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Independence, Relative Randomness, and PA Degrees

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Abstract We study pairs of reals that are mutually Martin-Löf random with respect to a common, not necessarily computable probability measure. We show that a generalized version of van Lambalgen's theorem holds for noncomputable probability measures, too. We study, for a given real A, the *independence spectrum* of A, the set of all B such that there exists a probability measure μ so that $\mu\{A, B\} = 0$ and (A, B) is $(\mu \times \mu)$-random. We prove that if A is computably enumerable (c.e.), then no Δ^0_2-set is in the independence spectrum of A. We obtain applications of this fact to PA degrees. In particular, we show that if A is c.e. and P is of PA degree so that $P \not\geq_T A$, then $A \oplus P \geq_T \emptyset'$.

1 Independence and Relative Randomness

The property of independence is central to probability theory. Given a probability space with measure μ, we call two measurable sets $\mathcal{A}$ and $\mathcal{B}$ *independent* if

$$\mu\mathcal{A} = \frac{\mu(\mathcal{A} \cap \mathcal{B})}{\mu\mathcal{B}}.$$

The idea behind this definition is that if event $\mathcal{B}$ occurs, it does not make event $\mathcal{A}$ any more or less likely. This paper considers a similar notion, that of relative randomness. The theory of algorithmic randomness provides a means of defining which elements of Cantor space (2^ω) are random. We call $A \in 2^\omega$ *Martin-Löf random* if A is not an element of any effective null set. We denote the class of all Martin-Löf random reals by MLR.[1]

We say that A is *Martin-Löf random relative* to B, or $A \in \mathrm{MLR}(B)$ if A is not an element of any null set effective in B. Relative randomness is analogous to independence because if $A \in \mathrm{MLR}(B)$, then not only is A a random real but *even given* the information in B, we cannot capture A in an effective null set. If we start

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with the assumption that A and B are both Martin-Löf random, then the following theorem of van Lambalgen establishes that relative randomness is symmetrical.

Theorem 1.1 (van Lambalgen [12]) *If* $A, B \in \mathrm{MLR}$, *then* $A \in \mathrm{MLR}(B)$ *if and only if* $B \in \mathrm{MLR}(A)$ *if and only if* $A \oplus B \in \mathrm{MLR}$.

We can extend the notion of relative randomness to any probability measure. We take $\mathcal{P}(2^\omega)$ to be the set of all Borel probability measures on Cantor space. Endowed with the weak-$*$ topology, $\mathcal{P}(2^\omega)$ becomes a compact metrizable space. The measures that are a finite, rational-valued, linear combination of Dirac measures form a countable dense subset, and one can choose a metric on $\mathcal{P}(2^\omega)$ that is compatible with the weak-$*$ topology so that the distance between those basic measures is a computable function, and with respect to which $\mathcal{P}(2^\omega)$ is complete. In other words, $\mathcal{P}(2^\omega)$ can be given the structure of an *effective Polish space*. We can represent measures via Cauchy sequences of basic measures. This allows for coding measures as reals, and one can show that there exists a continuous, surjective mapping $\rho : 2^\omega \to \mathcal{P}(2^\omega)$ such that for any $X \in 2^\omega$,

$$\rho^{-1}\big(\{\rho(X)\}\big) \quad \text{is a } \Pi^0_1(X)\text{-class.}$$

For details of this argument, see Day and Miller [2]. If $\mu \in \mathcal{P}(2^\omega)$, any real R with $\rho(R) = \mu$ is called a *representation* of μ.

We want to define randomness relative to a parameter with respect to a probability measure μ. Martin-Löf's framework easily generalizes to tests that have access to an oracle. However, our test should have access to *two* sources: the parameter of relative randomness and the measure (in the form of a representation).

Definition 1.2 Let R_μ be a representation of a measure μ, and let $A \in 2^\omega$.

1. An (R_μ, A)-*test* is given by a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ of uniformly $\Sigma^0_1(R_\mu \oplus A)$-classes $\mathcal{V}_n \subseteq 2^\omega$ such that for all n, $\mu(\mathcal{V}_n) \leq 2^{-n}$.
2. A real $X \in 2^\omega$ *passes* an (R_μ, A)-test $(\mathcal{V}_n)$ if $X \notin \bigcap_n \mathcal{V}_n$.
3. A real $X \in 2^\omega$ is (R_μ, A)-*random*, or R_μ-*random relative to* A, if it passes all (R_μ, A)-tests.

If, in the previous definition, $A = \emptyset$, we simply speak of an R_μ-*test* and of X being R_μ-*random*.

The previous definition defines randomness with respect to a specific representation. If X is random for one representation, it is not necessarily random for other representations. On the other hand, we can ask whether a real exhibits randomness with respect to *some* representation, so the following definition makes sense.

Definition 1.3 A real $X \in 2^\omega$ is μ-*random relative to* $A \in 2^\omega$, or simply μ-A-*random*, if there exists a representation R_μ of μ such that X is (R_μ, A)-random. We denote by $\mathrm{MLR}_\mu(A)$ the set of all μ-A-random reals.

For Lebesgue measure λ, we sometimes suppress the measure. Hence, in accordance with established notation, $\mathrm{MLR}(A)$ denotes the set of all Martin-Löf random reals.

A most useful property of the theory of Martin-Löf randomness is the existence of *universal tests*. Universal tests subsume all other tests. Furthermore, they can be defined uniformly with respect to any parameter. The construction can be extended to tests with respect to a measure μ. More precisely, there exists a uniformly c.e.

sequence $(U_n : n \in \mathbb{N})$ of sets $U_n \subseteq 2^{<\omega}$ such that if we set, for $R, A \in 2^\omega$,

$$\mathcal{U}_n^{R,A} = \{[\sigma] : \langle \sigma, \tau_0, \tau_1 \rangle \in U_n, \tau_0 \prec R, \tau_1 \prec A\},$$

then $(\mathcal{U}_n^{R,A})$ is an (R,A)-test and $X \in 2^\omega$ is (R,A)-random if and only if $X \notin \bigcap_n \mathcal{U}_n^{R,A}$. We call (U_n) a *universal oracle test*.

Since for any $R \in 2^\omega$, $\rho^{-1}(\rho(R))$ is $\Pi^0_1(R)$, we can eliminate the representation of a measure in a test for randomness by defining, for any $A \in 2^\omega$,

$$\widetilde{\mathcal{U}}_n^{R,A} = \bigcap_{S \in \rho^{-1}(\{\rho(R)\})} \mathcal{U}_n^{S,A}.$$

The resulting class $\widetilde{\mathcal{U}}_n^A$ is still $\Sigma^0_1(R)$, since $\rho^{-1}(\{\rho(R)\})$ is $\Pi^0_1(R)$ and hence compact.

Proposition 1.4 *For any $R, A \in 2^\omega$ with $\rho(R) = \mu$, a real X is μ-A-random if and only if*

$$X \notin \bigcap_n \widetilde{\mathcal{U}}_n^{R,A}.$$

Proof If X is μ-A-random, then it passes every (R_μ, A)-test for some representation R_μ of μ, in particular, the instance $(\mathcal{U}_n^{R_\mu,A})$ of the universal oracle test. Since $R_\mu \in \rho^{-1}(\{\rho(R)\})$, it follows that X passes $\widetilde{\mathcal{U}}_n^{R,A}$.

On the other hand, if for every representation R_μ of μ, X fails the test $(\mathcal{U}_n^{R,A})$, then $X \in \bigcap_n \widetilde{\mathcal{U}}_n^{R,A}$. □

Proposition 1.4 shows that the test $\widetilde{\mathcal{U}}_n^{R,A}$ is related to the concept of a *uniform test*, originally introduced by Levin [8], and further developed by Gács [4] and Hoyrup and Rojas [5]. Hence we call it a *uniform oracle test*. Note that if R, S are both representations of a measure μ, then the uniform oracle tests $(\widetilde{\mathcal{U}}_n^{R,A})_n$ and $(\widetilde{\mathcal{U}}_n^{S,A})_n$ are identical.

Definition 1.5 Take $A, B \in 2^\omega$ and $\mu \in \mathcal{P}(2^\omega)$. We say that A and B are *relatively random with respect to μ* if $A \in \mathrm{MLR}_\mu(B)$ and $B \in \mathrm{MLR}_\mu(A)$.

Note that the representations of μ witnessing randomness for A and B, respectively, do not have to be identical. If A and B are relatively random with respect to some measure μ, then μ might offer some information about the relationship between A and B. For example, we know that if A and B are relatively random with respect to Lebesgue measure, then any real they both compute must be K-trivial.

A trivial case of relative randomness occurs when A and B are both *atoms* of the underlying measure. A real X is an atom of a measure μ if $\mu\{X\} > 0$. A measure with no atoms is called *nonatomic*. Given this, perhaps the most obvious question to ask about relative randomness is the following.

Question 1.6 For which $A, B \in 2^\omega$ does there exist a measure μ such that A and B are relatively random with respect to μ and neither A nor B is an atom of μ?

This question is closely related to a theorem of Reimann and Slaman [10]. They proved that an element X of Cantor space is noncomputable if and only if there exists a measure μ such that X is μ-random and X is not an atom of μ.

Van Lambalgen's theorem shows that A and B are relatively random if and only if $A \oplus B \in \mathrm{MLR}$. If we take λ to be the uniform measure, then $A \oplus B \in \mathrm{MLR}$

if and only if the pair $(A, B) \in 2^\omega \times 2^\omega$ is Martin-Löf random with respect to the product measure $\lambda \times \lambda$, that is, $(A, B) \in \mathrm{MLR}_{\lambda\times\lambda}$. We begin our investigation into relative randomness by showing that van Lambalgen's theorem holds for any Borel probability measure on Cantor space.

Theorem 1.7 *Let $\mu \in \mathcal{P}(2^\omega)$, and let $A, B \in 2^\omega$. Then $(A, B) \in \mathrm{MLR}_{\mu\times\mu}$ if and only if $A \in \mathrm{MLR}_\mu$ and $B \in \mathrm{MLR}_\mu(A)$, that is, if and only if A and B are relatively random with respect to μ.*

Proof Let R be any representation of μ. First, let us consider if $B \notin \mathrm{MLR}_\mu(A)$. In this case we have that $B \in \bigcap_n \mathcal{U}_n^{R,A}$. We define an $(R, \emptyset)$-test for $2^\omega \times 2^\omega$ by $\mathcal{V}_n^R = \{[\tau] \times [\sigma] : \exists\eta \prec R(\langle\sigma, \eta, \tau\rangle \in U_n)\}$. This ensures that $(A, B) \in \bigcap_n \mathcal{V}_n^R$. By applying Fubini's theorem we can establish that

$$(\mu \times \mu)(\mathcal{V}_n^R) = \int_{2^\omega\times 2^\omega} \chi_{\mathcal{V}_n^R}(X, Y)\, d\mu \times d\mu$$
$$= \int_{2^\omega} \Big(\int_{2^\omega} \chi_{\mathcal{U}_n^{R,X}}(Y)\, d\mu(Y)\Big)\, d\mu(X)$$
$$\leq \int_{2^\omega} 2^{-n}\, d\mu(X) = 2^{-n}.$$

Hence (A, B) is not $(R, \emptyset)$-random. As this is true for any representation R of μ, we have $(A, B) \notin \mathrm{MLR}_{\mu\times\mu}$. The same argument shows a fortiori that if $A \notin \mathrm{MLR}_\mu$, then $(A, B) \notin \mathrm{MLR}_{\mu\times\mu}$.

To establish the other direction, assume that $(A, B) \notin \mathrm{MLR}_{\mu\times\mu}$. Again let R be any representation of μ.

Hence $(A, B) \in \bigcap_n \mathcal{V}_n^R$, where $(\mathcal{V}_n^R)$ is a universal R-test for $2^\omega \times 2^\omega$. Let

$$\mathcal{W}_n^{R,X} = \big\{Y : (X, Y) \in \mathcal{V}_n^R\big\}.$$

We have that $\mathcal{W}_n^{R,X}$ is a $\Sigma_1^0(R \oplus X)$-class and this is uniform in n. However, given any X, we do not know whether or not $\mu(\mathcal{W}_n^{R,X}) \leq f(n)$ for some decreasing computable function f such that $\lim_n f(n) = 0$. Hence we cannot necessarily turn this into a Martin-Löf test relative to X. In fact, it is not even necessarily true that $\liminf_n \mu(\mathcal{W}_n^{R,X}) = 0$. We will show that the failure to turn this into a Martin-Löf test for some $X \in 2^\omega$ implies that $X \notin \mathrm{MLR}_\mu$. This is a slight strengthening of the result that van Lambalgen obtained in his thesis. Van Lambalgen showed that if $\liminf_n \mu(\mathcal{W}_n^{R,X}) \neq 0$, then $X \notin \mathrm{MLR}_\mu$.

However, we can generalize the proof of van Lambalgen's theorem given in Nies [9]. We define another R-test by letting

$$\mathcal{T}_n^R = \big\{X \in 2^\omega : \mu(\mathcal{W}_{2n}^{R,X}) > 2^{-n}\big\}.$$

To see that $\mathcal{T}_n^R \leq 2^{-n}$, note that

$$(\mu \times \mu)\mathcal{V}_{2n}^R \geq \int_{\mathcal{T}_n^R\times 2^\omega} \chi_{\mathcal{V}_{2n}^R}(X, Y)\, d\mu \times d\mu$$
$$= \int_{\mathcal{T}_n^R} \int_{2^\omega} \chi_{\mathcal{W}_{2n}^{R,X}}(Y)\, d\mu(Y)\, d\mu(X)$$
$$\geq \int_{\mathcal{T}_n^R} 2^{-n}\, d\mu(X) = 2^{-n}\mu(\mathcal{T}_n^R).$$

Now as $2^{-2n} \geq (\mu \times \mu)\mathcal{V}^R_{2n}$, we have $\mu(\mathcal{T}^R_n) \leq 2^{-n}$. Hence $(\mathcal{T}^R_n)$ is an R-test. Assume that A is not R-random. Then A avoids all but finitely many of the sets $\mathcal{T}^R_n$. Hence for all but finitely many n we have $\mu\mathcal{W}^{R,A}_{2n} \leq 2^{-n}$, and so by modifying finitely many $\mathcal{W}^{R,A}_{2n}$ we can obtain an (R, A)-test that covers B. Therefore B is not R-random relative to A.

For all representations R of μ, we have shown that either A is not R-random or B is not R-random relative to A. However, to prove the theorem, it is essential that we get the *same* outcome for all representations; that is, if $(A, B) \notin \mathrm{MLR}_\mu$, then either for all representations R of μ, A is not R-random or for all representations R of μ, B is not R-random relative to A.

We can resolve this problem by taking our test $(\mathcal{V}^R_n)$ on the product space to be a uniform oracle test. (A uniform oracle test on $2^\omega \times 2^\omega$ can be defined analogously to the uniform oracle test on 2^ω defined above.) In this case we always obtain the same "projection tests" $(\mathcal{W}^{R,X}_n)$ (independent of R) and hence the same outcome for any representation of μ. □

Corollary 1.8 *If $A \geq_T B$ and $(A, B) \in \mathrm{MLR}_{\mu\times\mu}$, then B must be an atom of μ.*

Proof This holds because $B \in \mathrm{MLR}_\mu(A)$ if and only if B is an atom of μ. □

We note that we cannot extend one direction of van Lambalgen's theorem to product measures of the form $\mu \times \nu$. In particular, it is not true that if $A \in \mathrm{MLR}_\mu$ and $B \in \mathrm{MLR}_\nu(A)$, then $(A, B) \in \mathrm{MLR}_{\mu\times\nu}$. For example, it is possible to code a real B into a measure μ in such a way that B is computable from every representation of μ and choose a measure ν so that there exist $A \in \mathrm{MLR}_\mu$, $B \in \mathrm{MLR}_\nu(A)$, with $\mu\{A\} = \nu\{B\} = 0$. But for any such μ, ν we cannot have $(A, B) \in \mathrm{MLR}_{\mu\times\nu}$, since any $\mu \times \nu$-test has access to μ and hence can compute B.

Given any $X \in 2^\omega$, we will use $\mathcal{R}(X)$ to denote the set of reals Y such that X and Y are relatively random with respect to some measure μ and neither X nor Y are atoms of μ; that is,

$$\mathcal{R}(A) = \{B \in 2^\omega : (\exists\mu \in \mathcal{P}(2^\omega))[(A, B) \in \mathrm{MLR}_{\mu\times\mu}, \text{ and } \mu\{A\} = \mu\{B\} = 0]\}.$$

We call $\mathcal{R}(A)$ the *independence spectrum of A*.

The following proposition lists some basic properties of the independence spectrum.

Proposition 1.9 *For all $A, B \in 2^\omega$ the following hold:*

(1) $A \in \mathcal{R}(B)$ if and only if $B \in \mathcal{R}(A)$;
(2) $B \in \mathcal{R}(A)$ implies that $A \mid_T B$;
(3) if A is noncomputable and ν is a nonatomic measure with a computable representation, then $\mathcal{R}(A)$ has ν-measure 1;
(4) if $A \in \mathrm{MLR}$, then $\mathrm{MLR}(A) \subsetneq \mathcal{R}(A)$.

Proof Condition (1) is by definition, and (2) is by Corollary 1.8.

For (3) suppose that A is noncomputable and ν is a computable measure with $\nu\{A\} = 0$. There is a measure μ such that A is not an atom of μ and $A \in \mathrm{MLR}_\mu$, say, via a representation R_μ. Let $\kappa = (\mu + \nu)/2$. There exists a representation $R_\kappa \leq_T R_\mu$, as ν is computable. We claim that A is R_κ-random. For if not, then A fails some R_κ-test $(\mathcal{W}^{R_\kappa}_n)$. We have

$$\mu\mathcal{W}^{R_\kappa}_n = 2\kappa\mathcal{W}^{R_\kappa}_n - \nu\mathcal{W}^{R_\kappa}_n \leq 2\kappa\mathcal{W}^{R_\kappa}_n \leq 2^{n-1}.$$

Since $R_\kappa \leq_T R_\mu$, $(\mathcal{W}_{n+1}^{R_\kappa})$ would define an R_μ-test that covers A, contradicting the assumption that A is R_μ-random. Furthermore, by assumption on μ and ν, $\kappa\{A\} = 0$. Hence

$$\big(\mathrm{MLR}_\kappa(A) \setminus \{B : \kappa\{B\} \neq 0\}\big) \subseteq \mathcal{R}(A)$$

by van Lambalgen's theorem.

Now $\nu(\mathrm{MLR}_\kappa(A)) = 1$ because the complement of $\mathrm{MLR}_\nu(A)$ is a κ-null set and hence a ν-null set (ν is absolutely continuous with respect to κ by definition). Moreover, the set of atoms of κ is countable and so has ν-measure 0 by the assumption that ν is nonatomic. This gives us

$$\nu\big(\mathrm{MLR}_\kappa(A) \setminus \{B : \kappa\{B\} \neq 0\}\big) = 1$$

and thus $\nu\mathcal{R}(A) = 1$.

For (4) suppose that A is Martin-Löf random. By the definition of $\mathcal{R}(A)$ and Theorem 1.7 we have $\mathrm{MLR}(A) \subseteq \mathcal{R}(A)$.

On the other hand, A is not computable, and hence by (3), $\mathcal{R}(A)$ has measure 1 for any computable, nonatomic measure. Let ν be a computable, nonatomic measure orthogonal to Lebesgue measure (e.g., the $(1/3, 2/3)$-Bernoulli measure). Since $\nu\mathcal{R}(A) = 1$, $\mathcal{R}(A)$ has to contain a ν-random element X. But X cannot be relatively Martin-Löf random. Therefore, $\mathrm{MLR}(A) \subsetneq \mathcal{R}(A)$. □

The proposition shows that, outside the upper and lower cone of a real A, the complement of $\mathcal{R}(A)$ is rather small measurewise. On the other hand, the above properties leave open the possibility that $\mathcal{R}(A)$ is just the set of reals that are Turing incomparable with A. We will now establish that this is not necessarily the case.

Proposition 1.10 *Let R be a representation of a measure μ. If $A \in 2^\omega$ is such that*

1. *A is c.e.,*[2]
2. *A is R-random, and*
3. *A is not an atom of μ,*

then $R \oplus A \geq_T R'$.

Proof Given such an R and A, let A_s be a computable approximation to A. We define the function $f \leq_T A \oplus R$ by

$$f(x) = \min\big\{s : (\exists m \leq s)\big(A_s \restriction m = A \restriction m \wedge \mu_s[A \restriction m] < 2^{-x}\big)\big\}.$$

In this definition we take $\mu_s[\sigma]$ to be an R-computable approximation to $\mu[\sigma]$ from above. Note that f is well defined because A is not an atom of μ. We claim that if g is any partial function computable in R, then for all but finitely many $x \in \mathrm{dom}(g)$, we have $f(x) > g(x)$. To establish this claim, let g be an R-computable partial function. We will build an R-test $\{U_n\}_{n\in\omega}$ by defining U_n to be

$$\big\{X \in 2^\omega : (\exists x > n)(\exists m)\big(g(x)\downarrow \wedge \mu[A_{g(x)} \restriction m] < 2^{-x} \wedge X \succ (A_{g(x)} \restriction m)\big)\big\}.$$

Because any $x \in \mathrm{dom}(g)$ adds a single open set ($[A_{g(x)} \restriction m]$ for some m) of measure less than 2^{-x} to those U_n with $n < x$, we have constructed a valid test. Now if $g(x) \downarrow\geq f(x)$, then by definition of f, there is some $m \leq f(x)$ such that $\mu[A \restriction m] < 2^{-x}$ and $A \restriction m = A_{f(x)} \restriction m = A_{g(x)} \restriction m$. Here we use the fact that A is c.e. Thus for all $n < x$, $A \in U_n$. Because A is R-random, we have $f(x) > g(x)$ for all but finitely many x in $\mathrm{dom}(g)$.

Let $g(x)$ be the R-computable partial function with domain R' such that $g(x)$ is the unique s such that $x \in R'_{s+1} \setminus R'_s$. For almost all x, we have $x \in R'$ if and only if $x \in R'_{f(x)}$, and so $R' \leq_T A \oplus R$. □

Theorem 1.11 *Let R be a representation of a measure μ. If*

1. *A is c.e.,*
2. *A is μ-random, and*
3. *A is not an atom of μ,*

then $R \oplus A \geq_T \emptyset'$.

Proof Note the following characteristics of the previous proof. First, the totality of f does not depend on the fact that A is R-random; it only depends on the fact that A is not an atom of μ. The construction is uniform, so there is a single index e such that $\Phi_e(A \oplus \hat{R})$ is total if $\hat{R}$ is any representation of μ. Additionally, if A is $\hat{R}$-random, then for all but finitely many x, $\Phi_e(A \oplus \hat{R}; x) \geq g(x)$, where g is any $\hat{R}$-computable partial computable function.

Let R be any representation of μ. The set $\{A \oplus \hat{R} : \hat{R}$ is a representation of $\mu\}$ is a $\Pi^0_1(A \oplus R)$-class, and Φ_e is total on this class. From $A \oplus R$ we can compute a function f that dominates $\Phi_e(A \oplus \hat{R})$, where A is $\hat{R}$-random. As f dominates any $\hat{R}$-computable partial function, we have $A \oplus R \geq_T \emptyset'$. □

Corollary 1.12 *If A is c.e. and $B \leq_T \emptyset'$, then $B \notin \mathcal{R}(A)$.*

The question remains, however, how big the independence spectrum of a real can be outside its upper and lower cones.

Question 1.13 Is the set of all X such that $X \mid_T A$ and $X \notin \mathcal{R}(A)$ countable?

2 Computably Enumerable Sets and PA Degrees

We will now give two (somehow unexpected) applications of Theorem 1.11 to the interaction between c.e. sets and sets of PA degree. Recall that a set $A \subseteq \mathbb{N}$ is of *PA degree* if it is Turing equivalent to a set coding a complete extension of Peano arithmetic (PA). PA degrees have many interesting computability-theoretic properties. For instance, a set is of PA degree if and only if it computes a path through every nonempty Π^0_1-class. However, a complete degree-theoretic characterization of the PA degrees is still not known. If $A \geq_T \emptyset'$, then A is of PA degree. On the other hand, Gödel's first incompleteness theorem implies that no c.e. set can be a complete extension of PA. Jockusch and Soare [6] showed, moreover, that if a set is of incomplete c.e. degree, it cannot be of PA degree. This also follows from the Arslanov completeness criterion, since any PA degree computes a fixed-point free function (see [1]).

It seems therefore worthwhile to gain a complete understanding how c.e. sets and PA degrees are related. The crucial fact that links Theorem 1.11 to PA degrees is a result by Day and Miller [2]. They showed that every set of PA degree computes a representation of a *neutral measure*. Such a measure has the property that *every* real is random with respect to it, that is, $2^\omega = \mathrm{MLR}_\mu$. The existence of neutral measures was first established by Levin [8].

Our first result shows that below $\emptyset'$, c.e. sets and PA degrees behave quite complementary.[3]

Corollary 2.1 (to Theorem 1.11) *If A is a c.e. set and P a set of* PA *degree such that $P \not\geq_{\mathrm{T}} A$, then $P \oplus A \geq_{\mathrm{T}} \emptyset'$.*

Proof By the result of Day and Miller [2] mentioned above, P computes a representation R_μ of a neutral measure μ and $A \in \mathrm{MLR}_\mu$. Day and Miller [2] also showed that a real X is an atom of a neutral measure if every representation of the measure computes X. Now because $P \not\geq_{\mathrm{T}} A$, we have that A is not an atom of μ. Thus all hypotheses of Theorem 1.11 are satisfied, and we have $P \oplus A \geq_{\mathrm{T}} \emptyset'$. □

Corollary 2.1 strengthens a result due to Kučera and Slaman (unpublished). Recall that a function $f : \mathbb{N} \to \mathbb{N}$ is *diagonally noncomputable* if $f(n) \neq \varphi_n(n)$ for all n, where φ_n denotes, as usual, the nth partial computable function. Kučera and Slaman constructed a low$_2$ c.e. set so that $A \oplus f \equiv_{\mathrm{T}} \emptyset'$ for any diagonally noncomputable function $f \leq_{\mathrm{T}} \emptyset'$. It is well known that every PA degree computes a $\{0, 1\}$-valued diagonally noncomputable function. Hence the set constructed by Kučera and Slaman joins any PA degree below $\emptyset'$ to $\emptyset'$. As we will see below (Corollary 2.4), Corollary 2.1 yields that this is in fact true for any nonlow c.e. set.

One can ask which kind of incomplete c.e. sets *can* be bounded by PA degrees below $\emptyset'$. This question was first raised by Kučera [7].

For which incomplete c.e. sets A does there exist a set P of PA degree such that $A <_{\mathrm{T}} P <_{\mathrm{T}} \emptyset'$?

We can use Corollary 2.1 to completely answer this question. We say that a set B is of *PA degree relative* to a set A, written $B \gg A$ (see Simpson [11]), if B computes a path through every nonempty $\Pi^0_1(A)$-class. One well-known fact we will make use of is the following. If P is of *PA* degree, then there exists a set Q of *PA* degree such that $P \gg Q$. One way to prove this fact is to observe that the Π^0_1-class

$$\big\{(A, B) \in 2^\omega \times 2^\omega \colon A \in \mathrm{DNR}_2 \wedge B \in \mathrm{DNR}_2(A)\big\}$$

is nonempty, where DNR_2 and $\mathrm{DNR}_2(A)$ are the classes of $\{0, 1\}$-valued diagonally noncomputable functions and $\{0, 1\}$-valued diagonally noncomputable functions relative to A, respectively.

Theorem 2.2 *If A is a c.e. set, then the following are equivalent.*

(1) A is low.
(2) There exist P, $P \gg A$, and P is low.
(3) There exists P of PA degree such that $\emptyset' >_{\mathrm{T}} P >_{\mathrm{T}} A$.

Proof (1) ⇒ (2): There is a (nonempty) $\Pi^0_1(A)$-class of sets $B \gg A$. Relativize the low basis theorem to find $P \gg A$ and $P' \equiv_{\mathrm{T}} A'$. As A is low, so is P.

(2) ⇒ (3): This is clear.

(3) ⇒ (1): Take any Q of PA degree such that $P \gg Q$. Now $Q \geq_{\mathrm{T}} A$ because otherwise $Q \oplus A \geq \emptyset'$, but this is impossible because $P \geq_{\mathrm{T}} Q \oplus A$ and $P \not\geq_{\mathrm{T}} \emptyset'$. Hence $P \gg A$. But now we have that $\emptyset'$ is c.e. in A and also that $\emptyset'$ computes a DNR function relative to A. Hence by relativizing Arslanov's completeness criterion we have $A' \equiv_{\mathrm{T}} \emptyset'$. □

Observe that in the proof of (3) ⇒ (1), showing $P \gg A$ only used the facts that $P \not\geq_T \emptyset'$ and $P \geq_T A$. Hence we get the following corollary.

Corollary 2.3 *If P is a set of PA degree and A is a c.e. set such that $P \geq_T A$ and $P \not\geq_T \emptyset'$, then $P \gg A$.*

We also obtain the strengthening of the Kučera-Slaman result mentioned above.

Corollary 2.4 *If A is a nonlow c.e. set, then $A \oplus P \equiv_T \emptyset'$ for all $P \leq_T \emptyset'$ of PA degree.*

Notes

1. For a comprehensive presentation of the theory of Martin-Löf randomness, see the monographs by Downey and Hirschfeldt [3] and Nies [9].

2. We mean here, of course, that A is computably enumerable viewed as a subset of $\mathbb{N}$, by identifying a subset of $\mathbb{N}$ with the real given by its characteristic sequence.

3. After the authors announced the result presented in Corollary 2.1, proofs not involving measure-theoretic arguments have been found independently by A. Kučera and J. Miller.

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