Notre Dame Journal of Formal Logic
Volume 54, Numbers 3–4, 2013

Elimination of Hyperimaginaries and Stable Independence in Simple CM-Trivial Theories

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Abstract In a simple CM-trivial theory every hyperimaginary is interbounded with a sequence of finitary hyperimaginaries. Moreover, such a theory eliminates hyperimaginaries whenever it eliminates finitary hyperimaginaries. In a supersimple CM-trivial theory, the independence relation is stable.

1 Introduction

An important notion introduced by Shelah for a first-order theory is that of an *imaginary* element: the class of a finite tuple by a ∅-definable equivalence relation. The construction obtained by adding all imaginary elements to a structure does not change its basic model-theoretic properties but introduces a convenient context and language to talk about quotients (by definable equivalence relations) and *canonical parameters* of definable sets. In the context of a stable theory it also ensures the existence of *canonical bases* for arbitrary complete types, generalizing the notion of a field of definition of an algebraic variety.

The generalization of stability theory to the wider class of simple theories necessitated the introduction of hyperimaginaries, classes of countable tuples modulo ∅-type-definable equivalence relations. Although the relevant model theory for hyperimaginaries has been reasonably well understood (see Hart, Kim, and Pillay [5]), they cannot simply be added as extra sorts to the underlying structure, since inequality of two hyperimaginaries amounts to nonequivalence, and thus a priori is an open, but not a closed condition. While hyperimaginary elements are needed for the general theory, all known examples of a simple theory *eliminate* them in the sense that they are interdefinable (or at least interbounded) with a sequence of ordinary imaginaries; the latter condition is called *weak elimination*. The following question has thus been asked (and even been conjectured).

Received December 12, 2011; accepted January 13, 2012
2010 Mathematics Subject Classification: Primary 03C45
Keywords: simple, CM-trivial, hyperimaginaries, stable forking
 10.1215/00294527-2143970

Question Do all simple theories eliminate hyperimaginaries?

The answer is positive for stable theories (see Pillay and Poizat [13]) and for supersimple theories (see Buechler, Pillay, and Wagner [1]). Among nonsimple theories, the relation of being infinitely close in a nonstandard real closed field gives rise to noneliminable hyperimaginaries; Casanovas and the second author have constructed noneliminable hyperimaginaries in a theory without the strict order property in [3].

A hyperimaginary is *finitary* if it is the class of a finite tuple modulo a type-definable equivalence relation. Kim [6] has shown that small theories eliminate finitary hyperimaginaries, and a result of Lascar and Pillay [9] states that bounded hyperimaginaries can be eliminated in favor of finitary bounded ones. We shall show that in a CM-trivial simple theory all hyperimaginaries are interbounded with sequences of finitary hyperimaginaries. We shall deduce that in such a theory hyperimaginaries can be eliminated in favor of finitary ones. In particular, a small CM-trivial simple theory eliminates hyperimaginaries. However, even the question of whether all one-based simple theories eliminate hyperimaginaries is still open.

Elimination of hyperimaginaries is closely related to another question, the *stable forking conjecture*.

Question In a simple theory, if $a \not\mathop{\smile\!\!\!\!\!|}_B M$ for some model M containing B, is there a stable formula in $\mathrm{tp}(a/M)$ which forks over B?

If we do not require M to be a model or to contain B, this is called *strong stable forking*. Every known simple theory has stable forking; Kim [7] has shown that one-based simple theories with elimination of hyperimaginaries have stable forking. Kim and Pillay [8] have strengthened this to show that one-based simple theories with weak elimination of imaginaries hyperimaginaries have strong stable forking; on the other hand, pseudo-finite fields (which are supersimple of SU-rank 1) do not. Conversely, stable forking implies weak elimination of hyperimaginaries (see [8]).

While we shall not attack the stable forking conjecture as such, we shall show in the last section that the independence relation $x \mathop{\smile\!\!\!\!\!|}_{y_1} y_2$ is stable, meaning that it cannot order an infinite indiscernible sequence.

2 Preliminaries

As usual, we shall work in the monster model $\mathfrak{C}$ of a complete first-order theory (with infinite models), and all sets of parameters and all sequences of elements will live in $\mathfrak{C}^{\mathrm{eq}}$. Given any sequences a, b and any set of parameters A, we write $a \equiv_A b$ whenever a and b have the same type over A. We shall write $a \equiv^{\mathrm{s}}_A b$ if in addition a and b lie in the same class modulo all A-definable finite equivalence relations (i.e., if a and b have the same *strong type* over A), and $a \equiv^{\mathrm{Ls}}_A b$ if they lie in the same class modulo all A-invariant bounded equivalence relations (i.e., if a and b have the same *Lascar strong type* over A). Recall that a theory is G-compact over a set A iff $\equiv^{\mathrm{Ls}}_A$ is type-definable over A (in which case it is the finest bounded equivalence relation type-definable over A). A theory T is G-compact whenever it is G-compact over any A. In particular, simple theories are G-compact (see [6]).

Definition 2.1 A hyperimaginary h is *finitary* if $h \in \mathrm{dcl}^{\mathrm{heq}}(a)$ for some finite tuple a of imaginaries, and *quasi-finitary* if $h \in \mathrm{bdd}(a)$ for some finite tuple a of imaginaries.

Definition 2.2 A hyperimaginary h is *eliminable* if it is interdefinable with a sequence $e = (e_i : i \in I)$ of imaginaries, that is, if there is such a sequence e with $\mathrm{dcl}^{\mathrm{heq}}(e) = \mathrm{dcl}^{\mathrm{heq}}(h)$. A theory T eliminates (finitary/quasi-finitary) hyperimaginaries if all (finitary/quasi-finitary) hyperimaginaries are eliminable in all models of T.

Remark 2.3 ([9, Corollary 1.5]) If $h \in \mathrm{dcl}^{\mathrm{heq}}(a)$, then there is a type-definable equivalence relation E on $\mathrm{tp}(a)$ such that h and the class a_E of a modulo E are interdefinable.

Lemma 2.4 *Let e be a finitary hyperimaginary. If T eliminates finitary hyperimaginaries, then $T(e)$ eliminates finitary hyperimaginaries.*

Proof Let a be a finite tuple with $e \in \mathrm{dcl}^{\mathrm{heq}}(a)$, and let h be a finitary hyperimaginary over e. So there is a finite tuple b with $h \in \mathrm{dcl}^{\mathrm{heq}}(eb) \subseteq \mathrm{dcl}^{\mathrm{heq}}(ab)$. Then there is a type-definable equivalence relation E over $\emptyset$ such that e and a_E are interdefinable, and a type-definable equivalence relation F_a over a such that h and b_{F_a} are interdefinable. Moreover, F_a only depends on the E-class of a; that is, if $a'Ea$, then $F_{a'} = F_a$.

Type-define an equivalence relation by

$$xy\bar{E}uv \Leftrightarrow xEu \wedge yF_x v.$$

It is easy to see that h is interdefinable with $(ab)_{\bar{E}}$ over e. Moreover, $(ab)_{\bar{E}}$ is clearly finitary and hence eliminable in T. So h is eliminable in $T(e)$. □

The following fact appears in [9, Proof of Proposition 2.2], but was first stated as such in [1, Lemma 2.17].

Fact 2.5 Let h be a hyperimaginary, and let a be a sequence of imaginaries such that $a \in \mathrm{bdd}(h)$ and $h \in \mathrm{dcl}^{\mathrm{heq}}(a)$. Then, h is eliminable.

Fact 2.6 ([1, Lemma 2.18]) Let h, e be hyperimaginaries with $h \in \mathrm{bdd}(e)$. Then the set of e-conjugates of h is interdefinable with a hyperimaginary h'.

Fact 2.7 ([9, Theorem 4.15]) A bounded hyperimaginary is interdefinable with a sequence of finitary hyperimaginaries.

Proposition 2.8 *If T eliminates finitary hyperimaginaries, then T eliminates quasi-finitary hyperimaginaries.*

Proof Let h be a quasi-finitary hyperimaginary, and let a be a finite tuple of imaginaries such that $h \in \mathrm{bdd}(a)$. By Evans and Hrushovski [4, Lemma 1.4] there is $a' \equiv_h a$ with $\mathrm{acl}^{\mathrm{eq}}(a) \cap \mathrm{acl}^{\mathrm{eq}}(a') = \mathrm{acl}^{\mathrm{eq}}(h)$. Let h' be the hyperimaginary corresponding to the set of aa'-conjugates of h. Then h' is aa'-invariant and hence finitary. It is thus interdefinable with a sequence e of imaginaries.

On the other hand, $h \in \mathrm{bdd}(a) \cap \mathrm{bdd}(a')$, as are all its aa'-conjugates. Thus $h' \in \mathrm{bdd}(a) \cap \mathrm{bdd}(a')$, and so $e \subseteq \mathrm{acl}^{\mathrm{eq}}(a) \cap \mathrm{acl}^{\mathrm{eq}}(a') = \mathrm{acl}^{\mathrm{eq}}(h)$. Hence $e \subseteq \mathrm{acl}^{\mathrm{eq}}(h)$ and $h \in \mathrm{bdd}(h') = \mathrm{bdd}(e)$. By Fact 2.7, there is a sequence h'' of finitary hyperimaginaries interdefinable with h over e. By Lemma 2.4 and elimination of finitary hyperimaginaries we see that h'' is interdefinable over e with a sequence e' of imaginaries. So $h \in \mathrm{dcl}^{\mathrm{heq}}(ee')$ and $e' \in \mathrm{dcl}^{\mathrm{eq}}(eh)$. Moreover, $ee' \in \mathrm{acl}^{\mathrm{eq}}(h)$ since $e \in \mathrm{acl}^{q}(h)$. Hence h is eliminable by Fact 2.5. □

The following remarks and lemmas will need G-compactness.

Remark 2.9 Let T be G-compact over a set A. The following are equivalent:

(1) $a \equiv^{\mathrm{Ls}}_A b$ iff $a \equiv^{\mathrm{s}}_A b$ for all sequences a, b;
(2) $\mathrm{Aut}(\mathfrak{C}/\,\mathrm{bdd}(A)) = \mathrm{Aut}(\mathfrak{C}/\,\mathrm{acl}^{\mathrm{eq}}(A))$;
(3) $\mathrm{bdd}(A) = \mathrm{dcl}^{\mathrm{heq}}(\mathrm{acl}^{\mathrm{eq}}(A))$.

Proof This is an easy exercise. □

Remark 2.10 Let T be a G-compact theory, and assume further that $a \equiv^{\mathrm{Ls}}_A b \Leftrightarrow a \equiv^{\mathrm{s}}_A b$ for all sequences a, b and for any set A. Let now h be a hyperimaginary, and let e be a sequence of imaginaries such that h and e are interbounded. Then h is eliminable.

Proof It follows from Remark 2.9 that $\mathrm{bdd}(e) = \mathrm{dcl}^{\mathrm{heq}}(\mathrm{acl}^{\mathrm{eq}}(e))$. Fix an enumeration $\bar{e}$ of $\mathrm{acl}^{\mathrm{eq}}(e)$, and observe that $h \in \mathrm{dcl}^{\mathrm{heq}}(\bar{e})$ and $\bar{e} \in \mathrm{bdd}(h)$. Then apply Fact 2.5 to eliminate h. □

It turns out for G-compact theories that elimination of hyperimaginaries can be decomposed as weak elimination of hyperimaginaries plus the equality between Lascar strong types and strong types over parameter sets.

Fact 2.11 ([2, Proposition 18.27]) Assume that T is G-compact. Then T eliminates all bounded hyperimaginaries iff $a \equiv^{\mathrm{Ls}} b \Leftrightarrow a \equiv^{\mathrm{s}} b$ for all sequences a, b.

Proof The proof in Casanovas [2] is nice and intuitive; however, we will give another one using Remark 2.10. If T eliminates bounded hyperimaginaries, then $\mathrm{Aut}(\mathfrak{C}/\,\mathrm{bdd}(\emptyset)) = \mathrm{Aut}(\mathfrak{C}/\,\mathrm{acl}^{\mathrm{eq}}(\emptyset))$. By Remark 2.9 we get Lstp $=$ stp. For the other direction, let $e \in \mathrm{bdd}(\emptyset)$, and let $\bar{a}$ be an enumeration of $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$. It is clear that e and $\bar{a}$ are interbounded. By Remark 2.10, e is eliminable. □

Lemma 2.12 *Suppose that T is G-compact, and assume further that T eliminates finitary hyperimaginaries. Then $a \equiv^{\mathrm{Ls}}_A b$ iff $a \equiv^{\mathrm{s}}_A b$ for all sequences a, b and for any set A.*

Proof Since T is G-compact, it is enough to check the condition for finite A. But then $T(A)$ eliminates finitary hyperimaginaries by Remark 2.4 and hence all bounded hyperimaginaries by Fact 2.7. Now applying Fact 2.11 we obtain $a \equiv^{\mathrm{Ls}} b$ iff $a \equiv^{\mathrm{s}} b$ in $T(A)$. □

3 Elimination of Hyperimaginaries in Simple Theories

In this section T will be a simple theory. Recall that the *canonical base* of a over b, denoted $\mathrm{Cb}(a/b)$, is the smallest definably closed subset C of $\mathrm{bdd}(b)$ such that $a \perp_C b$ and $\mathrm{tp}(a/C)$ is Lascar strong.

Lemma 3.1 *For any a and any $h \in \mathrm{bdd}(c)$ we have $\mathrm{Cb}(a/h) \subseteq \mathrm{dcl}(ac) \cap \mathrm{bdd}(h)$. Therefore, the canonical base of the type of an imaginary finite tuple over a quasi-finitary hyperimaginary is finitary. Furthermore, if $b \in \mathrm{Cb}(a/c)$, then $\mathrm{dcl}(ab) \cap \mathrm{bdd}(b) \subseteq \mathrm{Cb}(a/c)$. In particular, if $c \in \mathrm{dcl}(a)$, then $\mathrm{Cb}(a/c) = \mathrm{dcl}(a) \cap \mathrm{bdd}(c)$.*

Proof Since $h \in \mathrm{bdd}(c)$, equality of Lascar strong types over c refines equality of Lascar strong types over h, and the class of a modulo the former is clearly in $\mathrm{dcl}(ac)$. So the class of a modulo the latter is in $\mathrm{dcl}(ac)$, and $\mathrm{Cb}(a/h) \in \mathrm{dcl}(ac) \cap \mathrm{bdd}(h)$. As a consequence, if a is a finite tuple and h is a quasi-finitary hyperimaginary bounded over some finite tuple c, then $\mathrm{Cb}(a/h)$ is definable over the finite tuple ac.

For the second assertion put $b' = \mathrm{dcl}(ab) \cap \mathrm{bdd}(b)$. Since $b' \in \mathrm{dcl}(ab)$, there is an equivalence relation E on $\mathrm{tp}(a/b)$ type-definable over b such that b' is interdefinable over b with a_E. As $b' \in \mathrm{bdd}(b)$ and $b \in \mathrm{Cb}(a/c)$, the E-class of a is bounded over $\mathrm{Cb}(a/c)$; as $\mathrm{tp}(a/\mathrm{Cb}(a/c))$ is Lascar strong, $a_E \in \mathrm{Cb}(a/c)$.

The "in particular" clause is essentially [1, Remark 3.8]. If $c \in \mathrm{dcl}(a)$, then clearly $c \in \mathrm{Cb}(a/c)$; the assertion follows. □

Recall the definition of CM-triviality.

Definition 3.2 A simple theory T is *CM-trivial* if for every tuple a and for any sets $A \subseteq B$ with $\mathrm{bdd}(aA) \cap \mathrm{bdd}(B) = \mathrm{bdd}(A)$ we have $\mathrm{Cb}(a/A) \subseteq \mathrm{bdd}(\mathrm{Cb}(a/B))$.

Remark 3.3 As in Pillay [11, Corollary 2.5], in the definition of CM-triviality we may take $A \subseteq B$ to be models of the ambient theory and a to be a tuple from the home sort. Therefore, it makes no difference in the definition of CM-triviality whether we consider hyperimaginaries or just imaginaries.

Now we characterize canonical bases in simple CM-trivial theories in terms of finitary hyperimaginaries.

Proposition 3.4 *Assume that the theory is simple CM-trivial. If a is a finite imaginary tuple, then*

$$\mathrm{bdd}\big(\mathrm{Cb}(a/B)\big) = \mathrm{bdd}\big(\mathrm{Cb}(a/b) : b \in X\big),$$

where X is the set of all finitary $b \in \mathrm{bdd}(\mathrm{Cb}(a/B))$.

Proof Since $\mathrm{Cb}(a/b) \subseteq \mathrm{bdd}(b) \subseteq \mathrm{bdd}(\mathrm{Cb}(a/B))$ for $b \in X$, we have

$$\mathrm{bdd}\big(\mathrm{Cb}(a/b) : b \in X\big) \subseteq \mathrm{bdd}\big(\mathrm{Cb}(a/B)\big).$$

For the reverse inclusion, for every $b \in X$ let $\hat{b}$ be a real tuple with $\mathrm{Cb}(a/b) \in \mathrm{dcl}(\hat{b})$; we choose them such that

$$(\hat{b} : b \in X) \mathop{\perp}_{(\mathrm{Cb}(a/b):b\in X)} aB,$$

whence $(\hat{b} : b \in X) \perp_B a$.

Now, if $a \not\perp_{(\hat{b}:b\in X)} B$, then there is a finite tuple $b' \in B \cup \{\hat{b} : b \in X\}$ and a formula $\varphi(x, b') \in \mathrm{tp}(a/B, \hat{b} : b \in X)$ which divides over $(\hat{b} : b \in X)$. Put $\bar{b} = \mathrm{bdd}(ab') \cap \mathrm{bdd}(B, \hat{b} : b \in X)$. Then $\bar{b}$ is a quasi-finitary hyperimaginary, and by CM-triviality

$$\mathrm{Cb}(a/\bar{b}) \subseteq \mathrm{bdd}\big(\mathrm{Cb}(a/B, \hat{b} : b \in X)\big) = \mathrm{bdd}\big(\mathrm{Cb}(a/B)\big).$$

Since $\mathrm{Cb}(a/\bar{b})$ is finitary by Lemma 3.1, it belongs to X. Note that $b' \in \bar{b}$; but $a \perp_{\mathrm{Cb}(a/\bar{b})} \bar{b}$, so $\varphi(x, b')$ cannot divide over $\mathrm{Cb}(a/\bar{b})$, and even less over $(\hat{b} : b \in X)$ as this contains $\widehat{\mathrm{Cb}(a/\bar{b})}$. Thus, $a \perp_{(\hat{b}:b\in X)} B$, whence $a \perp_{(\mathrm{Cb}(a/b):b\in X)} B$ by transitivity. Therefore,

$$\mathrm{Cb}(a/B) \subseteq \mathrm{bdd}\big(\mathrm{Cb}(a/b) : b \in X\big).$$

□

Question The same proof will work without assuming CM-triviality if for every finite tuple $b \in B$ there is some quasi-finitary hyperimaginary $\bar{b} \in \operatorname{bdd}(B)$ with $b \in \operatorname{dcl}(\bar{b})$ such that $\operatorname{Cb}(a/\bar{b}) \subseteq \operatorname{bdd}(\operatorname{Cb}(a/B))$. Is this true in general?

We can now state (and prove) the main result.

Theorem 3.5 *Let T be a simple CM-trivial theory. Then every hyperimaginary is interbounded with a sequence of finitary hyperimaginaries.*

Proof By Lemma 3.1 every hyperimaginary is interbounded with a canonical base. Since $\operatorname{Cb}(A/B)$ is interdefinable with $\bigcup\{\operatorname{Cb}(\bar{a}/B) : \bar{a} \in A \text{ finite}\}$, it is enough to show that canonical bases of types of finite tuples are interbounded with sequences of finitary hyperimaginaries. This is precisely Proposition 3.4. □

Corollary 3.6 *A simple CM-trivial theory eliminates hyperimaginaries whenever it eliminates finitary ones.*

Proof By Theorem 3.5 every hyperimaginary is interbounded with a sequence of finitary hyperimaginaries and so with a sequence of imaginaries. Since T is simple, it is G-compact, whence Lstp = stp over any set by Lemma 2.12. We conclude that every hyperimaginary is eliminable by Remark 2.10. □

Corollary 3.7 *Every small simple CM-trivial theory eliminates hyperimaginaries.*

Proof A small simple theory eliminates finitary hyperimaginaries by [6]. Now apply Corollary 3.6. □

4 Stable Independence for CM-Trivial Theories

Recall that an $\emptyset$-invariant relation $R(x, y)$ is *stable* if there is no infinite indiscernible sequence $(a_i, b_i : i < \omega)$ such that $R(a_i, b_j)$ holds if and only if $i < j$. In this section, we shall show that independence is a stable relation, even with varying base set. We hope that this will help elucidate the stable forking problem.

Theorem 4.1 *In a supersimple CM-trivial theory, the relation $R(x; y_1 y_2)$ given by $x \mathop{\smile\!\!\!\!\!\!\mid}_{y_1} y_2$ is stable.*

Proof Suppose not. Then there is an indiscernible sequence $I = (a_i : i \in \mathbb{Q})$ and tuples b, c such that

- $I^+ = (a_i : i > 0)$ is indiscernible over $I^- bc$,
- $I^- = (a_i : i < 0)$ is indiscernible over $I^+ bc$, and
- $a_i \mathop{\smile\!\!\!\!\!\!\mid}_c b$ if and only if $i > 0$.

We consider limit types with respect to the cut at zero. Put

$$p = \lim(I/I), \qquad p^+ = \lim(I^+/Ibc), \qquad \text{and} \qquad p^- = \lim(I^-/Ibc).$$

By finite satisfiability, p^+ and p^- are both nonforking extensions of p, which is Lascar strong. Let

$$A = \operatorname{Cb}(p) = \operatorname{Cb}(p^+) = \operatorname{Cb}(p^-) \in \operatorname{bdd}(I^+) \cap \operatorname{bdd}(I^-).$$

As p^+ and p^- do not fork over A, we have

$$a_i \mathop{\smile\!\!\!\!\!\!\mid}_A I^+ bc \quad \text{for all } i < 0 \qquad \text{and} \qquad a_i \mathop{\smile\!\!\!\!\!\!\mid}_A I^- bc \quad \text{for all } i > 0.$$

We consider first $e_0 = \mathrm{bdd}(a_1 c) \cap \mathrm{bdd}(Ac)$. Then

$$\mathrm{bdd}(a_1 e_0) \cap \mathrm{bdd}(Ae_0) = e_0.$$

Put $A_0 = \mathrm{Cb}(a_1/e_0)$. By CM-triviality,

$$A_0 \in e_0 \cap \mathrm{bdd}\big(\mathrm{Cb}(a_1/Ae_0)\big) \subseteq \mathrm{bdd}(a_1 c) \cap \mathrm{bdd}(A),$$

since $a_1 \perp_A e_0$ implies $\mathrm{Cb}(a_1/Ae_0) \subseteq \mathrm{bdd}(A)$.

Note that $a_1 \perp_c b$ yields $a_1 \perp_{A_0 c} b$. Moreover, $c \in e_0$, so $a_1 \perp_{A_0} e_0$ implies $a_1 \perp_{A_0} c$, whence $a_1 \perp_{A_0} cb$ by transitivity. On the other hand, suppose $bc \perp_{A_0} a_{-1}$. Then $b \perp_{A_0 c} a_{-1}$; as $b \perp_c a_1$ implies $b \perp_c A_0$, we obtain $b \perp_c A_0 a_{-1}$, contradicting $a_{-1} \not\perp_c b$. Therefore, $bc \not\perp_{A_0} a_{-1}$. Since I remains indiscernible over A_0, and both I^+ and I^- remain indiscernible over $A_0 bc$, we may add A_0 to the parameters and suppose $c = \emptyset$ (replacing b by bc).

Fact 4.2 ([14, Theorem 5.2.18]) In a supersimple theory, for any finitary a there are some $B \perp a$ and a hyperimaginary finite tuple $\bar{a}$ of independent realizations of regular types over B, such that $\bar{a}$ is domination-equivalent with a over B.

By Fact 4.2 there are $B \perp a_1$ and an independent tuple $\bar{a}_1$ of realizations of regular types over B such that $\bar{a}_1$ is domination-equivalent with a_1 over B. Since $B \perp a_1$ and I is indiscernible, we may assume by Wagner [14, Theorem 2.5.4] that $Ba_i \equiv Ba_1$ for all $i \in \mathbb{Q}$, and $B \perp I$. So there are $\bar{a}_i$ for $i \in \mathbb{Q}$ with $Ba_i\bar{a}_i \equiv Ba_1\bar{a}_1$. We can also assume $B \perp_I b$, whence $B \perp Ib$. In particular $b \perp_{a_i} B$, so for $i > 0$ we obtain $b \perp Ba_i$ and thus $b \perp_B a_i$, while for $i < 0$ we have $b \not\perp a_i B$ and $b \perp B$, whence $b \not\perp_B a_i$. By domination equivalence, $\bar{a}_i \perp_B b$ for $i > 0$, whereas $\bar{a}_i \not\perp_B b$ for $i < 0$.

By compactness and Ramsey we may suppose in addition that $\bar{I} = (\bar{a}_i : i \in \mathbb{Q})$ is B-indiscernible, $\bar{I}^+ = (\bar{a}_i : i > 0)$ is indiscernible over $Bb\bar{I}^-$, and $\bar{I}^- = (\bar{a}_i : i < 0)$ is indiscernible over $Bb\bar{I}^+$. We shall add B to the parameters and suppress it from the notation. We may further assume that $\bar{a}'_{-1} \perp b$ for any proper subtuple $\bar{a}'_{-1} \subseteq \bar{a}_{-1}$.

Claim *All the regular types in $\bar{a}_i$ are nonorthogonal.*

Proof of Claim Consider $c, c' \in \bar{a}_{-1}$, and put $\bar{c} = \bar{a}_{-1} \setminus \{c, c'\}$. Then $\bar{c}c \perp b$ and $\bar{c}c' \perp b$ by minimality, whence $c \perp b\bar{c}$ and $c' \perp b\bar{c}$, as $\bar{a}_{-1} = \bar{c}cc'$ is an independent tuple.

Suppose $c \perp_{b\bar{c}} c'$. Then $c \perp b\bar{c}c'$, whence $c \perp_{\bar{c}c'} b$ and finally $b \perp \bar{c}cc'$, contradicting $b \not\perp \bar{a}_{-1}$. So $\mathrm{tp}(c/b\bar{c})$ and $\mathrm{tp}(c'/b\bar{c})$ are nonorthogonal; as they do not fork over $\emptyset$ we get $\mathrm{tp}(c)$ nonorthogonal to $\mathrm{tp}(c')$. The claim now follows, as all $\bar{a}_i$ have the same type over $\emptyset$. □

Let $w_{\mathcal{P}}(\cdot)$ denote the weight with respect to that nonorthogonality class $\mathcal{P}$ of regular types. Then $\bar{a}_i$ is $\mathcal{P}$-semiregular; since $\bar{a}_{-1} \not\perp b$ we obtain

$$w_{\mathcal{P}}(\bar{a}_{-1}) > w_{\mathcal{P}}(\bar{a}_{-1}/b)$$

by Pillay [12, Lemma 7.1.14]. (The proof works just as well for the simple case.)

We again consider limit types with respect to the cut at zero. Put

$$\bar{p} = \lim(\bar{I}/\bar{I}), \qquad \bar{p}^+ = \lim(\bar{I}^+/\bar{I}b), \qquad \text{and} \qquad \bar{p}^- = \lim(\bar{I}^-/\bar{I}b).$$

Once more, $\bar{p}$ is Lascar strong, and $\bar{p}^+$ and $\bar{p}^-$ are nonforking extensions of $\bar{p}$ by finite satisfiability; let

$$\bar{A} = \mathrm{Cb}(\bar{p}) = \mathrm{Cb}(\bar{p}^+) = \mathrm{Cb}(\bar{p}^-) \in \mathrm{bdd}(\bar{I}^+) \cap \mathrm{bdd}(\bar{I}^-).$$

As before,

$$\bar{a}_i \mathop{\smile\!\!\!\!\!\mid}_{\bar{A}} \bar{I}^+ b \quad \text{for all } i < 0, \qquad \text{and} \qquad \bar{a}_i \mathop{\smile\!\!\!\!\!\mid}_{\bar{A}} \bar{I}^- b \quad \text{for all } i > 0.$$

Put $e_1 = \mathrm{bdd}(\bar{a}_{-1}b) \cap \mathrm{bdd}(\bar{A}b)$. Then

$$\mathrm{bdd}(\bar{a}_{-1}e_1) \cap \mathrm{bdd}(\bar{A}e_1) = e_1.$$

Let $A_1 = \mathrm{Cb}(\bar{a}_{-1}/e_1)$. By CM-triviality,

$$A_1 \in e_1 \cap \mathrm{bdd}\big(\mathrm{Cb}(\bar{a}_{-1}/\bar{A}e_1)\big) \subseteq \mathrm{bdd}(\bar{a}_{-1}b) \cap \mathrm{bdd}(\bar{A}),$$

since $\bar{a}_{-1} \mathop{\smile\!\!\!\!\!\mid}_{\bar{A}} e_1$ implies $\mathrm{Cb}(a_{-1}/\bar{A}e_1) \subseteq \mathrm{bdd}(\bar{A})$.

As $b \in e_1$ and $\bar{a}_{-1} \mathop{\smile\!\!\!\!\!\mid}_{A_1} e_1$ we obtain $\bar{a}_{-1} \mathop{\smile\!\!\!\!\!\mid}_{A_1} b$. Moreover, $\bar{a}_1 \equiv_{A_1} \bar{a}_{-1}$, since $A_1 \subseteq \mathrm{bdd}(\bar{A})$ and $\bar{I}$ remains indiscernible over $\bar{A}$. Therefore,

$$w_{\mathcal{P}}(\bar{a}_{-1}/A_1b) = w_{\mathcal{P}}(\bar{a}_{-1}/A_1) = w_{\mathcal{P}}(\bar{a}_1/A_1).$$

Recall that $A_1 \subseteq \mathrm{bdd}(\bar{a}_{-1}b)$. Then

$$\begin{aligned} w_{\mathcal{P}}(\bar{a}_{-1}/b) &= w_{\mathcal{P}}(\bar{a}_{-1}A_1/b) \\ &= w_{\mathcal{P}}(\bar{a}_{-1}/A_1b) + w_{\mathcal{P}}(A_1/b) \\ &= w_{\mathcal{P}}(\bar{a}_1/A_1) + w_{\mathcal{P}}(A_1/b) \\ &\geq w_{\mathcal{P}}(\bar{a}_1/A_1b) + w_{\mathcal{P}}(A_1/b) \\ &= w_{\mathcal{P}}(\bar{a}_1A_1/b) \geq w_{\mathcal{P}}(\bar{a}_1/b) \\ &= w_{\mathcal{P}}(a_1) = w_{\mathcal{P}}(a_{-1}) > w_{\mathcal{P}}(\bar{a}_{-1}/b). \end{aligned}$$

This final contradiction proves the theorem. □

Remark 4.3 Note that the proof uses only the conclusion of Fact 4.2. The theorem thus still holds for simple CM-trivial theories with finite weights (strongly simple theories) and enough regular types, for instance, CM-trivial simple theories, without dense forking chains.

Question By Palacín and Wagner [10, Theorem 4.20] it is sufficient to assume that every regular type is CM-trivial, as this implies global CM-triviality. However, for a regular type p a more general notion of CM-triviality is often more appropriate, namely,

$$\mathrm{cl}_p(aA) \cap \mathrm{cl}_p(B) = \mathrm{cl}_p(A) \Rightarrow \mathrm{Cb}\big(a/\,\mathrm{cl}_p(A)\big) \subseteq \mathrm{cl}_p\big(\mathrm{Cb}\big(a/\,\mathrm{cl}_p(B)\big)\big).$$

If this holds for all regular types p, is independence still stable?

Corollary 4.4 *An ω-categorical supersimple CM-trivial theory has stable forking.*

Proof Suppose $A \not\mathop{\smile\!\!\!\!\!\mid}_B C$. Then there are finite tuples $\bar{a} \in A$ and $\bar{c} \in C$ with $\bar{a} \not\mathop{\smile\!\!\!\!\!\mid}_B \bar{c}$. By supersimplicity, there is a finite $\bar{b} \in B$ with $\bar{a}\bar{c} \mathop{\smile\!\!\!\!\!\mid}_{\bar{b}} B$. Thus $\bar{a} \not\mathop{\smile\!\!\!\!\!\mid}_{\bar{b}} \bar{c}$. By ω-categoricity there is a formula $\varphi(\bar{x}, \bar{y}_1\bar{y}_2)$ which holds if and only if $\bar{x} \not\mathop{\smile\!\!\!\!\!\mid}_{\bar{y}_1} \bar{y}_2$. Then φ is stable by Theorem 4.1, and $\varphi(\bar{x}, \bar{b}\bar{c}) \in \mathrm{tp}(\bar{a}/\bar{b}\bar{c})$. □

Let Σ be an $\emptyset$-invariant family of types. Recall the definition of Σ-closure:

$$\mathrm{cl}_\Sigma(A) = \{a : \mathrm{tp}(a/A) \text{ is } \Sigma\text{-analyzable}\}.$$

Fact 4.5 ([14, Lemmas 3.5.3, 3.5.5]) If $\mathrm{dcl}(AB) \cap \mathrm{cl}_\Sigma(A) \subseteq \mathrm{bdd}(A)$, then $B \downarrow_A \mathrm{cl}_\Sigma(A)$. If $A \downarrow_B C$, then $A \downarrow_{\mathrm{cl}_\Sigma(B)} C$.

Corollary 4.6 *In a supersimple CM-trivial theory the relation $R(x; y_1 y_1)$ given by $x \downarrow_{\mathrm{cl}_\Sigma(y_1)} y_2$ is stable.*

Proof Suppose not. Then there is an indiscernible sequence $I = (a_i : i \in \mathbb{Q})$ and tuples b, c such that

- $I^+ = (a_i : i > 0)$ is indiscernible over $I^- bc$,
- $I^- = (a_i : i < 0)$ is indiscernible over $I^+ bc$, and
- $a_i \downarrow_{\mathrm{cl}_\Sigma(c)} b$ if and only if $i > 0$.

Put $c' = \mathrm{dcl}(bc) \cap \mathrm{cl}_\Sigma(c)$. By Fact 4.5 we have $b \downarrow_{c'} \mathrm{cl}_\Sigma(c)$, so by transitivity $a_i \downarrow_{c'} b$ for $i > 0$. Suppose $a_i \downarrow_{c'} b$ for $i < 0$. Since $\mathrm{cl}_\Sigma(c') = \mathrm{cl}_\Sigma(c)$, Fact 4.5 yields $a_i \downarrow_{\mathrm{cl}_\Sigma(c)} b$, a contradiction. Thus $a_i \downarrow_{c'} b$ if and only if $i > 0$, contradicting Theorem 4.1. □

To conclude the paper we prove a version of Corollary 4.6 without the assumption of CM-triviality but for a particular $\emptyset$-invariant family, namely, the family $\mathcal{P}$ of all non-one-based types.

Fact 4.7 ([10, Corollary 5.2]) In a simple theory $a \downarrow_{\mathrm{cl}_{\mathcal{P}}(a) \cap \mathrm{bdd}(b)} b$ for all tuples a and b, where $\mathcal{P}$ is the family of all non-one-based types.

Theorem 4.8 *In a simple theory, the relation $R(x; y_1 y_2)$ given by $x \downarrow_{\mathrm{cl}_{\mathcal{P}}(y_1)} y_2$ is stable, where $\mathcal{P}$ is the family of all non-one-based types.*

Proof Suppose not. Then there is an indiscernible sequence $I = (a_i : i \in \mathbb{Q})$ and tuples b, c such that

- $I^+ = (a_i : i > 0)$ is indiscernible over $I^- bc$,
- $I^- = (a_i : i < 0)$ is indiscernible over $I^+ bc$, and
- $a_i \downarrow_{\mathrm{cl}_{\mathcal{P}}(c)} b$ if and only if $i > 0$.

As before, we consider limit types with respect to the cut at zero. Let

$$p = \lim(I/I), \qquad p^+ = \lim(I^+/Ib), \qquad \text{and} \qquad p^- = \lim(I^-/Ib).$$

By finite satisfiability, p^+ and p^- are both nonforking extensions of p, which is Lascar strong. Let

$$A = \mathrm{Cb}(p) = \mathrm{Cb}(p^+) = \mathrm{Cb}(p^-) \in \mathrm{bdd}(I^+) \cap \mathrm{bdd}(I^-).$$

As in the proof of Theorem 4.1 we have

$$a_i \mathop{\downarrow}_A I^+ bc \quad \text{for all } i < 0 \qquad \text{and} \qquad a_i \mathop{\downarrow}_A I^- bc \quad \text{for all } i > 0.$$

We consider first $e = \mathrm{cl}_{\mathcal{P}}(a_1) \cap \mathrm{bdd}(A)$. Then $a_1 \downarrow_e A$ by Fact 4.7; since I remains indiscernible over $\mathrm{bdd}(A)$ we have $a_{-1} \equiv_{\mathrm{bdd}(A)} a_1$, whence $e = \mathrm{cl}_{\mathcal{P}}(a_{-1}) \cap \mathrm{bdd}(A)$ and $a_{-1} \downarrow_e A$. On the other hand, since $e \in \mathrm{bdd}(A)$ and $a_i \downarrow_A bc$ for $i \in \mathbb{Q}$ we obtain

$$a_1 \mathop{\downarrow}_e bc \qquad \text{and} \qquad a_{-1} \mathop{\downarrow}_e bc.$$

Now put $c' = \mathrm{dcl}(bc) \cap \mathrm{cl}_{\mathscr{P}}(c)$; note that $\mathrm{cl}_{\mathscr{P}}(c') = \mathrm{cl}_{\mathscr{P}}(c)$. Then $b \mathop{\smile\!\!\!\!\!\!\mid}_{c'} \mathrm{cl}_{\mathscr{P}}(c)$ by Fact 4.5. Moreover, $a_1 \mathop{\smile\!\!\!\!\!\!\mid}_{\mathrm{cl}_{\mathscr{P}}(c)} b$ yields $\mathrm{cl}_{\mathscr{P}}(a_1) \mathop{\smile\!\!\!\!\!\!\mid}_{\mathrm{cl}_{\mathscr{P}}(c)} b$ by Fact 4.5, whence $\mathrm{cl}_{\mathscr{P}}(a_1) \mathop{\smile\!\!\!\!\!\!\mid}_{c'} b$. Thus $e \mathop{\smile\!\!\!\!\!\!\mid}_{c'} b$, and hence $e \mathop{\smile\!\!\!\!\!\!\mid}_{c'} bc$ since $c \subseteq c'$. But now $c' \subseteq \mathrm{dcl}(bc)$ and $a_{-1} \mathop{\smile\!\!\!\!\!\!\mid}_{e} bc$ imply that $a_{-1} \mathop{\smile\!\!\!\!\!\!\mid}_{c'} bc$. Hence $a_{-1} \mathop{\smile\!\!\!\!\!\!\mid}_{\mathrm{cl}_{\mathscr{P}}(c)} b$ by Fact 4.5, as $\mathrm{cl}_{\mathscr{P}}(c') = \mathrm{cl}_{\mathscr{P}}(c)$. This contradiction finishes the proof. □

Remark 4.9 If the theory is supersimple, we can take $\mathscr{P}$ to be the family of all non-one-based *regular* types.

Remark 4.10 Theorem 4.8 generalizes the fact that independence is stable in a one-based theory. For a true generalization of Theorem 4.1 to arbitrary theories, one should take $\mathscr{P}$ to be the family of all 2-ample types. This is work in progress.

References

[1] Buechler, S., A. Pillay, and F. O. Wagner, "Supersimple theories," *Journal of the American Mathematical Society*, vol. 14 (2001), pp. 109–24. MR 1800350. DOI 10.1090/S0894-0347-00-00350-7. 542, 543, 545

[2] Casanovas, E., *Simple Theories and Hyperimaginaries*, vol. 39 of *Lecture Notes in Logic*, Cambridge University Press, Cambridge, 2011. Zbl 1243.03045. MR 2814891. DOI 10.1017/CBO9781139003728. 544

[3] Casanovas, E., and F. O. Wagner, "The free roots of the complete graph," *Proceedings of the American Mathematical Society*, vol. 132 (2004), pp. 1543–48. Zbl 1040.03028. MR 2053363. DOI 10.1090/S0002-9939-03-07193-4. 542

[4] Evans, D. M., and E. Hrushovski, "On the automorphism groups of finite covers," *Annals of Pure and Applied Logic*, vol. 62 (1993), pp. 83–112. MR 1226301. DOI 10.1016/0168-0072(93)90168-D. 543

[5] Hart, B., B. Kim, and A. Pillay, "Coordinatisation and canonical bases in simple theories," *Journal of Symbolic Logic*, vol. 65 (2000), pp. 293–309. Zbl 0945.03051. MR 1782121. DOI 10.2307/2586538. 541

[6] Kim, B., "A note on Lascar strong types in simple theories," *Journal of Symbolic Logic*, vol. 63 (1998), pp. 926–36. Zbl 0961.03030. MR 1649069. DOI 10.2307/2586720. 542, 546

[7] Kim, B., "Simplicity, and stability in there," *Journal of Symbolic Logic*, vol. 66 (2001), 822–36. Zbl 1002.03026. MR 1833481. DOI 10.2307/2695047. 542

[8] Kim, B., and A. Pillay, "Around stable forking," *Fundamenta Mathematicae*, vol. 170 (2001), pp. 107–18. MR 1881371. DOI 10.4064/fm170-1-6. 542

[9] Lascar, D., and A. Pillay, "Hyperimaginaries and automorphism groups," *Journal of Symbolic Logic*, vol. 66 (2001), pp. 127–43. Zbl 1002.03027. MR 1825177. DOI 10.2307/2694914. 542, 543

[10] Palacín, D., and F. O. Wagner, "Ample thoughts," *Journal of Symbolic Logic*, vol. 78 (2013), pp. 489–510. 548, 549

[11] Pillay, A., "The geometry of forking and groups of finite Morley rank," *Journal of Symbolic Logic*, vol. 60 (1995), pp. 1251–59. Zbl 0845.03016. MR 1367208. DOI 10.2307/2275886. 545

[12] Pillay, A., *Geometric Stability Theory*, vol. 32 of *Oxford Logic Guides*, Oxford University Press, New York, 1996. Zbl 0871.03023. MR 1429864. 547

[13] Pillay, A., and B. Poizat, "Pas d'imaginaires dans l'infini!" *Journal of Symbolic Logic*, vol. 52 (1987), pp. 400–403. Zbl 0631.03014. MR 0890448. DOI 10.2307/2274390. 542

[14] Wagner, F. O., *Simple Theories*, vol. 503 of *Mathematics and Its Applications*, Kluwer, Dordrecht, 2000. MR 1747713. 547, 549

Acknowledgments

The first author was partially supported by research project MTM 2008-01545 of the Spanish government and research project 2009SGR 00187 of the Catalan government. The second author was partially supported by the research project ANR-09-BLAN-0047 of the Agence Nationale de la Recherche.

This work was partially done while the first author was visiting the Institut Camille Jordan of Lyon. The first author wishes to express his gratitude to the members of Lyon logic group for their hospitality. He also would like to thank Enrique Casanovas for his valuable comments.

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