

Baire Categoricity and Σ_1^0 -Induction

Stephen G. Simpson

Abstract We investigate the reverse-mathematical status of a version of the Baire category theorem known as BCT. In a 1993 paper Brown and Simpson showed that BCT is provable in RCA_0 . We now show that BCT is equivalent to RCA_0 over RCA_0^* .

Consider the following version of the Baire category theorem.

Definition 1 Let BCT be the statement that, in any complete separable metric space, the intersection of any countable sequence of dense open sets is dense. Thus BCT is essentially the usual statement of the Baire category theorem for complete separable metric spaces.

The purpose of this paper is to prove a new result concerning the reverse-mathematical status of BCT. From Brown and Simpson [1, Theorem 2.1] we already know that BCT is provable in RCA_0 . Here, of course, RCA_0 is the usual base theory for reverse mathematics (see [5]), consisting of Δ_1^0 -comprehension plus Σ_1^0 -induction. We now prove that BCT is logically equivalent to RCA_0 over the weaker base theory RCA_0^* . The system RCA_0^* was first introduced in Simpson and Smith [6]. Two recent papers making use of RCA_0^* are Simpson and Yokoyama [8] and Yokoyama [10].

In addition to BCT itself, we consider the following special case of BCT.

Definition 2 The *Cantor space* is the space $\{0, 1\}^{\mathbb{N}}$ of infinite sequences of 0's and 1's. We endow $\{0, 1\}$ with the discrete topology and $\{0, 1\}^{\mathbb{N}}$ with the product topology. Let $\text{BCT}(\{0, 1\}^{\mathbb{N}})$ be the statement that BCT holds for the Cantor space.

More precisely, let $\{0, 1\}^*$ be the set of finite sequences of 0's and 1's. For $\sigma \in \{0, 1\}^*$ and $X \in \{0, 1\}^{\mathbb{N}}$, we write $\sigma \subset X$ to mean that σ is an initial segment of X . We also write $\llbracket \sigma \rrbracket = \{X \in \{0, 1\}^{\mathbb{N}} \mid \sigma \subset X\}$. Note that the sets $\llbracket \sigma \rrbracket$

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where $\sigma \in \{0, 1\}^*$ form a basis for the topology of $\{0, 1\}^{\mathbb{N}}$. For $\sigma, \tau \in \{0, 1\}^*$, we write $\sigma \subset \tau$ to mean that σ is a proper initial segment of τ . We say that $D \subseteq \{0, 1\}^*$ is *dense in* $\{0, 1\}^*$ if for all $\sigma \in \{0, 1\}^*$ there exists $\tau \in D$ such that $\sigma \subset \tau$. Thus the dense open sets in $\{0, 1\}^{\mathbb{N}}$ are just the sets of the form $\llbracket D \rrbracket = \bigcup_{\tau \in D} \llbracket \tau \rrbracket$, where D is dense in $\{0, 1\}^*$. We say that $X \in \{0, 1\}^{\mathbb{N}}$ *meets* D if there exists $\tau \in D$ such that $\tau \subset X$. Within RCA_0^* let $\text{BCT}(\{0, 1\}^{\mathbb{N}})$ be the statement that for all sequences of dense sets $D_i \subseteq \{0, 1\}^*$, $i \in \mathbb{N}$, and all $\sigma \in \{0, 1\}^*$ there exists $X \in \{0, 1\}^{\mathbb{N}}$ such that $\sigma \subset X$ and X meets D_i for each $i \in \mathbb{N}$.

Theorem 1 *The following are pairwise equivalent over RCA_0^* :*

1. RCA_0 ,
2. BCT ,
3. $\text{BCT}(\{0, 1\}^{\mathbb{N}})$,
4. *for all finite sequences of dense sets $D_i \subseteq \{0, 1\}^*$, $1 \leq i \leq n$, there exists $X \in \{0, 1\}^{\mathbb{N}}$ such that X meets D_i for each $i = 1, \dots, n$.*

Proof We reason in RCA_0^* . The implication $1 \Rightarrow 2$ is already known (see [1, Theorem 2.1]). The implications $2 \Rightarrow 3$ and $3 \Rightarrow 4$ are obvious.

It remains to prove $4 \Rightarrow 1$. For this purpose we use the following lemma from [8]. Within RCA_0^* a set $C \subseteq \mathbb{N}$ is defined to be *infinite* if it is not finite, or equivalently, it is *unbounded*, that is, $\forall n \exists c (n < c \in C)$.

Lemma 1 *Over RCA_0^* the following are equivalent:*

1. RCA_0 ,
2. *each infinite subset of \mathbb{N} includes arbitrarily large finite subsets.*

Proof This is [8, Lemma 3.2]. □

We now prove $4 \Rightarrow 1$. Assume 4. By Lemma 1 it suffices to prove that each infinite subset of \mathbb{N} includes arbitrarily large finite subsets. Given an infinite set $C \subseteq \mathbb{N}$, for each $i \in \mathbb{N}$ let D_i be the set of strings in $\{0, 1\}^*$ of the form

$$\sigma \wedge \langle 1 \rangle \wedge \underbrace{\langle 0, \dots, 0 \rangle}_c \wedge \langle 1 \rangle \wedge \underbrace{\langle 0, \dots, 0 \rangle}_i \wedge \langle 1 \rangle, \quad (1)$$

where $c \in C$ and c is greater than the length of σ . The sequence of sets $\langle D_i \mid i \in \mathbb{N} \rangle$ exists by Δ_1^0 -comprehension. Since C is infinite, each D_i is dense in $\{0, 1\}^*$. Given $n \in \mathbb{N}$, apply 4 to obtain $X \in \{0, 1\}^{\mathbb{N}}$ such that X meets D_i for each $i = 1, \dots, n$. By Σ_1^0 -bounding (see [6]) plus Δ_1^0 -comprehension, there exists a finite sequence of strings τ_i , $1 \leq i \leq n$, such that $\tau_i \in D_i$ and $\tau_i \subset X$ for each $i = 1, \dots, n$. Consider the finite sequence c_1, \dots, c_n , where τ_i is as in (1) with $c = c_i$. For $i \neq j$, we have $\tau_i \neq \tau_j$; hence $\tau_i \subset \tau_j$ or $\tau_j \subset \tau_i$; hence $c_i < c_j$ or $c_j > c_i$; hence $c_i \neq c_j$. Thus $\{c_1, \dots, c_n\}$ is a finite subset of C of cardinality n . □

Theorem 2 *BCT is not Π_1^0 -conservative over RCA_0^* .*

Proof Recall from [5, Section X.4] and [6] that RCA_0^* is RCA_0 with Σ_1^0 -induction weakened to Σ_0^0 -induction plus *natural number exponentiation*, that is, the assertion that m^n exists for all $m, n \in \mathbb{N}$. It is known that RCA_0^* is Π_2^0 -equivalent to elementary function arithmetic (see [6]) and hence is much weaker than RCA_0 , which is Π_2^0 -equivalent to primitive recursive arithmetic (see [5, Section IX.3]). Since primitive recursive arithmetic proves the consistency of elementary function arithmetic (see,

e.g., [4], [5, Theorems II.8.11, IX.3.16]), it follows that RCA_0 is not Π_1^0 -conservative over RCA_0^* . This fact together with Theorem 1 gives Theorem 2. \square

Remarks

1. Beyond RCA_0^* one may consider even weaker base theories for reverse mathematics. In this direction there is the following result of Fernandes [2]: BCT is Π_1^1 -conservative over Σ_1^b -NIA + ∇_1^b -CA. Note that Σ_1^b -NIA + ∇_1^b -CA is “feasible”; that is, it does not include natural number exponentiation.
2. Actually, Fernandes [2] showed that BCT is conservative over Σ_1^b -NIA + ∇_1^b -CA not only for Π_1^1 -sentences but also for sentences of the form $(\forall X)(\exists \text{ unique } Y)\Phi$, where Φ is arithmetical. And Yamazaki [9] showed that Π_∞^0 -BCT is conservative over RCA_0 for this same class of sentences, which arose previously in connection with Tanaka’s conjecture (see [7, Theorem 4.18]).
3. Our Theorems 1 and 2 were inspired by Fernandes [2, Proposition 1] and Hirschfeldt, Shore, and Slaman [3, Theorem 4.3].

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Department of Mathematics
Pennsylvania State University
University Park
State College, Pennsylvania 16802
USA
t20@psu.edu
<http://www.personal.psu.edu/t20>