max_{\sqsubset}(M_2)$. Thus, the stronger version of the third axiom is fulfilled. Obviously, it is the case only if, after Rescher, we admit that $\sqsubset$ is not transitive in U.

As was mentioned before, the stronger version (A3$_s$) of the third axiom may be more intuitive; however (A3) is strong enough to prove all facts (from Section 4.1) that are used in the rest of the paper. The following remark provides an informal motivation for (A3) and (A3$_s$).

Remark 4.15 Succinctly, the axioms (A3$_s$) and (A3) break the universe up into $\mathbb{MCT}_{\mathcal{U}}$'s that intersect only on their *extreme boundaries*.

Let $M, M' \in \mathbb{MCT}_{\mathcal{U}}$, $M \neq M'$, $x, y \in M$, $x, z_1, z_2 \in M'$, and $y \sqsubset x \sqsubset z_1 \sqsubset z_2$. Then, in relation to the remarks in [3], x is *maximal* in M (resp., *minimal* in M') with respect to $\sqsubset$ such that we can say correctly that x *has* y (resp., z_1 *has* x and z_2 *has* x). For example, the sentences below are correct.

- This orchestra (z_2) has a violin section (z_1). $z_1 \sqsubset z_2$
- This orchestra (z_2) has a violinist (x). $x \sqsubset z_2$

- This violinist (x) has a heart (y). $y \sqsubset x$

However, the following sentences are semantically weird.

- This orchestra has this violinist's arm.
- This violin section has this violinist's heart.

We can assume that an orchestra is a *system*, in which its musicians and conductor are *minimal* elements with respect to $\sqsubset$. One can say that it is also a *closed system*, since all musicians are pairwise disjoint (exterior). Additionally, the fifth axiom, given below, is fulfilled. It says that: for any $M \in \mathbb{MCT}_{\mathcal{U}}$ all members of the set $\min_{\sqsubset}(M)$ are pairwise in the relation $\wr$ (what is equivalent to $\wr^{M} \subseteq \wr$; see Lemma 8.1). Moreover, any musician is also a closed system of their body parts, in which the musician is a *maximal* element with respect to $\sqsubset$.

5 Two Versions of the Fourth Axiom: Conditions of Separation

Let $\mathcal{U} = \langle U, \sqsubset\rangle$ satisfy axioms (A1)–(A3). Further, we will consider the following conditions of separation (supplementation):

$$\forall_{x,y\in U}\big(x \not\sqsubseteq y \Longrightarrow \exists_{z\in U}(z \sqsubseteq x \wedge z \wr y)\big), \tag{SSP}$$

$$\forall_{x,y\in U}\big(y \sqsubset x \Longrightarrow \exists_{z\in U}(z \sqsubset x \wedge z \wr y)\big), \tag{WSP}$$

$$\forall_{x,y\in U}\big(x \between y \Longrightarrow \exists_{z\in U}(z \sqsubset x \wedge z \wr y)\big), \tag{$\mathrm{SP}_{\between}$}$$

and its "partial versions" for every M from $\mathbb{MCT}_{\mathcal{U}}$:

$$\forall_{x,y\in M}\big(x \not\sqsubseteq y \Longrightarrow \exists_{z\in M}(z \sqsubseteq x \wedge z \wr^{M} y)\big), \tag{SSP^M}$$

$$\forall_{x,y\in M}\big(y \sqsubset x \Longrightarrow \exists_{z\in M}(z \sqsubset x \wedge z \wr^{M} y)\big), \tag{WSP^M}$$

$$\forall_{x,y\in M}\big(x \between y \Longrightarrow \exists_{z\in M}(z \sqsubset x \wedge z \wr^{M} y)\big). \tag{$\mathrm{SP}^M_{\between}$}$$

The above conditions are characteristic for the notion of being part of in its mereological use, i.e., when it is transitive (see, e.g., [5], [7]). In this case the sentences (SSP) and (WSP) are called by Simons in [7], respectively, *strong supplementation principle* and *weak supplementation principle*. Notice that $\sqsubset|_M$ is transitive for every $M \in \mathbb{MCT}_{\mathcal{U}}$.

Before giving the forth axiom we will analyze relations that hold between the above conditions of separation.

First, notice that by definitions, ($\mathrm{irr}_{\sqsubset}$), and ($\mathrm{as}_{\sqsubset}$), using (2), we obtain the following.

Lemma 5.1 *We have the following.*

1. *(SSP), ($\mathrm{as}_{\sqsubset}$), and definitions entail (WSP).*
2. *(SSP) and definitions entail ($\mathrm{SP}_{\between}$).*
3. *(WSP), ($\mathrm{SP}_{\between}$), and definitions entail (SSP).*
4. *The claims 1–3 do hold also in their "partial versions", that is, when (SSP), (WSP), and ($\mathrm{SP}_{\between}$) are replaced, respectively, by (SSP^M), (WSP^M), and ($\mathrm{SP}^M_{\between}$).*

Second, from (A1)–(A3) we obtain the following.

Lemma 5.2 *Let $\mathcal{U}$ satisfy (A1)–(A3). Then*

1. *if (SSP) holds, then (SSP^M) holds for every M from $\mathbb{MCT}_{\mathcal{U}}$;*
2. *if (SSP^M) holds for every M from $\mathbb{MCT}_{\mathcal{U}}$, then ($\mathrm{SP}_{\between}$) also holds.*

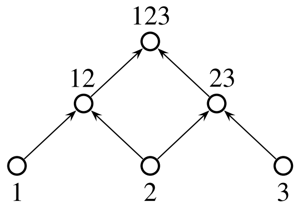

Figure 2

Proof *Ad* 1. Let $M \in \mathbb{MCT}_{\mathcal{U}}$, $x, y \in M$, and $x \not\sqsubseteq y$. We have that either $x \wr^M y$ or $x \bigcirc^M y$. In the first case we have our thesis. In the second case, for some $u \in M$, $u \sqsubseteq x$ and $u \sqsubseteq y$. Hence, by the assumption, $x \neq u$, that is, $u \sqsubset x$. Moreover, by (SSP), there exists part z of x, which has no common part with y. Hence, by the assumption, $x \neq z$, that is, $z \sqsubset x$. Thus, by Corollary 4.8(1), $z \in M$. Moreover, $z \wr^M y$, since $\wr \subseteq \wr^M$.

Ad 2. Let $x \between y$. Then, by Lemma 4.11, $\mathrm{M}^x = \mathrm{M}^y$ and $x \between^{\mathrm{M}^x} y$. Moreover, since $x \not\sqsubseteq y$, so by ($\mathtt{SSP}^{\mathrm{M}^x}$) for some $z \in \mathrm{M}^x$, $z \sqsubseteq x$ and $z \wr^{\mathrm{M}^x} y$. Hence $z \neq x$, so $z \sqsubset x$. We show that $z \wr y$. Indeed, suppose that for some u_0 we have $u_0 \sqsubseteq z$ and $u_0 \sqsubseteq y$. Then $u_0 \in \mathsf{P}(y) \subseteq \mathrm{M}^y = \mathrm{M}^x$. And this leads to a contradiction: $z \bigcirc^{\mathrm{M}^x} y$. □

Fact 5.3 There is a frame $\mathcal{U}$ for which:

(a) the axioms (A1)–(A3) are true,
(b) for every M from $\mathbb{MCT}_{\mathcal{U}}$ the condition ($\mathtt{SSP}^M$) holds,
(c) (WSP) and (SSP) are false.

Thus (WSP), and in consequence (SSP), do not follow from (A1)–(A3) and ($\mathtt{SSP}^M$) for every $M \in \mathbb{MCT}_{\mathcal{U}}$.

Proof We consider a frame $\mathcal{U} = \langle U, \sqsubset \rangle$ that is depicted on the diagram in Figure 2. Of course, for this diagram the relation $\sqsubset$, represented again by $\rightarrow$, is locally transitive and acyclic.

We have that $\mathbb{MCT}_{\mathcal{U}} = \{M_1, M_2\}$, where $M_1 = \{123, 12, 23\}$ and $M_2 = \{12, 23, 1, 2, 3\}$. So the family $\mathbb{MCT}_{\mathcal{U}}$ meets (A3), ($\mathrm{A3_s}$), ($\mathtt{SSP}^{M_1}$), and ($\mathtt{SSP}^{M_2}$).

Moreover, we have $23 \sqsubset 123$, but there is no $z \in U$ such that $z \sqsubset 123$ and $z \wr 23$, since $12 \bigcirc 23$. Hence (WSP) and (SSP) are false in $\mathcal{U}$. □

We introduce two versions of the *fourth axiom*:

(A4) $\mathcal{U}$ satisfies (SSP) (in a strong version),
($\mathrm{A4_w}$) ($\mathtt{SSP}^M$) holds for every $M \in \mathbb{MCT}_{\mathcal{U}}$ (in a weak version).

We prove that on the basis of (A1)–(A3) the axiom (A4) is stronger than ($\mathrm{A4_w}$) (see Lemma 5.2 and Fact 5.3). In Section 8 we assume the fifth axiom, (A5), which makes axioms (A4) and ($\mathrm{A4_w}$) equivalent (see Theorem 8.5).

From (A4) we obtain the *proper parts principle* (see [7]):

$$\forall_{x,y \in U} \big(\emptyset \neq \mathsf{PP}(x) \subseteq \mathsf{PP}(y) \Longrightarrow x \sqsubseteq y\big). \tag{PPP}$$

Indeed, let $\emptyset \neq \mathsf{PP}(x) \subseteq \mathsf{PP}(y)$. Then $x \bigcirc y$. Suppose that $x \not\sqsubseteq y$. Then, by (SSP), for some z, $z \sqsubseteq x$ and $z \wr y$. Hence $z \neq x$, so $z \sqsubset x$. Thus also $z \sqsubset y$, a contradiction.

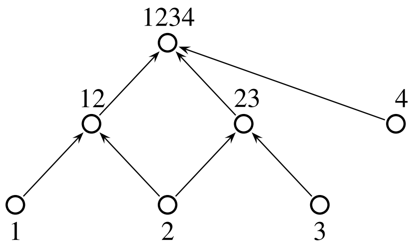

Figure 3

Similarly, from (A4$_w$) for every $M \in \mathbb{MCT}_{\mathcal{U}}$ we obtain the following "partial version" of *PPP*:

$$\forall_{x,y \in M}\big(\emptyset \neq \mathsf{PP}(x) \cap M \subseteq \mathsf{PP}(y) \Longrightarrow x \sqsubseteq y\big). \tag{$\mathtt{PPP}^M$}$$

Now for any $x \in U$ and $M \in \mathbb{MCT}_{\mathcal{U}}$ we put

$$\mathsf{O}(x) := \{y \in U : y \bigcirc x\},$$
$$\mathsf{O}^M(x) := \{y \in U : y \bigcirc^M x\}.$$

Notice that $\mathsf{P}(x) \subseteq \mathsf{O}(x)$, so from (SSP) we have the following sentence:

$$\begin{aligned} &\forall_{x,y \in U}\big(\mathsf{P}(x) \subseteq \mathsf{O}(y) \Longrightarrow x \sqsubseteq y\big), \\ &\forall_{x,y \in U}\big(\mathsf{O}(x) \subseteq \mathsf{O}(y) \Longrightarrow x \sqsubseteq y\big). \end{aligned} \tag{$\mathtt{SSP}_{\bigcirc}$}$$

Finally, since for each $M \in \mathbb{MCT}_{\mathcal{U}}$ the pair $\langle M, \sqsubset|_M \rangle$ is a separative strict partial order, by ($\mathtt{SSP}^M$), Lemma 4.2(4), and $\mathsf{P}(x) \cap M \subseteq \mathsf{O}^M(x)$ we obtain

$$\begin{aligned} &\forall_{x,y \in M}\big(x \sqsubseteq y \iff \mathsf{P}(x) \cap M \subseteq \mathsf{O}^M(y)\big), \\ &\forall_{x,y \in M}\big(x \sqsubseteq y \iff \mathsf{O}^M(x) \subseteq \mathsf{O}^M(y)\big). \end{aligned} \tag{$\mathtt{SSP}^M_\star$}$$

Referring to Lemma 4.2 and endnote 4 let us notice the following fact.

Fact 5.4 There is a frame $\mathcal{U}$ for which:

(a) the axioms (A1)–(A4) are true,
(b) for some $M \in \mathbb{MCT}_{\mathcal{U}}$, $\bigcirc^M \subsetneq \bigcirc|_M$, $\between^M \subsetneq \between|_M$, and $\wr|_M \subsetneq \wr^M$.

Proof This frame is depicted in the diagram in Figure 3. Of course, for this diagram the relation $\sqsubset$ is locally transitive and acyclic. We have $\mathbb{MCT}_{\mathcal{U}} = \{M_1, M_2\}$, where $M_1 = \{1234, 12, 23, 4\}$ and $M_2 = \{12, 23, 1, 2, 3, 4\}$. So in this frame the family $\mathbb{MCT}_{\mathcal{U}}$ satisfies axioms (A3) and (A3$_s$). Moreover, (SSP) is also true in this frame. Finally, $12 \bigcirc 23$, but $12 \wr^{M_1} 23$. □

6 "Partial Principle of Monotonicity": An Equivalent Version of (A4$_w$)

Let $\mathcal{U} = \langle U, \sqsubset \rangle$ satisfy axioms (A1)–(A3). To make the expressions shorter we put for any $M \in \mathbb{MCT}_{\mathcal{U}}$ and $S \in 2^U$,

$$\bigcup \mathsf{O}(S) := \{x \in U : \exists_{u \in S}\, u \bigcirc x\},$$
$$\bigcup \mathsf{O}^M(S) := \{x \in U : \exists_{u \in S}\, u \bigcirc^M x\}.$$

Of course, $\bigcup \mathsf{O}(\emptyset) = \emptyset = \bigcup \mathsf{O}^M(\emptyset)$, $\bigcup \mathsf{O}^M(S) \subseteq M$, and $\bigcup \mathsf{O}^M(S) \subseteq \bigcup \mathsf{O}(S)$.

The lemma below indicates a condition equivalent to the axiom ($\mathrm{A4_w}$).

Lemma 6.1 *For every $M \in \mathbb{MCT}_{\mathcal{U}}$,*

1. *the transitivity of $\sqsubset|_M$ and ($\mathtt{SSP}^M$) entail the following "partial principle of monotonicity":*

$$\forall_{x,y\in M}\forall_{S\in 2^M}\big(S \subseteq \mathsf{P}(y) \wedge \mathsf{P}(x)\cap M \subseteq \bigcup \mathsf{O}^M(S) \Longrightarrow x \sqsubseteq y\big); \qquad (\mathtt{MP}^M)$$

2. *($\mathtt{SSP}^M$) follows from ($\mathtt{MP}^M$) and the reflexivity of $\sqsubseteq|_M$.*

Proof *Ad* 1. Let $x, y \in M$, $S \subseteq M \cap \mathsf{P}(y)$ and $\forall_{z\in M}(z \sqsubseteq x \Rightarrow \exists_{u\in S}\, u \bigcirc^M z)$. Then $\forall_{z\in M}(z \sqsubseteq x \Rightarrow \exists_{u\in M}(u \sqsubseteq y \wedge u \bigcirc^M z))$. Hence, by Lemma 4.2(4), we obtain $\forall_{z\in M}(z \sqsubseteq x \Rightarrow z \bigcirc^M y)$. Hence $x \sqsubseteq y$, by ($\mathtt{SSP}^M$).

Ad 2. For any $x, y \in M$ we put $S := \{y\}$. □

From Lemma 6.1 we obtain the following.

Theorem 6.2 *On the basis of (A1)–(A3),*

$$(\mathrm{A4_w}) \iff (\mathtt{MP}^M) \text{ holds, for every } M \in \mathbb{MCT}_{\mathcal{U}}.$$

7 Mereological Sums

The crucial notion of mereology is that of mereological sum. By defining it we want to say what it means that some object is *a mereological sum* (or *a collective class*) *of* all members of some distributive class of objects.

Remark 7.1 In mereology, assuming that the relation $\sqsubset$ is transitive on U, we consider the following definition of being a sum of all elements of a given set (see, e.g., [5], [7]):

$$x \,\mathsf{sum_{cl}}\, S \overset{\text{df}}{\iff} S \subseteq \mathsf{P}(x) \subseteq \bigcup \mathsf{O}(S). \qquad (\mathrm{df}\,\mathsf{sum_{cl}})$$

If $x \,\mathsf{sum_{cl}}\, S$, then we say that x is a mereological sum of all members of the set S. Since we have not assumed the transitivity of the relation $\sqsubset$, we have to modify the above definition.

Suppose that the frame $\mathcal{U} = \langle U, \sqsubset\rangle$ satisfies axioms (A1)–(A3) and ($\mathrm{A4_w}$). We define the following binary relation sum in $U \times 2^U$:

$$x \,\mathsf{sum}\, S \overset{\text{df}}{\iff} S \subseteq \mathsf{P}(x) \wedge \forall_{M\in\mathbb{MCT}_{\mathcal{U}}}\big(\mathsf{P}(x) \subseteq M \Longrightarrow \mathsf{P}(x) \subseteq \bigcup \mathsf{O}^M(S)\big). \qquad (\mathrm{df}\,\mathsf{sum})$$

By Corollary 3.4 the above definition of mereological sum is correct.

Notice that we obtain the following.

Lemma 7.2 *For any $x \in U$ and $S \in 2^U$, we have the following:*

1. *if $x \,\mathsf{sum_{cl}}\, S$ or $x \,\mathsf{sum}\, S$, then $S \neq \emptyset$;*
2. *$x \,\mathsf{sum_{cl}}\, \{x\}$, $x \,\mathsf{sum_{cl}}\, \mathsf{P}(x)$, $x \,\mathsf{sum}\, \{x\}$, and $x \,\mathsf{sum}\, \mathsf{P}(x)$;*
3. *$x \,\mathsf{sum_{cl}}\, \mathsf{PP}(x)$ iff $\mathsf{PP}(x) \neq \emptyset$ iff $x \,\mathsf{sum}\, \mathsf{PP}(x)$.*

Proof By the reflexivity of $\sqsubseteq$, we have $\{x\} \subseteq \mathsf{P}(x) \neq \emptyset$, $\mathsf{P}(x) \subseteq \mathsf{O}(x)$, and $\mathsf{P}(x) \subseteq \mathsf{O}^M(x)$, for any $M \in \mathbb{MCT}_{\mathcal{U}}$ such that $\mathsf{P}(x) \subseteq M$.

Ad 1. If $x \,\mathsf{sum_{cl}}\, S$, then $\emptyset \neq \mathsf{P}(x) \subseteq \bigcup \mathsf{O}(S)$ and consequently $S \neq \emptyset$, since $\bigcup \mathsf{O}(\emptyset) = \emptyset$. If $x \,\mathsf{sum}\, S$, then by Corollary 3.4 for some $M \in \mathbb{MCT}_{\mathcal{U}}$ we have $\mathsf{P}(x) \subseteq M$. So $\emptyset \neq \mathsf{P}(x) \subseteq \bigcup \mathsf{O}^M(S)$, and consequently $S \neq \emptyset$, since $\bigcup \mathsf{O}^M(\emptyset) = \emptyset$.

Ad 2. We have $\{x\} \subseteq \mathsf{P}(x) \subseteq \mathsf{O}(x) = \bigcup \mathsf{O}(\{x\}) \subseteq \bigcup \mathsf{O}(\mathsf{P}(x))$. So $x \,\mathsf{sum_{cl}}\, \{x\}$ and $x \,\mathsf{sum_{cl}}\, \mathsf{P}(x)$. Moreover, for any $M \in \mathbb{MCT}_{\mathcal{U}}$ such that $\mathsf{P}(x) \subseteq M$, $\{x\} \subseteq \mathsf{P}(x) \subseteq \mathsf{O}^M(x) = \bigcup \mathsf{O}^M(\{x\}) \subseteq \bigcup \mathsf{O}^M(\mathsf{P}(x))$. So $x \,\mathsf{sum}\, \{x\}$ and $x \,\mathsf{sum}\, \mathsf{P}(x)$.

Ad 3. If $\mathsf{PP}(x) \neq \emptyset$, then $\mathsf{PP}(x) \subseteq \mathsf{P}(x) \subseteq \bigcup \mathsf{O}^M(\mathsf{PP}(x)) \subseteq \bigcup \mathsf{O}(\mathsf{PP}(x))$, for any $M \in \mathbb{MCT}_{\mathcal{U}}$ such that $\mathsf{P}(x) \subseteq M$. So $x \,\mathsf{sum_{cl}}\, \mathsf{PP}(x)$ and $x \,\mathsf{sum}\, \mathsf{PP}(x)$. □

Remark 7.3

1. If we suppose that in $\mathcal{U} = \langle U, \sqsubset \rangle$ the relation $\sqsubset$ is transitive, then $\mathbb{MCT}_{\mathcal{U}} = \{U\}$. So $\bigcirc^M = \bigcirc^U = \bigcirc$, and we obtain the "classical definition" of mereological sum, that is, we obtain $\mathsf{sum} = \mathsf{sum_{cl}}$.
2. In the general case the relation $\mathsf{sum_{cl}}$ is not suitable even for axioms (A1)–(A4) and (A3$_{\mathrm{s}}$). For example, in the frame in Figure 3, we have $1234 \,\mathsf{sum_{cl}}\, \{23, 4\}$.

Moreover, for axioms (A1)–(A3), (A3$_{\mathrm{s}}$), and (A4$_{\mathrm{w}}$) in the frame in Figure 2, we have $123 \,\mathsf{sum_{cl}}\, \{12\}$, $12 \,\mathsf{sum_{cl}}\, \{12\}$, $123 \,\mathsf{sum_{cl}}\, \{23\}$, and $23 \,\mathsf{sum_{cl}}\, \{23\}$.

3. We obtain $\mathsf{sum} = \mathsf{sum_{cl}}$, if we assume the fifth axiom as we do in Section 8 (see Theorem 9.1).

The definition of mereological sum is so complicated, since the case $\mathsf{PP}(x) = \emptyset$ is possible. The following lemma indicates a simplified version of the definition of mereological sum.

Lemma 7.4 *For all $x \in U$ and $S \in 2^U$, we have the following.*

1. *If $\mathsf{PP}(x) = \emptyset$, then $x \,\mathsf{sum}\, S$ iff $S = \{x\}$ iff $x \,\mathsf{sum_{cl}}\, S$.*
2. *If $\mathsf{PP}(x) \neq \emptyset$, then $\mathsf{P}(x) \subseteq \mathrm{M}^x$ and, moreover,*

$$x \,\mathsf{sum}\, S \iff S \subseteq \mathsf{P}(x) \subseteq \bigcup \mathsf{O}^{\mathrm{M}^x}(S).$$

Proof *Ad* 1. Let $\mathsf{PP}(x) = \emptyset$. Then $\mathsf{P}(x) = \{x\}$ and by Lemma 7.2(1), if $x \,\mathsf{sum_{cl}}\, S$ or $x \,\mathsf{sum}\, S$, then $\emptyset \neq S \subseteq \mathsf{P}(x) = \{x\}$. Moreover, by Lemma 7.2(2), $x \,\mathsf{sum_{cl}}\, \{x\}$ and $x \,\mathsf{sum}\, \{x\}$.

Ad 2. Let $\mathsf{PP}(x) \neq \emptyset$. Then M^x is the unique set in $\mathbb{MCT}_{\mathcal{U}}$ which includes $\mathsf{P}(x)$, by Corollary 4.5. □

So the following corollary holds.

Fact 7.5 We have $\mathsf{sum} \subseteq \mathsf{sum_{cl}}$.

Proof Suppose that $x \,\mathsf{sum}\, S$. If $\mathsf{PP}(x) = \emptyset$, then $x \,\mathsf{sum_{cl}}\, S$, by Lemma 7.4(1). If $\mathsf{PP}(x) \neq \emptyset$ then, by Lemma 7.4(2), $\emptyset \neq S \subseteq \mathsf{P}(x) \subseteq \mathrm{M}^x$ and $\mathsf{P}(x) \subseteq \bigcup \mathsf{O}^{\mathrm{M}^x}(S)$. So also in this case we have that $x \,\mathsf{sum_{cl}}\, S$, because $\bigcup \mathsf{O}^{\mathrm{M}^x}(S) \subseteq \bigcup \mathsf{O}(S)$. □

Remark 7.6 From (A1)–(A4) it does not follow that $\mathsf{sum_{cl}} \subseteq \mathsf{sum}$. For example, for the frame in Figure 3 we have $1234 \,\mathsf{sum_{cl}}\, \{23, 4\}$, but $\neg\, 1234 \,\mathsf{sum}\, \{23, 4\}$.

By definitions and Lemma 7.2(1), for any $x \in U$ and $S \in 2^U$ we can prove the following equivalence:

$$x \,\mathsf{sum}\, S \iff \emptyset \neq S \subseteq \mathsf{P}(x) \wedge \forall_{M \in \mathbb{MCT}_{\mathcal{U}}} \big(\mathsf{P}(x) \subseteq M \Rightarrow \mathsf{PP}(x) \subseteq \bigcup \mathsf{O}^M(S)\big).$$

Indeed, for $\Rightarrow$ we use Lemma 7.2(1) and (dfsum). For $\Leftarrow$ suppose that $\mathsf{P}(x) \subseteq M \in \mathbb{MCT}_{\mathcal{U}}$. Then there is $u_0 \in S \subseteq \mathsf{P}(x)$; so for some $u \in S \subseteq M$ we have $u \bigcirc^M x$. So, by (dfsum), $x \,\mathsf{sum}\, S$.

Thus, by the above fact and Lemma 7.4 we have the following.

Fact 7.7 If $\mathsf{PP}(x) \neq \emptyset$, then

$$x \,\mathsf{sum}\, S \iff \emptyset \neq S \subseteq \mathsf{P}(x) \wedge \mathsf{PP}(x) \subseteq \bigcup \mathsf{O}^{\mathrm{M}^x}(S).$$

Now notice that, by the above facts and Lemmas 5.1 and 7.2(2), we obtain the following.

Fact 7.8 The axioms (A1)–(A3) and ($\mathrm{A4_w}$) entail

$$\forall_{x,y\in U}\big(x \,\mathsf{sum}\, \{y\} \iff x = y\big).$$

Proof Let $x \,\mathsf{sum}\, \{y\}$ and $x \neq y$. Then $y \sqsubset x$ and $y, x \in \mathrm{M}^x$. Moreover, by ($\mathtt{WSP}^{\mathrm{M}^x}$), for some $z \in \mathrm{M}^x$, $z \sqsubset x$, and $z \wr^{\mathrm{M}^x} y$. This is a contradiction, because $z \sqsubset x$ and $x \,\mathsf{sum}\, \{y\}$ entail $y \bigcirc^{\mathrm{M}^x} z$. □

Remark 7.9

1. From (A1)–(A3) and ($\mathtt{WSP}$) we obtain

$$\forall_{x,y\in U}\big(x \,\mathsf{sum_{cl}}\, \{y\} \iff x = y\big).$$

2. But for (A1)–(A3), ($\mathrm{A3_s}$), and ($\mathrm{A4_w}$), for the frame in Figure 2, we have $123 \,\mathsf{sum_{cl}}\, \{12\}$, $12 \,\mathsf{sum_{cl}}\, \{12\}$, $123 \,\mathsf{sum_{cl}}\, \{23\}$, and $23 \,\mathsf{sum_{cl}}\, \{23\}$.

As the second corollary from Lemma 6.1 we have the following fact.

Fact 7.10 The axioms (A1)–(A3) and ($\mathrm{A4_w}$) entail

$$\forall_{x,y\in U}\forall_{S\in 2^U}(x \,\mathsf{sum}\, S \wedge y \,\mathsf{sum}\, S \Longrightarrow x = y). \tag{f-sum}$$

Proof Let $x \,\mathsf{sum}\, S$ and $y \,\mathsf{sum}\, S$. If $\mathsf{PP}(x) = \emptyset$ or $\mathsf{PP}(y) = \emptyset$, then $S = \{x\}$ or $S = \{y\}$, by Lemma 7.4. Hence $x = y$, by Fact 7.8.

Let $\mathsf{PP}(x) \neq \emptyset \neq \mathsf{PP}(y)$, and suppose that $x \neq y$. Then $\emptyset \neq S \subseteq \mathsf{P}(x) \cap \mathsf{P}(y) \subseteq \mathrm{M}^x \cap \mathrm{M}^y$. We show that $\mathrm{M}^x = \mathrm{M}^y$.

Suppose that $\mathrm{M}^x \neq \mathrm{M}^y$. Then either $\mathsf{P}(x) \cap \mathsf{P}(y) \subseteq \min_{\sqsubset}(\mathrm{M}^x) \cap \max_{\sqsubset}(\mathrm{M}^y)$ or $\mathsf{P}(x) \cap \mathsf{P}(y) \subseteq \max_{\sqsubset}(\mathrm{M}^x) \cap \max_{\sqsubset}(\mathrm{M}^y)$, by (A3′). Hence $y \not\sqsubset x$ and $x \not\sqsubset y$. But $x \bigcirc y$, so $x \between y$, by (1) and definitions. Therefore $\mathrm{M}^x = \mathrm{M}^y$, by Lemma 4.11.

Thus we obtain $x, y \in \mathrm{M}^x$, $S \subseteq \mathsf{P}(x)$, $\mathrm{M}^x \cap \mathsf{P}(x) \subseteq \bigcup \mathsf{O}^{\mathrm{M}^x}(S)$, $S \subseteq \mathsf{P}(y)$, and $\mathrm{M}^x \cap \mathsf{P}(y) \subseteq \bigcup \mathsf{O}^{\mathrm{M}^x}(S)$. Hence $x \sqsubseteq y$ and $y \sqsubseteq x$, by Lemma 6.1. So, by ($\mathrm{anti}_{\sqsubseteq}$), we obtain a contradiction: $x = y$. □

For any $x \in U$ and $S \in 2^U$, we say that x is the *supremum* of all elements of a set S (we will write for short: $x \,\mathsf{sup}\, S$) iff x is the least upper bound of all elements of S. Formally, we define the binary relation sup in $U \times 2^U$:

$$x \,\mathsf{sup}\, S \overset{\mathrm{df}}{\iff} S \subseteq \mathsf{P}(x) \wedge \forall_{y\in U}\big(S \subseteq \mathsf{P}(y) \Rightarrow x \sqsubseteq y\big).$$

From ($\mathrm{anti}_{\sqsubseteq}$) we obtain that for any $x, y \in U$ and $S \in 2^U$,

$$x \,\mathsf{sup}\, S \wedge y \,\mathsf{sup}\, S \Longrightarrow x = y. \tag{f-sup}$$

Lemma 7.11 *For all $x \in U$ and $S \in 2^U$ we have the following:*

1. $x \,\mathsf{sup}\, \emptyset$ iff $\forall_{u\in U}\, x \sqsubseteq u$;

2. *if* $\mathsf{PP}(x)=\emptyset$, *then:* $x \operatorname{\mathsf{sup}} S$ *iff* $S=\{x\}$ *or both* $S=\emptyset$ *and* $\forall_{u\in U}\, x\sqsubseteq u$;
3. *if* $\mathsf{PP}(x)\neq\emptyset$, *then* $\mathsf{P}(x)\subseteq \mathrm{M}^x$ *and, moreover,*

$$x \operatorname{\mathsf{sup}} S \iff S\subseteq \mathsf{P}(x)\wedge \forall_{y\in\mathrm{M}^x}\big(S\subseteq \mathsf{P}(y)\Rightarrow x\sqsubseteq y\big).$$

Proof *Ad* 2. Let $\mathsf{PP}(x)=\emptyset$. For $\Rightarrow$, if $x \operatorname{\mathsf{sup}} S$, then $S\subseteq\mathsf{P}(x)=\{x\}$. So if $S\neq\{x\}$, then $S=\emptyset$. For $\Leftarrow$, first $x \operatorname{\mathsf{sup}}\{x\}$. Second, we use 1.

Ad 3. Let $\mathsf{PP}(x)\neq\emptyset$. Then M^x is the unique set in $\mathbb{MCT}_{\mathcal{U}}$ which includes $\mathsf{P}(x)$. $\Rightarrow$ By definitions. For $\Leftarrow$, notice that, by assumptions, $S\neq\emptyset$. Indeed, if $S=\emptyset$, then for any $y\in\mathrm{M}^x$, $x\sqsubseteq y$, and consequently we have a contradiction: $\mathsf{PP}(x)\neq\emptyset$, and for any $y\in\mathsf{PP}(x)\subseteq\mathrm{M}^x$, $x\sqsubseteq y$. We show that for any u such that $S\subseteq\mathsf{P}(u)$ either $u\in\mathrm{M}^x$ or $x\sqsubseteq u$. Suppose that $S\subseteq\mathsf{P}(u)$, $u\notin\mathrm{M}^x$, and $x\not\sqsubseteq u$. Then $u\not\sqsubset x$ and $\emptyset\neq S\subseteq\mathsf{PP}(x)\cap\mathsf{PP}(u)$, so $x \between u$. Hence, by Lemma 4.11, $\mathrm{M}^x=\mathrm{M}^u$ and $u\in\mathrm{M}^x$. □

In [4, p. 78] it is proved that if $\sqsubset$ is transitive, then (SSP) is equivalent to the inclusion $\mathsf{sum}_{\mathsf{cl}}\subseteq\mathsf{sup}$. Below we generalize it without assuming the transitivity of $\sqsubset$.

Theorem 7.12 *On the basis of (A1)–(A3), the axiom (A4$_w$) is equivalent to the following inclusion:* $\mathsf{sum}\subseteq\mathsf{sup}$.

Proof $\Rightarrow$ Let $x \operatorname{\mathsf{sum}} S$. We consider two cases.

First, let $\mathsf{PP}(x)=\emptyset$. Then $S=\{x\}$ and $x \operatorname{\mathsf{sup}}\{x\}$, by Lemmas 7.4 and 7.11.

Second, let $\mathsf{PP}(x)\neq\emptyset$. Then $\mathsf{P}(x)\subseteq\mathrm{M}^x$ and $S\subseteq\mathsf{P}(x)\subseteq\bigcup\mathsf{O}^{\mathrm{M}^x}(S)$, by Lemma 7.4(2). Suppose that $y\in\mathrm{M}^x$ and $S\subseteq\mathsf{P}(y)$. Then, since $\mathsf{P}(x)\subseteq\bigcup\mathsf{O}^{\mathrm{M}^x}(S)$, we have $x\sqsubseteq y$, by (MP^{M}). Therefore $x \operatorname{\mathsf{sup}} S$, by Lemma 7.11(3).

$\Leftarrow$ Let $M\in\mathbb{MCT}_{\mathcal{U}}$, $x,y\in M$ and $\mathsf{P}(x)\cap M\subseteq\mathsf{O}^M(y)$. We prove that $x\sqsubseteq y$, so we obtain (SSP^{M}). We consider two cases.

First, let $\mathsf{PP}(x)=\emptyset$ or $\mathsf{PP}(x)\nsubseteq M$. Then $x\in\min_{\sqsubset}(M)$, by the assumption or Corollary 4.8(2). Hence $x\sqsubseteq y$, by (r$_\sqsubseteq$), definitions, and assumptions.

Second, let $\emptyset\neq\mathsf{PP}(x)\subseteq M$. Then $\mathsf{P}(x)\subseteq M$, so $M=\mathrm{M}^x$. Now, by Lemma 7.4(2), we prove that $x \operatorname{\mathsf{sum}} S$, where $S:=\mathrm{M}^x\cap\mathsf{P}(x)\cap\mathsf{P}(y)$. Since $S\subseteq\mathsf{P}(x)$, we must show that $\mathsf{P}(x)\subseteq\bigcup\mathsf{O}^{\mathrm{M}^x}(S)$. Let $z\in\mathsf{P}(x)$. Then, by the assumption, $z \mathrel{\bigcirc^{\mathrm{M}^x}} y$; that is, for some $u\in\mathrm{M}^x$ we have $u\sqsubseteq z$ and $u\sqsubseteq y$. So also $u\sqsubseteq x$, by transitivity of $\sqsubseteq|_{\mathrm{M}^x}$. Hence $u\in S$ and $u \mathrel{\bigcirc^{M}} z$, by (r$_\sqsubseteq$).

Now, by the assumption, $x \operatorname{\mathsf{sup}} S$. Therefore $x\sqsubseteq y$, since $S\subseteq\mathsf{P}(y)$. □

From (f-sup) and Theorem 7.12 we obtain the following.

Corollary 7.13 *The axioms (A1)–(A3) and (A4$_w$) entail*

$$\forall_{x,y\in U}\forall_{S\in 2^U}\,(x \operatorname{\mathsf{sum}} S\wedge y \operatorname{\mathsf{sup}} S\Longrightarrow x=y).$$

Remark 7.14 1. There is a separative strict partial order $\langle U,\sqsubset\rangle$ such that the following sentence is false:

$$\forall_{S\in 2^U\smallsetminus\{\emptyset\}}\forall_{x\in U}\,(x \operatorname{\mathsf{sup}} S\Rightarrow x \operatorname{\mathsf{sum}} S). \tag{$*$}$$

Thus ($*$) does not follow from (A1)–(A4) (nor from (A1)–(A5); see Section 8). We consider a separative strict partial order which is depicted in the diagram in Figure 4. Notice that the set $\{1,2\}$ has the upper bound equal to 123, while it has no mereological sum.

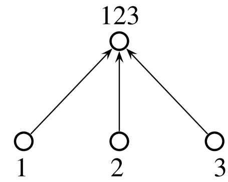

Figure 4

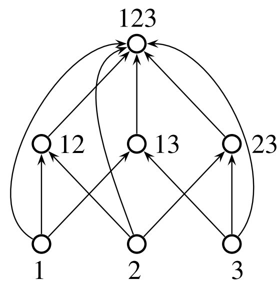

Figure 5

2. Lack of (∗) among the theorems of a theory seems to be a good option, if we want to have a theory free from existential assumptions. The diagram in Figure 4 shows that in the absence of (∗) there does not have to exist an object being a mereological sum of two objects (e.g., of 1 and 2; of 2 and 3; of 1 and 3), even in the case when these objects are parts of some third object (e.g., 123 in the structure considered). On the other hand, the presence of (∗) in a theory would entail existence of sums of the listed pairs of objects. Thus, enriching a theory with (∗) would force us to "close" the structure from Figure 4 with sums of 1 and 2; 2 and 3; 1 and 3. The structure obtained in this way would be the one in Figure 5.

8 The Fifth Axiom

By Lemma 4.2 for every $M \in \mathbb{MCT}_{\mathcal{U}}$ we have $\wr|_M \subseteq \wr^M$, $\bigcirc^M \subseteq \bigcirc|_M$, and all members of the set $\min_{\sqsubset}(M)$ are pairwise in the relation $\wr^M$. Before introducing the fifth axiom let us notice the following.

Lemma 8.1 *The axioms (A1)–(A3) imply that for any $M \in \mathbb{MCT}_{\mathcal{U}}$ the following conditions are equivalent:*

(a) all members of the set $\min_{\sqsubset}(M)$ are pairwise in the relation $\wr$,
(b) $\wr^M \subseteq \wr|_M$,
(c) $\bigcirc|_M \subseteq \bigcirc^M$.
(d) $\between|_M \subseteq \between^M$.

Proof (a) ⇒ (b) Let $x \wr^M y$ (so $x, y \in M$). If $\neg\, x \wr|_M y$, then $x \bigcirc|_M y$. So $x, y \in \min_{\sqsubset}(M)$, by Lemma 4.10. Hence, by (a), we have $x \wr y$, and consequently also $x \wr|_M y$. Therefore $x \wr|_M y$.

(b) $\Rightarrow$ (a) By Lemma 4.2(3), all members of the set $\min_{\sqsubset}(M)$ are pairwise in the relation $\wr^{M}$, so we use (b).

(b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) Lemma 4.2(2). □

We now assume the following *fifth axiom*:

(A5) every M from $\mathbb{MCT}_{\mathcal{U}}$, satisfies the (equivalent) conditions from Lemma 8.1.

Example 8.2 Referring again to Example 4.14, the members of C (cells) are minimal in the set M_1. All cells are pairwise in the relation $\wr$. Similarly, it is also the case for the members of $N \cup A$ (nuclei of cells and other parts of cells) which are minimal in the set M_2.

Fact 8.3 There is frame for which

(a) the axioms (A1)–(A4) are true;

(b) (A5) is false.

Thus, axioms (A1)–(A4) do not imply (A5).

Proof This frame is depicted in the diagram in Figure 3. Axiom (A5) is false in this frame, since $12, 23 \in \min_{\sqsubset}(M_1)$, but $12 \bigcirc 23$. □

Now we show that the axioms (A1)–(A3) and (A5) entail that the axioms (A4) and ($\mathrm{A4_w}$) are equivalent. To this end we prove the following lemma.

Lemma 8.4 *The axioms (A1)–(A3), ($A4_w$), and (A5) entail* (WSP)*.*

Proof Let $y \sqsubset x$. Then $x, y \in \mathrm{M}^x$, by Lemma 4.3. Moreover, $x \not\sqsubseteq y$, by ($\mathrm{as}_{\sqsubset}$). Hence, by ($\mathtt{SSP}^{\mathrm{M}^x}$), there is $z \in \mathrm{M}^x$ such that $z \sqsubseteq x$ and $z \wr^{\mathrm{M}^x} y$. We have that $z \neq x$, that is, $z \sqsubset x$. Moreover, $z \wr y$, by the fourth axiom and Lemma 8.1. □

Thus, by Lemmas 5.1, 5.2, and 8.4 we obtain the following.

Theorem 8.5 *The axioms (A1)–(A3) and (A5) imply that (A4) and ($A4_w$) are equivalent.*

Now notice what follows.

Theorem 8.6

1. *The axioms (A1)–(A3) and (A5) imply that for any $M \in \mathbb{MCT}_{\mathcal{U}}$,*

$$\bigcirc|_M = \bigcirc^M \qquad \textit{and} \qquad \wr|_M = \wr^M .$$

2. *The axioms (A1)–(A5) imply that for any $M \in \mathbb{MCT}_{\mathcal{U}}$,*

$$\forall_{x,y \in M}\big(x \sqsubseteq y \iff \forall_{z \in M}(z \bigcirc x \Rightarrow z \bigcirc y)\big). \qquad (\mathtt{SSP}^{M}_{\star\star})$$

Proof *Ad* 1. This is proved by Lemmas 4.2(1) and 8.1.

Ad 2. This is proved by ($\mathtt{SSP}^{M}_{\star}$) and 1. □

Remark 8.7 If we suppose that in an irreflexive frame $\mathcal{U} = \langle U, \sqsubset \rangle$ the relation $\sqsubset$ is transitive, then $\mathbb{MCT}_{\mathcal{U}} = \{U\}$, so this frame automatically fulfills the axioms (A1)–(A3) and (A5) (see Remark 4.1). Assuming then an additional axiom (SSP) (i.e., (A4)), we will consider a theory of these separative strict partial orders in which

Figure 6

the following condition holds:

$$\forall_{x,y\in U}\big(x \sqsubseteq y \iff \forall_{z\in U}(z \bigcirc x \Rightarrow z \bigcirc y)\big).$$

In this theory we assume "the classical definition" of mereological sum (see Remark 7.1).

Remark 8.8 Notice that (SSP) is not entailed by ($\mathrm{irr}_{\sqsubset}$) and ($\mathrm{t}_{\sqsubset}$) nor by (A1)–(A3) and (A5). Hence (A4) is independent from (A1)–(A3) and (A5) (see Remarks 4.1 and 8.7). Indeed, we consider a strict partial order which is depicted in a diagram on the left in Figure 6. This frame is not separative; that is, it does not satisfy (WSP) or (SSP).

Moreover, in the right part of Figure 6 we consider another frame $\mathcal{U}$ in which (A1)–(A3) and (A5) are true but (A4) is false. In this case $\mathbb{MCT}_{\mathcal{U}} = \{\{2,3\},\{1,2\}\}$.

9 Mereological Sums for Axioms (A1)–(A5)

By Lemma 7.4 and Theorem 8.6, in a theory based on axioms (A1)–(A5) we have the "classical definition" of mereological sum.

Theorem 9.1 *The axioms (A1)–(A5) imply*

$$\mathsf{sum} = \mathsf{sum}_{\mathsf{cl}}.$$

Proof $\subseteq$: This is proved by Fact 7.5.

$\supseteq$: Let $x\,\mathsf{sum}_{\mathsf{cl}}\,S$; that is, let $S \subseteq \mathsf{P}(x) \subseteq \bigcup \mathsf{O}(S)$. First, if $\mathsf{PP}(x) = \emptyset$, then we use Lemma 7.4(1). Second, if $\mathsf{PP}(x) \neq \emptyset$, then $S \subseteq \mathsf{P}(x) \subseteq \mathrm{M}^x$. Hence $\forall_{y\in\mathrm{M}^x}(y \sqsubseteq x \Rightarrow \exists_{u\in S}\, u \bigcirc|_{\mathrm{M}^x}\, y)$. So $\mathsf{P}(x) \subseteq \bigcup \mathsf{O}^{\mathrm{M}^x}(S)$, by Theorem 8.6(1). We finish the proof by applying Lemma 7.4(2). □

Notes

1. Theories presented in the paper do not have any axioms postulating existence of mereological sums (or collective sets). A sum postulated by such axioms may seem to be an ad hoc object. For example, it does not have to be the case that two objects being both parts of some third object have their mereological sum (for details see Remark 7.14). In the theories presented in this paper, only those mereological sums exist which are obtainable solely by the suitable definitions.

2. This is not, however, Leśniewski's approach. As Tarski notes, "it should be emphasized that mereology, as it was conceived by its author (i.e., Leśniewski), is not to be regarded

as a formal theory where primitive notions may admit many different interpretations" (see [8, p. 334]).

3. We say that a path $(u_0, u_1, \dots, u_n)$ is included in a subset X iff $\{u_0, u_1, \dots, u_n\} \subseteq X$. Moreover, we will say that a point u is a member of a path $(u_0, u_1, \dots, u_n)$ iff $u \in \{u_0, u_1, \dots, u_n\}$.

4. Note that $\bigcirc^M$ (resp., $\between^M$, $\wr^M$) does not have to be identical with $\bigcirc|_M$ (resp., $\between|_M$, $\wr|_M$) (see, e.g., diagrams in Figures 2 and 3).

5. The quantifier $\exists^1$ means "there is exactly one." Throughout the paper we use the description operator ι to form the expression $\ulcorner(\iota M)\varphi(M)\urcorner$, which is the individual constant "the only set M such that $\varphi(M)$." To use it, first we have to prove that there exists exactly one set M such that $\varphi(M)$, formally, $\ulcorner\exists^1_M \varphi(M)\urcorner$; that is, the formula $\varphi(M)$ must fulfill the following condition: $\ulcorner\exists_M(\varphi(M) \wedge \forall_{M'}(\varphi(M) \Rightarrow M = M'))\urcorner$.

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