

Classical Consequences of Continuous Choice Principles from Intuitionistic Analysis

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Abstract The *sequential form* of a statement

$$\forall \xi (B(\xi) \rightarrow \exists \zeta A(\xi, \zeta)) \quad (\dagger)$$

is the statement

$$\forall \xi (\forall n B(\xi_n) \rightarrow \exists \zeta \forall n A(\xi_n, \zeta_n)).$$

There are many classically true statements of the form (\dagger) whose proofs lack uniformity, and therefore the corresponding sequential form is not provable in weak classical systems. The main culprit for this lack of uniformity is of course the law of excluded middle. Continuing along the lines of Hirst and Mummert, we show that if a statement of the form (\dagger) satisfying certain syntactic requirements is provable in some weak intuitionistic system, then the proof is necessarily sufficiently uniform that the corresponding sequential form is provable in a corresponding weak classical system. Our results depend on Kleene's realizability with functions and the Lifschitz variant thereof.

1 Introduction

In [1], Brouwer introduced the continuity theorem, which states that every function on the unit interval is (uniformly) continuous. While many other principles of intuitionistic analysis are classically valid (e.g., the fan theorem and the bar theorem), Brouwer's continuity theorem contradicts the law of excluded middle. Indeed, were the equality of two real numbers decidable, then the characteristic function of the singleton $\{0\}$ would be an example of a discontinuous function defined on the unit interval.

Still, many formal systems of constructive analysis either satisfy Brouwer's continuity theorem or are compatible with it. In fact, variants of the continuity theorem

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are often combined with the (classically valid) choice principles to yield continuous choice principles of the following form:

If for every ξ there is a ζ such that $A(\xi, \zeta)$, then there is a continuous function F such that $A(\xi, F(\xi))$ holds for all ξ .

When ξ and ζ are interpreted as varying over the unit interval (or the real numbers or Cantor space or Baire space), this enforces a highly constructive strength to the existential quantifier. Indeed, the continuity of F allows us to effectively translate finitary information about the argument ξ into finitary information about a witness ζ to the statement $A(\xi, \zeta)$. Thus, even in very weak systems where infinitary constructions are hardly formalizable, one can still use F to simultaneously transform an infinite sequence of arguments $\langle \xi_0, \xi_1, \dots \rangle$ into a corresponding sequence of witnesses $\langle \zeta_0, \zeta_1, \dots \rangle$ such that $A(\xi_n, \zeta_n)$ holds for every n .

This general idea was exploited by Hirst and Mummert [4] to show that if $A(\xi, \zeta)$ has a special syntactic form, then

$$\text{E-HA}^\omega + \text{AC} \vdash \forall \xi \exists \zeta A(\xi, \zeta)$$

implies

$$\text{RCA}^\omega \vdash \forall \xi \exists \zeta \forall n A(\xi_n, \zeta_n),$$

where E-HA^ω is a system Heyting arithmetic with extensional higher types that is used in proof theory (cf. Kohlenbach [7]), AC is the full axiom of choice, and RCA^ω is a variant with higher types of Friedman's classical system of recursive comprehension that is used in reverse mathematics (cf. Kohlenbach [6]).

The results of Hirst and Mummert are based on Kreisel's modified realizability and Gödel's Dialectica interpretation. In this paper, we use Kleene's realizability with functions and a Lifschitz variant thereof due to van Oosten [10] to obtain similar results. Our first result (Corollary 3.9) shows in particular that if $A(\xi, \zeta)$ satisfies certain syntactic requirements, then

$$\text{EL} + \text{GC} \vdash \forall \xi \exists \zeta A(\xi, \zeta),$$

implies

$$\text{RCA} \vdash \forall \xi \exists \zeta \forall n A(\xi_n, \zeta_n),$$

where EL is a system of intuitionistic analysis described in the next section and GC is a strong continuous choice principle that implies Brouwer's continuity theorem. Our second result (Corollary 4.9) is similar except that it incorporates the weak König lemma (WKL).¹ If $A(\xi, \zeta)$ satisfies certain syntactic requirements, then

$$\text{EL} + \text{WKL} + \text{GC}_L \vdash \forall \xi \exists \zeta A(\xi, \zeta)$$

implies

$$\text{RCA} + \text{WKL} \vdash \forall \xi \exists \zeta \forall n A(\xi_n, \zeta_n),$$

where GC_L is a weakening of GC that does not imply continuous choice but still implies Brouwer's continuity theorem. This result is very interesting since WKL is not generally recognized as a constructive principle.

2 The Systems EL and RCA

Our base system for intuitionistic analysis is a minor variant of the system EL described by Troelstra [9, Section 1.9.10]. This is a system with two sorts: numbers and (unary) functions. We will generally use roman letters a, b, c, \dots to range over number terms and Greek letters $\alpha, \beta, \gamma, \dots$ to range over function terms. The terms

of the language are built as follows:

- number variables are number terms;
- function variables are function terms;
- the zero constant 0 is a number term;
- the successor constant σ is a function term;
- if t_1, \dots, t_k are number terms and f is a symbol for a k -ary primitive recursive function, then $f(t_1, \dots, t_k)$ is a number term;
- if t is a number term and τ is a function term, then the evaluation $\tau(t)$ is a number term;
- if t is a number term and x is a number variable, then $\lambda x.t$ is a function term;
- if t is a number term and τ is a function term, then $Rt\tau$ is a function term.

The only atomic relation in our language is equality for the number sort; equality for the function sort is defined by extensionality:

$$\alpha = \beta \leftrightarrow \forall x(\alpha(x) = \beta(x)).$$

Formulas are built in the usual way for intuitionistic systems, except that we think of the disjunction $A \vee B$ as an abbreviation for

$$\exists x((x = 0 \rightarrow A) \wedge (x \neq 0 \rightarrow B)).$$

Since equality for the number sort is decidable, this is equivalent to the usual intuitionistic disjunction (see [9, Section 1.3.7]).

In addition to the usual intuitionistic logic axioms, our base systems have the usual equality axioms and the defining axioms for all primitive recursive functions. Of course, for this to make sense, the zero and successor constants must satisfy

$$\sigma(x) \neq 0 \wedge (\sigma(x) = \sigma(y) \rightarrow x = y) \quad (\text{SA})$$

and the induction scheme

$$\forall x(A(x) \rightarrow A(\sigma(x))) \rightarrow \forall x(A(0) \rightarrow A(x)), \quad (\text{IA})$$

where $A(x)$ is any formula. The last two term formation rules are governed by the λ -conversion scheme

$$(\lambda x.t)(t') = t[x/t'] \quad (\text{CON})$$

and the recursion scheme

$$(Rt\tau)(0) = t \wedge (Rt\tau)(\sigma(t')) = \tau((Rt\tau)(t')). \quad (\text{REC})$$

Moreover, we have the following choice scheme:

$$\forall x \exists y A(x, y) \rightarrow \exists \alpha \forall x A(x, \alpha(x)), \quad (\text{QF-AC}^{0,0})$$

where $A(x, y)$ is a quantifier-free formula. The system EL_0 is defined in exactly the same way, except that IA is replaced by the quantifier-free induction axiom QF-IA.

Although not part of our base systems, we will often make use of the *Markov principle*

$$\neg \neg \exists x(\alpha(x) = 0) \rightarrow \exists x(\alpha(x) = 0). \quad (\text{M})$$

This principle is a simple consequence of the *law of excluded middle* (LEM), which distinguishes classical systems from intuitionistic systems. We define RCA and RCA_0 to be the classical systems $\text{EL} + \text{LEM}$ and $\text{EL}_0 + \text{LEM}$, respectively. These are function-based systems which are equivalent to the set-based system of recursive comprehension (with full induction and just Σ_1^0 -induction, respectively) traditionally used in reverse mathematics (see Simpson [8]).

Since our basic systems have symbols for all primitive recursive functions, pairs and sequences of numbers can be encoded in the usual manner. The length of a finite sequence x is denoted by $|x|$. We write $\langle x_0, \dots, x_{n-1} \rangle$ for the finite sequence of length n whose $(i + 1)$ th term is x_i . The concatenation of x and y is denoted $x \hat{\ } y$. We will often view functions as infinite sequences of numbers. If α is a function, we write $\bar{\alpha}n$ for the finite initial segment $\langle \alpha(0), \dots, \alpha(n - 1) \rangle$.

For pairs and sequences of functions, we use the following encoding schemes. Define

$$\pi_0 = \lambda n.2n, \quad \pi_1 = \lambda n.2n + 1.$$

If α, β are two functions, then $\langle \alpha, \beta \rangle$ denotes the unique function such that $\langle \alpha, \beta \rangle \pi_0 = \alpha$ and $\langle \alpha, \beta \rangle \pi_1 = \beta$. In a similar fashion, any function α can also be viewed as an infinite sequence of functions where the $(m + 1)$ th such function is

$$\alpha_m = \lambda n.\alpha(2^m(2n + 1) - 1).$$

When it makes sense, we will write $\langle \alpha_m \rangle_{m=0}^\infty$ for the unique α whose $(m + 1)$ th component is α_m . Number–function pairs are encoded by concatenation; that is, $\langle n \rangle \hat{\ } \alpha$ denotes the unique function such that $(\langle n \rangle \hat{\ } \alpha)(0) = n$ and $(\langle n \rangle \hat{\ } \alpha)\sigma = \alpha$.

2.1 Kleene’s second algebra in EL Our results of Section 3 depend on Kleene’s realizability with functions. The base system EL is tailored to formalize this notion of realizability. To do this, we need to discuss the representation of partial continuous maps inside EL.

A function α encodes a partial continuous map from functions to numbers defined by

$$\alpha(\beta) = \alpha(\bar{\beta}n) - 1,$$

where n is the unique number such that

$$\alpha(\bar{\beta}n) \neq 0 \wedge \forall m < n(\alpha(\bar{\beta}m) = 0);$$

if there is no such n , then $\alpha(\beta)$ is undefined. We write $\alpha(\beta)\downarrow$ when $\alpha(\beta)$ is defined, and we write $\alpha(\beta)\uparrow$ when $\alpha(\beta)$ is undefined.

Similarly, α encodes a partial continuous map from functions to functions defined by

$$\alpha|\beta = \lambda n.\alpha(\langle n \rangle \hat{\ } \beta),$$

provided that $\alpha(\langle n \rangle \hat{\ } \beta)\downarrow$ for every n . We write $\alpha|\beta\downarrow$ when $\alpha|\beta$ is defined, and we write $\alpha|\beta\uparrow$ when $\alpha|\beta$ is undefined. We use the left associative convention for $|$, that is, we will write $\alpha|\beta|\gamma$ for $(\alpha|\beta)|\gamma$. Consequently, $\alpha|\beta|\gamma\downarrow$ abbreviates $\alpha|\beta\downarrow \wedge (\alpha|\beta)|\gamma\downarrow$, and so on.

Every partial continuous map F from functions to functions whose domain is a G_δ -set admits a representation of the form $F(\xi) = \varphi|\xi$. We will write $\Lambda\xi.F(\xi)$ for a function φ that represents F in this way. There are always multiple choices for φ , but in all instances of this fact that we will use there is a natural choice of φ that can be read from the description of F .

2.2 Compact sets of functions in EL Our results of Section 4 depend on the Lifschitz variant of realizability with functions due to van Oosten. To formalize this notion of realizability, we need to introduce an encoding of compact sets of functions.

Every function α encodes a compact set of functions defined by

$$[\alpha] = \{\xi \leq \alpha\pi_0 : \alpha\pi_1(\xi)\uparrow\}.$$

Formally, we think of $\xi \in [\alpha]$ as an abbreviation for the statement

$$\forall n(\xi(n) \leq \alpha\pi_0(n) \wedge \alpha\pi_1(\bar{\xi}n) = 0).$$

We will write $[\alpha] \neq \emptyset$ to assert that $[\alpha]$ is inhabited: $\exists \xi(\xi \in [\alpha])$.

For a sound theory of compact sets, we will make frequent use of the *weak König lemma*:

$$T(\alpha, \beta) \wedge \forall n \exists x(|x| = n \wedge \alpha(x) = 0) \rightarrow \exists \xi \leq \beta \forall n(\alpha(\bar{\xi}n) = 0), \quad (\text{WKL})$$

where $T(\alpha, \beta)$ says that $\{x : \alpha(x) = 0\}$ is a tree bounded by β :

$$\forall x, y(\alpha(x \hat{\ } y) = 0 \rightarrow \alpha(x) = 0 \wedge y \leq \beta(|x|)).$$

With this axiom, the statement $[\alpha] \neq \emptyset$ is equivalent to a Π_1^0 -formula.

Van Oosten [10] shows that many properties of compact sets can be formalized in the theory $\text{EL} + \text{WKL} + \text{M}$. In particular, [10, Lemma 5.7] will be useful.

Lemma 2.1 *There is a function term ι such that $\text{EL} + \text{WKL} + \text{M}$ proves that*

$$\forall \xi \in [\alpha](\varphi|\xi \downarrow) \rightarrow \iota|\langle \varphi, \alpha \rangle \downarrow \wedge \forall \zeta(\zeta \in [\iota|\langle \varphi, \alpha \rangle] \leftrightarrow \exists \xi \in [\alpha](\zeta = \varphi|\xi)).$$

In other words, if $[\alpha] \subseteq \text{dom } \varphi$, then $[\iota|\langle \varphi, \alpha \rangle] = \{\varphi|\xi : \xi \in [\alpha]\}$.

It is unclear whether the Markov principle M is necessary to establish this and other lemmas from [10].

3 Classical Consequences of GC

Troelstra's *generalized continuity principle* is the scheme

$$\forall \xi(B(\xi) \rightarrow \exists \zeta A(\xi, \zeta)) \rightarrow \exists \alpha \forall \xi(B(\xi) \rightarrow \alpha|\xi \downarrow \wedge A(\xi, \alpha|\xi)), \quad (\text{GC})$$

where $B(\xi)$ is in N_K (defined below) and $A(\xi, \zeta)$ is arbitrary. One immediate consequence of GC is that if $A(\xi, \zeta)$ defines the graph of a total function, then this function must be continuous. It follows that GC is plainly false in the classical system RCA.

However, we will momentarily define two classes of formulas N_K and Γ_K such that consequences of GC of the form

$$\forall \xi(B(\xi) \rightarrow \exists \zeta A(\xi, \zeta)),$$

where $B(\xi)$ is in N_K and $A(\xi, \zeta)$ is in Γ_K are not only consequences of RCA, but the sequential form

$$\forall \xi(\forall n B(\xi_n) \rightarrow \exists \zeta \forall n A(\xi_n, \zeta_n))$$

is also a consequence of RCA.

The proof of this fact relies on Kleene's realizability with functions (see Kleene and Vesley [5]), which is defined next.

Definition 3.1 We have the following:

- $\alpha \text{ r f } A$ is A for atomic A ,
- $\alpha \text{ r f } (A \wedge B)$ is $\alpha\pi_0 \text{ r f } A \wedge \alpha\pi_1 \text{ r f } B$,
- $\alpha \text{ r f } (A \rightarrow B)$ is $\forall \xi(\beta \text{ r f } A \rightarrow \alpha|\xi \downarrow \wedge \alpha|\xi \text{ r f } B)$,
- $\alpha \text{ r f } \forall x A$ is $\forall x(\alpha_x \text{ r f } A)$,
- $\alpha \text{ r f } \forall \xi A$ is $\forall \xi(\alpha|\xi \downarrow \wedge \alpha|\xi \text{ r f } A)$,
- $\alpha \text{ r f } \exists x A$ is $\alpha\sigma \text{ r f } A[x/\alpha(0)]$,
- $\alpha \text{ r f } \exists \xi A$ is $\alpha\pi_1 \text{ r f } A[\xi/\alpha\pi_0]$.

Note that $\alpha \text{ rff } A$ never involves existential quantifiers, except to say that $\alpha|\xi\downarrow$, in which case the scope of the existential quantifier is quantifier-free. It follows that $\alpha \text{ rff } A$ always belongs to the class \mathbf{N}_K .²

Definition 3.2

- If A is quantifier-free, then A , $\exists xA$, and $\exists\xi A$ are in \mathbf{N}_K .
- If A, B are in \mathbf{N}_K , then so are $A \wedge B$, $A \rightarrow B$, $\forall xA$, and $\forall\xi A$.

In fact, the formulas of \mathbf{N}_K are precisely the formulas which realize themselves in the sense of [9, Lemma 3.3.8].

Lemma 3.3 *If $B(\xi) \in \mathbf{N}_K$, then*

$$\text{EL} \vdash \exists\alpha(\alpha \text{ rff } B(\xi)) \leftrightarrow B(\xi).$$

In fact, there is a function term ω_B such that

$$\text{EL} \vdash B(\xi) \leftrightarrow \omega_B|\xi\downarrow \wedge \omega_B|\xi \text{ rff } B(\xi).$$

Stated in full generality, B could depend on more than one argument (and hence so would ω_B). However, this more general statement can be derived from Lemma 3.3 by packing all the arguments into one.

Kleene's realizability with functions was given the following characterization by Troelstra [9, Theorem 3.3.11].

Theorem 3.4 (Characterization of rff) *For every formula A ,*

- $\text{EL} + \text{GC} \vdash A \leftrightarrow \exists\alpha(\alpha \text{ rff } A)$,
- $\text{EL} + \text{GC} \vdash A \Leftrightarrow \text{EL} \vdash \exists\alpha(\alpha \text{ rff } A)$.

If A has the property that $\text{EL} \vdash \exists\alpha(\alpha \text{ rff } A) \rightarrow A$, then we have

$$\text{EL} + \text{GC} \vdash A \Leftrightarrow \text{EL} \vdash A.$$

Thus $\text{EL} + \text{GC}$ is conservative over EL for formulas with this property. Lemma 3.3 shows that every formula in \mathbf{N}_K has this property, but so do many other formulas.

Definition 3.5

- Quantifier-free formulas are in Γ_K .
- If A, B are in Γ_K , then so are $A \wedge B$, $\forall xA$, $\forall\xi A$, $\exists xA$, and $\exists\xi A$.
- If A is in \mathbf{N}_K and B is in Γ_K , then $A \rightarrow B$ is in Γ_K .

The following fact is implicit in [9, Theorem 3.6.18].

Lemma 3.6 *If $A \in \Gamma_K$, then $\text{EL} \vdash \exists\alpha(\alpha \text{ rff } A) \rightarrow A$.*

Thus, by the characterization of rff, it follows that GC is conservative over EL for formulas in Γ_K .

Together, the above results imply the following.

Proposition 3.7 *Suppose that $B(\xi) \in \mathbf{N}_K$ and that $A(\xi, \zeta) \in \Gamma_K$. If*

$$\text{EL} + \text{GC} \vdash \forall\xi(B(\xi) \rightarrow \exists\zeta A(\xi, \zeta)),$$

then

$$\text{EL} \vdash \exists\alpha\forall\xi(B(\xi) \rightarrow \alpha|\xi\downarrow \wedge A(\xi, \alpha|\xi)).$$

Proof By Theorem 3.4, we know that

$$\text{EL} \vdash \exists \beta (\beta \text{ rff } \forall \xi (B(\xi) \rightarrow \exists \zeta A(\xi, \zeta))).$$

Work in EL, and assume that $\beta \text{ rff } \forall \xi (B(\xi) \rightarrow \exists \zeta A(\xi, \zeta))$. Unpacking Definition 3.1, we see that

$$\gamma \text{ rff } B(\xi) \rightarrow \beta|\xi|\gamma\downarrow \wedge (\beta|\xi|\gamma)\pi_1 \text{ rff } A(\xi, (\beta|\xi|\gamma)\pi_0).$$

Since $B(\xi) \in N_K$, it follows from Lemma 3.3 that there is a term ω_B such that $B(\xi) \leftrightarrow \omega_B|\xi\downarrow \wedge \omega_B|\xi \text{ rff } B(\xi)$. Finally, since $A(\xi, \zeta) \in \Gamma_K$, it follows from Lemma 3.6 that $\alpha = \Lambda \xi. (\beta|\xi|(\omega_B|\xi))\pi_0$, as required. \square

By the deduction theorem, the above result also holds when EL is replaced by $\text{EL} + \Delta$, where Δ is any collection of sentences from N_K .

Definition 3.8 Let CN be the set of all sentences A from N_K such that $\text{RCA} \vdash A$. In other words, CN consists of all consequences of the law of excluded middle which belong to the syntactic class N_K .

Note that CN includes the Markov principle M.

Our uniformization result for this section is the following.

Corollary 3.9 *Suppose that $B(\xi)$ is from N_K and that $A(\xi, \zeta)$ is from Γ_K . If*

$$\text{EL} + \text{GC} + \text{CN} \vdash \forall \xi (B(\xi) \rightarrow \exists \zeta A(\xi, \zeta)),$$

then

$$\text{RCA} \vdash \forall \xi (\forall n B(\xi_n) \rightarrow \exists \zeta \forall n A(\xi_n, \zeta_n)).$$

Proof Suppose that

$$\text{EL} + \text{GC} + \text{CN} \vdash \forall \xi (B(\xi) \rightarrow \exists \zeta A(\xi, \zeta)).$$

By Lemma 3.7, we know that

$$\text{EL} + \text{CN} \vdash \exists \alpha \forall \xi (B(\xi) \rightarrow \alpha|\xi\downarrow \wedge A(\xi, \alpha|\xi)).$$

Now work in RCA, which extends $\text{EL} + \text{CN}$. Given α such that

$$\forall \xi (B(\xi) \rightarrow \alpha|\xi\downarrow \wedge A(\xi, \alpha|\xi)),$$

if ξ is such that $\forall n B(\xi_n)$, then $\zeta = \langle \alpha|\xi_n \rangle_{n=0}^\infty$ is such that $\forall n A(\xi_n, \zeta_n)$. It follows that

$$\text{RCA} \vdash \forall \xi (\forall n B(\xi_n) \rightarrow \exists \zeta \forall n A(\xi_n, \zeta_n)). \quad \square$$

Note that this proof gives much more than the conclusion of the theorem requires. Indeed, Proposition 3.7 is a much stronger result than Corollary 3.9. Nevertheless, Corollary 3.9 has several uses, and its proof constitutes a nice warm-up for the next section.

Remark 3.10 In reverse mathematics, it is traditional to use the base system RCA_0 , which postulates only Σ_1^0 -induction, rather than the system RCA, which postulates full induction. Unfortunately, following the proof-theoretic tradition, Troelstra assumes full induction throughout (see [9]). However, a close inspection of Troelstra's arguments shows that this assumption is not necessary to establish the characterization and conservation results for rff. Therefore, Proposition 3.7 and Corollary 3.9 have analogues with EL and RCA replaced by EL_0 and RCA_0 , respectively.

4 Classical Consequences of GC_L

Van Oosten's *Lifschitz generalized continuity principle* is the scheme

$$\forall \xi (B(\xi) \rightarrow \exists \zeta A(\xi, \zeta)) \\ \rightarrow \exists \alpha \forall \xi (B(\xi) \rightarrow \alpha | \xi \downarrow \wedge [\alpha | \xi] \neq \emptyset \wedge \forall \zeta \in [\alpha | \xi] A(\xi, \zeta)), \quad (GC_L)$$

where $B(\xi)$ is in N_L (defined below) and $A(\xi, \zeta)$ is arbitrary. Unlike GC, which offers a single witness for $\exists \zeta A(\xi, \zeta)$, GC_L offers a nonempty compact set of witnesses for $\exists \zeta A(\xi, \zeta)$. The parameter for this compact set varies continuously with ξ , but there is no general way to continuously select a single element from this compact set. Thus, GC implies GC_L , but the converse is false.

Nevertheless, GC_L still implies Brouwer's continuity theorem. Indeed, if $A(\xi, \zeta)$ describes the graph of a total function, then the compact set of witnesses produced by GC_L must be a singleton set. Since it is possible to continuously extract the unique element of a singleton set from its parameter (see [10, Lemma 5.3]), this shows that $A(\xi, \zeta)$ describes the graph of a continuous function. Like GC, it follows that GC_L is also classically false.

Similarly to the case of GC, we will define two classes of formulas N_L and Γ_L such that consequences of $EL + WKL + M + GC_L$ of the form

$$\forall \xi (B(\xi) \rightarrow \exists \zeta A(\xi, \zeta)),$$

where $B(\xi)$ is in N_L and $A(\xi, \zeta)$ is in Γ_L , are not only consequences of $RCA + WKL$, but the sequential form

$$\forall \xi (\forall n B(\xi_n) \rightarrow \exists \zeta \forall n A(\xi_n, \zeta_n))$$

is also a consequence of $RCA + WKL$. The proof of this fact relies on Lifschitz realizability with functions, which was introduced by van Oosten [10].

Definition 4.1 We have the following:

- $\alpha \text{ lrf } A$ is A for atomic A ,
- $\alpha \text{ lrf } (A \wedge B)$ is $\alpha \pi_0 \text{ lrf } A \wedge \alpha \pi_1 \text{ lrf } B$,
- $\alpha \text{ lrf } (A \rightarrow B)$ is $\forall \beta (\beta \text{ lrf } A \rightarrow \alpha | \beta \downarrow \wedge \alpha | \beta \text{ lrf } B)$,
- $\alpha \text{ lrf } \forall x A$ is $\forall x (\alpha_x \text{ lrf } A)$,
- $\alpha \text{ lrf } \forall \xi A$ is $\forall \xi (\alpha | \xi \downarrow \wedge \alpha | \xi \text{ lrf } A)$,
- $\alpha \text{ lrf } \exists x A$ is $[\alpha] \neq \emptyset \wedge \forall \beta \in [\alpha] (\beta \sigma \text{ lrf } A[x/\beta(0)])$,
- $\alpha \text{ lrf } \exists \xi A$ is $[\alpha] \neq \emptyset \wedge \forall \beta \in [\alpha] (\beta \pi_1 \text{ lrf } A[\xi/\beta \pi_0])$.

The analogue of the class N_K is the broader class N_L .³

Definition 4.2

- If A is quantifier-free, then $A, \exists x A, \exists \xi A$ are in N_L .
- If A is quantifier-free and τ is a function term in which ξ does not occur, then $\exists \xi \leq \tau \forall z A$ is in N_L . Similarly, if A is quantifier-free and t is a number term in which x does not occur, then $\exists x \leq t \forall z A$ is in N_L .
- If A, B are in N_L , then so are $A \wedge B, A \rightarrow B, \forall x A$, and $\forall \xi A$.

With the aid of the second clause, the disjunction of one or more Π_1^0 -statements can be formulated in N_L . Thus, statements like the dichotomy law for Cauchy real numbers (discussed in Section 5) can be expressed in N_L but not in N_K .

Again, the formula $\alpha \text{ rrf } A$ is always in N_L . In fact, the formulas of N_L are precisely the formulas which realize themselves in the following sense (see [10, Lemma 5.12]).

Lemma 4.3 *If $B(\xi) \in N_L$, then*

$$\text{EL} + \text{WKL} + \text{M} \vdash \exists \alpha (\alpha \text{ rrf } B(\xi)) \leftrightarrow B(\xi).$$

In fact, there is a function term ω_B such that

$$\text{EL} + \text{WKL} + \text{M} \vdash B(\xi) \leftrightarrow \omega_B | \xi \downarrow \wedge \omega_B | \xi \text{ rrf } B(\xi).$$

Again, there is a more general form of this which allows B to have more than one parameter, but this can be derived from the above by packing all arguments into one.

Lifschitz realizability with functions was characterized by van Oosten [10, Theorem 5.15].⁴

Theorem 4.4 (Characterization of rrf) *For every formula A ,*

- (a) $\text{EL} + \text{WKL} + \text{M} + \text{GC}_L \vdash A \leftrightarrow \exists \alpha (\alpha \text{ rrf } A)$,
- (b) $\text{EL} + \text{WKL} + \text{M} + \text{GC}_L \vdash A \Leftrightarrow \text{EL} + \text{WKL} + \text{M} \vdash \exists \alpha (\alpha \text{ rrf } A)$.

Again, it is unclear whether M is necessary for this characterization of rrf .

The class Γ_L is defined as follows.

Definition 4.5

- Quantifier-free formulas are in Γ_L .
- If A, B are in Γ_L , then so are $A \wedge B$, $\forall x A$, $\forall \xi A$, $\exists x A$, and $\exists \xi A$.
- If A is in N_L and B is in Γ_L , then $A \rightarrow B$ is in Γ_L .

Together with the characterization of rrf , the following fact shows that GC_L is conservative over $\text{EL} + \text{WKL} + \text{M}$ for formulas in Γ_L .

Lemma 4.6 *If $A \in \Gamma_L$, then $\text{EL} + \text{WKL} + \text{M} \vdash \exists \alpha (\alpha \text{ rrf } A) \rightarrow A$.*

Proof sketch The proof of this lemma is a straightforward induction on the complexity of A . We prove only the implication case.

Work in $\text{EL} + \text{WKL} + \text{M}$. Suppose that $\alpha \text{ rrf } (B \rightarrow A)$, where A is from Γ_L and B is from N_L . We need to show that $B \rightarrow A$. By definition of rrf , we then have that if $\beta \text{ rrf } B$, then $\alpha | \beta \downarrow$ and $\alpha | \beta \text{ rrf } A$. Assume B . By Lemma 4.3, there is a function term ω such that $\omega \text{ rrf } B$. It follows that $\alpha | \omega \downarrow$ and $\alpha | \omega \text{ rrf } A$. Therefore A , by the induction hypothesis. \square

Together, the above results imply the following.

Proposition 4.7 *Suppose that $B(\xi) \in N_L$ and that $A(\xi, \zeta) \in \Gamma_L$. If*

$$\text{EL} + \text{WKL} + \text{M} + \text{GC}_L \vdash \forall \xi (B(\xi) \rightarrow \exists \zeta A(\xi, \zeta)),$$

then

$$\text{EL} + \text{WKL} + \text{M} \vdash \exists \alpha \forall \xi (B(\xi) \rightarrow \alpha | \xi \downarrow \wedge [\alpha | \xi] \neq \emptyset \wedge \forall \zeta \in [\alpha | \xi] A(\xi, \zeta)).$$

Proof By Theorem 4.4, we know that

$$\text{EL} + \text{WKL} + \text{M} \vdash \exists \beta (\beta \text{ rrf } \forall \xi (B(\xi) \rightarrow \exists \zeta A(\xi, \zeta))).$$

Work in $\text{EL} + \text{WKL} + \text{M}$, and assume that $\beta \text{ rrf } \forall \xi (B(\xi) \rightarrow \exists \zeta A(\xi, \zeta))$. Unpacking the definition of rrf , we see that if $\gamma \text{ rrf } B(\xi)$, then $\beta | \xi \downarrow$, $[\beta | \xi | \gamma] \neq \emptyset$, and

$$\forall \zeta \in [\beta | \xi | \gamma] (\zeta \pi_1 \text{ rrf } A(\xi, \zeta \pi_0)).$$

Since $B(\xi) \in \mathbf{N}_L$, it follows from Lemma 4.3 that there is a term ω_B such that

$$B(\xi) \leftrightarrow \omega_B | \xi \downarrow \wedge \omega_B | \xi \uparrow \text{rf } B(\xi).$$

Finally, since $A(\xi, \zeta) \in \Gamma_L$, it follows from Lemma 4.6 that

$$\alpha = \Lambda \xi. \iota \{ \Lambda \zeta. \zeta \pi_0. \beta | \xi | (\omega_B | \xi) \}$$

is as required, where ι is as in Lemma 2.1. \square

As for Proposition 3.7, we can add to the theories in Proposition 4.7 any collection of sentences from \mathbf{N}_L .

Definition 4.8 Let \mathbf{CN}_L be the collection of all sentences A from \mathbf{N}_L such that $\text{RCA} + \text{WKL} \vdash A$. In other words, \mathbf{CN}_L consists of all consequences of the law of excluded middle which belong to the syntactic class \mathbf{N}_L .

Note that \mathbf{CN}_L includes the Markov principle M as well as the lesser limited principle of omniscience (LLPO) (see Section 5).

Corollary 4.9 *Suppose that $B(\xi)$ is from \mathbf{N}_L and that $A(\xi, \zeta)$ is from Γ_L . If*

$$\text{EL} + \text{WKL} + \text{GC}_L + \mathbf{CN}_L \vdash \forall \xi (B(\xi) \rightarrow \exists \zeta A(\xi, \zeta)),$$

then

$$\text{RCA} + \text{WKL} \vdash \forall \xi (\forall n B(\xi_n) \rightarrow \exists \zeta \forall n A(\xi_n, \zeta_n)).$$

Proof Suppose that

$$\text{EL} + \text{WKL} + \text{GC}_L + \mathbf{CN}_L \vdash \forall \xi (B(\xi) \rightarrow \exists \zeta A(\xi, \zeta)).$$

By Proposition 4.7, we know that

$$\text{EL} + \text{WKL} + \mathbf{CN}_L \vdash \exists \alpha \forall \xi (B(\xi) \rightarrow \alpha | \xi \downarrow \wedge [\alpha | \xi] \neq \emptyset \wedge \forall \zeta \in [\alpha | \xi] A(\xi, \zeta)).$$

Now work in $\text{RCA} + \text{WKL}$, which extends $\text{EL} + \text{WKL} + \mathbf{CN}_L$. Find α such that if $B(\xi)$, then

$$\alpha | \xi \downarrow \wedge [\alpha | \xi] \neq \emptyset \wedge \forall \zeta \in [\alpha | \xi] A(\xi, \zeta).$$

If ξ is such that $\forall n B(\xi_n)$, then

$$\forall n (\alpha | \xi_n \downarrow \wedge [\alpha | \xi_n] \neq \emptyset).$$

By [8, Lemma VIII.2.4], we can find a ζ such that $\zeta_n \in [\alpha | \xi_n]$ for every n . It then follows that $\forall n A(\xi_n, \zeta_n)$. We have just shown that

$$\text{RCA} + \text{WKL} \vdash \forall \xi (\forall n B(\xi_n) \rightarrow \exists \zeta \forall n A(\xi_n, \zeta_n)). \quad \square$$

Remark 4.10 As in Remark 3.10, it would be desirable to eliminate the induction assumptions from Corollary 4.9. Unfortunately, van Oosten's arguments from [10] do appear to make some use of this inductive assumption. Close inspection reveals that these uses are limited to Π_1^0 -bounding; therefore, Corollary 4.9 does have an analogue with $\text{RCA} + \text{WKL}$ replaced by $\text{RCA}_0 + \text{WKL} + \text{B}\Pi_1^0$.

5 Applications

To compare the earlier results of Hirst and Mummert with ours, it is useful to compare the syntactic restrictions involved, specifically [4, Theorem 3.6], since the syntactic conditions for [4, Theorem 5.6] are even more restrictive.

The analogue of N_K and N_L for Hirst and Mummert are \exists -free formulas: formulas built in the usual manner but without the use of existential quantifiers or disjunctions. The \exists -free formulas are a proper subset of N_K and hence N_L since some existential quantifiers are allowed by the first clause of Definition 3.2, and still more are allowed by the second clause of Definition 4.2.

The analogue of Γ_K and Γ_L for Hirst and Mummert is the class Γ_1 , which is defined in exactly the same way, except that hypotheses of conditionals are restricted to \exists -free formulas. Thus, Γ_1 is also a proper subset of Γ_K and hence Γ_L .

To illustrate the difference, consider this familiar statement:

Every $(n \times n)$ -matrix with nonzero determinant has an inverse.

The “nonzero determinant” hypothesis is not expressible by an \exists -free formula, since to say that a Cauchy real or complex number (see below) is apart from zero requires an existential quantifier. However, this hypothesis is expressible in N_K . Since EL proves that every $(n \times n)$ -matrix with nonzero determinant has an inverse, it follows that the sequential form of the above statement is provable in RCA. The reader should not feel too enlightened by this simple example since the obvious proof is nothing more than Cramer’s rule.

On the other hand, the results of Hirst and Mummert allow for higher types, while ours only involve first-order and second-order types. Therefore, there is a vast sea of statements for which the results of Hirst and Mummert apply but ours do not. Still, the nonprovability examples that Hirst and Mummert give are all second-order, so they all have equivalents in our context. In particular, neither $EL + GC + CN$ nor $EL + WKL + GC_L + CN_L$ prove that every $(n \times n)$ -matrix has a Jordan canonical form.

5.1 Trichotomy and dichotomy for Cauchy reals A *Cauchy real* is a rational-valued function α such that $|\alpha(s) - \alpha(t)| \leq 2^{-s}$ for all $s < t$. We write $\alpha \in \mathbb{R}^C$ to abbreviate the statement that α is a Cauchy real. If $\alpha, \beta \in \mathbb{R}^C$, then we define

$$\alpha = \beta \leftrightarrow \forall s (|\alpha(s) - \beta(s)| \leq 2^{1-s}).$$

We also define

$$\alpha > \beta \leftrightarrow \exists s (\alpha(s) - \beta(s) > 2^{1-s})$$

and

$$\alpha \leq \beta \leftrightarrow \forall s (\alpha(s) - \beta(s) \leq 2^{1-s}).$$

Note that $\alpha \leq \beta \leftrightarrow \neg(\alpha > \beta)$ and $\alpha > \beta \rightarrow \neg(\alpha \leq \beta)$, but the implication $\neg(\alpha \leq \beta) \rightarrow \alpha > \beta$ is equivalent to the Markov principle M.

The *trichotomy law*

$$\alpha < \beta \vee \alpha = \beta \vee \alpha > \beta$$

and the formally weaker *dichotomy law*

$$\alpha \leq \beta \vee \alpha \geq \beta$$

are both consequences of the law of excluded middle. However, over EL_0 these are respectively equivalent to the *limited principle of omniscience*

$$\exists n(\xi(n) \neq 0) \vee \forall n(\xi(n) = 0) \quad (\text{LPO})$$

and the *lesser limited principle of omniscience*

$$\neg(\exists n(\xi(n) \neq 0) \wedge \exists n(\zeta(n) \neq 0)) \rightarrow \forall n(\xi(n) = 0) \vee \forall n(\zeta(n) = 0) \quad (\text{LLPO})$$

(see [2] for details).

Proposition 5.1 *The following equivalent statements are both provable in RCA_0 , but neither is provable in $EL + GC + CN$:*

- (a) *the dichotomy law for Cauchy reals,*
- (b) *the lesser limited principle of omniscience.*

Proof The dichotomy law can be stated as

$$\forall \alpha, \beta (\alpha, \beta \in \mathbb{R}^C \rightarrow \exists y ((y = 0 \rightarrow \alpha \leq \beta) \wedge (y \neq 0 \rightarrow \alpha \geq \beta))).$$

Inspection shows that this is in the form required for Corollary 3.9. Dorais, Hirst, and Shafer [2] have shown that the corresponding sequential form is equivalent to WKL over RCA_0 ; it follows that the dichotomy law is not provable in $EL + GC + CN$. \square

Proposition 5.2 *The following equivalent statements are both provable in RCA , but neither is provable in $EL + WKL + GC_L + CN_L$:*

- (a) *the trichotomy law for Cauchy reals,*
- (b) *the limited principle of omniscience.*

Proof For all $\alpha, \beta \in \mathbb{R}^C$, the trichotomy law can be stated as

$$\exists y ((y = 0 \rightarrow \alpha < \beta) \wedge (y = 1 \rightarrow \alpha > \beta) \wedge (y > 1 \rightarrow \alpha = \beta)).$$

This statement is in the form required for Corollary 4.9. Dorais, Hirst, and Shafer [2] have shown that the corresponding sequential form is equivalent to arithmetic comprehension over RCA_0 ; it follows that the trichotomy law is not provable in $EL + WKL + GC_L + CN_L$. \square

5.2 Dedekind reals and Cauchy reals A *Dedekind real* is a decidable set δ of rationals such that

$$\exists p, q \in \mathbb{Q} (p \in \delta \wedge q \notin \delta) \wedge \forall p, q \in \mathbb{Q} (p \in \delta \wedge q \notin \delta \rightarrow p < q).$$

We write $\delta \in \mathbb{R}^D$ to abbreviate the fact that δ is a Dedekind real. We say that a Cauchy real α and a Dedekind real δ are equivalent when

$$\forall s \in \mathbb{N} \forall p, q \in \mathbb{Q} (p \in \delta \wedge q \notin \delta \rightarrow \neg(\alpha(s) + 2^{1-s} < p < q < \alpha(s) - 2^{1-s})).$$

Proposition 5.3

- (a) EL_0 *proves that every Dedekind real has an equivalent Cauchy real.*
- (b) $EL_0 + WKL$ *proves that every Cauchy real has an equivalent Dedekind real.*

Proof The proof of part (a) is straightforward, so we only prove part (b).

Suppose that α is a Cauchy real. Fix an enumeration $\langle q_i \rangle_{i=0}^\infty$ of \mathbb{Q} . Let $R(\alpha, x)$ denote the statement

$$\begin{aligned} \forall i < |x| ((x(i) = 0 \vee x(i) = 1) \\ \wedge (\exists s \leq |x| (q_i < \alpha(s) - 2^{1-s}) \rightarrow x(i) = 1) \\ \wedge (\exists s \leq |x| (q_i > \alpha(s) + 2^{1-s}) \rightarrow x(i) = 0)). \end{aligned}$$

Then the decidable set $\delta = \{q_i : \xi(i) = 1\}$ is a Dedekind real equivalent to α if and only if $\forall n R(\alpha, \bar{\xi}n)$. Since $\forall n \exists x (|x| = n \wedge R(\alpha, x))$, it follows from WKL that there is a Dedekind real δ which is equivalent to α . \square

Of course, RCA_0 proves that every Cauchy real has an equivalent Dedekind real. However, the usual proof of this fact is nonuniform since it relies on first deciding whether or not the Cauchy real represents a rational number. Such lack of uniformity is actually necessary, as the next proposition shows.

Proposition 5.4 *The system $\text{EL} + \text{GC} + \text{CN}$ does not prove that every Cauchy real has an equivalent Dedekind real.*

Proof Formally, the statement that every Cauchy real has an equivalent Dedekind real is as follows: for every $\alpha \in \mathbb{R}^C$ there is a $\delta \in \mathbb{R}^D$ such that

$$\forall p, q \in \mathbb{Q} \forall s \in \mathbb{N} (p \in \delta \wedge q \notin \delta \rightarrow \neg(\alpha(s) + 2^{1-s} < p \vee q < \alpha(s) - 2^{1-s})).$$

Inspection shows that this has the right form for Corollary 3.9. However, Hirst [3] has shown that the sequential form of this statement is equivalent to WKL over RCA_0 . It follows that the statement is not provable in $\text{EL} + \text{GC} + \text{CN}$. \square

Note that dichotomy is trivially true for Dedekind reals (simply check in which half 0 is). Thus, Proposition 5.4 is actually a corollary of Proposition 5.1.

5.3 The fundamental theorem of algebra Cauchy complex numbers are pairs $\langle \xi_0, \xi_1 \rangle$, where $\xi_0, \xi_1 \in \mathbb{R}^C$. These are intended to represent the real and imaginary parts of the complex number. Thus, we write $\xi \in \mathbb{C}^C$ to abbreviate $\xi\pi_0, \xi\pi_1 \in \mathbb{R}^C$. Addition and multiplication on complex numbers are defined as usual; it is not difficult to check that EL_0 proves that \mathbb{C}^C is a field. However, $\text{EL} + \text{GC}$ does not prove that \mathbb{C}^C is algebraically complete.

Proposition 5.5 *$\text{EL} + \text{GC} + \text{CN}$ does not prove that every complex number has a square root.*

Proof Suppose, on the contrary, that $\text{EL} + \text{GC}$ does prove that every complex number has a square root. It follows from Proposition 3.7 that EL proves the existence of some α such that

$$(\forall \xi)(\xi \in \mathbb{C}^C \rightarrow \alpha|\xi \downarrow \wedge \alpha|\xi \in \mathbb{C}^C \wedge (\alpha|\xi)^2 = \xi).$$

Since this statement is in Γ_K , it follows that EL proves the existence of such an α . This is impossible since the axioms of EL are classically valid and there is no total continuous function on the complex numbers that selects one of the two square roots of its argument. \square

The use of Proposition 3.7 instead of Corollary 3.9 was necessary for this argument since RCA_0 does prove the sequential form

$$\forall \xi (\forall n (\xi_n \in \mathbb{C}^{\mathbb{C}}) \rightarrow \exists \zeta \forall n (\zeta_n \in \mathbb{C}^{\mathbb{C}} \wedge \xi_n = \zeta_n^2)).$$

In particular, the converse of Corollary 3.9 is false.

While the fundamental theorem of algebra is not provable in $\text{EL} + \text{GC} + \text{CN}$, it is provable in $\text{EL} + \text{WKL}$.

Proposition 5.6 $\text{EL} + \text{WKL}$ proves that

$$\forall \xi_1, \dots, \xi_n \in \mathbb{C}^{\mathbb{C}} \exists \zeta \in \mathbb{C}^{\mathbb{C}} (\zeta^n + \xi_1 \zeta^{n-1} + \dots + \xi_n = 0).$$

This is because EL proves that for any coefficients $\xi_1, \dots, \xi_n \in \mathbb{C}^{\mathbb{C}}$, there is a function α such that

$$[\alpha] = \{\zeta \in \mathbb{C}^{\mathbb{C}} : \zeta^n + \xi_1 \zeta^{n-1} + \dots + \xi_n = 0\}.$$

Then, by proving the existence of approximate roots, $\text{EL} + \text{WKL}$ proves that $[\alpha] \neq \emptyset$.

Notes

1. Note that in the reverse mathematics literature, WKL is normally used as an abbreviation for RCA together with the weak König lemma. We will avoid this practice since we also want to consider the weak König lemma in intuitionistic systems.
2. Elements of N_K are called “almost negative formulas” by Troelstra [9].
3. Elements of N_L are called “ $\text{B}\Sigma_2^1$ -negative formulas” by van Oosten.
4. Note that the statement of [10, Theorem 5.15(ii)] has a typo, which is corrected in our statement of Theorem 4.4(b).

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