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Algebraization, Transcendence, and D-Group Schemes

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Abstract We present a conjecture in Diophantine geometry concerning the construction of line bundles over smooth projective varieties over $\overline{\mathbb{Q}}$. This conjecture, closely related to the Grothendieck period conjecture for cycles of codimension 1, is also motivated by classical algebraization results in analytic and formal geometry and in transcendence theory. Its formulation involves the consideration of D-group schemes attached to abelian schemes over algebraic curves over $\overline{\mathbb{Q}}$. We also derive the Grothendieck period conjecture for cycles of codimension 1 in abelian varieties over $\overline{\mathbb{Q}}$ from a classical transcendence theorem à la Schneider–Lang.

0 Foreword

My aim, in this largely expository article, is to present a conjecture in Diophantine geometry, concerning the construction of line bundles over smooth projective varieties over $\overline{\mathbb{Q}}$. This conjecture is motivated by the classical Grothendieck period conjecture (cf. Section 5.1) and by the philosophy, already advocated in diverse places (see, e.g., Bost [13], Chambert-Loir [28], Bost and Chambert-Loir [16], Gasbarri [42]), that various results in Diophantine approximation and transcendence theory are arithmetic counterparts, valid in varieties over number fields, or rather in their model of finite type over $\mathbb{Z}$, of geometric algebraicity criteria concerning formal objects inside algebraic varieties over some (algebraically closed) field k.

Most of the presently known results in transcendence appear actually to be analogues of geometric algebraicity criteria concerning germs $\widehat{V}$ of formal subvarieties along a projective subvariety Y of some ambient variety X over k—by such a $\widehat{V}$ we mean a smooth formal subscheme $\widehat{V}$ of the completion $\widehat{X}_Y$ admitting Y as the

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scheme of definition. (Any such $\widehat{V}$ may be written as the limit

$$\widehat{V} = \varinjlim_i V_i$$

of the successive infinitesimal neighborhoods V_i, $i \in \mathbb{N}$, of Y in $\widehat{V}$, which are closed subschemes of X, of support $|V_i| = Y$.) These criteria assert that, if Y is smooth, *of dimension at least one*, and if the normal bundle $N_Y \widehat{V}$ of Y in $\widehat{V}$ satisfies some suitable positivity condition, then $\widehat{V}$ is algebraic—roughly speaking, this means that $\widehat{V}$ is a "branch" along Y of some subvariety W of X containing Y.

When the base field k is the field $\mathbb{C}$ of complex numbers, that kind of result may be stated in the following terms, which avoid an explicit appeal to formal geometry and so may look more familiar. In the situation when $k = \mathbb{C}$, any germ of a $\mathbb{C}$-analytic submanifold $\mathcal{V}$ of X along Y defines a smooth formal germ $\widehat{V} := \widehat{\mathcal{V}}_Y$ along Y (namely, the limit $\lim_i \mathcal{V}_i$ of the successive infinitesimal neighborhood of Y inside $\mathcal{V}$; these are projective analytic subspaces in X, which may be identified to projective subschemes over $\mathbb{C}$). Then the above-mentioned algebraicity criteria assert that, *when the normal bundle of Y in $\mathcal{V}$* satisfies a suitable positivity condition, for instance, when it *is ample, then $\mathcal{V}$ is contained in some algebraic subvariety W of X of the same* (complex) *dimension as $\mathcal{V}$*. That type of geometric result goes back to Andreotti [4].

In transcendence theory, one deals with algebraicity criteria concerning smooth formal germs of subvarieties $\widehat{V}$ through some K-rational point P in a variety X over a number field K. According to a viewpoint that goes back to Kronecker, it is appropriate to consider a model $\mathcal{X}$ of X of finite type over the ring of integers $\mathcal{O}_K$ of K (hence over $\mathbb{Z}$), in which P extends to a point $\mathcal{P}$ in $\mathcal{X}(\mathcal{O}_K)$. The algebraicity criteria established in transcendence turn out to deal with a formal germ in the completion $\widehat{\mathcal{X}}_{\mathcal{P}}$ along the "arithmetic curve" $\mathcal{P} \simeq \operatorname{Spec} \mathcal{O}_K$. In this Kroneckerian perspective, transcendence results are indeed algebraicity criteria concerning formal germs along *curves*, analogue to the geometric algebraicity criteria à la Andreotti.

It turns out that, in the context of analytic and formal geometry, algebraicity criteria have been established that concern not only subvarieties but also coherent sheaves (e.g., line bundles or vector bundles), notably by Grothendieck [45], [47] in the context of formal geometry. In their most basic geometric version, for instance, the algebraization results in [47] (also presented in Hartshorne [50]) deal with germs of formal (or analytic) vector bundles along suitable ample projective subvarieties Y of some algebraic variety X over some base field k. Their validity requires Y to be *of dimension at least two*. The Kroneckerian viewpoint mentioned above—in which the arithmetic counterpart of a surface over some base field is an "arithmetic surface" that is an integral model of a curve over a number field—leads one to expect that one could formulate, and possibly establish, some significant arithmetic algebraization criterion, concerning *formal line or vector bundles over the completion $\widehat{X}_Y$ of some algebraic variety X over a number field along some projective curve Y.*

In this article, I present a conjectural transcendence statement of this kind (Conjecture 7.3 infra), the validity of which would actually imply some new cases of the classical Grothendieck period conjecture.

An interesting feature of this conjectural statement is that it introduces differential algebraic groups in a classical Diophantine context, concerning algebraic varieties over number fields. Recall that the role of differential algebra in Diophantine geometry over function fields is well established since the work of Manin [71]–[73] on algebraic curves over function fields, culminating with his proof of the geometric Mordell conjecture, and has more recently considerably expanded, in a series of works initiated by the contributions of Buium [22]–[24], Buium and Voloch [26], and Hrushovski [55], which make conspicuous the role of differential algebraic groups in the Diophantine geometry of abelian varieties over function fields.[1] The occurrence of nonlinear differential algebraic groups over curves over number fields in Conjecture 7.3, which reflects the 2-dimensional nature of the problem at hand, has appeared to me worthy of attention, and I took the opportunity of the Oléron conference to present it to experts in model theory and differential algebra gathered on the occasion of Anand Pillay's 60th birthday.

Actually, although the content of this work has presently no explicit link with model theory, it turns out to involve several of the mathematical themes so successfully explored by Anand Pillay during recent years, notably the interplay between the analytic geometry of compact complex manifolds and algebraic geometry, and the study of algebraic D-groups, especially in relation to abelian varieties and their universal vector extensions. This article is dedicated to him, as a token of appreciation and confidence in his mathematical vision.

This paper, like my oral presentation in Oléron, is to a large extent expository: I seriously attempted to discuss the classical facts relevant to the formulation of Conjecture 7.3 in a form accessible to mathematicians of diverse backgrounds (with possibly a limited success, notably in the last sections of this article). Especially I tried to avoid any real knowledge of formal geometry, by putting forward the analytic variants of diverse results usually formulated in terms of formal geometry or by translating statements in formal geometry into equivalent statements involving systems of successive thickenings, to stay in the realm of algebraic geometry. I also tried to present various themes from some unconventional point of view, for instance in emphasizing the role of moduli spaces of vector bundles with integrable connections.

However, besides Conjecture 7.3 itself, I also included some original content, notably in Section 5 a proof of the Grothendieck period conjecture in codimension 1 for abelian varieties. Readers interested in this result may read Sections 4 and 5 independently of the rest of the article.

1 Algebraization of Analytic Objects, I

1.1 Algebraization of compact Riemann surfaces and of projective analytic sets Algebraization of analytic objects (such as varieties and their morphisms, vector bundles, coherent sheaves, etc.) is a central theme in the development of algebraic and analytic geometry at least since the 1830s. Already recognizable in the pioneering work of Abel and Jacobi on elliptic functions and elliptic curves, it appears in a form familiar to modern mathematicians in the work of Puiseux and Riemann.

For instance, in the first part of his memoir on abelian functions [87]—devoted to a systematic study of what today would be called "compact Riemann surfaces realized as a finite covering of the projective complex line $\mathbb{P}^1(\mathbb{C})$"—Riemann establishes the *algebraicity* of any pair (C, ν) where C is *a compact connected Riemann surface* and $\nu : C \longrightarrow \mathbb{P}^1(\mathbb{C})$ is *a ramified analytic covering* (or equivalently, a nonconstant $\mathbb{C}$-analytic map).

Namely, he proves that, for any such pair (C, ν), there exists an irreducible polynomial P in $\mathbb{C}[X, Y]$ (of positive degree in Y) and an isomorphism from C to the compact Riemann surface associated to the plane algebraic curve of equation $P(X, Y) = 0$ such that, through this isomorphism, the map ν (seen as a meromorphic function on C) gets identified with the meromorphic function defined by the first coordinate X. To achieve this, Riemann constructs a suitable meromorphic function on C (which ultimately will become the second coordinate Y) by appealing to the Dirichlet principle.

An important step in the development of algebraization theorems has been the theorem of Chow [29], which asserts that *any closed* $\mathbb{C}$*-analytic subset* X *of the projective space* $\mathbb{P}^N(\mathbb{C})$ *is algebraic.* In other words, there exists a finite family $(P_\alpha)_{1\leq\alpha\leq A}$ of homogeneous polynomials in $\mathbb{C}[X_0, \ldots, X_N]$ such that, for any point $(x_0 : \cdots : x_N)$ in $\mathbb{P}^N(\mathbb{C})$,

$$(x_0 : \cdots : x_N) \in X \iff \text{for } \alpha = 1, \ldots, A,\ P_\alpha(x_0, \ldots, x_N) = 0.$$

The statement of Chow's theorem clearly did not come as a surprise at the time of the publication of [29] (see, e.g., H. Cartan's summary of [29] in *Mathematical Reviews* [MR 0033093]). A significant point in [29] is the formal rigor of its proofs—based on some algebraicity criterion formulated in terms of intersections numbers with algebraic subvarieties of $\mathbb{P}^N(\mathbb{C})$—which links the theme of algebraization of analytic objects to the development of rigorous foundations for algebraic topology and geometry, in the line of earlier works by Lefschetz, van der Waerden, and Chevalley.

1.2 Algebraization of line bundles over complex projective varieties Actually, more than forty years before Chow's work, a remarkable variation on this theme of algebraization was initiated by Poincaré and Lefschetz during their investigation of *algebraic cycles on complex surfaces* by means of the so-called normal functions. Motivated by techniques and problems of the Italian school of algebraic geometry and by Picard's contributions to the theory of algebraic surfaces, they basically established the following theorem, when $\dim X = 2$.

Let X *be a smooth closed* $\mathbb{C}$*-analytic subvariety of* $\mathbb{P}^N(\mathbb{C})$ (necessarily algebraic, according to Chow's theorem). *Then any analytic line bundle* L *over* X *is algebraic.*

This result was extended by Hodge ([54, pp. 214–216]) to higher-dimensional smooth projective varieties. Kodaira and Spencer [61] gave a new "modern" proof of this theorem in 1953, in what probably constitutes the first application of sheaf theory and cohomological techniques to projective complex varieties.

Let us formulate a few comments on the content of the Poincaré–Lefschetz–Hodge theorem.

We shall denote $\mathcal{O}_X^{\mathrm{an}}$ and $\mathcal{C}_X$ (resp., $\mathcal{O}_X$) the sheaf of analytic and complex-valued continuous functions (resp., of regular functions) on X equipped with the usual "analytic" topology (resp., with the Zariski topology).

Recall that, for any analytic line bundle L over X, there exist an open covering $\mathcal{U} := (U_\alpha)_{\alpha\in A}$ of X (in the analytic topology) and, for every $\alpha \in A$, an analytic trivialization of L over U_α:

$$s_\alpha : \mathcal{O}_{U_\alpha}^{\mathrm{an}} \xrightarrow{\sim} L_{U_\alpha}.$$

By comparing the trivializations, namely, by introducing the functions $\varphi_{\alpha\beta}$ in $\mathcal{O}_X^{\mathrm{an}}(U_\alpha \cap U_\beta)^*$ defined by

$$s_\alpha = \varphi_{\alpha\beta} s_\beta \quad \text{over } U_\alpha \cap U_\beta,$$

one defines a 1-cocycle $(\varphi_{\alpha\beta})$ in $Z^1(\mathcal{U}, \mathcal{O}_X^{\mathrm{an}*})$. The class of this cocycle in $H^1(X, \mathcal{O}_X^{\mathrm{an}*})$ determines the isomorphism class of L, and any cohomology class in $H^1(X, \mathcal{O}_X^{\mathrm{an}*})$ arises through this construction from a suitable analytic line bundle L.

The line bundle L is *algebraic* precisely when the above covering $\mathcal{U} =: (U_\alpha)_{\alpha\in A}$, and trivializations $(s_\alpha)_{\alpha\in A}$, may be chosen in such a way that every U_α is *Zariski-open* in X and every function $\varphi_{\alpha\beta} s_\beta$ is *regular*[2] over $U_\alpha \cap U_\beta$; then $(\varphi_{\alpha\beta})$ defines a 1-cocycle in $Z^1(\mathcal{U}, \mathcal{O}^*)$.

The above formulation of the theorem of Poincaré, Lefschetz, and Hodge, in terms of algebraicity of analytic line bundles, is basically its "modern" formulation by Kodaira and Spencer. Let us recall how it translates into its "classical" formulation à la Lefschetz and Hodge, involving (co)homology classes of divisors. The following arguments, now classical, appear in Kodaira and Spencer [62].

Consider the short exact sequences of sheaves of abelian groups over X defined by the "exponential" map $\mathbf{e} := \exp(2\pi i)$:

$$0 \longrightarrow \mathbb{Z}_X \longrightarrow \mathcal{C}_X \xrightarrow{\mathbf{e}} \mathcal{C}_X^* \longrightarrow 0$$

and

$$0 \longrightarrow \mathbb{Z}_X \longrightarrow \mathcal{O}_X^{\mathrm{an}} \xrightarrow{\mathbf{e}} \mathcal{O}_X^{\mathrm{an}*} \longrightarrow 0.$$

The abelian group of isomorphism classes of topological (resp., analytic line) bundles over X is naturally identified with $H^1(X, \mathcal{C}_X^*)$ (resp., $H^1(X, \mathcal{O}_X^{\mathrm{an}*})$). The long exact sequences of cohomology groups associated to the above short exact sequences of sheaves fit into a commutative diagram:

$$\begin{array}{ccccccc}
H^1(X, \mathcal{C}_X) & \xrightarrow{\mathbf{e}} & H^1(X, \mathcal{C}_X^*) & \xrightarrow{\delta} & H^2(X, \mathbb{Z}) & \longrightarrow & H^2(X, \mathcal{C}_X) \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & H^1(X, \mathcal{O}_X^{\mathrm{an}*}) & \xrightarrow{\delta^{\mathrm{an}}} & H^2(X, \mathbb{Z}) & \longrightarrow & H^2(X, \mathcal{O}_X)
\end{array} \tag{1.1}$$

The exactness of the first line and the vanishing of $H^1(X, \mathcal{C}_X)$ and $H^2(X, \mathcal{C}_X)$ define an isomorphism

$$c_{1,\mathrm{top}} := \delta : H^1(X, \mathcal{C}_X^*) \xrightarrow{\sim} H^2(X, \mathbb{Z}), \tag{1.2}$$

which maps the isomorphism class of some topological line bundle L to its so-called *first Chern class*. The exactness of the second line in (1.1) precisely asserts that a class α in $H^2(X, \mathbb{Z})$ belongs to the image of δ^{an}—or equivalently, is the first Chern class $c_1(L)$ of some *analytic* line bundle—if and only if α belongs to the kernel

$$\ker\big(H^2(X, \mathbb{Z}) \longrightarrow H^2(X, \mathcal{O}_X^{\mathrm{an}})\big)$$

of the map induced by the inclusion of sheaves $\mathbb{Z}_X \hookrightarrow \mathcal{O}_X^{\mathrm{an}}$ or, equivalently, if the real cohomology class $\alpha_{\mathbb{R}}$ in $H^2(X, \mathbb{R})$ belongs to

$$\ker\big(H^2(X, \mathbb{R}) \longrightarrow H^2(X, \mathcal{O}_X^{\mathrm{an}})\big).$$

In the classical notation of Hodge theory, this is precisely the space $H^2(X, \mathbb{R}) \cap H^{1,1}(X)$ of real 2-cohomology classes on X of type $(1,1)$. In the case of surfaces, considered by Lefschetz, this space may be defined by the classical vanishing condition

$$\int_X \alpha \wedge \omega = 0$$

of the integrals along α of the global regular algebraic 2-forms ω on X.

Besides, an *algebraic* line bundle L may be described in terms of the divisor D of some nonzero rational section s: the section s establishes an isomorphism from L to the line bundle $\mathcal{O}(D)$, and the class $c_1(L) = c_1(\mathcal{O}(D))$ coincides with the class $[D]$ in $H^2(X,\mathbb{Z})$ Poincaré dual to the divisor D, seen as a codimension 1 algebraic cycle on X.

Taking the above facts into account, Kodaira and Spencer's version of the theorem of Poincaré, Lefschetz, and Hodge admits the following consequence, which is actually its original version due to Lefschetz and Hodge[3]: *a class α in $H^2(X,\mathbb{Z})$ is algebraic—namely, the class $[D]$ of some algebraic cycle D of codimension* 1 *on X—if and only if $\alpha_{\mathbb{R}}$ is of type* $(1,1)$.

1.3 GAGA The diverse algebraicity statements in the previous sections appear today as special instances of Serre's GAGA theorem (1956; see [92]).

To formulate Serre's results, consider a complex algebraic variety X. From any algebraic coherent sheaf F over X equipped with the Zariski topology—for example, an algebraic vector bundle E over X, defined by some 1-cocycle $(\varphi_{\alpha\beta}) \in Z^1((U_\alpha), \mathrm{GL}_N(\mathcal{O}_X))$, attached to some Zariski-open covering (U_α) of X, with values in invertible matrices of regular functions—we deduce an analytic coherent sheaf F^{an} on X equipped with the analytic topology; for instance, E^{an} is the analytic vector bundle defined by the cocycle $(\varphi_{\alpha\beta})$ seen as an analytic cocycle (i.e., as an element of $Z^1((U_\alpha), \mathrm{GL}_N(\mathcal{O}_X^{\mathrm{an}}))$). This is a straightforward consequence of the fact that the analytic topology of X is finer than its Zariski topology, and the fact that, for every Zariski-open subset U of X, $\mathcal{O}_X(U)$ is a subring of $\mathcal{O}_X^{\mathrm{an}}(U)$.

These facts also imply the existence of canonical "analytification maps" between cohomology groups:

$$H^i(X,F) \longrightarrow H^i(X^{\mathrm{an}}, F^{\mathrm{an}}). \tag{1.3}$$

Here X (resp., X^{an}) denotes the variety X equipped with the Zariski topology (resp., the underlying analytic space, which topologically is the set of complex points of X equipped with the usual "analytic" topology).

Serre's GAGA theorem is the conjunction of the following two statements.

GAGA comparison theorem *For any projective complex variety X and any coherent algebraic sheaf F on X, the "analytification maps" (1.3) are isomorphisms:*

$$H^i(X,F) \xrightarrow{\sim} H^i(X^{\mathrm{an}}, F^{\mathrm{an}}). \tag{1.4}$$

GAGA existence theorem *For any projective complex variety X and for any analytic coherent sheaf $\mathcal{F}$ on X^{an}, there exists some algebraic coherent sheaf F over X* (unique up to unique isomorphism) *such that $\mathcal{F}$ is isomorphic to F^{an}* (as an analytic coherent sheaf over X^{an}).

Let us stress that the projectivity assumption in the GAGA theorem is essential (see Section 2.3 for a discussion of counterexamples in the quasi-projective situation).

The Poincaré–Lefschetz–Hodge theorem is nothing but the special case of the GAGA existence theorem concerning line bundles over smooth varieties.

Chow's theorem also follows from the GAGA existence theorem—with the notation of paragraph (1.1), it follows from this theorem applied to $\mathcal{O}_X^{\mathrm{an}}$, seen as a coherent analytic sheaf over $\mathbb{P}^N(\mathbb{C})^{\mathrm{an}}$. Observe also that conversely, by considering graphs, Chow's theorem implies the comparison isomorphism (1.4) when $i = 0$ and F is a vector bundle.

Serre's proof of GAGA theorems is the archetype of "modern cohomological proofs" and, besides its considerable importance in itself, has also played an important role as a model for the development of cohomological techniques in algebraic and formal geometry.

To establish the GAGA comparison theorem, using that X may be embedded into some projective space $\mathbb{P}^N_{\mathbb{C}}$, one reduces to the special case $X = \mathbb{P}^N_{\mathbb{C}}$. In that case, Serre's proof relies on some "algebraic dévissage of F" by means of a left resolution by algebraic coherent sheaves that are direct sums of line bundles of the form $\mathcal{O}_{\mathbb{P}^N}(k)$, $k \in \mathbb{Z}$, combined with a direct computation of the algebraic and analytic cohomology groups in (1.4) when $F = \mathcal{O}_{\mathbb{P}^N}(k)$.

The proof of the GAGA existence theorem may be seen as a deep amplification and simplification of Kodaira and Spencer's proof in [61]. Besides the comparison theorem previously established, it relies on the finite dimensionality of the analytic cohomology groups $H^i(X^{\mathrm{an}}, \mathcal{F})$ attached to an arbitrary analytic coherent sheaf $\mathcal{F}$ on X. This result, of analytic nature, was established by Cartan and Serre [27] with X^{an} an arbitrary compact complex analytic space. Actually only the degree $i = 1$ case of the finiteness theorem of Cartan and Serre is used in the proof of the existence theorem. When X is smooth and $\mathcal{F}$ is a line bundle, it was established by Kodaira and Spencer as a consequence of the description of $H^i(X^{\mathrm{an}}, \mathcal{F})$ by means of harmonic forms and of the fact that elliptic differential operators on compact manifolds are Fredholm.

2 Algebraization of Analytic Objects, II: Comments and Applications

2.1 Un peu d'histoire I would like to stress that the content of the previous sections provides a very fragmentary image of the history of algebraization theorems, a topic especially rich in results and techniques, where the evolution of ideas over the long term seems rather difficult to untangle.

To illustrate this last point, let me indicate that algebraicity theorems à la Chow may be derived from Bézout-type bounds on intersection multiplicities. That line of argument appears, for instance, in Poincaré's survey article on abelian functions [82], when he proves that a compact complex torus embedded in a complex projective space is actually algebraic (see [82, Section 2, pp. 53–56]). It constitutes the central point in Chow's proof in [29] and, more recently, plays a key role in the work of Hrushovski and Zilber [56] on Zariski geometries (see [56, Section 7]). The influence of Poincaré's work on [29] and [56] seems unclear, and [82] could be a striking example of double *plagiat par anticipation* by Poincaré.

Another approach, due to Serre, to Chow's theorem—which appears as an anonymous contribution in [5]—consists in deriving it from the fact that the transcendence degree over $\mathbb{C}$ of the field $\mathcal{M}(X)$ of meromorphic functions on some compact connected complex manifold X is at most its (complex) dimension:

$$\operatorname{degtr}_{\mathbb{C}} \mathcal{M}(X) \leq \dim X. \tag{2.1}$$

Indeed, if X is analytically embedded in $\mathbb{P}^N(\mathbb{C})$, its Zariski closure $\overline{X}^{\mathrm{Zar}}$ is irreducible, the field $\mathbb{C}(\overline{X}^{\mathrm{Zar}})$ of rational function on $\overline{X}^{\mathrm{Zar}}$ may be identified to a subfield of the field of meromorphic function $\mathcal{M}(X)$, and the upper bound (2.1) implies that the Zariski closure $\overline{X}^{\mathrm{Zar}}$ of X in $\mathbb{P}^N(\mathbb{C})$ has dimension at most $\dim X$ and hence is equal to $\dim X$. Besides, the irreducibility of $\overline{X}^{\mathrm{Zar}}$ implies its connectedness and the connectedness of its subset $\overline{X}^{\mathrm{Zar}}_{\mathrm{reg}}$ of smooth points in the analytic topology. This connectedness is a GAGA-type statement which goes back to Puiseux [84, Section I] in the case of plane curves; Puiseux's original proof actually extends to higher-dimensional varieties (see, e.g., Shafarevich [95, Section VII.2]) and probably constitutes, with other arguments in Puiseux [83] and [84], the first proof of such results satisfactory according to modern standards. The connectedness of $\overline{X}^{\mathrm{Zar}}_{\mathrm{reg}}$ and its density in $\overline{X}^{\mathrm{Zar}}$ for the analytic topology, together with the inclusion $X \subset \overline{X}^{\mathrm{Zar}}$ and the equality of dimension $\dim X = \dim \overline{X}^{\mathrm{Zar}}$, imply the equality $X = \overline{X}^{\mathrm{Zar}}$, that is, the algebraicity of X.

In turn, proofs of the upper bound (2.1) appear to have a complicated history—this bound seems to have been established for the first time in a completely satisfactory way by Serre [91, Section 3] and Thimm [99]. In [96], Siegel discusses the history of the question and gives an ingenious "elementary" proof, directly influenced by Poincaré's article [82][4] and actually very close to the proof in [91]. Conversely, as observed in Remmert [86], (2.1) is an easy consequence of Chow's theorem and Remmert's proper image theorem. In turn, both of these theorems may be derived from the fundamental extension theorems concerning complex analytic sets, due to Thullen, Remmert, and Stein (see, e.g., Mumford [77, Section 4A] or Gunning [48, Chapters K and M]).

Concerning the history of the Poincaré–Lefschetz–Hodge theorem, I refer to the classical analysis by Zariski and to the additional comments by Mumford in Zariski [103, Chapter VII].[5]

2.2 Algebraic de Rham cohomology In this section, we apply the GAGA comparison theorem to the study of the algebraic de Rham cohomology, in the "easy" case of projective smooth varieties. The formalism below seems to appear in printed form in the famous letter of Grothendieck [46] to Atiyah, although algebraic de Rham cohomology already occurs implicitly in diverse classical works on algebraic curves, surfaces, and abelian varieties. See Hartshorne [51] for a systematic presentation of the de Rham cohomology of algebraic varieties and for references.

2.2.1 Let X be a smooth projective complex algebraic variety. It is equipped with the algebraic de Rham complex

$$\Omega^\bullet_{X/\mathbb{C}} : 0 \longrightarrow \Omega^0_{X/\mathbb{C}} = \mathcal{O}_X \xrightarrow{d} \Omega^1_{X/\mathbb{C}} \xrightarrow{d} \Omega^2_{X/\mathbb{C}} \xrightarrow{d} \cdots, \tag{2.2}$$

and the hypercohomology groups of this complex of sheaves over X equipped with the Zariski topology define the *algebraic de Rham cohomology groups* of X:

$$H^i_{\mathrm{dR}}(X/\mathbb{C}) := \mathbb{H}^i(X, \Omega^\bullet_{X/\mathbb{C}}).$$

By "analytification," the algebraic de Rham complex (2.2) becomes the analytic de Rham complex of the $\mathbb{C}$-analytic manifold X^{an}:

$$\Omega^\bullet_{X^{\mathrm{an}}} : 0 \longrightarrow \Omega^0_{X^{\mathrm{an}}} = \mathcal{O}^{\mathrm{an}}_{X^{\mathrm{an}}} \xrightarrow{d} \Omega^1_{X^{\mathrm{an}}} \xrightarrow{d} \Omega^2_{X^{\mathrm{an}}} \xrightarrow{d} \cdots. \tag{2.3}$$

The hypercohomology groups of $\Omega^\bullet_{X^{\mathrm{an}}}$ define the *analytic de Rham cohomology groups* of X^{an} $\mathbb{H}^i(X^{\mathrm{an}};\Omega^\bullet_{X^{\mathrm{an}}})$, and "analytification" defines canonical $\mathbb{C}$-linear maps:

$$\mathbb{H}^i(X,\Omega^\bullet_{X/\mathbb{C}}) \longrightarrow \mathbb{H}^i(X^{\mathrm{an}},\Omega^\bullet_{X^{\mathrm{an}}}). \tag{2.4}$$

The algebraic (resp., analytic) de Rham cohomology groups are related to the algebraic (resp., analytic) "Hodge cohomology groups" $H^q(X,\Omega^p_{X/\mathbb{C}})$ (resp., $H^q(X^{\mathrm{an}},\Omega^p_{X^{\mathrm{an}}})$) by the usual spectral sequences

$$E_1^{p,q} = H^q(X,\Omega^p_{X/\mathbb{C}}) \Rightarrow \mathbb{H}^{p+q}(X,\Omega^\bullet_{X^{\mathrm{an}}})$$
$$\big(\text{resp., } E_1^{p,q} = H^q(X^{\mathrm{an}},\Omega^p_{X^{\mathrm{an}}}) \Rightarrow \mathbb{H}^{p+q}(X^{\mathrm{an}},\Omega^\bullet_{X^{\mathrm{an}}})\big).$$

The formation of these spectral sequences is compatible with analytification. Consequently, from the GAGA comparison isomorphisms

$$H^q(X,\Omega^p_{X/\mathbb{C}}) \xrightarrow{\sim} H^q(X^{\mathrm{an}},\Omega^p_{X^{\mathrm{an}}}),$$

we deduce that the analytification maps (2.4) from algebraic to analytic de Rham cohomology groups are isomorphisms.

Besides, according to the analytic Poincaré lemma, the inclusion of the locally constant sheaf $\mathbb{C}_{X^{\mathrm{an}}}$ into $\mathcal{O}^{\mathrm{an}}_{X^{\mathrm{an}}}$ defines a quasi-isomorphism of a complex of sheaves on X^{an},

$$\mathbb{C}_{X^{\mathrm{an}}} \xrightarrow{\text{q.i.}} \Omega^\bullet_{X^{\mathrm{an}}},$$

and consequently an isomorphism of (hyper)cohomology groups:

$$H^i(X^{\mathrm{an}},\mathbb{C}) \xrightarrow{\sim} \mathbb{H}^i(X^{\mathrm{an}},\Omega^\bullet_{X^{\mathrm{an}}}). \tag{2.5}$$

The isomorphisms (2.4) and (2.5) define by composition an isomorphism of finite-dimensional $\mathbb{C}$-vector spaces:

$$\begin{aligned} H^i_{\mathrm{dR}}(X/\mathbb{C}) &\longrightarrow H^i(X^{\mathrm{an}},\mathbb{C}),\\ \beta &\longmapsto \beta^{\mathrm{an}}. \end{aligned} \tag{2.6}$$

2.2.2 Observe that the definition of the algebraic de Rham cohomology makes sense for any smooth projective variety X_0 defined over an arbitrary base field k. Indeed, we may consider the algebraic de Rham complex

$$\Omega^\bullet_{X_0/k} : 0 \longrightarrow \Omega^0_{X_0/k} = \mathcal{O}_{X_0} \xrightarrow{d} \Omega^1_{X_0/k} \xrightarrow{d} \Omega^2_{X_0/k} \xrightarrow{d} \cdots \tag{2.7}$$

and define

$$H^i_{\mathrm{dR}}(X_0/k) := \mathbb{H}^i(X_0,\Omega^\bullet_{X_0/k}).$$

These are finite-dimensional k-vector spaces, and when k is a subfield of $\mathbb{C}$, this construction defines a natural "form over k" of the cohomology with complex coefficients $H^i(X^{\mathrm{an}};\mathbb{C})$ of the $\mathbb{C}$-analytic manifold X^{an} attached to a complex algebraic variety $X := X_0 \otimes_k \mathbb{C}$ deduced from X_0 by extending the base field from k to $\mathbb{C}$. Indeed, by composing a straightforward base change isomorphism and the comparison isomorphism (2.6), we obtain a canonical isomorphism

$$H^i_{\mathrm{dR}}(X_0/k)\otimes_k \mathbb{C} \xrightarrow{\sim} H^i_{\mathrm{dR}}(X/\mathbb{C}) \xrightarrow{\sim} H^i(X^{\mathrm{an}},\mathbb{C}). \tag{2.8}$$

2.2.3 Example I: Smooth projective curves Let X_0 be a smooth, projective, geometrically connected curve, of genus g, over k. Then $H^i_{\mathrm{dR}}(X_0/k)$ vanishes if $i > 2$ and is a canonically isomorphic to k when $i = 0$ or 2. The first de Rham cohomology group $H^1_{\mathrm{dR}}(X_0/k)$ is a $2g$-dimensional k-vector space. It may be identified with the quotient of the space of meromorphic 1-forms over X_0/k of the second kind (that is, with vanishing residues) by its subspace $dk(X_0)$ formed by the differentials of rational functions $k(X_0)$ over X_0.

For instance, when k is a field of characteristic not equal to 2, 3, if X_0 is an elliptic curve E of plane equation

$$y^2 = 4x^3 - g_2 x - g_3,$$

then $H^1_{\mathrm{dR}}(E/k_0)$ is a 2-dimensional k-vector space with basis $([\alpha], [\beta])$, where $\alpha := dx/y$ and $\beta := x \cdot dx/y$.

2.2.4 Example II: The first Chern class in algebraic de Rham cohomology The morphism of sheaves of abelian groups over X_0,

$$\begin{aligned} d\log : \mathcal{O}^*_{X_0} &\longrightarrow \Omega^1_{X_0/k}, \\ \varphi &\longmapsto d\varphi/\varphi, \end{aligned}$$

takes its values in the subsheaf $\Omega^{1\mathrm{closed}}_{X_0/k}$ of closed 1-forms. Therefore it defines a morphism of complex of sheaves

$$d\log : \mathcal{O}^*_{X_0} \longrightarrow \Omega^\bullet_{X_0/k}[1]$$

and finally of (hyper)cohomology groups

$$H^1(X_0, \mathcal{O}^*_{X_0}) \longrightarrow \mathbb{H}^1\big(X_0, \Omega^\bullet_{X_0/k}[1]\big) = \mathbb{H}^2(X_0, \Omega^\bullet_{X_0/k}).$$

The map so defined will be denoted

$$c_{1,\mathrm{dR}} : \mathrm{Pic}(X_0) := H^1(X_0, \mathcal{O}^*_{X_0}) \longrightarrow H^2_{\mathrm{dR}}(X_0/k).$$

It sends the class of the line bundle L over X_0 defined by a cocycle $(\varphi_{\alpha\beta})$ in $Z^1(\mathcal{U}, \mathcal{O}^*_{X_0})$ to the class of the (hyper)cocycle $(d\varphi_{\alpha\beta}/\varphi_{\alpha\beta})$ in $Z^1(\mathcal{U}, \Omega^{1\,\mathrm{closed}}_{X_0/k})$, identified to a subspace of $Z^2(\mathcal{U}, \Omega^\bullet_{X_0/k})$.

This construction of the first Chern class in algebraic de Rham cohomology is compatible with the topological first Chern class defined in (1.2).

Lemma 2.1 *Assume that k is a subfield of $\mathbb{C}$, and consider a smooth projective variety X_0 over k, the complex algebraic projective variety $X := X_0 \otimes_k \mathbb{C}$, and the associated C-analytic manifold X^{an}, as in 2.2.2. Let L be a line bundle over X_0, let $L_{\mathbb{C}}$ be the algebraic line bundle over X deduced from L by extension of scalars from k to $\mathbb{C}$, and let $L^{\mathrm{an}}_{\mathbb{C}}$ be the associated analytic line bundle over X^{an}.*

The morphism

$$\begin{aligned} H^i_{\mathrm{dR}}(X_0/k) \longrightarrow H^i_{\mathrm{dR}}(X/\mathbb{C}) &\xrightarrow{\sim} H^i(X^{\mathrm{an}}, \mathbb{C}), \\ \alpha \longmapsto \alpha_{\mathbb{C}} := \alpha \otimes_k 1_{\mathbb{C}} &\longmapsto \alpha^{\mathrm{an}}_{\mathbb{C}} \end{aligned}$$

maps $c_{1,\mathrm{dR}}(L)$ to $2\pi i\, c_{1,\mathrm{top}}(L^{\mathrm{an}}_{\mathbb{C}})$.

To prove this lemma, it is enough to consider the case $k = \mathbb{C}$. Then it follows from the fact that the composite morphism of sheaves over X^{an},

$$\mathcal{O}^{\mathrm{an}} \xrightarrow{\mathbf{e}} \mathcal{O}^{\mathrm{an}*} \xrightarrow{d\log} \Omega^1_{X^{\mathrm{an}}},$$

is[6] $2\pi i d$.

2.2.5 Amplification: Modules with integrable connections and de Rham cohomology
In the last sections of this article, we shall use a generalization of the previous results, concerning cohomology with coefficients not only in $\mathbb{C}$ but in local systems of finite-dimensional $\mathbb{C}$-vector spaces.

Let (E, ∇) be a "module with integrable connection" over X, namely, a vector bundle E over X equipped with a connection

$$\nabla : E \longrightarrow E \otimes_{\mathcal{O}_X} \Omega^1_{X/\mathbb{C}}$$

with vanishing curvature. Then ∇ canonically extends to morphisms of sheaves over X,

$$\nabla : E \otimes_{\mathcal{O}_X} \Omega^l_{X/\mathbb{C}} \longrightarrow E \otimes_{\mathcal{O}_X} \Omega^{l+1}_{X/\mathbb{C}},$$

which satisfy the Leibniz rule—namely, for any sections ω of $\Omega^k_{X/\mathbb{C}}$ and α of $E \otimes_{\mathcal{O}_X} \Omega^*_{X/\mathbb{C}}$,

$$\nabla(\omega \wedge \alpha) = d\omega \wedge \alpha + (-1)^k \omega \wedge \nabla\alpha$$

—and the relation

$$\nabla \circ \nabla = 0.$$

Consequently, we may define

$$H^i_{\mathrm{dR}}\big(X/\mathbb{C}, (E, \nabla)\big) := \mathbb{H}^i\big(X, (\Omega^\bullet_{X/\mathbb{C}} \otimes_{\mathcal{O}_X} E, \nabla)\big). \tag{2.9}$$

By analytification, we obtain a complex of sheaves $(\Omega^\bullet_{X^{\mathrm{an}}} \otimes_{\mathcal{O}^{\mathrm{an}}_X} E^{\mathrm{an}}, \nabla)$ on X^{an} from $(\Omega^\bullet_{X/\mathbb{C}} \otimes_{\mathcal{O}_X} E, \nabla)$, and we may define

$$H^i_{\mathrm{dR}}\big(X^{\mathrm{an}}, (E^{\mathrm{an}}, \nabla)\big) := \mathbb{H}^i\big(X^{\mathrm{an}}, (\Omega^\bullet_{X^{\mathrm{an}}} \otimes_{\mathcal{O}^{\mathrm{an}}_X} E^{\mathrm{an}}, \nabla)\big). \tag{2.10}$$

An application of GAGA similar to the one in paragraph 2.2.1 shows that (2.9) and (2.10) are finite-dimensional vector spaces and that the analytification morphisms

$$H^i_{\mathrm{dR}}\big(X/\mathbb{C}, (E, \nabla)\big) \longrightarrow H^i_{\mathrm{dR}}\big(X^{\mathrm{an}}, (E^{\mathrm{an}}, \nabla)\big) \tag{2.11}$$

are isomorphisms.

Besides, the "analytic de Rham complex with coefficients" $(\Omega^\bullet_{X^{\mathrm{an}}} \otimes_{\mathcal{O}^{\mathrm{an}}_X} E^{\mathrm{an}}, \nabla)$ is a resolution of the local constant sheaf E^h of finite-dimensional complex vector spaces (of dimension the rank of E) defined by the $\mathbb{C}$-analytic sections of E^{an} which are "horizontal," that is, in the kernel of ∇. In other words, we have an "analytic Poincaré lemma with coefficients" over X^{an},

$$E^h \xrightarrow{\text{q.i}} (\Omega^\bullet_{X^{\mathrm{an}}} \otimes_{\mathcal{O}^{\mathrm{an}}_X} E^{\mathrm{an}}, \nabla),$$

and consequently an isomorphism of (hyper)cohomology groups:

$$H^i(X^{\mathrm{an}}, E^h) \xrightarrow{\sim} H^i_{\mathrm{dR}}\big(X^{\mathrm{an}}, (E^{\mathrm{an}}, \nabla)\big). \tag{2.12}$$

The isomorphisms (2.11) and (2.12) define by composition an isomorphism

$$H^i_{\mathrm{dR}}\big(X/\mathbb{C}, (E, \nabla)\big) \xrightarrow{\sim} H^i(X^{\mathrm{an}}, E^h).$$

When $X = X_0 \times_k \mathbb{C}$ and (E, ∇) are defined over some subfield k of $\mathbb{C}$, we may define

$$H^i_{\mathrm{dR}}\big(X_0/k, (E, \nabla)\big) := \mathbb{H}^i\big(X_0, (\Omega^\bullet_{X_0/k} \otimes_{\mathcal{O}_{X_0}} E, \nabla)\big).$$

It is a finite-dimensional k-vector space, which defines a natural "form over k" of the cohomology $H^i(X^{\mathrm{an}}, E^h)$ with coefficients in the local system E^h.

2.3 Algebraic and analytic structures, and moduli spaces of vector bundles with integrable connections

2.3.1 Applied to graphs of morphisms, Chow's theorem shows that, for any two *projective* complex varieties X_1 and X_2 (say *smooth* for simplicity), the analytification map defines a bijection:

$$\left\{\begin{matrix}\text{morphisms } \varphi : X_1 \to X_2 \\ \text{of complex algebraic varieties}\end{matrix}\right\} \xrightarrow{\sim} \left\{\begin{matrix}\text{morphisms } \psi : X_1^{\mathrm{an}} \to X_2^{\mathrm{an}} \\ \text{of complex analytic manifolds}\end{matrix}\right\}$$
$$\varphi \longmapsto \varphi^{\mathrm{an}}$$

(see, e.g., [77, Section 4B] for details).

In particular, X_1 and X_2 are isomorphic as complex algebraic varieties if and only if X_1^{an} and X_2^{an} are isomorphic as complex analytic manifolds. Moreover, for any smooth projective complex algebraic variety X, the algebraic variety structure of X is uniquely determined by the structure of the $\mathbb{C}$-analytic manifold X^{an} it induces.

This does not hold anymore for general quasi-projective varieties. In this section, we want to discuss a remarkable family of counterexamples, namely, of pairs (X_1, X_2) of smooth quasi-projective complex algebraic varieties such that X_1^{an} and X_2^{an} are "naturally" isomorphic complex manifolds, although X_1 and X_2 are not algebraically isomorphic.

The GAGA existence theorem will actually play a crucial role in the construction of these counterexamples, which are built from moduli spaces of vector bundles with integrable connections of a given rank N on a smooth projective variety M, and from spaces of representations of degree N of the fundamental group of M^{an}. When $N = 1$, these spaces have been classically considered by Severi and Conforto, and then by Rosenlicht and Serre, during the decades around 1950. For arbitrary $N \geq 1$, they have been investigated thoroughly by Simpson [97], [98] (see also Le Potier [69] for a survey).

2.3.2 Let M be a smooth connected projective complex algebraic variety, and let o be a (complex) point of X. Choose a positive integer N, and consider the following kinds of data:

(i) 3-uples (E, ∇, ψ) consisting in *a vector bundle E of rank N over M, an integrable connection ∇ on E, and a "rigidification"* ψ of E at o, namely, an isomorphism of $\mathbb{C}$-vector spaces

$$\psi : E_o \xrightarrow{\sim} \mathbb{C}^N;$$

(ii) *representations of degree N,*

$$\rho : \Gamma \longrightarrow \mathrm{GL}_N(\mathbb{C})$$

of the fundamental group $\Gamma := \pi_1(M^{\mathrm{an}}, o)$ of the complex analytic manifold M^{an} with base point o.

Observe that we may consider $\mathbb{C}$-analytic versions of data of type (i), namely,

(i)$^{\mathrm{an}}$ 3-uples $(E^{\mathrm{an}}, \nabla^{\mathrm{an}}, \psi)$ consisting in *an analytic vector bundle E of rank N over M^{an}, an integrable analytic connection ∇^{an} on E^{an}, and a rigidification* ψ of E^{an} at o.

The notion of isomorphisms between two data of type (i), or between two data of type (i)$^{\mathrm{an}}$, is defined in the obvious manner as an isomorphism of (algebraic or analytic) vector bundles, compatible with the connections and rigidifications. Observe that, when such an isomorphism exists, it is actually unique.

Through analytification, any data (E, ∇, ψ) of type (i) determines a data $(E^{\mathrm{an}}, \nabla^{\mathrm{an}}, \psi)$ of type (i)$^{\mathrm{an}}$. Conversely, GAGA theorems show that any data of type (i)$^{\mathrm{an}}$ may be obtained by analytification from some data of type (i), that is, uniquely determined (up to unique algebraic isomorphism).[7]

In turn, to any data of type (i)$^{\mathrm{an}}$ is associated its monodromy representation in the fiber E_0 of the flat vector bundle $(E^{\mathrm{an}}, \nabla^{\mathrm{an}})$, which may be identified to a $\mathrm{GL}_N(\mathbb{C})$-representation by means of the rigidification ψ:

$$\rho : \Gamma \longrightarrow \mathrm{GL}(E_o) \xrightarrow{\psi.\psi^{-1}} \mathrm{GL}_N(\mathbb{C}).$$

Conversely, we may introduce the universal covering $(\tilde{M}, \tilde{o})$ of the pointed connected complex manifold (M^{an}, o)—it is a Γ-covering of M^{an}—and the trivial vector bundle $\tilde{E} := \tilde{M} \times \mathbb{C}^N$ of rank N over $\tilde{M}$, equipped with the "trivial" integrable analytic connection $\tilde{\nabla} := d \otimes \mathrm{Id}_{\mathbb{C}^N}$. If $\rho : \Gamma \longrightarrow \mathrm{GL}_N(\mathbb{C})$ denotes an arbitrary representation, the action of Γ on $\mathbb{C}^N$ defined by ρ makes $(\tilde{E}, \tilde{\nabla})$ a Γ-equivariant analytic vector bundle with integrable connection, which moreover is naturally rigidified at $\tilde{o}$. This equivariant rigidified vector bundle with integrable connection over $(\tilde{M}, \tilde{o})$ descends to some rigidified vector bundle of rank N with integrable connection $(E^{\mathrm{an}}, \nabla^{\mathrm{an}}, \psi)$ on the pointed complex manifold (M^{an}, o).

These last two constructions are clearly inverse of each other and establish a natural bijection between (isomorphism classes) of data of type (i)$^{\mathrm{an}}$ and representations of type (ii). Combined with the above GAGA correspondence between data of type (i) and (i)$^{\mathrm{an}}$, this becomes a natural bijection between (isomorphism classes) of data of type (i) and representations of type (ii).

2.3.3 The set of (isomorphism classes) of data of type (i) coincides with the set of complex points $\mathbf{MIC}_N(M, o)(\mathbb{C})$ of some quasi-projective scheme $\mathbf{MIC}_N(M, o)$ over $\mathbb{C}$, which represents the functor which maps a $\mathbb{C}$-scheme (of finite type) S to the isomorphism classes of "data of type (i) over S," defined as 3-uples (E, ∇, ψ) where E denotes a locally free coherent sheaf of rank N over $M \times S$, ∇ denotes an integrable connection on E, relative to the projection $M \times S \to S$, and ψ denotes a rigidification $E_{|o \times S} \xrightarrow{\sim} \mathcal{O}_S^{\oplus N}$.

At this level of generality, the existence of the quasi-projective scheme $\mathbf{MIC}_N(M, o)$ representing this functor is one of the main results of Simpson in [97] and [98], where it is denoted $\mathbf{R}_{\mathrm{DR}}(M, o, N)$. A central point in the construction of $\mathbf{MIC}_N(M, o)$ is the fact that the vector bundles E of rank N over M admitting an integrable connection ∇ constitute a bounded family (see [69, Lemme 9] for a concise presentation of Simpson's argument in this specific situation).

The set of representations of type (ii) coincides with the set of complex points $\mathbf{Rep}_N(\Gamma)(\mathbb{C})$ of the quasi-projective (actually affine) scheme $\mathbf{Rep}_N(\Gamma)$ over $\mathbb{C}$ which represents the functor which sends a $\mathbb{C}$-scheme of finite type S to the set of representations

$$\rho : \Gamma \longrightarrow \mathrm{GL}_N\big(\Gamma(S, \mathcal{O}_S)\big).$$

The existence of the scheme $\mathbf{Rep}_N(\Gamma)$ is a straightforward consequence of the existence of a finite presentation for the fundamental group Γ (see, e.g., [98, Section 5], where this scheme is denoted $\mathbf{R}(\Gamma, N)$ or $\mathbf{R}_{\mathrm{B}}(M, o, N)$).

The bijection constructed in Section 2.3.2, by associating the monodromy representation of its analytification to some data of type (i), defines a bijection

$$\mathbf{MIC}_N(M,o)(\mathbb{C}) \xrightarrow{\sim} \mathbf{Rep}_N(\Gamma)(\mathbb{C}), \tag{2.13}$$

which turns out to be defined by a canonical isomorphism of $\mathbb{C}$-analytic spaces

$$\mathrm{mon}_o : \mathbf{MIC}_N(M,o)^{\mathrm{an}} \xrightarrow{\sim} \mathbf{Rep}_N(\Gamma)^{\mathrm{an}} \tag{2.14}$$

(cf. [98, Section 7]; this formally expresses the fact that the construction in Section 2.3.2 “analytically depends on parameters” in an arbitrary analytic space).

2.3.4 However, in general, the analytic isomorphism (2.14) is *not* induced by an algebraic isomorphism from $\mathbf{MIC}_N(M,o)$ to $\mathbf{Rep}_N(\Gamma)$.

This is already the case when M is a smooth connected projective curve C of positive genus g and $N = 1$. Then

$$\mathrm{Pic}^{\natural}(C) := \mathbf{MIC}_1(C,o)$$

may be identified with the *universal vector extension* $E(\mathrm{Pic}_0(C))$ of the connected Picard variety $\mathrm{Pic}_0(C)$ of C (see, e.g., Messing [76], Mazur [75], Bost and Künnemann [17]). Actually, $\mathrm{Pic}^{\natural}(C)$ classifies pairs (L,∇) consisting in a line bundle L of degree zero over C and a (necessarily integrable) connection ∇ over L. The tensor product of line bundles with connections induces a structure of algebraic groups on $\mathrm{Pic}^{\natural}(C)$. It fits into the following exact sequence of connected commutative group schemes over $\mathbb{C}$, which displays it as a vector extension of $\mathrm{Pic}_0(C)$:

$$\begin{array}{ccccccccc} 0 \longrightarrow \Omega^1(C) \longrightarrow & \mathrm{Pic}^{\natural}(C) & \longrightarrow \mathrm{Pic}_0(C) \longrightarrow 0, \\ \alpha \longmapsto & [(\mathcal{O}_C, d+\alpha)], & \\ & [(L,\nabla)] & \longmapsto \quad [L]. \end{array} \tag{2.15}$$

Besides, the representation space $\mathbf{Rep}_1(\pi_1(C^{\mathrm{an}},o))$ may be identified with the torus

$$H^1(C^{\mathrm{an}},\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{G}_m \simeq \mathbb{G}_m^{2g},$$

and the monodromy isomorphism (2.14) takes the form of an isomorphism of complex Lie groups:

$$\mathrm{Pic}^{\natural}(C)^{\mathrm{an}} \xrightarrow{\sim} \mathbb{C}^{*2g}.$$

However, the description of $\mathrm{Pic}^{\natural}(C)$ as a vector extension of an abelian variety easily implies that every morphism of algebraic variety from $\mathrm{Pic}^{\natural}(C)$ to $\mathbb{G}_m$ is constant. A fortiori, the algebraic varieties $\mathbf{MIC}_1(C,o) = \mathrm{Pic}^{\natural}(C)$ and $\mathbf{Rep}_1(\pi_1(C^{\mathrm{an}},o)) \simeq \mathbb{G}_m^{2g}$ are not isomorphic.[8]

2.3.5 For later reference, let us indicate diverse variants of the previous constructions.

First of all, for any base field of characteristic zero and any pointed connected smooth pointed variety (M,o) over k, the construction of the quasi-projective scheme $\mathbf{MIC}_N(M,o)$ makes sense over k: it classifies data of type (i) over varying k-schemes S. This follows from a straightforward generalization of the arguments in [97], or (say, when k is a subfield of $\mathbb{C}$) from a descent argument.

When $N = 1$, the tensor product of line bundles with (necessarily integrable) connections makes the quasi-projective scheme $\mathbf{MIC}_1(M,o)$ a group scheme, necessarily smooth over k. Moreover, its connected component $\mathbf{MIC}_1(M,o)_0$ may be identified with the universal vector extension $E(\mathrm{Pic}_0(M))$ of the connected Picard variety $\mathrm{Pic}_0(M)$ of M. Indeed, the obvious analogue of the short exact sequence (2.15) still holds in this setting (see, e.g., [17, Appendix B]).

When M is the abelian variety $\hat{A}$ dual to some abelian variety A over k, this construction identifies the universal vector extension $E(A)$ of A to the k-algebraic group

$$\mathrm{Pic}^{\natural}(\hat{A}) := \mathbf{MIC}_1(\hat{A}, 0_{\hat{A}}),$$

which classifies line bundles with (necessarily integrable) connections over A, and the short exact sequence (2.15) becomes the extension defining $E(A)$:

$$0 \longrightarrow \mathbb{E}_{\hat{A}} := (\mathrm{Lie}\,\hat{A})^{\vee} \longrightarrow E(A) \xrightarrow{p_A} A \longrightarrow 0.$$

Second, it is convenient to have at one's disposal diverse generalizations of the moduli spaces $\mathbf{MIC}_N(M, o)$. For instance, if (M, o, o') denotes a connected smooth projective variety over k, endowed with two (possibly equal) "base points" o and o' in $M(k)$, we may construct a quasi-projective scheme $\mathbf{MIC}_N(M, o, o')$ that classifies vector bundles E of rank N over M, equipped with an integrable connection ∇ and with rigidifications $\psi : E_o \xrightarrow{\sim} k^N$ and $\psi' : E_{o'} \xrightarrow{\sim} k^N$ at o and o' (cf. [97, Remark, p. 109]). Thanks to the morphism

$$F : \mathbf{MIC}_N(M, o, o') \longrightarrow \mathbf{MIC}_N(M, o)$$

defined by forgetting the rigidifications ψ' at o' and to the action by composition of $\mathrm{GL}_{N,k}$ on these rigidifications, $\mathbf{MIC}_N(M, o, o')$ becomes a $\mathrm{GL}_{N,k}$-torsor over $\mathbf{MIC}_N(M, o)$. When $N = 1$, the tensor product again makes $\mathbf{MIC}_N(M, o, o')$ a commutative algebraic group over k, and the above structure of the $\mathrm{GL}_{N,k}$-torsor becomes an extension of commutative algebraic groups:

$$0 \longrightarrow \mathbb{G}_{m,k} \longrightarrow \mathbf{MIC}_1(M, o, o') \longrightarrow \mathbf{MIC}_1(M, o) \longrightarrow 0. \tag{2.16}$$

When $M = \hat{A}$ as above, $o = 0_{\hat{A}}$, and o' is a point P in $\hat{A}(k)$ parameterizing some line bundle L over A (equipped with a rigidification $\epsilon : k \simeq L_{0_A}$ and algebraically equivalent to zero), one gets an extension

$$0 \longrightarrow \mathbb{G}_{m,k} \longrightarrow \mathbf{MIC}_1(\hat{A}, 0_A, P) \longrightarrow E(A) \longrightarrow 0, \tag{2.17}$$

which may be described as follows. The $\mathbb{G}_{m,k}$-torsor $L^{\times}$ over A, deduced from the total space of L by deleting its zero section may be endowed with a unique structure of k-algebraic group which makes the diagram

$$0 \longrightarrow \mathbb{G}_{m,k} \xrightarrow{\epsilon} L^{\times} \longrightarrow A \longrightarrow 0 \tag{2.18}$$

a short exact sequence of commutative algebraic groups over k, and the extension (2.17) coincides with the pullback of the extension (2.18) by $p_A : E(A) \longrightarrow A$.

3 Algebraization of Formal Objects

3.1 A theorem of Grauert and Grothendieck Since the work of Zariski [102] on "holomorphic functions" and its amplification in Grothendieck's new foundations of algebraic geometry (see [45]), *formal schemes* and coherent sheaves over them play a central role in modern algebraic geometry. Grothendieck notably established some comparison and existence theorems that relate algebraic and formal geometry over a suitable complete "adic" base ring (cf. Grothendieck [45], [44], Illusie [58]). In SGA2 (see [47]), motivated by some earlier work of Grauert, he also used formal geometry to investigate the classical Lefschetz theorems comparing the geometry of projective varieties and of their hyperplane sections.

In the remainder of this article, we shall be concerned by the algebraization theorems of "Lefschetz type" established in SGA2 rather than by the earlier "fundamental" comparison and existence theorems discussed in [45], [44], and [58].

For the sake of simplicity, we first state a (weaker) analytic version of these theorems of Lefschetz type in a special simple case.

Theorem 3.1 (Grauert, Grothendieck [47]) *Let $X \hookrightarrow \mathbb{P}^N_{\mathbb{C}}$ be a smooth projective complex variety of dimension d, and let $Y := X \cap \mathbb{P}^{N-1}_{\mathbb{C}}$ be a hyperplane section of X of dimension $d-1$.*

Gr1: *If $d \geq 2$, then for every algebraic vector bundle E over X, the restriction map*

$$\Gamma(X, E) \longrightarrow \{\textit{germs of analytic sections of } E \textit{ along } Y\}$$

is an isomorphism.

Gr2: *If $d \geq 3$, any germ of analytic vector bundle $\mathcal{E}$ on some analytic neighborhood of Y in X "extends" to some coherent sheaf E over X.*

Observe that, like GAGA, this theorem decomposes into two parts: a "comparison theorem" **Gr1**, and an "existence theorem" **Gr2**.

Observe also that, according to Serre's GAGA, the vector bundle E in **Gr1** and its space of global sections $\Gamma(X, E)$ may be equivalently taken in the algebraic or in the analytic category. The same remark applies to the coherent sheaf E the existence of which is asserted in **Gr2**. Accordingly, when the conclusion of **Gr2** holds, we shall say that $\mathcal{E}$ is *algebraizable*.

Let us emphasize that the assumptions on the dimension d are crucial in Theorem 3.1.

Indeed, **Gr1** trivially fails for $X = \mathbb{P}^1$, $Y = \{\text{point}\}$, and $E = \mathcal{O}_X$.

The existence theorem **Gr2** already fails for line bundles when X is the projective plane $\mathbb{P}^2_{\mathbb{C}}$ and $Y = \mathbb{P}^1_{\mathbb{C}}$ a projective line in X. This follows from Proposition 3.2 below, which is a simple consequence of **Gr1**.

Let X_∞ denote a projective line in X distinct from Y, and let us consider the affine plane $\mathbb{A}^2_{\mathbb{C}} := X \setminus X_\infty$ and the affine line $\mathbb{A}^1_{\mathbb{C}} := \mathbb{A}^2_{\mathbb{C}} \cap Y$. Choose affine coordinates (x, y) on $\mathbb{A}^2_{\mathbb{C}}$ such that $\mathbb{A}^1_{\mathbb{C}} = (x = 0)$. For any converging power series f in $\mathbb{C}\{T\}$, the equation

$$y = f(x)$$

defines a germ T_f of smooth analytic curve in $X = \mathbb{P}^2_{\mathbb{C}}$ transverse to $Y = \mathbb{P}^1_{\mathbb{C}}$.

Proposition 3.2 *The germ of analytic line bundle $\mathcal{O}^{\mathrm{an}}(T_f)$ along $\mathbb{P}^1_{\mathbb{C}}$ in $\mathbb{P}^2_{\mathbb{C}}$ is algebraizable if and only if the series f belongs to $\mathbb{C}T + \mathbb{C}$.*

Observe also that Theorem 3.1 admits striking elementary geometric applications. For instance, it implies that *any germ of analytic hypersurface along $\mathbb{P}^2_{\mathbb{C}}$ in $\mathbb{P}^3_{\mathbb{C}}$ extends to a global algebraic hypersurface, defined by the vanishing of some homogeneous polynomial in $\mathbb{C}[X_0, X_1, X_2, X_3]$.*

3.2 Formal geometry In SGA2, Theorem 3.1 is stated and proved in a more general formulation, which (i) concerns *formal* sections and vector bundles instead of analytic germs of sections and vector bundles, (ii) makes sense over an arbitrary base field k—indeed, over an arbitrary Noetherian base S—instead of $\mathbb{C}$, and (iii) holds under regularity assumptions weaker than the smoothness of X, formulated in terms of depth. In this paragraph, we want to give some indication of the generalizations (i) and (ii), while keeping minimal the prerequisites from formal geometry.

Recall (see, e.g., [58]) that, for any Noetherian scheme X and any closed subscheme Y in X, a coherent formal sheaf $\mathcal{E}$ over the formal scheme $\widehat{X}_Y$, by completion of X along Y, is nothing else than the data of a system $(\mathcal{E}_n)_{n\in\mathbb{N}}$ of coherent sheaves on the successive infinitesimal neighborhoods Y_n of Y in X ($Y_0 := Y$; Y_n is defined by the $(n+1)$th power $\mathcal{I}_Y^{n+1}$ of the ideal sheaf $\mathcal{I}_Y$ of Y in $\mathcal{O}_X$), equipped with isomorphisms

$$\mathcal{E}_{n+1|Y_n} \xrightarrow{\sim} \mathcal{E}_n. \tag{3.1}$$

The coherent formal sheaf $\mathcal{E}$ is locally free—and then called a *vector bundle*—if and only if, for every n, $\mathcal{E}_n$ is a locally free coherent sheaf of $\mathcal{O}_{Y_n}$-modules.

By definition, the space of sections of $\mathcal{E}$ over $\widehat{X}_Y$ is precisely the projective limit

$$\Gamma(\widehat{X}_Y, \mathcal{E}) := \varprojlim_n \Gamma(Y_n, \mathcal{E}_n),$$

defined by means of the isomorphisms (3.1) and of the induced projective system of spaces of sections:

$$\Gamma(Y_{n+1}, \mathcal{E}_{n+1}) \xrightarrow{\cdot_{|Y_n}} \Gamma(Y_n, \mathcal{E}_{n+1|Y_n}) \xrightarrow{\sim} \Gamma(Y_n, \mathcal{E}_n).$$

A coherent sheaf E over X defines a formal coherent sheaf $E_{|\widehat{X}_Y} := (E_{|Y_n})$ over $\widehat{X}_Y$. A formal coherent sheaf on $\widehat{X}_Y$ will be called *algebraizable* if, up to isomorphism, it is of the form $E_{|\widehat{X}_Y}$ for some coherent sheaf E over X.

Using these definitions, we may state a generalized version of Theorem 3.1 valid for a smooth projective scheme over an arbitrary base field k.

Theorem 3.3 *Let $X \hookrightarrow \mathbb{P}_k^N$ be a smooth projective scheme over k, of pure dimension d, and let $Y := X \cap \mathbb{P}_k^{N-1}$ be some hyperplane section, of dimension $d-1$.*

Gr1: *If $d \geq 2$, then for any vector bundle E over X, the restriction map*

$$\Gamma(X, E) \longrightarrow \Gamma(\widehat{X}_Y, E_{|\widehat{X}_Y}) := \varprojlim_n \Gamma(Y_n, E_{|Y_n})$$

is an isomorphism.

Gr2: *If $d \geq 3$, then any vector bundle $\mathcal{E}$ over $\widehat{X}_Y$ is algebraizable.*

Like the proof of Serre's GAGA and of Grothendieck's comparison and existence theorems in [45], [44], and [58], the proofs in SGA2 are cohomological. For instance, a key point in the proof of **Gr2** is that, since $d \geq 3$, the Cartier divisor Y has depth at least 2 and the ampleness of $\mathcal{O}_X(Y)_{|Y}$ implies that, for every vector bundle E_0 over Y, the cohomology group $H^1(Y, E_0 \otimes \mathcal{O}_X(-Y)_{|Y}^{\otimes n})$ vanishes for n a sufficiently large positive integer (lemma of Enriques–Severi–Zariski). This implies that, for any vector bundle $\mathcal{E} = (E_n)$ over $\widehat{X}_Y$, the system $(H^1(Y_n, E_n))$ is essentially constant, and consequently

$$H^1(\widehat{X}_Y, \mathcal{E}) = \varprojlim_n H^1(Y_n, \mathcal{E}_n)$$

is a *finite-dimensional* k-vector space. The finite dimensionality of a first cohomology group plays the same role here as in the proofs of the Poincaré–Lefschetz–Hodge theorem by Kodaira and Spencer, and of the GAGA existence theorem by Serre.

Let us also indicate that the results in SGA2 have been extended in diverse directions by Raynaud [85] and Faltings [39] and that, besides the original cohomological proofs, it is possible to give more "classical" proofs of Theorems 3.1 and 3.3, based on Theorem 3.4 infra and its formal variant, which ultimately rely on the use of "auxiliary polynomials," familiar in Diophantine approximation and transcendence.

3.3 A theorem of Andreotti and Hartshorne Let us mention that diverse algebraization results concerning formal meromorphic functions along subvarieties have also been established, notably by Hironaka and Matsumura [53], Faltings [40] and [41], and Chow [30].

We want to discuss briefly an algebraization result, concerning formal germs along curves, that is related both to the results in [53], [40], [41], and [30] and to the Grauert–Grothendieck Theorems 3.1 and 3.3. For the sake of simplicity, we state it in the analytic framework, in which situation it goes back to Andreotti [4].

Theorem 3.4 *Let $C \hookrightarrow \mathbb{P}^N_{\mathbb{C}}$ be a smooth connected projective complex algebraic curve, and let $\mathcal{V}$ be a germ of a smooth $\mathbb{C}$-analytic submanifold along C in $\mathbb{P}^N(\mathbb{C})$.*
If the normal bundle $N_C\mathcal{V}$ to C in $\mathcal{V}$ is ample, then $\mathcal{V}$ is algebraic.

Observe that the normal bundle $N_C\mathcal{V}$ is an analytic vector bundle over C, which by GAGA defines an algebraic vector bundle over C. When $\dim \mathcal{V} = 2$, it is a line bundle, and its ampleness is equivalent to the positivity of its degree $\deg_C N_C\mathcal{V}$.

In Theorem 3.4, the algebraicity of $\mathcal{V}$ precisely means that the dimension $\dim \overline{\mathcal{V}}^{\mathrm{Zar}}$ of its Zariski closure $\overline{\mathcal{V}}^{\mathrm{Zar}}$ in $\mathbb{P}^N_{\mathbb{C}}$, which is at least equal to the complex dimension $\dim \mathcal{V}$ of the complex manifold $\mathcal{V}$, actually coincides with $\dim \mathcal{V}$. This is equivalent to the fact that the germ $\mathcal{V}$ is a "branch" along C of some (irreducible) algebraic subset of $\mathbb{P}^N_{\mathbb{C}}$ containing C.

Here again, Theorem 3.4 admits a formal generalization, valid over any base field, where $\mathcal{V}$ is a smooth formal subscheme containing C of the formal completion of $\mathbb{P}^N_k$ along a smooth projective k-curve. It may also be extended to higher-dimensional situations: the curve C may be replaced by any smooth projective subvariety Y of dimension at least 1. This condition is similar to the dimension condition in the assertions **Gr1** in Theorems 3.1 and 3.3. Actually **Gr1** may be derived from Theorem 3.4 and its higher-dimensional and formal generalization by considering the graphs of analytic or formal sections (see [16]).

In its analytic (resp., formal) form, Theorem 3.4 is a direct consequence—by the "anonymous" argument recalled in Section 2.1—of a result of Andreotti [4] (resp., of Hartshorne [49]) which asserts that the field of meromorphic functions (resp., of formal meromorphic functions) on $\mathcal{V}$ is a field of transcendence degree at most $\dim \mathcal{V}$ over $\mathbb{C}$ (resp., over k).

Theorem 3.4 may also be established by directly estimating the Hilbert function of the Zariski closure of $\mathcal{V}$, with no recourse to the (formal) meromorphic functions (see Bost [13, Section 3.3], and [15]). This type of argument may be seen as a geometric counterpart of the use of auxiliary polynomials in Diophantine approximation and transcendence proofs.

Algebraization criteria in the style of Theorem 3.4 have been recently reconsidered in Bogomolov and McQuillan [11] and [13] in relation to algebraicity properties of leaves of algebraic foliations (see Kebekus, Solá Conde, and Toma [60] for geometric applications and references, and see Bost [14] for similar geometric applications to groups schemes over projective curves).

3.4 Algebraization over function fields The above algebraization theorems concerning formal "objects" over projective varieties on some base field k may be used to derive algebraization theorems over projective varieties on function fields of the form $k(C)$, where C denotes some projective variety over k.

We illustrate this general principle by formulating an application of Theorem 3.4 to the algebraicity of formal germs in varieties over the function field $\mathbb{C}(C)$ defined by some smooth projective complex curve C. The details of its derivation, which is straightforward, will be left to the reader, as well as the derivation from the formal variant of Theorem 3.4 of a similar algebraicity criterion for formal germs in varieties over a general function field $k(C)$.

Let C be a smooth projective complex curve, and let $\pi : \mathcal{X} \to C$ be a projective complex variety fibered over C. (In other words, π is a flat surjective morphism of complex schemes.)

Let $K := \mathbb{C}(C)$ be the function field of C, and let $X := \mathcal{X}_K$ be the generic fiber of π. It is a projective K-variety, and conversely, any projective K-variety may be realized as the generic fiber of a suitable projective model $\mathcal{X}$ fibered over C as above.

Let P be a K-point of X. By the projectivity of π, it extends to a section $\mathcal{P}$ of π over C.

Consider a smooth formal germ of a subvariety through P in X,

$$\widehat{V} := \varinjlim_i V_i,$$

namely, a smooth formal subscheme of the completion $\widehat{X}_P$. Here again it is said to be *algebraic* when its Zariski closure $\overline{\widehat{V}}^{\mathrm{Zar}\, X}$ in the K-scheme X has the same dimension as $\widehat{V}$.

The V_i's are zero-dimensional subschemes of $X = \mathcal{X}_K$ supported by P. Their closures in $\mathcal{X}$,

$$\mathcal{V}_i := \overline{V_i}^{\mathrm{Zar}\, \mathcal{X}},$$

are one-dimensional subschemes of $\mathcal{X}$ with support $\mathcal{P}$ and constitute an inductive system

$$\mathcal{V}_0 = \mathcal{P} \hookrightarrow \mathcal{V}_1 \hookrightarrow \mathcal{V}_2 \hookrightarrow \cdots \hookrightarrow \mathcal{V}_i \hookrightarrow \mathcal{V}_{i+1} \hookrightarrow \cdots .$$

In general this system $(\mathcal{V}_i)_{i\in\mathbb{N}}$ does *not* define a formal subscheme of the completion $\hat{\mathcal{X}}_{\mathcal{P}}$ smooth over C. However, it is the case when there exists a germ $\mathcal{V}$ of analytic submanifold of $\mathcal{X}^{\mathrm{an}}$ along $\mathcal{P}$ that "extends" $(\mathcal{V}_i)_{i\in\mathbb{N}}$ in the sense that $\mathcal{V}_i$ is the ith infinitesimal neighborhood of $\mathcal{P}$ in $\mathcal{V}$.

Corollary 3.5 *With the above notation, if $\widehat{V}$ extends to a germ $\mathcal{V}$ of a smooth analytic submanifold of $\mathcal{X}^{\mathrm{an}}$ along $\mathcal{P}$ and if the normal bundle $N_{\mathcal{P}}\mathcal{V}$ to $\mathcal{P}$ in $\mathcal{V}$ is ample, then $\widehat{V}$ is algebraic.*

A generalization of this corollary, formulated in terms of formal geometry only, holds when the base field $\mathbb{C}$ is replaced by an arbitrary base field k. Namely, *$\widehat{V}$ is algebraic when it extends to a formal subscheme $\hat{\mathcal{V}}$ of $\hat{\mathcal{X}}_{\mathcal{P}}$ smooth over the base curve C and when the normal bundle $N_{\mathcal{P}}\hat{\mathcal{V}}$ is ample.*

4 Algebraization and Transcendence

Various classical results in transcendence theory and Diophantine approximation may be rephrased in geometric terms as algebraization results, asserting the algebraicity of certain formal or analytic subvarieties inside algebraic varieties defined over number fields, provided that suitable arithmetic and analytic conditions are satisfied (see, e.g., [13], [15], Chambert-Loir [28], Gasbarri [42]).

In this article, we are concerned with transcendence results of "Schneider–Lang type" in the line of the classical theorems of Schneider (see [89], [90]) about the transcendence of values of abelian functions and of their modern amplification by Lang (see [65], [66], [67]). We shall content ourselves with two instances of these transcendence theorems, whose proofs involve only elementary analytic techniques. We refer the reader to Bombieri [12], Waldschmidt [100], Demailly [37], Gasbarri [42], and Herblot [52] for more general higher-dimensional situations and references to related work.

In the remainder of the article, $\overline{\mathbb{Q}}$ will denote the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$—or equivalently, an algebraic closure of $\mathbb{Q}$ equipped with some preferred embedding in $\mathbb{C}$.

4.1 Algebraicity of leaves of rank 1 algebraic foliations Let K be a number field embedded in $\mathbb{C}$, let X be a smooth quasi-projective variety over K, and let $L \hookrightarrow T_{X/K}$ be a subvector bundle of rank 1 of its tangent bundle.

By base field extension from K to $\mathbb{C}$ and analytification, we obtain a complex analytic manifold $X^{\mathrm{an}}_{\mathbb{C}}$ and an analytic subvector bundle $L^{\mathrm{an}}_{C} \hookrightarrow T_{X^{\mathrm{an}}_{C}}$. Since $L^{\mathrm{an}}_{\mathbb{C}}$ has rank 1, it is integrable (in other words, its sheaf of sections is stable under Lie bracket) and defines a $\mathbb{C}$-analytic foliation of $X^{\mathrm{an}}_{\mathbb{C}}$. Consider some analytic leaf $\mathcal{F}$ of this foliation—it is a connected Riemann surface, equipped with an injective analytic immersion into $X^{\mathrm{an}}_{\mathbb{C}}$—and assume that, for some closed discrete subset Δ of $\mathbb{C}$, we are given a nonconstant analytic map:

$$f : \mathbb{C} \setminus \Delta \longrightarrow \mathcal{F}.$$

The map f defines an analytic map from $\mathbb{C} \setminus \Delta$ into the quasi-projective complex variety $X^{\mathrm{an}}_{\mathbb{C}} \hookrightarrow \mathbb{P}^N(\mathbb{C})$. As such, it is said to be *meromorphic* on $\mathbb{C}$ when it extends to an analytic map, which we will still denote f, from $\mathbb{C}$ to $\mathbb{P}^N(\mathbb{C})$. When this holds, it is said to be *of order at most* ρ for some $\rho \in \mathbb{R}_+$ when, for every $\epsilon > 0$, it admits an analytic lift[9]

$$F = (F_0, \ldots, F_N) : \mathbb{C} \longrightarrow \mathbb{C}^{N+1} \setminus \{0\}$$

such that

$$\log^+ \max_{0 \leq i \leq N} \left|F_i(t)\right| = O\left(|t|^{\rho+\epsilon}\right) \quad \text{when } |t| \to +\infty.$$

Here is a first instance of a transcendence theorem à la Schneider–Lang (see, e.g., [52], notably Section 6, for a proof and for a discussion of earlier variants).

Theorem 4.1 *Let K, X, $\mathcal{F}$, Δ, and f be as above. If*

(1) f is meromorphic of finite order at most ρ, and

(2) there exists a subset A of $\mathbb{C} \setminus \Delta$ such that $f(A) \subset X(K)$, whose cardinality $|A|$ satisfies

$$|A| > 2\rho[K : \mathbb{Q}],$$

then $\mathcal{F}$ is algebraic.

Here the algebraicity of $\mathcal{F}$ precisely means that the Riemann surface $\mathcal{F}$, injectively immersed in $X_{\mathbb{C}}^{\mathrm{an}}$, is actually a (necessarily closed and smooth) complex algebraic curve in $X_{\mathbb{C}}$. It is equivalent to the algebraicity of the formal germ $\widehat{\mathcal{F}}_{f(z)}$ of $\mathcal{F}$ through $f(z)$, for any $z \in A$. The formal germ $\widehat{\mathcal{F}}_{f(z)} \hookrightarrow \widehat{X}_{\mathbb{C},f(z)}$ is indeed defined[10] over K, and consequently its Zariski closure in $X_{\mathbb{C}}$ is also. Finally, when conditions (1) and (2) hold, $\mathcal{F}$ is the set of complex points of some smooth closed K-curve in X.

Classically a transcendence theorem à la Schneider–Lang like Theorem 4.1 is rather expressed in the following contrapositive formulation: *if f is meromorphic of finite order ρ and if $\mathcal{F}$ is not algebraic, then the cardinality of the subset $f^{-1}(X(K))$ of $\mathbb{C} \setminus \Delta$ is at most $2\rho[K:\mathbb{Q}]$.*

A simple but nontrivial instance of Theorem 4.1 arises when

$$X := \mathbb{A}^1 \times \mathbb{G}_m,$$
$$L := (\partial/\partial x + y\partial/\partial y)\mathcal{O}_X$$

(where x and y denote the standard coordinates on $\mathbb{A}^1 \times \mathbb{G}_m \hookrightarrow \mathbb{A}^2$), and $\mathcal{F}$ is the image of

$$f : \mathbb{C} \longrightarrow X_{\mathbb{C}}^{\mathrm{an}}, \qquad t \longmapsto (t, e^t).$$

Clearly f is of order at most 1 and $\mathcal{F}$ is not algebraic, and Theorem 4.1 asserts that, for any number field K in $\mathbb{C}$, the intersection $f^{-1}(X(K))$ is finite, of cardinality at most $2[K:\mathbb{Q}]$. Besides, if for some z in K, $f(z)$ belongs to $X(K)$, then for any $n \in \mathbb{Z}$, $f(nz)$ belongs to $X(K)$. Consequently in this case Theorem 4.1 boils down to the *theorem of Hermite and Lindemann*, which asserts that *for any nonzero complex number z, (z, e^z) does not belong to $\overline{\mathbb{Q}}^2$.*

4.2 Algebraic Lie subalgebras Let G be a (quasi-projective) algebraic group over $\overline{\mathbb{Q}}$, and let $\operatorname{Lie} G$ denote its Lie algebra. Observe that

$$\operatorname{Lie} G_{\mathbb{C}} := \operatorname{Lie} G \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \simeq \operatorname{Lie}(G_{\mathbb{C}})$$

may be identified with the Lie algebra of the complex Lie group $G_{\mathbb{C}}^{\mathrm{an}}$. In particular, we may consider the exponential map of this Lie group:

$$\exp_{G_{\mathbb{C}}} : \operatorname{Lie} G_{\mathbb{C}} \longrightarrow G_{\mathbb{C}}^{\mathrm{an}}.$$

It is a $\mathbb{C}$-analytic map, étale at zero, and of finite order.

We may also consider the formal variant of this exponential map:

$$\widehat{\exp}_G : (\operatorname{Lie} G)_0^{\wedge} \xrightarrow{\sim} \widehat{G}_e,$$

which is an isomorphism between the formal completion of $\operatorname{Lie} G$ at zero—defined as the formal spectrum of the completion of the symmetric algebra $\operatorname{Sym}^{\bullet}(\operatorname{Lie} G)^{\vee}$,

$$(\operatorname{Lie} G)_0^{\wedge} := \operatorname{Spf}\big[\operatorname{Sym}^{\bullet}(\operatorname{Lie} G)^{\vee}\big]^{\wedge}$$

—and the formal completion $\widehat{G}_e$ of G at its unit element e.

A $\overline{\mathbb{Q}}$-Lie subalgebra V of $\operatorname{Lie} G$ will be called *algebraic* when the formal subgroup $\widehat{\exp}_G V_0^{\wedge}$ that it defines may be algebraized, or equivalently, when *there exists a $\overline{\mathbb{Q}}$-algebraic subgroup H of G such that $V = \operatorname{Lie} H$.*

Transcendence techniques à la Schneider–Lang may be used to derive "arithmetic criteria" for a Lie subalgebra of $\operatorname{Lie} G$ to be algebraic. For instance, when G is commutative—so that any $\overline{\mathbb{Q}}$-vector subspace of $\operatorname{Lie} G$ is a Lie subalgebra—they lead to the following result, which appears as a vast generalization of Schneider's original result in [89] (see Lang [68, Chapter IV, Section 4, Theorem 2] when G is a linear group or an abelian variety, and see [100, Théorème 5.2.1] for a general commutative algebraic group G).

Theorem 4.2 *For any commutative algebraic group G over $\overline{\mathbb{Q}}$ and any $\overline{\mathbb{Q}}$-vector subspace V of* $\operatorname{Lie} G$*, the following two conditions are equivalent:*

(1) V is an algebraic Lie subalgebra;
(2) there exists a family $(w_i)_{i\in I}$ of element of $V_{\mathbb{C}}$ such that, for any $i \in I$,

$$\exp_{G_{\mathbb{C}}} w_i \in G(\overline{\mathbb{Q}}),$$

which generates the $\mathbb{C}$-vector space $V_{\mathbb{C}}$.

The direct implication (1) $\Rightarrow$ (2) is straightforward. The converse implication (2) $\Rightarrow$ (1) is a transcendence statement. Consider, for instance, the case where $G = \mathbb{G}_m^2$. Then the (connected) algebraic subgroups of G are defined by monomial equations, and consequently the algebraic Lie subalgebras V of

$$\operatorname{Lie} G = \operatorname{Lie} \mathbb{G}_m^2 = \overline{\mathbb{Q}} \cdot x\partial/\partial x \oplus \overline{\mathbb{Q}} \cdot y\partial/\partial y$$

are precisely the $\overline{\mathbb{Q}}$-vector subspaces of $\operatorname{Lie} G$ which are $\mathbb{Q}$-rational in the basis $(x\partial/\partial x, y\partial/\partial y)$. Therefore Theorem 4.2 for $G = \mathbb{G}_m^2$ becomes the *theorem of Gelfand and Schneider*, which asserts that *for any α in $\overline{\mathbb{Q}}^*$ and any* nonzero *complex number* $\log \alpha$ *such that* $\exp(\log \alpha) = \alpha$*, and for any β in $\overline{\mathbb{Q}} \setminus \mathbb{Q}$, $\alpha^\beta := \exp(\beta \log \alpha)$ does not belong to $\overline{\mathbb{Q}}$.*

Observe also that, when $\dim V = 1$, Theorem 4.2 follows from Theorem 4.1 applied to the translation invariant subvector bundle L in $T_{G/\overline{\mathbb{Q}}}$ such that $L_e = V$. (Choose K large enough to have G and V defined over K.) In general, Theorem 4.2 may be seen as an algebraic integrability criterion for translation-invariant algebraic foliations on the algebraic groups G.

Let me point out that Theorem 4.2 is now subsumed in stronger transcendence results on commutative algebraic groups, such as the theorems of Baker on linear forms in logarithms and the analytic subgroup theorem of Wüstholz. The reader may find a recent survey of these results in the monograph by Baker and Wüstholz [6].

4.3 Morphisms of commutative algebraic groups In the remainder of the article, we shall use a corollary of Theorem 4.2 which describes morphisms of connected commutative algebraic groups over $\overline{\mathbb{Q}}$ in terms of Lie theoretic data. This type of consequence was already pointed out by Bertrand in [8, Section 5, Proposition 2B], where Theorem 4.2 is applied in a similar way to investigate the ring of endomorphisms of a commutative algebraic group.

If G is a connected commutative algebraic group over $\mathbb{C}$, we may introduce its group of "periods"

$$\operatorname{Per} G := \ker \exp_G,$$

defined as the kernel of its exponential map. It is a discrete subgroup of its Lie algebra $\operatorname{Lie} G$ and fits into an exact sequence of commutative complex Lie groups

$$0 \longrightarrow \operatorname{Per} G \hookrightarrow \operatorname{Lie} G \xrightarrow{\exp_G} G^{\mathrm{an}} \longrightarrow 0.$$

We shall say that G *satisfies condition* **LP** when the group of periods $\mathrm{Per}\, G$ generates $\mathrm{Lie}\, G$ as a complex vector space.

Observe that this condition is preserved by isogenies and by forming quotients and products, and is satisfied by the multiplicative group $\mathbb{G}_{m\mathbb{C}}$, complex abelian varieties, and universal vector extensions. Actually, a connected commutative algebraic group G over $\mathbb{C}$ satisfies condition **LP** precisely when G is "almost semiabelian" or "antiadditive" in the sense of Bertrand and Pillay [9, Section 3.1], namely, when the torsion points of $G(\mathbb{C})$ are Zariski-dense in G or, equivalently, when there is no nontrivial morphism of algebraic groups from G to the additive group $\mathbb{G}_{a\mathbb{C}}$ (cf. [9, Appendix I]). In particular condition **LP** is a purely algebraic condition, invariant under the automorphisms of the field $\mathbb{C}$.

Corollary 4.3 *Let G_1 and G_2 be connected commutative algebraic groups over $\overline{\mathbb{Q}}$.*

(1) For any φ in the $\mathbb{Z}$-module $\mathrm{Hom}_{\mathrm{gp}/\overline{\mathbb{Q}}}(G_1, G_2)$ of morphisms of algebraic groups over $\overline{\mathbb{Q}}$ from G_1 to G_2, the $\overline{\mathbb{Q}}$-linear map

$$\mathrm{Lie}\,\varphi := D\varphi(e) : \mathrm{Lie}\, G_1 \longrightarrow \mathrm{Lie}\, G_2$$

satisfies

$$(\mathrm{Lie}\,\varphi)_{\mathbb{C}}(\mathrm{Per}\, G_{1\mathbb{C}}) \subset \mathrm{Per}\, G_{2\mathbb{C}}.$$

The map

$$\begin{aligned} &\mathrm{Lie} : \mathrm{Hom}_{\mathrm{gp}/\overline{\mathbb{Q}}}(G_1, G_2) \\ &\quad \longrightarrow \left\{\psi \in \mathrm{Hom}_{\overline{\mathbb{Q}}}(\mathrm{Lie}\, G_1, \mathrm{Lie}\, G_2) | \psi_{\mathbb{C}}(\mathrm{Per}\, G_{1\mathbb{C}}) \subset \mathrm{Per}\, G_{2\mathbb{C}}\right\} \end{aligned} \tag{4.1}$$

so defined is an injective morphism of $\mathbb{Z}$-modules.

(2) When the group $G_{1\mathbb{C}}$ satisfies condition **LP**, *then the morphism (4.1) is bijective.*

Proof Assertion (1) follows from identification of $(\mathrm{Lie}\,\varphi)_{\mathbb{C}}$ with the differential $\mathrm{Lie}\,\varphi_{\mathbb{C}} := D\varphi_{\mathbb{C}}(e)$ of the complexification $\varphi_{\mathbb{C}} : G_{1\mathbb{C}} \to G_{2\mathbb{C}}$ of the morphism of $\overline{\mathbb{Q}}$-algebraic groups φ, together with the commutativity of the diagram:

$$\begin{array}{ccc} \mathrm{Lie}\, G_{1\mathbb{C}} & \xrightarrow{\mathrm{Lie}\,\varphi_{\mathbb{C}}} & \mathrm{Lie}\, G_{2\mathbb{C}} \\ {\scriptstyle \exp_{G_{1\mathbb{C}}}}\downarrow & & \downarrow{\scriptstyle \exp_{G_{2\mathbb{C}}}} \\ G_{1\mathbb{C}}^{\mathrm{an}} & \xrightarrow{\varphi_{\mathbb{C}}} & G_{2\mathbb{C}}^{\mathrm{an}} \end{array}$$

To prove (2), assume that condition **LP** is satisfied by $G_{1\mathbb{C}}$, and consider some $\overline{\mathbb{Q}}$-linear map

$$\psi : \mathrm{Lie}\, G_1 \longrightarrow \mathrm{Lie}\, G_2$$

such that $\psi_{\mathbb{C}}(\mathrm{Per}\, G_{1\mathbb{C}}) \subset \mathrm{Per}\, G_{2\mathbb{C}}$. We need to establish the existence of a morphism of $\overline{\mathbb{Q}}$-algebraic groups $\varphi : G_1 \longrightarrow G_2$ such that

$$\psi = \mathrm{Lie}\,\varphi. \tag{4.2}$$

To achieve this, we will apply Theorem 4.2 to the group $G := G_1 \times G_2$ and to the subspace V of

$$\mathrm{Lie}\, G = \mathrm{Lie}\, G_1 \oplus \mathrm{Lie}\, G_2$$

defined as the graph of ψ.

Indeed, as G is commutative, V is a Lie subalgebra of $\operatorname{Lie} G$. Moreover, the complex vector space $V_{\mathbb{C}}$ is the graph of $\psi_{\mathbb{C}}$ and therefore contains

$$\widetilde{\operatorname{Per} G_{1\mathbb{C}}} := \{(\gamma, \psi_{\mathbb{C}}(\gamma)), \gamma \in \operatorname{Per} G_{1\mathbb{C}}\},$$

which is included in $\operatorname{Per} G_{1\mathbb{C}} \times \operatorname{Per} G_{2\mathbb{C}} = \operatorname{Per} G_{\mathbb{C}}$. Besides, the condition **LP** on $G_{1\mathbb{C}}$ shows that $\widetilde{\operatorname{Per} G_{1\mathbb{C}}}$ generates this $\mathbb{C}$-vector space. According to Theorem 4.2, V is algebraic and is the Lie algebra of some connected $\overline{\mathbb{Q}}$-algebraic subgroup H of G.

The first projection $p := \operatorname{pr}_{1|H} : H \longrightarrow G_1$ is étale. Moreover, $H_{\mathbb{C}}^{\mathrm{an}}$ is the image by $\exp_{G_{\mathbb{C}}}$ of $V_{\mathbb{C}}$. This immediately implies that $p_{\mathbb{C}} : H_{\mathbb{C}} \longrightarrow G_{1\mathbb{C}}$ is injective, and finally that p is an isomorphism. In other words, H is the graph of some morphism φ of algebraic groups from G_1 to G_2. Clearly it satisfies (4.2). □

4.4 Transcendence theorems and the analogy between number fields and functions fields Theorems 4.1 and 4.2 may be seen as arithmetic counterparts of algebraization theorems such as Andreotti's Theorem 3.4, or **Gr1** in Theorems 3.1 and 3.3, or more specifically, of their consequences concerning algebraization over function fields, such as Corollary 3.5 and its formal variant. The role of the function field $\mathbb{C}(C)$ or $k(C)$ is now played by $\overline{\mathbb{Q}}$ or by a number field K over which the geometric data X and L, or G and V, are defined.

Observe that the so-called Kronecker dimension of K—namely, the Krull dimension of $\operatorname{Spec} \mathcal{O}_K$—is *one* and that the algebraization Theorems 4.1 and 4.2, which are algebraicity criteria for smooth formal germs of subvarieties through K-rational *points*, isomorphic to $\operatorname{Spec} K$, are indeed algebraization theorems concerning smooth formal germs along some *arithmetic curves* $\operatorname{Spec} \mathcal{O}_K$ in some integral model of the given K-variety.

The classical proofs of Theorems 4.1 and 4.2 may be understood in a way that makes this geometric analogy precise. This geometric approach even suggests the formulation and the proof of new transcendence theorems, as demonstrated by the recent works of Gasbarri [42] and Herblot [52] who have established sophisticated generalizations of previously known transcendence theorems à la Schneider–Lang. I might also refer the reader to [28] and [15] for discussions of this geometric approach and of some of its applications in the framework of Diophantine results à la Chudnovsky (see [31], [32]) instead of Schneider–Lang. The arithmetic counterparts of the ampleness conditions in the geometric theorem of Andreotti and Hartshorne and **Gr1** appear more clearly in this somewhat simpler framework.

At the present stage, in this analogy, there is no known counterpart in transcendence theory of the general existence theorems, such as **Gr2** in Theorems 3.1 and 3.3. This absence appears especially regrettable when one considers the important geometric applications of these theorems: we have discussed at length several consequences of the GAGA existence theorem in Sections 1.2, 2.2, and 2.3; as demonstrated in [47], **Gr2** is the key to a modern approach to "Lefschetz-type theorems" which compare invariants, such as their fundamental group or their Picard group, of projective varieties to the ones of their hyperplane section.

The dimension condition

$$\dim Y \geq 2$$

in **Gr2** leads one, in a Kroneckerian perspective, to expect a suitable arithmetic counterpart of **Gr2** to be an algebraization criterion concerning formal line or vector bundles over the completion $\widehat{X}_Y$ of some algebraic variety X over a number field K, along a smooth projective embedded curve Y over K or, if one prefers, over the completion $\widehat{\mathcal{X}}_{\mathcal{Y}}$ of some scheme of finite type $\mathcal{X}$ over Spec $\mathcal{O}_K$ along a projective arithmetic surface $\mathcal{Y}$.

In the spirit of transcendence theorems à la Schneider–Lang, like Theorems 4.1 and 4.2, this criterion would also require some "differential algebraic" conditions (comparable to the occurrence of algebraic foliations in these theorems) and some "analytic control" on the considered formal vector bundles.

The remainder of this article is devoted to presenting such a criterion, in a conjectural form, and its relation to the Grothendieck period conjecture in codimension 1.

The proof of this last conjecture for abelian varieties may actually be derived from Theorem 4.2 and its Corollary 4.3. As it provides a further illustration of the "concrete geometric content" of transcendence theorems à la Schneider–Lang, we begin by a discussion of this material in Section 5. Then, in Sections 6.1–6.5, we review the formalism of D-group schemes and of their extensions that will be used in the last part to formulate our conjectural algebraization criterion.

5 The Grothendieck Period Conjecture for Cycles of Codimension 1 in Abelian Varieties

5.1 Grothendieck's conjecture GPC$^1(X)$ Let X be a smooth projective algebraic variety over $\overline{\mathbb{Q}}$, let $X_{\mathbb{C}}$ denote the smooth complex projective variety $X \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$, and let X^{an} be the corresponding compact complex manifold.

As discussed in Section 2.2, the Picard groups of X, $X_{\mathbb{C}}$, and $X_{\mathbb{C}}^{\mathrm{an}}$—which classify the algebraic lines bundles over X and $X_{\mathbb{C}}$, and the analytic line bundles over $X_{\mathbb{C}}^{\mathrm{an}}$—fit into the following commutative diagram:

$$
\begin{array}{ccc}
\mathrm{Pic}(X) & \xrightarrow{c_{1\,\mathrm{dR}/\overline{\mathbb{Q}}}} & H^2_{\mathrm{dR}}(X/\overline{\mathbb{Q}}) \\
\big\downarrow & & \big\downarrow {\scriptstyle .\otimes_{\overline{\mathbb{Q}}} 1_{\mathbb{C}}} \\
\mathrm{Pic}(X_{\mathbb{C}}) & \xrightarrow{c_{1\,\mathrm{dR}/\mathbb{C}}} & H^2_{\mathrm{dR}}(X_{\mathbb{C}}/\mathbb{C}) \\
\big\downarrow {\scriptstyle .^{\mathrm{an}}} & & \big\downarrow {\scriptstyle .^{\mathrm{an}}} \\
\mathrm{Pic}(X^{\mathrm{an}}_{\mathbb{C}}) & \xrightarrow{c^{\mathrm{an}}_{1\,\mathrm{dR}}} & H^2_{\mathrm{dR}}(X^{\mathrm{an}}_{\mathbb{C}}/\mathbb{C}) \\
\big\downarrow {\scriptstyle c_{1\,\mathrm{top}}} & & \big\downarrow {\scriptstyle \text{de Rham isomorphism}} \\
H^2(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{Z}) & \xrightarrow{2\pi i(.\otimes_{\mathbb{Z}} 1_{\mathbb{C}})} & H^2(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{C})
\end{array}
$$

The upper vertical arrows are induced by the field extension $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. The map $\mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(X_{\mathbb{C}})$ maps the class of some line bundle L over X to the class of the line bundle $L_{\mathbb{C}}$ over $X_{\mathbb{C}}$, and is injective but not surjective when the connected Picard variety $\mathrm{Pic}_0(X/\overline{\mathbb{Q}})$ has positive dimension.[11] However, since any line bundle over $X_{\mathbb{C}}$ is algebraically equivalent to some line bundle defined over $\overline{\mathbb{Q}}$, the images of $\mathrm{Pic}(X)$ and $\mathrm{Pic}(X_{\mathbb{C}})$ by the first Chern class coincide. The map $H^2_{\mathrm{dR}}(X/\overline{\mathbb{Q}}) \longrightarrow H^2_{\mathrm{dR}}(X_{\mathbb{C}}/\mathbb{C})$

induces an isomorphism $H^2_{\mathrm{dR}}(X/\overline{\mathbb{Q}}) \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \xrightarrow{\sim} H^2_{\mathrm{dR}}(X_{\mathbb{C}}/\mathbb{C})$. The image in $H^2_{\mathrm{dR}}(X_{\mathbb{C}}/\mathbb{C})$ of an element α in $H^2_{\mathrm{dR}}(X/\overline{\mathbb{Q}})$ will be denoted $\alpha \otimes_{\overline{\mathbb{Q}}} 1_{\mathbb{C}}$.

The two middle vertical arrows $\cdot^{\mathrm{an}}$, defined by analytification, are isomorphisms according to GAGA. The analytification isomorphism $H^2_{\mathrm{dR}}(X_{\mathbb{C}}/\mathbb{C}) \xrightarrow{\sim} H^2_{\mathrm{dR}}(X^{\mathrm{an}}_{\mathbb{C}}/\mathbb{C})$ will be noted as an equality.

The image of some class $\beta \in H^2(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{Z})$ by the natural map $H^2(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{Z}) \longrightarrow H^2(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{C})$ (defined by extending the coefficients from $\mathbb{Z}$ to $\mathbb{C}$) will be denoted $\beta \otimes_{\mathbb{Z}} 1_{\mathbb{C}}$, and the image of some class γ in $H^2_{\mathrm{dR}}(X^{\mathrm{an}}_{\mathbb{C}}/\mathbb{C})$ by the de Rham isomorphism will be denoted γ^{B}.

We may define the subgroup $H^2_{\mathrm{Gr}}(X)$ of "Grothendieck's classes" in $H^2_{\mathrm{dR}}(X/\overline{\mathbb{Q}}) \oplus H^2(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{Z})$ by the condition that, for any $\alpha \in H^2_{\mathrm{dR}}(X/\overline{\mathbb{Q}})$ and any $\beta \in H^2(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{Z})$:

$$(\alpha, \beta) \in H^2_{\mathrm{Gr}}(X) \Longleftrightarrow (\alpha \otimes_{\overline{\mathbb{Q}}} 1_{\mathbb{C}})^{\mathrm{B}} = 2\pi i \beta \otimes_{\mathbb{Z}} 1_{\mathbb{C}}. \tag{5.1}$$

The commutativity of the diagram above shows that the algebraic and topological first Chern classes define a morphism of abelian groups:

$$\begin{aligned} c_{1\mathrm{dRB}} : \mathrm{Pic}(X) &\longrightarrow H^2_{\mathrm{Gr}}(X), \\ [L] &\longmapsto (c_{1\mathrm{dR}}(L), c_{1\mathrm{top}}(L^{\mathrm{an}}_{\mathbb{C}})). \end{aligned}$$

The classical Grothendieck period conjecture[12] leads one to conjecture that *the morphism* $c_{1\mathrm{dRB}}$ *is onto*, namely, that *a class* γ *in* $H^2(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{Z})$ *such that* $2\pi i \cdot \gamma \otimes_{\mathbb{Z}} 1\mathbb{C}$ *is* $\overline{\mathbb{Q}}$*-rational in*

$$H^2(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{C}) \simeq H^2_{\mathrm{dR}}(X/\overline{\mathbb{Q}}) \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$$

is algebraic in the sense of Section 1.2.

This conjectural assertion may be called *the Grothendieck period conjecture in codimension* 1 for the smooth projective variety X over $\overline{\mathbb{Q}}$ and will be denoted $\mathrm{GPC}^1(X)$ in the remainder of the article.

Conjecture $\mathrm{GPC}^1(X)$ admits a $\mathbb{Q}$-rational version, a priori weaker, that asserts the surjectivity of the map

$$c_{1\mathrm{dRB}\mathbb{Q}} : \mathrm{Pic}(X)_{\mathbb{Q}} \longrightarrow H^2_{\mathrm{Gr}}(X)_{\mathbb{Q}}$$

deduced from $c_{1\mathrm{dRB}}$ by tensoring with $\mathbb{Q}$. (The tensor product $H^2_{\mathrm{Gr}}(X)_{\mathbb{Q}} := H^2_{\mathrm{Gr}}(X) \otimes \mathbb{Q}$ may be identified with the $\mathbb{Q}$-vector subspace of $H^2_{\mathrm{dR}}(X/\overline{\mathbb{Q}}) \oplus H^2(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{Q})$ defined by the right-hand side of (5.1), with $\cdot \otimes_{\mathbb{Z}} \cdot$ replaced by $\cdot \otimes_{\mathbb{Q}} \cdot$.) A special feature of the codimension 1 case of the Grothendieck period conjecture is that this rational version of the conjecture—which is the one that appears in the references in note 12—actually implies the above "integral" version. Indeed, for any positive integer n, a class γ in $H^2(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{Z})$ is algebraic if $n\gamma$ is algebraic.

More generally, for any positive integer k, we may consider the Grothendieck period conjecture in codimension k, $\mathrm{GPC}^k(X)$: it asserts that any class γ in $H^{2k}(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{Q})$ such that $(2\pi i)^k \gamma \otimes_{\mathbb{Q}} 1_{\mathbb{C}}$ is $\overline{\mathbb{Q}}$-rational in $H^{2k}(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{C}) \simeq H^{2k}_{\mathrm{dR}}(X/\overline{\mathbb{Q}}) \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$ is algebraic. See [1, Section 7.5] for a discussion of the close relation between the original version of the Grothendieck period conjecture and the fullness conjecture for the "de Rham–Betti realization," namely, the conjunction of conjectures $\mathrm{GPC}^k(X)$ for all smooth projective varieties X over $\overline{\mathbb{Q}}$ and all integers k.[13] To my knowledge, the known results concerning these conjectures may be summarized as follows.

(i) The original Grothendieck period conjecture is known to be valid for a motive in the Tannakian category generated by the Tate motive (transcendence of π) or for an elliptic curve with complex multiplication (Chudnovsky).

(ii) The fullness of the de Rham–Betti realization is known for H^1 (cf. [1, Section 7.2.3], where it is derived from the transcendence results in Wüstholz [101]; this fullness is basically the content of Theorem 5.3 infra and, as shown in the next paragraphs, may be derived from Schneider–Lang's Theorem 4.2 and its Corollary 4.3).

In the next sections, we shall establish the validity of Grothendieck's period conjecture in codimension 1 for abelian varieties.

Theorem 5.1 *For any abelian variety A over $\overline{\mathbb{Q}}$,* $\mathrm{GPC}^1(A)$ *holds.*

The proof of Theorem 5.1 will be based on the "transcendental" characterization of algebraic Lie subalgebras in Theorem 4.2, via its Corollary 4.3 applied to universal vector extensions of abelian varieties, and on the identification of the Néron–Severi group of an abelian variety with the group of symmetric morphisms from the abelian variety to its dual (cf. [15, Theorem 6.4]). We present the details of this proof in Section 5.4. As a preliminary, in Section 5.2 we recall classical facts concerning abelian varieties, their duality, and their universal vector extensions, and in Section 5.3 we introduce the elementary, but convenient, formalism of the category $\mathcal{C}_{\mathrm{dRB}}$ of the "de Rham–Betti realizations" (in the spirit of the realization categories à la Deligne and Jannsen [59]; see also [1, Section 7.5]).

5.2 Abelian varieties, duality, and universal extensions In this section, we work over an algebraically closed field k of characteristic zero.

5.2.1 Dual abelian varieties and de Rham (co)homology If A is an abelian variety over k, we shall denote $\hat{A} := \mathrm{Pic}_0(A/k)$ the dual abelian variety. The group $\hat{A}(k)$ of its k-rational points may be identified with the subgroup $\mathrm{Pic}^0(A)$ of $\mathrm{Pic}(A)$ of isomorphism classes of line bundles algebraically equivalent to zero or, equivalently, with the kernel of

$$c_{1\mathrm{dR}} : \mathrm{Pic}(A) \longrightarrow H^2_{\mathrm{dR}}(A/k).$$

To any morphism $\varphi : A \longrightarrow B$ of abelian varieties over k is attached the dual morphism $\hat{\varphi} : \hat{B} \longrightarrow \hat{A}$. It maps the class of some line bundle L over B algebraically equivalent to zero to the class of $\varphi^*(L)$. This construction is additive and (contravariantly) functorial.

Let $\mathcal{P}_A$ denote the Poincaré line bundle over $A \times \hat{A}$. Its restriction to $0_A \times \hat{A}$ is trivial, and for any $\hat{a} \in \hat{A}(k)$, the isomorphism class of its restriction to $A \times \hat{a}$ is precisely $\hat{a}$ itself, and these properties characterize $\mathcal{P}_A$ up to isomorphism. By mapping a point a in $A(k)$ to the class $\iota_A(a)$ of $\mathcal{P}_{A|a\times\hat{A}}$, ones defines a canonical isomorphism

$$\iota_A : A \xrightarrow{\sim} \hat{\hat{A}},$$

which is sometimes written as an equality.

Recall that the following "biduality" properties are satisfied (cf. [7, Section V.1] or Coleman [33, Section 1]). For any $\varphi : A \longrightarrow B$ as above, $\hat{\hat{\varphi}} : \hat{\hat{A}} \longrightarrow \hat{\hat{B}}$ and φ (or more exactly, $\iota_B \circ \varphi \circ \iota_A$) coincide. Moreover, under the composite isomorphism

$$\begin{aligned} A \times \hat{A} &\xrightarrow{\sigma} \hat{A} \times A \xrightarrow{Id_{\hat{A}} \times \iota_A} \hat{A} \times \hat{\hat{A}}, \\ (a, \hat{a}) &\longmapsto (\hat{a}, a), \end{aligned}$$

the Poincaré bundle $\mathcal{P}_A$ of A becomes the Poincaré bundle $\mathcal{P}_{\hat{A}}$ of $\hat{A}$:

$$\big((\mathrm{Id}_{\hat{A}} \times \iota_A) \circ \sigma\big)^* \mathcal{P}_{\hat{A}} \xrightarrow{\sim} \mathcal{P}_A. \tag{5.2}$$

Moreover, $c_{1\mathrm{dR}}(\mathcal{P}_A)$ belongs to the Künneth component $H^1_{\mathrm{dR}}(A/k) \otimes H^1_{\mathrm{dR}}(\hat{A}/k)$ of $H^2(A \times \hat{A}/k)$. If we define

$$H_{1\mathrm{dR}}(A/k) := H^1_{\mathrm{dR}}(A/k)^\vee = \mathrm{Hom}_k\big(H^1_{\mathrm{dR}}(A/k), k\big),$$

then $c_{1\mathrm{dR}}(\mathcal{P}_A)$ defines an element ϖ_A in

$$H_{1\mathrm{dR}}(A/k)^\vee \otimes_k H^1_{\mathrm{dR}}(\hat{A}/k) \simeq \mathrm{Hom}_k\big(H_{1\mathrm{dR}}(A/k), H^1_{\mathrm{dR}}(\hat{A}/k)\big)$$

which actually is an isomorphism:

$$\varpi_A : H_{1\mathrm{dR}}(A/k) \xrightarrow{\sim} H^1_{\mathrm{dR}}(\hat{A}/k) = H_{1\mathrm{dR}}(\hat{A}/k)^\vee.$$

The duality isomorphism ϖ_A satisfies the following functoriality property.

Let $\varphi : A \longrightarrow B$ be a morphism of abelian varieties over k. It induces a k-linear map between de Rham cohomology groups:

$$H^1_{\mathrm{dR}}(\varphi) := \varphi^* : H^1_{\mathrm{dR}}(B/k) \longrightarrow H^1_{\mathrm{dR}}(A/k),$$

and then by duality, between homology groups:

$$H_{1\mathrm{dR}}(\varphi) := H^1_{\mathrm{dR}}(\varphi)^t : H_{1\mathrm{dR}}(A/k) \longrightarrow H_{1\mathrm{dR}}(B/k).$$

Then the dual morphism of abelian varieties

$$\hat{\varphi} : \hat{B} \longrightarrow \hat{A}$$

satisfies

$$H_{1\mathrm{dR}}(\hat{\varphi}) = \varpi_A^{\vee -1} \circ H_1(\varphi)^\vee \circ \varpi_B^\vee. \tag{5.3}$$

This follows from the isomorphism of line bundles over $A \times \hat{B}$:

$$(\mathrm{Id}_A \times \hat{\varphi})^* \mathcal{P}_A \simeq (\varphi \times \mathrm{Id}_{\hat{B}})^* \mathcal{P}_B$$

and from the implied equality between first Chern classes.

Observe however that the isomorphism

$$\varpi_{\hat{A}} : H_{1\mathrm{dR}}(\hat{A}/k) \xrightarrow{\sim} H^1_{\mathrm{dR}}(\hat{\hat{A}}/k) \simeq H_{1\mathrm{dR}}(\hat{\hat{A}}/k)^\vee$$

differs by a sign from the transpose of ϖ_A:

$$\varpi_{\hat{A}} = -H_{1\mathrm{dR}}(\iota_A)^\vee \circ \varpi_A^\vee. \tag{5.4}$$

This follows from the equality of first Chern classes implied by the isomorphism (5.2) and from the fact that switching the factors $A \simeq \hat{\hat{A}}$ and $\hat{A}$ introduces a sign in the Künneth morphism

$$H^1_{\mathrm{dR}}(A/k) \otimes_k H^1_{\mathrm{dR}}(\hat{A}/k) \hookrightarrow H^2_{\mathrm{dR}}(A \times \hat{A}/k).$$

5.2.2 Néron–Severi groups and symmetric morphisms To any line bundle L over A is attached a morphism of abelian varieties over k,

$$\varphi_L : A \longrightarrow \hat{A}$$

that is defined by

$$\varphi_L(a) := [\tau_a^* L \otimes L^\vee]$$

for any $a \in A(k)$, where τ_a denotes the translation by a on A. Moreover, φ_L is zero if and only if L is algebraically equivalent to zero and, for any two line bundles L_1 and L_2 on A, $\varphi_{L_1 \otimes L_2} = \varphi_{L_1} + \varphi_{L_2}$. Consequently this construction induces an injective morphism of $\mathbb{Z}$-modules:

$$NS(A) := \operatorname{Pic}(A)/\operatorname{Pic}_0(A) \longrightarrow \operatorname{Hom}_{\mathrm{gp}/k}(A, \hat{A}),$$
$$[L] \longmapsto \varphi_L.$$

Its image is the subgroup $\operatorname{Hom}_{\mathrm{gp}/k}(A, \hat{A})^{\mathrm{sym}}$ of *symmetric* morphisms, namely, the subgroup of morphisms $\varphi : A \longrightarrow \hat{A}$ such that

$$\hat{\varphi} \circ \iota_A = \varphi. \tag{5.5}$$

This actually holds for abelian schemes over an arbitrary base, as established by Nishi and Oda (cf. Oda [78, p. 77, note 2]).

Observe that, at the level of de Rham (co)homology groups, the symmetry condition (5.5) translates into a *skew-symmetry* condition on

$$\varpi_A^\vee \circ H_1(\varphi) : H_{1\mathrm{dR}}(A/k) \longrightarrow H_{1\mathrm{dR}}(A/k)^\vee.$$

Indeed the “duality” formulas (5.3) and (5.4) imply the relation

$$\varpi_A^\vee \circ H_1(\hat{\varphi} \circ \iota_A) = -\big(\varpi_A^\vee \circ H_1(\varphi)\big)^\vee. \tag{5.6}$$

In particular, when the base field k is $\mathbb{C}$, the above identification of $NS(A)$ with $\operatorname{Hom}_{\mathrm{gp}/k}(A, \hat{A})^{\mathrm{sym}}$ is basically the classical theory of Riemann forms attached to line bundles over complex abelian varieties.

5.2.3 Universal vector extensions (cf. Rosenlicht [88], Serre [17], [33], [75], [76], [93]).

For any abelian variety A over k, we shall denote $\mathbb{E}_A$ the k-vector space

$$\Gamma(A, \Omega^1_{A/k}) \simeq \Omega^1_{A/k, 0_A} \simeq (\operatorname{Lie} A)^\vee.$$

Observe that we have a canonical identification

$$\mathbb{E}_{\hat{A}} \simeq (\operatorname{Lie} \hat{A})^\vee \simeq H^1(A, \mathcal{O}_A)^\vee.$$

Let V be a finite-dimensional k-vector space, and let V^{gp} denote the associated k-vector group (namely the commutative algebraic group over K, such that the group $V^{\mathrm{gp}}(k)$ “is” the additive group $(V, +)$). Recall that any extension of commutative algebraic groups over k,

$$0 \longrightarrow V^{\mathrm{gp}} \longrightarrow G \longrightarrow A \longrightarrow 0 \tag{5.7}$$

of some abelian variety A over k by V^{gp} determines an $\mathcal{O}_A \otimes_k V$-torsor over A, and that this construction defines a canonical isomorphism[14]

$$\begin{aligned}\operatorname{Ext}^1_{\text{c-gp}/k}(A, V^{\mathrm{gp}}) &\xrightarrow{\sim} \operatorname{Ext}^1_{\mathcal{O}_A\text{-mod}}(\mathcal{O}_A, \mathcal{O}_A \otimes_k V) \\ &\simeq H^1(A, \mathcal{O}_A) \otimes_k V \simeq \operatorname{Hom}_k(\mathbb{E}_{\hat{A}}, V).\end{aligned} \tag{5.8}$$

Moreover, an extension (5.7) of commutative algebraic groups of an abelian variety by a vector group admits no nontrivial automorphism. Consequently, the isomorphism (5.8) with $V = \mathbb{E}_{\hat{A}}$ shows that to the element $\mathrm{Id}_{\mathbb{E}_{\hat{A}}}$ is canonically associated a vector extension of A by the vector group defined by $\mathbb{E}_{\hat{A}}$, which we shall denote

$$0 \longrightarrow \mathbb{E}_{\hat{A}} \hookrightarrow E(A) \xrightarrow{p_A} A \longrightarrow 0. \tag{5.9}$$

It is the *universal vector extension* of A: any vector extension (5.7) may be realized uniquely as a pushout of (5.9), namely, as the pushout by its "classifying element" in the right-hand side of (5.8).

5.2.4 The functor E Let $\varphi : A \longrightarrow B$ be a morphism of abelian varieties over k. We may consider the pullback by φ of the universal vector extension of B and use the universal property of the universal vector extension of A. We thus get the existence and unicity of a morphism $E(\varphi)$ of k-algebraic groups, which makes the following diagram commutative:

$$\begin{array}{ccc} E(A) & \xrightarrow{E(\varphi)} & E(B) \\ \downarrow p_A & & \downarrow p_B \\ A & \xrightarrow{\varphi} & B. \end{array}$$

The construction of $E(\varphi)$ is clearly additive and functorial in φ. Moreover, it is easily seen to be fully faithful.

Lemma 5.2 *For any two abelian varieties A and B over k, the morphism of $\mathbb{Z}$-modules*

$$\begin{aligned} \mathrm{Hom}_{\mathrm{gp}/k}(A, B) &\longrightarrow \mathrm{Hom}_{\mathrm{gp}/k}\big(E(A), E(B)\big), \\ \varphi &\longmapsto E(\varphi) \end{aligned} \tag{5.10}$$

is an isomorphism.

5.2.5 Biduality and universal vector extensions We shall also use that the biduality isomorphism

$$\iota_A : A(k) \xrightarrow{\sim} \hat{\hat{A}}(k) = \ker c_{1\mathrm{dR}} : H^1(\hat{A}, \mathcal{O}^*_{\hat{A}}) \longrightarrow H^1_{\mathrm{dR}}(\hat{A}, \Omega^\bullet_{\hat{A}/k})$$

may be lifted to an isomorphism

$$\iota_{E(A)} : E(A)(k) \xrightarrow{\sim} H^1(\hat{A}, \Omega^\times_{\hat{A}/k}),$$

where $\Omega^\times_{\hat{A}/k}$ denotes the complex

$$\mathcal{O}^*_{\hat{A}} \xrightarrow{d\log} \Omega^1_{\hat{A}/k} \xrightarrow{d} \Omega^2_{\hat{A}/k} \xrightarrow{d} \cdots,$$

which makes commutative the following diagram with exact lines:[15]

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{E}_{\hat{A}} & \longrightarrow & E(A)(k) & \xrightarrow{p_A} & A(k) & \longrightarrow & 0 \\ & & \simeq\downarrow & & \simeq\downarrow \iota_{E(A)} & & \simeq\downarrow \iota_A & & \\ 0 & \longrightarrow & H^1(\hat{A}, \sigma^{\geq 1}\Omega^\bullet_{\hat{A}/k}) & \longrightarrow & H^1(\hat{A}, \Omega^\times_{\hat{A}/k}) & \longrightarrow & \hat{\hat{A}}(k) & \longrightarrow & 0 \end{array} \tag{5.11}$$

(For constructing the second line, recall that $F^1 H^2_{\mathrm{dR}}(\hat{A}/k) := H^2(\hat{A}, \sigma^{\geq 1}\Omega^\bullet_{\hat{A}/k})$ injects into $H^2_{\mathrm{dR}}(\hat{A}/k)$ and that $c_{1\mathrm{dR}} : H^1(\hat{A}, \mathcal{O}^*_{\hat{A}}) \to H^1_{\mathrm{dR}}(\hat{A}, \Omega^\bullet_{\hat{A}/k})$ coincides with $d\log : H^1(\hat{A}, \mathcal{O}^*_{\hat{A}}) \to F^1 H^2_{\mathrm{dR}}(\hat{A}/k)$.)

Moreover, the "infinitesimal" version[16] of $\iota_{E(A)}$ defines an isomorphism

$$I_A := \mathrm{Lie}\,\iota_{E(A)} : \mathrm{Lie}\, E(A) \longrightarrow H^1(\hat{A}, \Omega^\bullet_{\hat{A}/k}) = H^1_{\mathrm{dR}}(\hat{A}/k),$$

and the infinitesimal version of (5.11) is an isomorphism of exact sequences of finite-dimensional k-vector spaces:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{E}_{\hat{A}} & \longrightarrow & \mathrm{Lie}\, E(A) & \xrightarrow{\mathrm{Lie}\, p_A} & \mathrm{Lie}\, A & \longrightarrow & 0 \\ & & = \big\downarrow & & \simeq \big\downarrow I_A & & \simeq \big\downarrow \mathrm{Lie}\,\iota_A & & \\ 0 & \longrightarrow & \mathbb{E}_{\hat{A}} & \longrightarrow & H^1_{\mathrm{dR}}(\hat{A}/k) & \longrightarrow & H^1(\hat{A}, \mathcal{O}_{\hat{A}}) & \longrightarrow & 0 \end{array} \tag{5.12}$$

(The second line defines the Hodge filtration on $H^1_{\mathrm{dR}}(\hat{A}/k)$.)

Finally, we get an isomorphism of k-vector spaces

$$J_A := \varpi_A^{-1} \circ I_A : \mathrm{Lie}\, E(A) \xrightarrow{\sim} H_{1\mathrm{dR}}(A/k).$$

It is easily checked to be functorial. Namely, for any morphism $\varphi : A \longrightarrow B$ of abelian varieties over k, the diagram

$$\begin{array}{ccc} \mathrm{Lie}\, E(A) & \xrightarrow{\mathrm{Lie}\, E(\varphi)} & \mathrm{Lie}\, E(B) \\ \simeq \big\downarrow J_A & & \simeq \big\downarrow J_B \\ H_{1\mathrm{dR}}(A/k) & \xrightarrow{H_{1\mathrm{dR}}(\varphi)} & H_{1\mathrm{dR}}(B/k) \end{array} \tag{5.13}$$

is commutative.

5.3 The category $\mathcal{C}_{\mathrm{dRB}}$

5.3.1 Definitions We define an additive category $\mathcal{C}_{\mathrm{dRB}}$—where $\mathcal{C}$ stands for "category" or "comparison" and dRB stands for "de Rham–Betti"—in the following way.

Its objects are triples

$$M = (M_{\mathrm{dR}}, M_{\mathrm{B}}, c_M),$$

where M_{dR} is a finite-dimensional $\overline{\mathbb{Q}}$-vector space, M_{B} is a free $\mathbb{Z}$-module of finite rank, and c_M is an isomorphism of $\mathbb{C}$-vector spaces:

$$c_M : M_{\mathrm{dR}} \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \xrightarrow{\sim} M_{\mathrm{B}} \otimes_{\mathbb{Z}} \mathbb{C}.$$

In other terms, an object M of $\mathcal{C}_{\mathrm{dRB}}$ may be seen as the data of the finite-dimensional $\mathbb{C}$-vector space

$$M_{\mathbb{C}} := M_{\mathrm{dR}} \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \simeq M_{\mathrm{B}} \otimes_{\mathbb{Z}} \mathbb{C},$$

together with a "$\overline{\mathbb{Q}}$-form" M_{dR} and a "$\mathbb{Z}$-form" M_{B} of $M_{\mathbb{C}}$.

If M and N are objects in $\mathcal{C}_{\mathrm{dRB}}$, the additive group of morphisms from M to N in $\mathcal{C}_{\mathrm{dRB}}$ is the subgroup $\mathrm{Hom}_{\mathrm{dRB}}(M, N)$ in $\mathrm{Hom}_{\overline{\mathbb{Q}}}(M_{\mathrm{dR}}, N_{\mathrm{dR}}) \oplus \mathrm{Hom}_{\mathbb{Z}}(M_{\mathrm{B}}, N_{\mathrm{B}})$

consisting of pairs of maps $\varphi = (\varphi_{dR}, \varphi_{\mathrm{B}})$ such that the following diagram is commutative:

$$\begin{array}{ccc} M_{\mathrm{dR}} \otimes_{\overline{\mathbb{Q}}} \mathbb{C} & \xrightarrow{\varphi_{\mathrm{dR}} \otimes_{\overline{\mathbb{Q}}} \mathrm{Id}_{\mathbb{C}}} & N_{\mathrm{dR}} \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \\ \simeq \big\downarrow c_M & & \simeq \big\downarrow c_N \\ M_{\mathrm{B}} \otimes_{\mathbb{Z}} \mathbb{C} & \xrightarrow{\varphi_{\mathrm{B}} \otimes_{\mathbb{Z}} \mathrm{Id}_{\mathbb{C}}} & N_{\mathrm{B}} \otimes_{\mathbb{Z}} \mathbb{C} \end{array}$$

These morphisms may be identified with the $\mathbb{C}$-linear maps $\varphi_{\mathbb{C}}$ from $M_{\mathbb{C}}$ to $N_{\mathbb{C}}$ which are compatible both to their $\overline{\mathbb{Q}}$-forms and their $\mathbb{Z}$-forms. The composition of these morphisms is the obvious one, defined by the composition of the "de Rham," "Betti," and "complex" realizations φ_{dR}, φ_{B}, and $\varphi_{\mathbb{C}}$, respectively.

The category $\mathcal{C}_{\mathrm{dRB}}$ is endowed with an internal tensor product, defined by

$$M \otimes N := (M_{\mathrm{dR}} \otimes_{\overline{\mathbb{Q}}} N_{\mathrm{dR}}, M_{\mathrm{B}} \otimes_{\mathbb{Z}} N_{\mathrm{B}}, c_M \otimes_{\mathbb{C}} c_N),$$

and with an internal duality functor, defined by

$$M^{\vee} := \big(\mathrm{Hom}_{\overline{\mathbb{Q}}}(M_{\mathrm{dR}}, \overline{\mathbb{Q}}), \mathrm{Hom}_{\mathbb{Z}}(M_{\mathrm{B}}, \mathbb{Z}), c^t\big)$$

and

$$\varphi^{\vee} := (\varphi_{\mathrm{dR}}^t, \varphi_{\mathrm{B}}^t) = (. \circ \varphi_{\mathrm{dR}}, . \circ \varphi_{\mathrm{B}}).$$

For any integer k, we denote $\mathbb{Z}(k)$ the object of $\mathcal{C}_{\mathrm{dRB}}$ defined by $\mathbb{Z}(k)_{\overline{\mathbb{Q}}} = \overline{\mathbb{Q}}$ and $\mathbb{Z}(k)_{\mathrm{B}} = (2\pi i)^k \mathbb{Z}$ in $\mathbb{Z}(k)_{\mathbb{C}} = \mathbb{C}$. Observe that $\mathbb{Z}(0)$ and the obvious isomorphism $\mathbb{Z}(0) \otimes \mathbb{Z}(0) \xrightarrow{\sim} \mathbb{Z}(0)$, mapping $1 \otimes 1$ to 1, define a unit object of $\mathcal{C}_{\mathrm{dRB}}$, which, endowed with $\otimes$ and $\cdot^{\vee}$ becomes a rigid tensor category. In particular, for any two objects M and N of $\mathcal{C}_{\mathrm{dRB}}$, we have a natural isomorphism:

$$\begin{aligned} \mathrm{Hom}_{\mathrm{dRB}}(M, N) &\xrightarrow{\sim} \mathrm{Hom}_{\mathrm{dR}}\big(\mathbb{Z}(0), M^{\vee} \otimes N\big), \\ (\varphi_{\mathrm{dR}}, \varphi_{\mathrm{B}}) &\longmapsto (1 \mapsto \varphi_{\mathrm{dR}}, 1 \mapsto \varphi_{\mathrm{B}}). \end{aligned} \tag{5.14}$$

Moreover, for every integer k, we get an identification

$$\mathrm{Hom}_{\mathrm{dRB}}\big(\mathbb{Z}(0), M \otimes \mathbb{Z}(k)\big) \xrightarrow{\sim} M_{\mathrm{dR}} \cap (2\pi i)^k M_{\mathrm{B}}, \tag{5.15}$$

where the intersection is taken in $M_{\mathbb{C}}$, by mapping a morphism $\varphi : \mathbb{Z}(0) \longrightarrow M \otimes \mathbb{Z}(k)$ to $\varphi_{\mathbb{C}}(1)$.

5.3.2 Examples, I: The (co)homology of smooth projective varieties over $\overline{\mathbb{Q}}$ For any smooth projective variety X over $\overline{\mathbb{Q}}$ and for any integer $i \geq 0$, the algebraic de Rham cohomology of X and the Betti cohomology of $X_{\mathbb{C}}^{\mathrm{an}}$ determine an object $H^i_{\mathrm{dRB}}(X)$ in $\mathcal{C}_{\mathrm{dRB}}$ defined as follows:

$$H^i_{\mathrm{dRB}} := \big(H^i_{\mathrm{dR}}(X/\overline{\mathbb{Q}}), H^i_{\mathrm{B}}(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{Z})/\mathrm{torsion}, c\big),$$

where c denotes the composition of the comparison isomorphism defined by the base change isomorphism, analytification, and the de Rham isomorphism

$$H^i_{\mathrm{dR}}(X/\overline{\mathbb{Q}}) \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \xrightarrow{\sim} H^i_{\mathrm{dR}}(X_{\mathbb{C}}/\mathbb{C}) \xrightarrow{\sim} H^i_{\mathrm{dR}}(X^{\mathrm{an}}_{\mathbb{C}}) \xrightarrow{\sim} H^i(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{C}),$$

and of the inverse of the isomorphism defined by extension of coefficients

$$\big(H^i(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{Z})/\mathrm{torsion}\big) \otimes_{\mathbb{Z}} \mathbb{C} \simeq H^i(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\sim} H^i(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{C}).$$

To a morphism

$$\varphi : X \longrightarrow Y$$

of smooth projective varieties over $\overline{\mathbb{Q}}$ is attached a morphism in "de Rham–Betti cohomology":

$$H^i_{\mathrm{dRB}}(\varphi) := \big(H^i_{\mathrm{dR}}(\varphi), H^i_B(\varphi)\big)$$

defined by the "pullback" morphisms

$$H^i_{\mathrm{dR}}(\varphi) := \varphi^* : H^i_{\mathrm{dR}}(Y/\overline{\mathbb{Q}}) \longrightarrow H^i_{\mathrm{dR}}(X/\overline{\mathbb{Q}})$$

and

$$H^i_{\mathrm{B}}(\varphi) := \varphi^{\mathrm{an}*}_{\mathbb{C}} : H^i(Y^{\mathrm{an}}_{\mathbb{C}}, \mathbb{Z})/\mathrm{torsion} \longrightarrow H^i(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{Z})/\mathrm{torsion}$$

in algebraic de Rham and Betti cohomology. This construction is clearly functorial.

Observe that, as an instance of (5.15), we have a natural identification

$$H^2_{\mathrm{Gr}}(X) \simeq \mathrm{Hom}_{\mathrm{dRB}}\big(\mathbb{Z}(0), H^2_{\mathrm{dRB}}(X) \otimes \mathbb{Z}(1)\big). \tag{5.16}$$

We shall also define the de Rham–Betti *homology* functor by duality in $\mathcal{C}_{\mathrm{dRB}}$:

$$H_{i\,\mathrm{dRB}}(X) := H^i_{\mathrm{dRB}}(X)^\vee \quad \text{and} \quad H_{i\,\mathrm{dRB}}(\varphi) := H^i_{\mathrm{dRB}}(\varphi)^\vee.$$

Observe that $H_{i\,\mathrm{dRB}}(X)_{\mathrm{B}}$ and $H_{i\,\mathrm{dRB}}(X)_{\mathbb{C}}$ may be identified with the Betti homology groups $H_i(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{Z})$ modulo torsion and $H_i(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{C})$ of $X^{\mathrm{an}}_{\mathbb{C}}$.

5.3.3 Examples, II: The homology of abelian varieties Let A be an abelian variety of dimension g over $\overline{\mathbb{Q}}$, and let $E(A)$ be its universal vector extension.

Consider the exponential map of the associated complex Lie group:

$$\exp_{E(A)_{\mathbb{C}}} : \mathrm{Lie}\, E(A)_{\mathbb{C}} \longrightarrow E(A)^{\mathrm{an}}_{\mathbb{C}}.$$

Its kernel, the group of periods $\mathrm{Per}\, E(A)_{\mathbb{C}}$ of $E(A)_{\mathbb{C}}$, is a free $\mathbb{Z}$-module of rank $2g$, and the inclusion $\mathrm{Per}\, E(A)_{\mathbb{C}} \hookrightarrow \mathrm{Lie}\, E(A)_{\mathbb{C}}$ extends to an isomorphism

$$\mathrm{Per}\, E(A)_{\mathbb{C}} \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\sim} \mathrm{Lie}\, E(A)_{\mathbb{C}}. \tag{5.17}$$

Consequently we may attach the following object of $\mathcal{C}_{\mathrm{dRB}}$ to the abelian variety A:

$$\mathrm{LiePer}\, E(A) := \big(\mathrm{Lie}\, E(A), \mathrm{Per}\, E(A)_{\mathbb{C}}, c\big),$$

where c denotes the inverse of the isomorphism (5.17).

As recalled in Section 5.2.5 above, the construction of $E(A)$ as the moduli space of line bundles with (integrable) connections over the dual abelian variety $\hat{A}$ provides a canonical isomorphism of $\overline{\mathbb{Q}}$-vector spaces:

$$I_A : \mathrm{Lie}\, E(A) \xrightarrow{\sim} H^1_{\mathrm{dR}}(\hat{A}/\overline{\mathbb{Q}}).$$

Moreover, the isomorphism of complex vector spaces

$$\begin{aligned} \mathrm{Lie}\, E(A)_{\mathbb{C}} &\xrightarrow{I_{A,\mathbb{C}} = I_{A_{\mathbb{C}}}} H^1_{\mathrm{dR}}(\hat{A}/\overline{\mathbb{Q}}) \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \\ &\simeq H^1_{\mathrm{dR}}(\hat{A}_{\mathbb{C}}/\mathbb{C}) \xrightarrow{\text{GAGA+deRham}} H^1(\hat{A}^{\mathrm{an}}_{\mathbb{C}}, \mathbb{C}) \end{aligned}$$

maps $\mathrm{Per}\, E(A)_{\mathbb{C}}$ onto $H^1(\hat{A}^{\mathrm{an}}_{\mathbb{C}}, 2\pi i \mathbb{Z})$. This follows from the description of $E(A)^{\mathrm{an}}_{\mathbb{C}}$ as $H^1(\hat{A}^{\mathrm{an}}_{\mathbb{C}}, \Omega^\times_{\hat{A}^{\mathrm{an}}_{\mathbb{C}}})$, where $\Omega^\times_{\hat{A}^{\mathrm{an}}_{\mathbb{C}}}$ denotes the complex $\mathcal{O}^{\mathrm{an}}_{\hat{A}^{\mathrm{an}}_{\mathbb{C}}} \xrightarrow{d\log} \Omega^1_{\hat{A}^{\mathrm{an}}_{\mathbb{C}}} \xrightarrow{d} \Omega^1_{\hat{A}^{\mathrm{an}}_{\mathbb{C}}} \xrightarrow{d} \cdots$.

In other words, I_A defines an isomorphism in $\mathcal{C}_{\mathrm{dRB}}$:

$$I_{A,\mathrm{dRB}} : \mathrm{LiePer}\, E(A) \xrightarrow{\sim} H^1_{\mathrm{dRB}}(\hat{A}) \otimes \mathbb{Z}(1).$$

Besides, the isomorphism $\varpi_{A,\mathrm{dR}}$ constructed in Section 5.2.1 above admits an obvious analogue $\varpi_{A_{\mathbb{C}},\mathrm{B}}$ involving the Betti (co)homology of $A^{\mathrm{an}}_{\mathbb{C}}$ and $\hat{A}^{\mathrm{an}}_{\mathbb{C}}$, which are defined by means of $c_{1\mathrm{B}}(\mathcal{P}_{A_{\mathbb{C}}})$. Up to a factor $2\pi i$ coming from the relation

$$c_{1\mathrm{dR}}(\mathcal{P}_A)_{\mathbb{C}} = 2\pi i\, c_{1\mathrm{B}}(\mathcal{P}_{A_{\mathbb{C}}}),$$

it is compatible with the isomorphism $\varpi_{A,\mathrm{dR}}$ in algebraic de Rham (co)homology. In other words, they define an isomorphism in $\mathcal{C}_{\mathrm{dRB}}$:

$$\varpi_{A,\mathrm{dRB}} := (\varpi_{A,\mathrm{dR}}, \varpi_{A_{\mathbb{C}},\mathrm{B}}) : H_{1,\mathrm{dRB}}(A) \xrightarrow{\sim} H^1_{\mathrm{dRB}}(\hat{A}) \otimes \mathbb{Z}(1).$$

Finally, we get a canonical isomorphism in $\mathcal{C}_{\mathrm{dRB}}$:

$$J_{A,\mathrm{dRB}} := \varpi^{-1}_{A,\mathrm{dRB}} \circ I_{A,\mathrm{dRB}} : \mathrm{LiePer}\, E(A) \xrightarrow{\sim} H_{1,\mathrm{dRB}}(A). \tag{5.18}$$

This construction is easily seen to be functorial in A. Namely, for any morphism $\varphi : A \longrightarrow B$ of abelian varieties over $\overline{\mathbb{Q}}$,

$$\mathrm{LiePer}\, E(\varphi) := \big(\mathrm{Lie}\, E(\varphi), \mathrm{Lie}\, E(\varphi)_{\mathbb{C}|\,\mathrm{Per}\, E(A)_{\mathbb{C}}}\big)$$

is an element of $\mathrm{Hom}_{\mathrm{dRB}}(\mathrm{LiePer}\, E(A), \mathrm{LiePer}\, E(B))$, and the following diagram commutes in $\mathcal{C}_{\mathrm{dRB}}$:

$$\begin{array}{ccc} \mathrm{LiePer}\, E(A) & \xrightarrow{\mathrm{LiePer}\, E(\varphi)} & \mathrm{LiePer}\, E(B) \\ \simeq \big\downarrow J_{A,\mathrm{dRB}} & & \simeq \big\downarrow J_{B,\mathrm{dRB}} \\ H_{1,\mathrm{dRB}}(A) & \xrightarrow{H_{1\mathrm{dRB}}(\varphi)} & H_{1\mathrm{dRB}}(B) \end{array}$$

5.3.4 Extensions For any two objects M and N in $\mathcal{C}_{\mathrm{dRB}}$, we may consider the set $\mathrm{Ext}^1_{\mathrm{dRB}}(M,N)$ of 1-extensions of M by N in $\mathcal{C}_{\mathrm{dRB}}$, namely, of diagrams in $\mathcal{C}_{\mathrm{dRB}}$ of the form

$$\mathcal{E} : \quad 0 \longrightarrow N \xrightarrow{\alpha} X \xrightarrow{\beta} M \longrightarrow 0$$

such that $\beta \circ \alpha = 0$ and the diagrams

$$\mathcal{E}_{\mathrm{dR}} : \quad 0 \longrightarrow N_{\mathrm{dR}} \xrightarrow{\alpha_{\mathrm{dR}}} X_{\mathrm{dR}} \xrightarrow{\beta_{\mathrm{dR}}} M_{\mathrm{dR}} \longrightarrow 0$$

and

$$\mathcal{E}_{\mathrm{B}} : \quad 0 \longrightarrow N_{\mathrm{B}} \xrightarrow{\alpha_{\mathrm{B}}} X_{\mathrm{B}} \xrightarrow{\beta_{\mathrm{B}}} M_{\mathrm{B}} \longrightarrow 0$$

are short exact sequences of $\overline{\mathbb{Q}}$-vector spaces and of $\mathbb{Z}$-modules, respectively.

Equipped with the Baer sum, $\mathrm{Ext}^1_{\mathrm{dRB}}(M,N)$ becomes an abelian group. Actually, for any extension $\mathcal{E}$ as above, we may choose a $\overline{\mathbb{Q}}$-linear splitting $\sigma_{\mathrm{dR}} : M_{\mathrm{dR}} \to X_{\mathrm{dR}}$ of $\mathcal{E}_{\mathrm{dR}}$ and a $\mathbb{Z}$-linear splitting $\sigma_{\mathrm{B}} : M_{\mathrm{B}} \to X_{\mathrm{B}}$ of $\mathcal{E}_{\mathrm{B}}$. Then $\sigma_{\mathrm{dR}\mathbb{C}} := \sigma_{\mathrm{dR}} \otimes_{\overline{\mathbb{Q}}} 1_{\mathbb{C}}$ and $\sigma_{\mathrm{B}\mathbb{C}} := \sigma_{\mathrm{B}} \otimes_{\mathbb{Z}} 1_{\mathbb{C}}$ are $\mathbb{C}$-linear splittings of

$$\mathcal{E}_{\mathbb{C}} : \quad 0 \longrightarrow N_{\mathbb{C}} \xrightarrow{\alpha_{\mathbb{C}}} X_{\mathbb{C}} \xrightarrow{\beta_{\mathbb{C}}} M_{\mathbb{C}} \longrightarrow 0,$$

and consequently $\sigma_{\mathrm{dR}\mathbb{C}} - \sigma_{\mathrm{B}\mathbb{C}}$ may be written $\alpha_{\mathbb{C}} \circ \varphi$ for some uniquely determined φ in $(M^\vee \otimes N)_{\mathbb{C}}$. The map

$$\begin{aligned} \mathrm{Ext}^1_{\mathrm{dRB}}(M,N) &\xrightarrow{\sim} (M^\vee \otimes N)_{\mathbb{C}} / \big[(M^\vee \otimes N)_{\mathrm{dR}} + (M^\vee \otimes N)_{\mathrm{B}}\big], \\ [\mathcal{E}] &\longmapsto [\varphi] \end{aligned} \tag{5.19}$$

so defined is easily seen to be an isomorphism of abelian groups.

In particular, we get the usual isomorphisms:

$$\mathrm{Ext}^1_{\mathrm{dRB}}(M,N) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{dRB}}\big(\mathbb{Z}(0), M^\vee \otimes N\big) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{dRB}}\big(M \otimes N^\vee, \mathbb{Z}(0)\big). \tag{5.20}$$

5.4 Abelian varieties over $\overline{\mathbb{Q}}$ satisfy GPC^1 We are now in position to complete the proof of Theorem 5.1.

As already observed, universal vector extensions of abelian varieties satisfy condition **LP**. Corollary 4.3 therefore implies that, for any two abelian varieties A and B over $\overline{\mathbb{Q}}$, the map

$$\mathrm{LiePer} : \mathrm{Hom}_{\mathrm{gp}/\overline{\mathbb{Q}}}\big(E(A), E(B)\big) \longrightarrow \mathrm{Hom}_{\mathrm{dRB}}\big(\mathrm{LiePer}\, E(A), \mathrm{LiePer}\, E(B)\big),$$
$$\psi \longmapsto \mathrm{LiePer}\,\psi := (\mathrm{Lie}\,\psi, \mathrm{Lie}\,\psi_{\mathbb{C}|\,\mathrm{Per}\,E(A)_{\mathbb{C}}})$$

is an isomorphism of $\mathbb{Z}$-modules.

Together with the isomorphism (5.10), which identifies morphisms between abelian varieties and between their universal vector extensions, this establishes the first assertion in the following theorem; the second assertion follows from the existence of a functorial isomorphism (5.18) between LiePer $E(A)$ and $H_{1,\mathrm{dRB}}(A)$.

Theorem 5.3 *For any two abelian varieties A and B over $\overline{\mathbb{Q}}$, the maps*

$$\mathrm{Hom}_{\mathrm{gp}/\overline{\mathbb{Q}}}(A,B) \longrightarrow \mathrm{Hom}_{\mathrm{dRB}}\big(\mathrm{LiePer}\, E(A), \mathrm{LiePer}\, E(B)\big),$$
$$\varphi \longmapsto \mathrm{LiePer}\, E(\varphi)$$

and

$$H_{1,\mathrm{dRB}} : \mathrm{Hom}_{\mathrm{gp}/\overline{\mathbb{Q}}}(A,B) \longrightarrow \mathrm{Hom}_{\mathrm{dRB}}\big(H_{1,\mathrm{dRB}}(A), H_{1,\mathrm{dRB}}(B)\big)$$

are isomorphisms of $\mathbb{Z}$-modules.

In other words, the realization functor $H_{1,\mathrm{dRB}}$ from the category of abelian varieties over $\overline{\mathbb{Q}}$ to the category $\mathcal{C}_{\mathrm{dRB}}$ is fully faithful (cf. [1, Section 7.5.3], where a "rational" version of this isomorphism is established, by a reference to some advanced transcendence results of Wüstholz [101]).

To complete the proof of Theorem 5.1, we consider an abelian variety A over $\overline{\mathbb{Q}}$ and we apply Theorem 5.3 to A and its dual abelian variety $\hat{A}$. In this way, we get an isomorphism

$$H_{1,\mathrm{dRB}} : \mathrm{Hom}_{\mathrm{gp}/\overline{\mathbb{Q}}}(A,\hat{A}) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{dRB}}\big(H_{1,\mathrm{dRB}}(A), H_{1,\mathrm{dRB}}(\hat{A})\big).$$

Composing this isomorphism with the transpose of

$$\varpi_{A,\mathrm{dRB}} : H_{1,\mathrm{dRB}}(A) \xrightarrow{\sim} H^1_{\mathrm{dRB}}(\hat{A}) \otimes \mathbb{Z}(1)$$

and with the natural identification (5.14), we get an isomorphism

$$\mathrm{Hom}_{\mathrm{gp}/\overline{\mathbb{Q}}}(A,\hat{A}) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{dRB}}\big(\mathbb{Z}(0), H^1_{\mathrm{dRB}}(A) \otimes H^1_{\mathrm{dRB}}(A) \otimes \mathbb{Z}(1)\big). \tag{5.21}$$

The discussion on signs in Section 5.2.2 (notably the identity (5.6)) shows that this isomorphism maps the subgroup of *symmetric* morphisms from A to $\hat{A}$ onto the subgroup of skew-symmetric, or *alternating*, elements[17] in $\mathrm{Hom}_{\mathrm{dRB}}(\mathbb{Z}(0), H^1_{\mathrm{dRB}}(A) \otimes H^1_{\mathrm{dRB}}(A) \otimes \mathbb{Z}(1))$:

$$\mathrm{Hom}_{\mathrm{gp}/\overline{\mathbb{Q}}}(A,\hat{A})^{\mathrm{sym}} \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{dRB}}\big(\mathbb{Z}(0), H^1_{\mathrm{dRB}}(A) \otimes H^1_{\mathrm{dRB}}(A) \otimes \mathbb{Z}(1)\big)^{\mathrm{alt}}. \tag{5.22}$$

The fact that the morphism of $\mathbb{Z}$-modules in (5.22) is an isomorphism is nothing but, in a disguised form, the validity of $\mathrm{GPC}^1(A)$. Indeed, by composition with the isomorphism

$$NS(A) := \mathrm{Pic}(A)/\,\mathrm{Pic}_0(A) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{gp}/\overline{\mathbb{Q}}}(A, \hat{A})^{\mathrm{sym}},$$
$$[L] \longmapsto \varphi_L,$$

the isomorphism (5.22) becomes the isomorphism

$$NS(A) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{dRB}}\big(\mathbb{Z}(0), H^1_{\mathrm{dRB}}(A) \otimes H^1_{\mathrm{dRB}}(A) \otimes \mathbb{Z}(1)\big)^{\mathrm{alt}}. \tag{5.23}$$

The "Betti" component of (5.23) takes its values in $(H^1_B(A_{\mathbb{C}}) \otimes_{\mathbb{Z}} H^1_B(A_{\mathbb{C}}))^{\mathrm{alt}}$ and is well known to coincide with the classical "Riemann form" of elements of the Néron–Severi group (see, e.g., Birkenhake and Lange [10, Chapter 2]). Consequently, after the identification of

$$\mathrm{Hom}_{\mathrm{dRB}}\big(\mathbb{Z}(0), H^1_{\mathrm{dRB}}(A) \otimes H^1_{\mathrm{dRB}}(A) \otimes \mathbb{Z}(1)\big)^{\mathrm{alt}}$$

and

$$\mathrm{Hom}_{\mathrm{dRB}}\big(\mathbb{Z}(0), H^2_{\mathrm{dRB}}(A) \otimes \mathbb{Z}(1)\big) = H^2_{\mathrm{Gr}}(A),$$

the isomorphism (5.22) may be read as asserting that the map

$$c_{1\mathrm{dRB}} : NS(A) \longrightarrow H^2_{\mathrm{Gr}}(A)$$

is an isomorphism. This is precisely the content of $\mathrm{GPC}^1(A)$.

5.5 $\overline{\mathbb{Q}}$-points of abelian varieties and extensions in $\mathcal{C}_{\mathrm{dRB}}$[18] Let A denote an abelian variety over $\overline{\mathbb{Q}}$.

Consider some line bundle L over A, algebraically equivalent to zero, equipped with some rigidification $\epsilon : k \simeq L_{0_A}$. Recall that the $\mathbb{G}_m$-torsor $L^\times \xrightarrow{\pi_L} A$ over A, deduced from the total space of L by deleting its zero section, may be endowed with a unique structure of a $\overline{\mathbb{Q}}$-algebraic group which makes the diagram

$$0 \longrightarrow \mathbb{G}_{m\overline{\mathbb{Q}}} \xrightarrow{\epsilon} L^\times \xrightarrow{\pi_L} A \longrightarrow 0$$

a short exact sequence of commutative $\overline{\mathbb{Q}}$-algebraic groups, and that this construction establishes an isomorphism of groups:

$$\hat{A}(\overline{\mathbb{Q}}) \xrightarrow{\sim} \mathrm{Ext}^1_{\text{c-gp}/\overline{\mathbb{Q}}}(A, \mathbb{G}_{m\overline{\mathbb{Q}}}).$$

The fiber product

$$E(L^\times) \simeq L^\times \times_A E(A)$$

defines a commutative $\overline{\mathbb{Q}}$-algebraic group which fits into the following commutative diagram with exact lines:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{G}_{m\overline{\mathbb{Q}}} & \xrightarrow{\tilde{\epsilon}} & E(L^\times) & \xrightarrow{\tilde{\pi}_L} & E(A) & \longrightarrow & 0 \\ & & \downarrow{=} & & \downarrow & & \downarrow{p_A} & & \\ 0 & \longrightarrow & \mathbb{G}_{m\overline{\mathbb{Q}}} & \xrightarrow{\epsilon} & L^\times & \xrightarrow{\pi_L} & A & \longrightarrow & 0 \end{array}$$

By considering the Lie algebra (over $\overline{\mathbb{Q}}$) and the periods (over $\mathbb{C}$) of the first line, we get a 1-extension in $\mathcal{C}_{\mathrm{dRB}}$:

$$0 \longrightarrow \mathbb{Z}(1) \xrightarrow{\mathrm{LiePer}\,\tilde{\epsilon}} \mathrm{LiePer}\,E(L^\times) \xrightarrow{\mathrm{LiePer}\,\tilde{\pi}_L} \mathrm{LiePer}\,E(A) \longrightarrow 0. \tag{5.24}$$

Thanks to the canonical isomorphisms in $\mathcal{C}_{\mathrm{dRB}}$,

$$\mathrm{LiePer}\, E(A) \xrightarrow{\sim} H_{1\mathrm{dRB}}(A) \xrightarrow{\sim} H^1_{\mathrm{dRB}}(\hat{A}) \otimes \mathbb{Z}(1) \xrightarrow{\sim} H_{1\mathrm{dRB}}(A)^\vee \otimes \mathbb{Z}(1),$$

its class defines an element in

$$\kappa_{\mathrm{dRB}}(L) \in \mathrm{Ext}^1_{\mathrm{dRB}}\big(H_{1\mathrm{dRB}}(A), \mathbb{Z}(1)\big) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{dRB}}\big(\mathbb{Z}(0), H_{1\mathrm{dRB}}(\hat{A})\big).$$

The proof of the following proposition is again an application of Corollary 4.3.

Proposition 5.4 *The morphism of abelian groups*

$$\kappa_{\mathrm{dRB}} : \hat{A}(\overline{\mathbb{Q}}) \longrightarrow \mathrm{Ext}^1_{\mathrm{dRB}}\big(\mathbb{Z}(0), H_{1\mathrm{dRB}}(\hat{A})\big)$$

is injective.

We leave the details to the reader and only emphasize that giving a direct description of the subgroup $\kappa_{\mathrm{dRB}}(\hat{A}(\overline{\mathbb{Q}}))$ of $\mathrm{Ext}^1_{\mathrm{dRB}}(\mathbb{Z}(0), H_{1\mathrm{dRB}}(\hat{A}))$ appears to be an intriguing and difficult issue.

6 D-group Schemes

In this part, we introduce D-schemes and D-group schemes in a geometric setting, suitable for the application to Diophantine geometry we want to discuss in the remainder of the article. These definitions are variants of the original definitions by Buium [20], [21], [25, Chapter 3] which make sense over some fixed differential base field (of characteristic zero). Here we shall consider D-schemes and group schemes over some smooth base variety instead: this framework is the one of Malgrange in [70], with the field of complex numbers replaced by some arbitrary field of characteristic zero.

For simplicity, we shall make smoothness and quasi-projectivity assumptions which actually could be relaxed in many places. Actually, on a base scheme of finite type over a field of characteristic zero, D-schemes are nothing but the "crystals in relative schemes" mentioned in a famous letter of Grothendieck to Tate.[19] The approach to D-schemes as "crystals," defined in terms of infinitesimal sites and stratifications, has much to recommend it (see, e.g., [98, Section 8]), but I have preferred to stick to a more naive approach in the spirit of classical differential geometry, at the expense of extra regularity assumptions, based on a definition of D-schemes that mimics the one of integrable Ehresmann connections on differentiable fiber bundles (see Ehresmann [38]).

In the following sections we denote k a fixed field *of characteristic zero*.

6.1 Basic definitions

Let S denote a smooth quasi-projective scheme over k.

6.1.1 D-schemes By a *D-scheme* over S we shall mean a pair $(X, \mathcal{F})$, where $X \xrightarrow{\pi} S$ is a smooth, quasi-projective scheme over S (hence over k) and $\mathcal{F}$ is an integrable[20] subvector bundle of the "absolute" tangent bundle $T_{X/k}$ of X such that

$$T_{X/k} = T_{X/S} \oplus \mathcal{F}.$$

This last condition means precisely that $\mathcal{F}$ determines a splitting of the exact sequence of vector bundles over the k-scheme X,

$$0 \longrightarrow T_{X/S} \hookrightarrow T_{X/k} \xrightarrow{D\pi} \pi^* T_{S/k} \longrightarrow 0,$$

defined by the differential of π or, equivalently, that the restriction of $D\pi$ to $\mathcal{F}$ is an isomorphism:

$$D\pi_{|\mathcal{F}} : \mathcal{F} \xrightarrow{\sim} \pi^* T_{S/k}. \tag{6.1}$$

A *morphism of D-schemes* over S,

$$\varphi : (X_1, \mathcal{F}_1) \longrightarrow (X_2, \mathcal{F}_2), \tag{6.2}$$

is a morphism of S-schemes $\varphi : X_1 \to X_2$ whose "absolute" differential

$$D\varphi : T_{X_1/k} \longrightarrow \varphi^* T_{X_2/k}$$

maps $\mathcal{F}_1$ to $\varphi^* \mathcal{F}_2$.

Observe that, if φ is a morphism of D-schemes over S from $(X_1, \mathcal{F}_1)$ to $(X_2, \mathcal{F}_2)$, then conditions (6.1) for $(X_1, \mathcal{F}_1)$ and $(X_2, \mathcal{F}_2)$ imply that $D\varphi$ maps $\mathcal{F}_1$ isomorphically onto $\varphi^* \mathcal{F}_2$.

Morphisms of D-schemes may be obviously composed and define the category of (smooth, quasi-projective) D-schemes over S. Clearly, this category admits finite products: $(S, T_{S/k})$ is a final object, and the product of two D-schemes $(X_1, \mathcal{F}_1)$ and $(X_2, \mathcal{F}_2)$ over S may be constructed as the D-scheme $(X, \mathcal{F})$ consisting of their product as schemes over S,

$$X := X_1 \times_S X_2,$$

equipped with the subvector bundle $\mathcal{F}$ of $T_{X/k}$ which is the "direct sum of $\mathcal{F}_1$ and $\mathcal{F}_2$ over $T_{S/k}$," formally defined as the kernel of the surjective morphism of vector bundles over X:

$$(D_{\pi_1}, -D_{\pi_2}) : (\mathcal{F}_1 \boxplus \mathcal{F}_2)_{|X} \longrightarrow \pi^* T_{S/k}.$$

(It lies inside the kernel of

$$(D_{\pi_1}, -D_{\pi_2}) : (T_{X_1/k} \boxplus T_{X_2/k})_{|X} \longrightarrow \pi^* T_{S/k},$$

which may be identified with $T_{X/k}$.)

A *closed D-subscheme* of a D-scheme $(X, \mathcal{F})$ over S is the image of a morphism of D-schemes with range $(X, \mathcal{F})$ that is also a closed immersion. Equivalently, it is a closed, smooth subscheme Y of X such that its tangent bundle $T_{Y/k}$, which may be identified to a subvector bundle of $T_{X/k|Y}$, contains $\mathcal{F}_{|Y}$.

A *horizontal section* of some D-scheme $(X, \mathcal{F})$ over S is a right inverse of the structural morphism $X \longrightarrow S$ in the category of D-schemes over S. In other words, it is a section $\mathcal{P}$ of this morphism over S, the differential of which, $D\mathcal{P} : T_{S/k} \longrightarrow \mathcal{P}^* T_{X/k}$, takes its values in $\mathcal{P}^* \mathcal{F}$, or equivalently, the image of which is a D-subscheme of $(X, \mathcal{F})$.

From the integrable subvector bundle $\mathcal{F}$ of $T_{X/k}$, the normal bundle $\mathcal{P}^* T_{X/S}$ of any horizontal section $\mathcal{P}$ inherits an integrable connection.

6.1.2 D-group schemes A (smooth, quasi-projective) *D-group scheme* over S is defined as a group object in the category of D-schemes over S.

A D-group scheme $\mathbf{G}$ over S may be identified with a pair $(G, \mathcal{F})$ where G is a smooth, quasi-projective group scheme over S and $\mathcal{F}$ a subvector bundle of $T_{G/k}$ which makes $(G, \mathcal{F})$ a D-scheme over S, in such a way that the graphs of the unit section e_G, of the inverse map, and of the composition map of the group scheme G become D-subschemes of the D-schemes G, G^2, and G^3 over S.

In intuitive terms, a D-group scheme may be thought as a smooth group scheme over S equipped with some "algebraic connection" compatible with its group structure.

Since its unit section e_G is horizontal, the relative Lie algebra $\mathrm{Lie}_S\, G := e_G^* T_{G/S}$ of the group scheme G over S underlying some D-group scheme $\mathbf{G}$ over S becomes endowed with a natural integrable connection. The so-defined module with integrable connection shall be denoted $\mathrm{Lie}_S\, \mathbf{G}$.

Assume that S is integral (or, equivalently, connected) of dimension s, and consider its field of rational functions $k(S)$. Let us choose some $k(S)$-basis $(v_1, \ldots, v_s)$ of the $k(S)$-vector space of rational sections of $T_{S/k}$ such that the Lie brackets $[v_i, v_j]$ all vanish.[21] Then the field $k(S)$ equipped with the derivations $(\delta_1, \ldots, \delta_s)$ becomes a differential field in the classical sense of Ritt and Kolchin. Let us finally choose a differential closure $(K; \delta_1, \ldots, \delta_s)$ of $(k(S); \delta_1, \ldots, \delta_s)$. Through the base changes

$$\mathrm{Spec}\, K \longrightarrow \mathrm{Spec}\, k(S) \hookrightarrow S,$$

any D-group scheme $(G, \mathcal{F})$ over S in our sense defines D-group schemes in the sense of Buium over the differential fields $(k(S); \delta_1, \ldots, \delta_s)$ and $(K; \delta_1, \ldots, \delta_s)$, and a Δ_0-group, that is, a differential algebraic group of finite dimension, in the sense of Kolchin, by considering the subgroup of the group $G(K)$ of K-points of G consisting of its "horizontal points." (We refer the reader to Buium [21, Chapter 5], Pillay [80], [81], and Bertrand and Pillay [9] for discussions of the relations between Buium's D-groups and differential algebraic groups.)

6.1.3 Extensions Let $\mathbf{G}_1 = (G_1, \mathcal{F}_1)$ and $\mathbf{G}_2 = (G_2, \mathcal{F}_2)$ be two commutative D-group schemes over S. An *extension* of $\mathbf{G}_1$ by $\mathbf{G}_2$ in the category of commutative D-group schemes over S is a diagram

$$0 \longrightarrow \mathbf{G}_2 \xrightarrow{i} \mathbf{G} \xrightarrow{p} \mathbf{G}_1 \longrightarrow 0 \tag{6.3}$$

in this category such that the underlying diagram of commutative group schemes over S,

$$0 \longrightarrow G_2 \xrightarrow{i} G \xrightarrow{p} G_1 \longrightarrow 0,$$

is a short exact sequence[22] (see Kowalski and Pillay [64] for related constructions).

The Baer sum of two extensions of $\mathbf{G}_1$ by $\mathbf{G}_2$ may be defined in an obvious way. Equipped with this operation, the set $\mathrm{Ext}^1_{\mathrm{c}D\text{-gp}/S}(\mathbf{G}_1, \mathbf{G}_2)$ of isomorphism classes of these extensions defines an abelian group, which satisfies the usual functorialities in S, $\mathbf{G}_1$, and $\mathbf{G}_2$.

We may apply the functor Lie_S to the extension (6.3). We obtain a short exact sequence of modules with integrable connections over S:

$$0 \longrightarrow \mathrm{Lie}_S\, \mathbf{G}_2 \xrightarrow{\mathrm{Lie}_S\, i} \mathrm{Lie}_S\, \mathbf{G} \xrightarrow{\mathrm{Lie}_S\, p} \mathrm{Lie}_S\, \mathbf{G}_1 \longrightarrow 0.$$

This construction defines an additive map, say, when S is projective:

$$\begin{aligned} \mathrm{Lie}^1_S : \mathrm{Ext}^1_{\mathrm{c}D\text{-gp}/S}(\mathbf{G}_1, \mathbf{G}_2) &\longrightarrow \mathrm{Ext}^1_{\mathrm{mic}/S}(\mathrm{Lie}_S\, \mathbf{G}_1, \mathrm{Lie}_S\, \mathbf{G}_2) \\ &\simeq H^1_{\mathrm{dR}}\big(S, (\mathrm{Lie}_S\, \mathbf{G}_1)^\vee \otimes \mathrm{Lie}_S\, \mathbf{G}_2\big), \end{aligned}$$

where we use the notation introduced in Section 2.2.5, formula (2.9).

6.1.4 Functoriality in S If $\varphi : S' \longrightarrow S$ is a morphism of projective schemes over k, then, from any D-scheme $(X, \mathcal{F})$ over S, we may deduce a D-scheme $(X', \mathcal{F}')$ over S' by "pulling it back" by φ as follows : X' is the smooth, quasi-projective S'-scheme defined as the fiber product $X \times_S S'$; if $\tilde{\varphi} : X' \longrightarrow X$ denotes the canonical "first projection" morphism and $D\tilde{\varphi} : T_{X'/k} \longrightarrow \tilde{\varphi}^* T_{X/k}$ its differential, the D-structure on X' over S' is defined by the integrable subvector bundle of $T_{X'/k}$

$$\mathcal{F}' := D\tilde{\varphi}^{-1}(\tilde{\varphi}^* \mathcal{F}).$$

This construction of "base change" is functorial, and transforms D-group schemes over S into D-group schemes over S'. It satisfies an obvious compatibility with the Lie algebra functor (from D-group schemes to modules with integrable connections) and the pullback of modules with integrable connections.

The D-schemes over $\operatorname{Spec} k$ are nothing but the smooth, quasi-projective schemes over k. A *constant* D-scheme over S is a D-scheme isomorphic to the pullback by the k-morphism $S \longrightarrow \operatorname{Spec} k$ of some smooth, quasi-projective schemes over k. In the remainder of the article, we shall denote $\mathbf{G}_{m,S}$ the constant multiplicative group scheme over S, defined as the pullback of the algebraic group $\mathbb{G}_{m,k}$. After the change of base $S \longrightarrow \operatorname{Spec} k$, the isomorphism

$$\begin{aligned} \operatorname{Lie} \mathbb{G}_{m,k} &\stackrel{\sim}{\longrightarrow} k, \\ X.\partial/\partial X &\longmapsto 1 \end{aligned}$$

becomes an isomorphism of modules with integrable connections:

$$\operatorname{Lie}_S \mathbf{G}_{m,S} \stackrel{\sim}{\longrightarrow} (\mathcal{O}_S, d).$$

6.1.5 Change of base fields If k' is a field extension of k, the extension of scalars from k to k' associates a D-scheme $(X_{k'}, \mathcal{F}_{k'})$ over $S_{k'}$, defined over the base field k', to any D-scheme $(X, \mathcal{F})$ over S. This operation satisfies obvious functoriality properties that we shall use freely in the remainder of the article. In particular, it attaches D-group schemes over $S_{k'}$ to D-group schemes over S and defines morphisms of extension groups:

$$\operatorname{Ext}^1_{\mathrm{c}D\text{-gp}/S}(\mathbf{G}_1, \mathbf{G}_2) \longrightarrow \operatorname{Ext}^1_{\mathrm{c}D\text{-gp}/S_{k'}}(\mathbf{G}_{1k'}, \mathbf{G}_{2k'}).$$

6.2 *D*-schemes and analytification When the base field k is $\mathbb{C}$, a D-scheme $(X, \mathcal{F})$ (resp., a D-group scheme $(G, \mathcal{F})$) over S determines, through analytification, a "D-analytic space" $(X^{\mathrm{an}}, \mathcal{F}^{\mathrm{an}})$ (resp., a "D-complex Lie group" $(G^{\mathrm{an}}, \mathcal{F}^{\mathrm{an}})$) over the complex manifold S^{an}. We shall omit the formal definitions of these notions—just "copy" the above ones in the analytic context—and content ourselves with a few observations.

First, after analytification, a D-scheme $(X, \mathcal{F})$ *projective* over S becomes locally constant in the analytic category. Namely, for any point s_0 of S^{an}, there exists an open neighborhood Ω of s_0 in S^{an} and an isomorphism of $\mathbb{C}$-analytic spaces over Ω,

$$\Psi_{s_0} : \Omega \times X^{\mathrm{an}}_{s_0} \stackrel{\sim}{\longrightarrow} X^{\mathrm{an}}_{\Omega}, \tag{6.4}$$

such that

$$\Psi_{s_0}(s_0, \cdot) = \mathrm{Id}_{X^{\mathrm{an}}_{s_0}} \tag{6.5}$$

and, for any (s, x) in $\Omega \times X^{\mathrm{an}}_{s_0}$,

$$\mathcal{F}_{\Psi_{s_0}(s,x)} = D\Psi_{s_0}(s, x)(T_s\Omega \oplus 0). \tag{6.6}$$

This follows from the analytic integrability of $\mathcal{F}^{\mathrm{an}}$, together with the properness of the structural morphism $X^{\mathrm{an}} \longrightarrow S^{\mathrm{an}}$ in the analytic topology. (Observe that conditions (6.5) and (6.6) uniquely determine Ψ_{s_0} for Ω connected.)

Second, as pointed out by Hamm (cf. [21, Chapter 2, 1.3]), a similar statement holds for any D-group scheme $(G, \mathcal{F})$ over S. Thus we get a (unique) isomorphism of complex Lie groups over[23] Ω (assumed to be small enough and connected),

$$\Psi_{s_0} : \Omega \times G^{\mathrm{an}}_{s_0} \xrightarrow{\sim} G^{\mathrm{an}}_{\Omega}, \tag{6.7}$$

which satisfy the initial condition (6.5) and the horizontality condition (6.6).

Consider in particular the case of a commutative D-group scheme $\mathbf{G} = (G, \mathcal{F})$ over S, with connected fibers. Then the “relative” exponential map

$$\exp_{G/S} : \mathrm{Lie}_S\, G \longrightarrow G^{\mathrm{an}}$$

defines a surjective morphism of complex Lie groups over S^{an}. It is compatible with the “horizontal” structures defined by the integrable connection on $\mathrm{Lie}_S\, \mathbf{G}$ and by (6.6), and consequently its kernel

$$\mathrm{Per}_S\, G := \ker \exp_{G/S}$$

is a local system (that is, a locally free sheaf) of $\mathbb{Z}$-modules of finite rank over S^{an}, which fits into a short exact sequence in the category of commutative complex Lie groups over S^{an}:

$$0 \longrightarrow \mathrm{Per}_S\, G \hookrightarrow \mathrm{Lie}_S\, G \xrightarrow{\exp_{G/S}} G^{\mathrm{an}} \longrightarrow 0.$$

This is even a short exact sequence of commutative D-complex Lie groups, which should be denoted

$$0 \longrightarrow \mathrm{Per}_S\, G \hookrightarrow \mathrm{Lie}_S\, \mathbf{G} \xrightarrow{\exp_{G/S}} \mathbf{G}^{\mathrm{an}} \longrightarrow 0.$$

This shows, in particular, that when s varies in S^{an}, the dimension of the complex subvector space of $\mathrm{Lie}\, G_s$ generated by its period lattice $\mathrm{Per}\, G_s$ is locally constant (in the analytic topology). Consequently, if S (hence S^{an}) is connected and if, for some $s_0 \in S$, G_{s_0} satisfies condition **LP** (cf. Section 4.3), then G_s satisfies **LP** for every s in S, and the structure of $\mathbf{G}$ as a D-group scheme over S is uniquely determined by its structure of a group scheme. Similarly, if $\mathbf{G}_1$ and $\mathbf{G}_2$ are two commutative D-groups schemes over S, and if $\mathbf{G}_1$ has connected fibers satisfying **LP**, then any morphism of group schemes from G_1 to G_2 is a morphism of D-group schemes from $\mathbf{G}_1$ to $\mathbf{G}_2$.

These remarks will apply to the D-group schemes associated to abelian schemes and to their extension by multiplicative groups considered in Sections 6.4 and 6.5 infra (see also [9, Lemma 3.4] for similar unicity statements in a more “differential algebraic” formulation).

Associating its local system of periods $\mathrm{Per}_S\, G$ to a D-group scheme $\mathbf{G}$ is a functorial construction (in S and $\mathbf{G}$). Applied to extensions, it defines a morphism of $\mathbb{Z}$-modules, for any two commutative D-groups schemes $\mathbf{G}_1$ and $\mathbf{G}_2$ with connected fibers over S:

$$\begin{aligned}\mathrm{Per}^1_S : \mathrm{Ext}^1_{cD\text{-gp}/S}(\mathbf{G}_1, \mathbf{G}_2) &\longrightarrow \mathrm{Ext}^1_{\text{Ab-Sheaves}/S^{\mathrm{an}}}(\mathrm{Per}_S\, G_1, \mathrm{Per}_S\, G_2)\\ &\simeq H^1\big(S^{\mathrm{an}}, (\mathrm{Per}_S\, G_1)^{\vee} \otimes \mathrm{Per}_S\, G_2\big).\end{aligned} \tag{6.8}$$

6.3 Moduli spaces of vector bundles with connections as D-schemes If the S-scheme X underlying some D-scheme $(X, \mathcal{F})$ as above is projective over S, then, locally in the étale topology of S, X is "constant" over S (namely, when k is algebraically closed, of the form $X_0 \times_k S$, after replacing S by some étale neighborhood of any given point of S). This follows from the representability of the Isom-functors in the projective case, together with the formal integrability of $\mathcal{F}$ and Artin's algebraization theorem (cf. Buium [20, Chapter II, Section 1] and Gillet [43, Section 3]).

This property is a refinement, which makes sense in pure algebraic geometry, of the local analytic triviality of projective D-schemes when $k = \mathbb{C}$. It strongly limits the possible constructions of smooth projective D-schemes.

It is remarkable that, in contrast, highly "nonconstant" smooth *quasi-projective* D-schemes arise naturally. Indeed the construction of the moduli spaces $\mathbf{MIC}_N(M, o)$ of vector bundles with connection recalled in Section 2.3.3 above, applied to smooth projective families of pointed projective varieties parameterized by S, provides quasi-projective D-schemes over S.

Namely, if M is a smooth, projective S-scheme with geometrically connected fibers, and if o denotes a section of M over S, then Simpson's techniques apply to this relative situation. They lead to the construction of a flat, quasi-projective S-scheme[24] $\mathbf{MIC}_N(M/S, o)$, the fiber of which over some point $s \in S(\overline{k})$ may be identified with the moduli space $\mathbf{MIC}_N(M_s, o(s))$. Formally, for any S-scheme Σ, $\mathbf{MIC}_N(M/S, o)(\Sigma)$ classifies vector bundles of rank N over $X_\Sigma := X \times_S \Sigma$, rigidified over o_Σ, and equipped with an integrable connection relative to Σ.

The S-scheme $\mathbf{MIC}_N(M/S, o)$ admits a canonical structure of D-scheme over S, which reflects its so-called crystalline nature. For general M and N, this scheme may actually not be smooth over S, and properly speaking it is not covered by the above definition of D-schemes (which should be replaced by a suitable definition in terms of the infinitesimal site and stratifications associated to X/k). However, in the remainder of the article, we shall be mainly concerned by the situation where $N = 1$, in which case $\mathbf{MIC}_1(M/S, o)$ is a *smooth*, quasi-projective, group scheme over S, and we allow ourselves to neglect this issue of regularity.

When $k = \mathbb{C}$, the D-scheme structure of $\mathbf{MIC}_N(M/S, o)$ may be described as follows. When s varies in the complex manifold S^{an}, the family of fundamental groups

$$\Gamma_s := \pi_1\big(M_s^{\mathrm{an}}, o(s)\big)$$

define a local system (i.e., locally constant sheaf) of groups on S^{an}. Over any simply connected open subset Ω in S^{an}, it may be trivialized: for any pair of points (s_0, s_1) in Ω, we get a canonical isomorphism

$$\gamma_{s_1,s_0} : \Gamma_{s_0} \xrightarrow{\sim} \Gamma_{s_1},$$

which clearly induces an isomorphism of representations spaces:

$$\begin{aligned} \Phi^{\mathrm{Rep}}_{s_1,s_0} : \mathbf{Rep}_N(\Gamma_{s_0}) &\xrightarrow{\sim} \mathbf{Rep}_N(\Gamma_{s_1}), \\ \rho &\longmapsto \rho \circ \gamma^{-1}_{s_0,s_1}. \end{aligned}$$

Moreover, the monodromy isomorphisms (2.14),

$$\mathrm{mon}_{o(s)} : \mathbf{MIC}_N(M/S, o)_s = \mathbf{MIC}_N\big(M_s, o(s)\big) \xrightarrow{\sim} \mathbf{Rep}_N(\Gamma_s)$$

and their inverses depend analytically on s, in the sense that, if s_0 denotes a base point in Ω, the bijection of sets

$$\begin{aligned} \Psi_{s_0} : \Omega \times \mathbf{Rep}_N(\Gamma_{s_0}) &\xrightarrow{\sim} \mathbf{MIC}_N(M/S, o)_\Omega, \\ (s, \rho) &\longmapsto \mathrm{mon}_s^{-1}\big(\Phi^{\mathrm{Rep}}_{s,s_0}(\rho)\big) \end{aligned} \tag{6.9}$$

is an isomorphism of $\mathbb{C}$-analytic spaces over Ω.

The D-scheme structure over s of $X := \mathbf{MIC}_N(M/S, o)$ is compatible with the "analytic trivialization" (6.9). Assume indeed that $\mathbf{MIC}_N(M/S, o)$ is smooth over S (for instance, suppose that $N = 1$); then the subvector bundle $\mathcal{F}$ of $T_{X/\mathbb{C}}$ which defines this structure becomes "horizontal" via the above isomorphism:

$$\text{for any } (s, \rho) \in \Omega \times \mathbf{Rep}_N(\Gamma_{s_0}),\ \mathcal{F}_{\Phi(s,\rho)} = D\Psi_{s_0}(s, \rho)(T_s\Omega \oplus 0).$$

It is quite remarkable that the *analytic* subvector bundle $\mathcal{F}$ of $T_{X/\mathbb{C}}$ defined through this formula in terms of the local analytic trivializations (6.9) of X over S is an *algebraic* subvector bundle of $T_{X/\mathbb{C}}$.

This is due to Grothendieck and to Mazur and Messing [75] when $N = 1$ (see also [21]), and to Simpson [98, Section 8]) in general. Basically their proof consists in considering the avatar in formal geometry (over the formal completion $\widehat{S}_{s_0}$ of S at s_0) of the local analytic trivialization of $\mathbf{MIC}_N(M/S, o)$ over Ω induced by (6.9):

$$\begin{aligned} \Psi^{\mathrm{MIC}}_{s_0} := \Psi_{s_0} \circ (\mathrm{Id}_\Omega \times \mathrm{mon}_{o(s_0)}) : \Omega \times \mathbf{MIC}_N(M/S, o)_0 \\ \xrightarrow{\sim} \mathbf{MIC}_N(M/S, o)_\Omega. \end{aligned} \tag{6.10}$$

It turns out that the formal analogue of (6.10) over $\widehat{S}_{s_0}$ may be directly constructed in (formal) algebraic geometry, with no recourse to analytic techniques, over any base field k of characteristic zero.

The existence of the local analytic trivializations $\Psi^{\mathrm{MIC}}_{s_0}$ is indeed a direct consequence of the following basic observation: if (E, ∇) is an analytic vector bundle with integrable connection over some connected analytic submanifold Y of some analytic manifold X, then (E, ∇) uniquely extends, as a vector bundle with integrable connection, over any sufficiently small open connected neighborhood of Y in X. This property admits a natural avatar in formal geometry, valid over any base field of characteristic zero, which implies the existence of a formal analogue of $\Psi^{\mathrm{MIC}}_{s_0}$. This construction, with s_0 varying in S, endows $\mathbf{MIC}_N(M/S, o)$ with a structure of a D-scheme over S.

6.4 Universal vector extensions as D-group schemes[25] The above discussion may be specialized to the case $N = 1$. Then $\mathbf{MIC}_1(M/S, o)$ is a smooth, quasi-projective group scheme over S—its group structure is induced by the tensor product of rigidified line bundles with connections—and its neutral component $\mathbf{MIC}_1(M/S, o)^0$ may be identified with the universal vector extension $E(\mathrm{Pic}_0(M/S))$ of the connected relative Picard variety $\mathrm{Pic}_0(M/S)$ of M over S. Moreover, the structure of a D-scheme over S on $\mathbf{MIC}_1(M/S, o)$ is compatible with its structure of a group scheme.

Let us introduce the relative Albanese variety of M over S, namely the abelian scheme over S defined as

$$\mathcal{A} := \widehat{\mathrm{Pic}_0(M/S)},$$

and the relative Albanese morphism

$$\alpha_o : M \longrightarrow \mathcal{A}$$

attached to the section o. It induces an isomorphism of group schemes over S (see, e.g., [17, Appendix B])

$$\alpha_o^* : \mathbf{MIC}_1(M/S, o)^0 \xrightarrow{\sim} \mathbf{MIC}_1(\mathcal{A}/S, 0_{\mathcal{A},0})^0,$$

compatible with their structure of D-schemes. Together with the identification of group schemes over S,

$$\mathbf{MIC}_1(\mathcal{A}/S, 0_{\mathcal{A}})^0 = \mathbf{MIC}_1(\mathcal{A}/S, 0_{\mathcal{A}}) \xrightarrow{\sim} E(\widehat{\mathcal{A}}),$$

this shows that (i) to study $\mathbf{MIC}_1(M/S, o)^0$, we may consider the case where M is some abelian scheme over S; and (ii) that the universal vector extension $E(\widehat{\mathcal{A}})$—hence by duality the universal vector extension of any abelian scheme over S—is endowed with a natural structure of D-group schemes that we shall denote $\mathbf{E}(\widehat{\mathcal{A}})$.

The analytic description of the D-structure on the moduli spaces $\mathbf{MIC}_N(M/S, o)$ boils down in the present situation to the following description of the D-group scheme $\mathbf{E}(\mathcal{B})$ defined by the universal vector extension $E(\mathcal{B})$ attached to some abelian scheme $\mathcal{B}$ (see also [75, Section 4.4]).

Assume that $k = \mathbb{C}$, and consider an abelian scheme over S, of relative dimension g,

$$\pi : \mathcal{B} \longrightarrow S.$$

As in Section 6.2, we may consider the analytic description of the complex Lie group $\mathcal{B}^{\mathrm{an}}$ over S^{an} as a quotient of $\mathrm{Lie}_S\, \mathcal{B}$ by its local system of periods:

$$0 \longrightarrow \mathrm{Per}_S\, \mathcal{B} \hookrightarrow \mathrm{Lie}_S\, \mathcal{B} \xrightarrow{\exp_{\mathcal{B}/S}} \mathcal{B}^{\mathrm{an}} \longrightarrow 0.$$

This local system $\mathrm{Per}_S\, \mathcal{B}$ is locally free of rank $2g$ and may be identified with the local systems of fundamental groups, of fiber at $s \in S$:

$$\pi_1(\mathcal{B}_s, 0_{\mathcal{B}_s}) \simeq H_1(\mathcal{B}_s^{\mathrm{an}}, \mathbb{Z}).$$

In the remainder of the article, we shall denote it $\mathcal{H}_{1\mathrm{B}}(\mathcal{B}^{\mathrm{an}}/S^{\mathrm{an}})$. In turn, the dual local system

$$\mathcal{H}^1_{\mathrm{B}}(\mathcal{B}^{\mathrm{an}}/S^{\mathrm{an}}) := \mathcal{H}_{1\mathrm{B}}(\mathcal{B}^{\mathrm{an}}/S^{\mathrm{an}})^{\vee}$$

may be identified with $R^1\pi_*^{\mathrm{an}}\mathbb{Z}_{\mathcal{B}^{\mathrm{an}}}$.

As discussed in Section 5.3.3, for any $s \in S^{\mathrm{an}}$, we have a canonical isomorphism,

$$J_{\mathcal{B}_s} : \mathrm{Lie}\, E(\mathcal{B}_s) \xrightarrow{\sim} H_{1\mathrm{dR}}(\mathcal{B}_s/\mathbb{C}) \simeq H_1(\mathcal{B}_s^{\mathrm{an}}, \mathbb{C}) \simeq H_1(\mathcal{B}_s^{\mathrm{an}}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}, \quad (6.11)$$

which sends $\mathrm{Per}\, \mathcal{B}_s$ isomorphically onto $H_1(\mathcal{B}_s^{\mathrm{an}}, \mathbb{Z})$. These isomorphisms depend analytically on $s \in S^{\mathrm{an}}$ and define isomorphisms $J_{\mathcal{B}}$ of analytic vector bundles and local systems over S^{an}, which fit into a commutative diagram:

$$\begin{array}{ccc} \mathrm{Per}_S\, E(\mathcal{B}) & \xrightarrow[\sim]{J_{\mathcal{B}}} & \mathcal{H}_{1\mathrm{B}}(\mathcal{B}^{\mathrm{an}}/S^{\mathrm{an}}) \\ \downarrow & & \downarrow \\ \mathrm{Lie}_S\, E(\mathcal{B}) & \xrightarrow[\sim]{J_{\mathcal{B}}} & \mathcal{H}_{1\mathrm{B}}(\mathcal{B}^{\mathrm{an}}/S^{\mathrm{an}}) \otimes_{\mathbb{Z}} \mathbb{C} \end{array}$$

where the vertical maps are the obvious injections. They induce an isomorphism of complex Lie groups over S^{an}:

$$J_{\mathcal{B}}^{\times} : E(\mathcal{B})^{\mathrm{an}} \xrightarrow{\sim} \mathcal{H}_{1\mathrm{B}}(\mathcal{B}^{\mathrm{an}}/S^{\mathrm{an}}) \otimes_{\mathbb{Z}} \mathbb{G}_{m\mathbb{C}}^{\mathrm{an}}$$

which makes the following diagram commutative:

$$\begin{array}{ccccccc} 0 \longrightarrow \mathcal{H}_{1\mathrm{B}}(\mathcal{B}^{\mathrm{an}}/S^{\mathrm{an}}) & \xrightarrow{J_{\mathcal{B}}^{-1}} & \mathrm{Lie}_S\, E(\mathcal{B}) & \xrightarrow{\exp_{E(\mathcal{B})/S}} & E(\mathcal{B})^{\mathrm{an}} & \longrightarrow 0 \\ \downarrow = & & \downarrow J_{\mathcal{B}} & & \downarrow J_{\mathcal{B}}^{\times} & \\ 0 \longrightarrow \mathcal{H}_{1\mathrm{B}}(\mathcal{B}^{\mathrm{an}}/S^{\mathrm{an}}) & \xrightarrow{.\otimes_{\mathbb{Z}} 1_{\mathbb{C}}} & \mathcal{H}_{1\mathrm{B}}(\mathcal{B}^{\mathrm{an}}/S^{\mathrm{an}}) \otimes_{\mathbb{Z}} \mathbb{C} & \xrightarrow{Id_{\mathcal{H}_{1\mathrm{B}}} \otimes_{\mathbb{Z}} \mathbf{e}} & \mathcal{H}_{1\mathrm{B}}(\mathcal{B}^{\mathrm{an}}/S^{\mathrm{an}}) \otimes_{\mathbb{Z}} \mathbb{G}_{m\mathbb{C}}^{\mathrm{an}} & \longrightarrow 0 \end{array} \tag{6.12}$$

(Recall that $\mathbf{e} := \exp(2\pi i \cdot)$.)

In (6.12), both lines are short exact sequences of commutative complex Lie groups over S^{an}, and the vertical arrows are isomorphisms. These isomorphisms are actually compatible with the D-structures in the analytic category: the connection on $\mathrm{Lie}_S\, \mathbf{E}(\mathcal{B})$ is the dual of the Gauss–Manin connection on $\mathcal{H}^1_{\mathrm{dR}}(\mathcal{B}/S)$ and is mapped by $J_{\mathcal{B}}$ to the connection on $\mathcal{H}_{1\mathrm{B}}(\mathcal{B}^{\mathrm{an}}/S^{\mathrm{an}}) \otimes_{\mathbb{Z}} \mathbb{C}$ which makes horizontal the sections of the local system $\mathcal{H}_{1\mathrm{B}}(\mathcal{B}^{\mathrm{an}}/S^{\mathrm{an}})$; the local analytic trivializations of $E(\mathcal{B})^{\mathrm{an}}$ induced by the D-structure of $\mathbf{E}(\mathcal{B})$ become, under the isomorphism $J_{\mathcal{B}}^{\times}$, the local trivializations of $\mathcal{H}_{1\mathrm{B}}(\mathcal{B}^{\mathrm{an}}/S^{\mathrm{an}}) \otimes_{\mathbb{Z}} \mathbb{G}_{m\mathbb{C}}^{\mathrm{an}}$ induced by local trivializations of $\mathcal{H}_{1\mathrm{B}}(\mathcal{B}^{\mathrm{an}}/S^{\mathrm{an}})$.

6.5 Extensions of abelian schemes by $\mathbb{G}_m$ and D-group schemes The construction of the algebraic groups $L^{\times}$ and $E(L^{\times})$ attached to some line bundle L algebraically equivalent to zero on some abelian variety A discussed in Section 5.5 extends to a relative situation.

Consider for instance an abelian scheme $\mathcal{B}$ over S as in Section 6.4. If $\mathcal{L}$ is a line bundle over $\mathcal{B}$, equipped with a rigidification along the zero section of $\mathcal{B}$,

$$\epsilon : \mathcal{O}_S \xrightarrow{\sim} 0_{\mathcal{B}}^{*}\mathcal{L},$$

and algebraically equivalent to zero on the fibers of $\mathcal{B}$—in other words, if $(\mathcal{L}, \epsilon)$ defines a section $\mathcal{P}$ over S over the dual abelian scheme $\widehat{\mathcal{B}}$—then the $\mathbb{G}_m$-torsor $\pi_{\mathcal{L}} : \mathcal{L}^{\times} \longrightarrow \mathcal{B}$, deduced from the total space of $\mathcal{L}$ by deleting its zero section, admits a unique structure of a commutative group scheme over S which makes the diagram

$$0 \longrightarrow \mathbb{G}_{mS} \xrightarrow{\epsilon} \mathcal{L}^{\times} \xrightarrow{\pi_{\mathcal{L}}} \mathcal{B} \longrightarrow 0 \tag{6.13}$$

an extension of smooth commutative group schemes over S. By pulling back this extension along the morphism

$$p_{\mathcal{B}} : E(\mathcal{B}) \longrightarrow \mathcal{B},$$

we define a smooth commutative group scheme

$$E(\mathcal{L}^{\times}) := \mathcal{L}^{\times} \times_{\mathcal{B}} E(\mathcal{B})$$

which fits into an short exact sequence of group schemes over S:

$$0 \longrightarrow \mathbb{G}_{mS} \xrightarrow{\epsilon'} E(\mathcal{L}^{\times}) \xrightarrow{\tilde{\pi}_{\mathcal{L}}} E(\mathcal{B}) \longrightarrow 0. \tag{6.14}$$

In the remainder of the article we shall use that $E(\mathcal{L}^{\times})$ may be canonically equipped with a D-structure, so that it becomes a commutative D-group scheme $\mathbf{E}(\mathcal{L}^{\times})$ over S and the extension of commutative group schemes (6.14) becomes an extension of commutative D-group schemes:

$$0 \longrightarrow \mathbf{G}_{mS} \xrightarrow{\epsilon'} \mathbf{E}(\mathcal{L}^{\times}) \xrightarrow{\tilde{\pi}_{\mathcal{L}}} \mathbf{E}(\mathcal{B}) \longrightarrow 0. \tag{6.15}$$

This construction is alluded to in Brylinski [19, (2.2.2.1)], and appears in a "differential algebraic context" in [9, Lemma 3.4(i)–(ii)] and in a "geometric context" in Andreatta and Barbieri-Viale [2] (see also Andreatta and Bertapelle [3]). The construction of the D-structure on $\mathbf{E}(\mathcal{L}^\times)$ and of the extension (6.15) may be understood as follows in terms of moduli spaces of vector bundles with integrable connections.

The construction of the relative moduli spaces $\mathbf{MIC}_N(M/S, o)$ and of their D-structure discussed in Section 6.3 directly extends to the moduli spaces $\mathbf{MIC}_N(M/S, o, o')$ of vector bundles equipped with a relative integrable connection rigidified along two sections o and o' of M over S. Besides, as explained in Section 6.4, $\mathbf{E}(\mathcal{B})$ may be identified with the D-group scheme $\mathbf{MIC}_1(\widehat{\mathcal{B}}/S, 0_{\widehat{\mathcal{B}}})$. The discussion of Section 2.3.5 may be extended to the relative case and allows one to identify $E(\mathcal{L}^\times)$ with $\mathbf{MIC}_1(\widehat{\mathcal{B}}/S, 0_{\widehat{\mathcal{B}}}, \mathcal{P})$, in a way compatible with their respective structure of $\mathbb{G}_m$-torsors over $E(\mathcal{B})$ and $\mathbf{MIC}_1(\widehat{\mathcal{B}}/S, 0_{\widehat{\mathcal{B}}})$. The canonical D-structure on $E(\mathcal{L}^\times)$ is the D-structure deduced from the one on $\mathbf{MIC}_1(\widehat{\mathcal{B}}/S, 0_{\widehat{\mathcal{B}}}, \mathcal{P})$ through this identification.

7 A Conjecture

In this final part, we consider the following geometric data: a smooth projective connected curve C over $\overline{\mathbb{Q}}$, and an abelian scheme over C, $\pi : \mathcal{A} \longrightarrow C$.

As before we denote $E(\mathcal{A})$ the universal vector extension of this abelian scheme. It is a smooth connected commutative group scheme over C, endowed with a canonical structure of a D-group scheme. If necessary, we shall use the notation $\mathbf{E}(\mathcal{A})$ to denote $E(\mathcal{A})$ considered as a D-group scheme over C, to distinguish it from the "plain" group scheme $E(\mathcal{A})$ over C.

As usual, we denote $\widehat{\mathcal{A}}$ the abelian scheme over C dual to $\mathcal{A}$.

We shall make the following simplifying assumption:

$$\textit{The vector bundle } \mathbb{E}_{\mathcal{A}} := (\mathrm{Lie}_C\, \mathcal{A})^\vee \textit{ is ample.} \tag{7.1}$$

Recall that, in general, $\mathbb{E}_{\mathcal{A}}$ is only semipositive. Condition (7.1) implies the vanishing of the $(\overline{\overline{\mathbb{Q}}(C)}/\overline{\mathbb{Q}})$-trace of the geometric generic fiber $\mathcal{A}_{\overline{\overline{\mathbb{Q}}(C)}}$ of $\mathcal{A}$ and shall ensure that the extensions of formal D-groups (7.6) and local systems (7.7) considered below have no nontrivial automorphisms (hence have their middle term defined, up to unique isomorphism, by their extension class).

7.1 A construction Suppose that we are given the following datum:

(i) *a section $\mathcal{P}$ over C of the dual abelian scheme $\mathcal{A}$.*

By the very definition of $\mathcal{A}$, it defines

(ii) *a line bundle $\mathcal{L}$ over $\mathcal{A}$, equipped with a rigidification $\epsilon : \mathcal{O}_C \xrightarrow{\sim} 0_{\mathcal{A}}^*\mathcal{L}$ along the zero section, algebraically equivalent to zero on the fiber of $\pi : \mathcal{A} \longrightarrow C$.*

As recalled above, the $\mathbb{G}_m$-torsor $\mathcal{L}^\times$ over $\mathcal{A}$ defines in a unique way

(iii) *an extension of smooth commutative group schemes over C,*

$$0 \longrightarrow \mathbb{G}_{m,S} \xrightarrow{\epsilon} \mathcal{L}^\times \longrightarrow \mathcal{A} \longrightarrow 0.$$

Finally, through the construction described in Section 6.5, we obtain

(iv) *an extension of commutative D-group scheme over C,*

$$0 \longrightarrow \mathbf{G}_{m\,S} \longrightarrow \mathbf{E}(\mathcal{L}^{\times}) \longrightarrow \mathbf{E}(\mathcal{A}) \longrightarrow 0.$$

These successive operations are easily seen to establish a bijective correspondence between the four kinds of data (i)–(iv) above, and to be additive.

Lemma 7.1 *The above construction defines isomorphisms of $\mathbb{Z}$-modules*

$$\widehat{\mathcal{A}}(C) \xrightarrow{\sim} \mathrm{Ext}^1_{\text{c-gp}/C}(\mathcal{A}, \mathbb{G}_{mS}) \xrightarrow{\sim} \mathrm{Ext}^1_{cD\text{-gp}}\big(\mathbf{E}(\mathcal{A}), \mathbf{G}_{m\,S}\big).$$

This would actually hold in the general situation considered in Section 6.5, without any further assumption on the base scheme S.

7.2 Lie^1_C and $\mathrm{Per}^1_{C^{\mathrm{an}}_{\mathbb{C}}}$ Recall that the dual of the module with integrable connection $\mathrm{Lie}_C\,\mathbf{E}(\mathcal{A})$ over C may be identified with the relative de Rham cohomology of $\mathcal{A}$ over C equipped with the Gauss–Manin connection $(\mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/C), \nabla_{GM})$, and the local system of periods $\mathrm{Per}_{C_{\mathbb{C}}}\,E(\mathcal{A})_{\mathbb{C}}$ over $C^{\mathrm{an}}_{\mathbb{C}}$ with the local system defined by the relative Betti first homology of $\mathcal{A}^{\mathrm{an}}_{\mathbb{C}}$ over $\mathbb{C}^{\mathrm{an}}_{\mathbb{C}}$, which we denote $\mathcal{H}^1_{\mathrm{B}}(\mathcal{A}^{\mathrm{an}}_{\mathbb{C}}/C^{\mathrm{an}}_{\mathbb{C}})$.

Besides, the module with integrable connection $\mathrm{Lie}_C\,\mathbf{G}_{m,C}$ over C may be identified with the trivial module with integrable connection $(\mathcal{O}_C, d)$, and the local system of periods $\mathrm{Per}_{C_{\mathbb{C}}}\,\mathbb{G}_{m,C_{\mathbb{C}}}$ over $C^{\mathrm{an}}_{\mathbb{C}}$ with the constant local system $\mathbb{Z}_{C^{\mathrm{an}}_{\mathbb{C}}}$.

Consequently the maps Lie^1_S and Per^1_S defined on extension classes of commutative D-group schemes in Sections 6.1.3 and 6.2 take here the following form:

$$\mathrm{Lie}^1_C : \mathrm{Ext}^1_{cD\text{-gp}/C}\big(\mathbf{E}(\mathcal{A}), \mathbf{G}_{m\,S}\big) \longrightarrow H^1_{\mathrm{dR}}\big(C, \big(\mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/C), \nabla_{GM}\big)\big)$$

and

$$\mathrm{Per}^1_{C^{\mathrm{an}}_{\mathbb{C}}} : \mathrm{Ext}^1_{cD\text{-gp}/C_{\mathbb{C}}}\big(\mathbf{E}(\mathcal{A})_{\mathbb{C}}, \mathbf{G}_{m\,S_{\mathbb{C}}}\big) \longrightarrow H^1\big(C^{\mathrm{an}}_{\mathbb{C}}, \mathcal{H}^1_{\mathrm{B}}(\mathcal{A}^{\mathrm{an}}_{\mathbb{C}}/C^{\mathrm{an}}_{\mathbb{C}})\big).$$

Observe that, after tensoring with $\mathbb{C}$, the range spaces of these two maps become canonically isomorphic. Indeed we have "elementary" isomorphisms defined by the base change from $\overline{\mathbb{Q}}$ to $\mathbb{C}$,

$$H^1_{\mathrm{dR}}\big(C, \big(\mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/C), \nabla_{GM}\big)\big) \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \xrightarrow{\sim} H^1_{\mathrm{dR}}\big(C_{\mathbb{C}}, \big(\mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}_{\mathbb{C}}/C_{\mathbb{C}}), \nabla_{GM}\big)\big) \quad (7.2)$$

and by extension of coefficients from $\mathbb{Z}$ to $\mathbb{C}$,

$$H^1\big(C^{\mathrm{an}}_{\mathbb{C}}, \mathcal{H}^1_{\mathrm{B}}(\mathcal{A}^{\mathrm{an}}_{\mathbb{C}}/C^{\mathrm{an}}_{\mathbb{C}})\big) \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\sim} H^1\big(C^{\mathrm{an}}_{\mathbb{C}}, \mathcal{H}^1_{\mathrm{B}}(\mathcal{A}^{\mathrm{an}}_{\mathbb{C}}/C^{\mathrm{an}}_{\mathbb{C}})_{\mathbb{C}}\big), \quad (7.3)$$

and the complex vector spaces in the right-hand sides of (7.2) and (7.3) may be identified by means of the comparison isomorphisms between Betti and algebraic de Rham cohomology (with coefficients) discussed in Section 2.2.5.

If $\mathcal{E}$ is an element of $\mathrm{Ext}^1_{cD\text{-gp}}(\mathbf{E}(\mathcal{A}), \mathbf{G}_{m\,S})$, we shall denote $\mathcal{E}_{\mathbb{C}}$ its "complexification" in the group $\mathrm{Ext}^1_{cD\text{-gp}}(\mathbf{E}(\mathcal{A})_{\mathbb{C}}, \mathbf{G}_{m\,S_{\mathbb{C}}})$ (in the sense of Section 6.1.5).

The following lemma is proved in the same way as Lemma 2.1, which compared the first Chern classes in de Rham and Betti cohomology (see also the discussion in Section 7.3.3 infra).

Lemma 7.2 *For any extension class $\mathcal{E}$ in $\mathrm{Ext}^1_{cD\text{-gp}}(\mathbf{E}(\mathcal{A}), \mathbf{G}_{m\,C})$, the equality*

$$(\mathrm{Lie}^1_C\,\mathcal{E}) \otimes_{\overline{\mathbb{Q}}} 1_{\mathbb{C}} = 2\pi i\,(\mathrm{Per}^1_{C^{\mathrm{an}}_{\mathbb{C}}}\,\mathcal{E}_{\mathbb{C}}) \otimes_{\mathbb{Z}} 1_{\mathbb{C}} \quad (7.4)$$

holds in

$$H^1_{\mathrm{dR}}\big(C, \big(\mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/C), \nabla_{GM}\big)\big) \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \simeq H^1\big(C^{\mathrm{an}}_{\mathbb{C}}, \mathcal{H}^1_{\mathrm{B}}(\mathcal{A}^{\mathrm{an}}_{\mathbb{C}}/C^{\mathrm{an}}_{\mathbb{C}})\big) \otimes_{\mathbb{Z}} \mathbb{C}.$$

7.3 A conjecture

7.3.1 We finally arrive at the formulation of the conjecture which constitutes the aim of this article.

Conjecture 7.3 *Any pair of classes of extensions* (α, β) *with* α *in* $H^1_{\mathrm{dR}}(C, (\mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/C), \nabla_{GM}))$ *and* β *in* $H^1(C^{\mathrm{an}}_{\mathbb{C}}, \mathcal{H}^1_{\mathrm{B}}(\mathcal{A}^{\mathrm{an}}_{\mathbb{C}}/C^{\mathrm{an}}_{\mathbb{C}}))$ *which satisfies the compatibility relation*

$$\alpha \otimes_{\overline{\mathbb{Q}}} 1_{\mathbb{C}} = 2\pi i \beta \otimes_{\mathbb{Z}} 1_{\mathbb{C}} \tag{7.5}$$

in

$$H^1_{\mathrm{dR}}\big(C, (\mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/C), \nabla_{GM})\big) \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \simeq H^1\big(C^{\mathrm{an}}_{\mathbb{C}}, \mathcal{H}^1_{\mathrm{B}}(\mathcal{A}^{\mathrm{an}}_{\mathbb{C}}/C^{\mathrm{an}}_{\mathbb{C}})\big) \otimes_{\mathbb{Z}} \mathbb{C}$$

is of the form $(\mathrm{Lie}_S\,\mathcal{E}, \mathrm{Per}^1_{C^{\mathrm{an}}_{\mathbb{C}}}\,\mathcal{E}_{\mathbb{C}})$ *for some class* $\mathcal{E}$ *in* $\mathrm{Ext}^1_{cD\text{-gp}}(\mathbf{E}(\mathcal{A}), \mathbf{G}_{mC})$ *and hence is obtained from some section* $\mathcal{P}$ *of the dual abelian scheme* $\widehat{\mathcal{A}}$ *over* C.

The class $\mathcal{E}$ and the section $\mathcal{P}$, if they exist, are uniquely determined by these conditions.

By using the Leray–Serre spectral sequence to analyze the group $H^2_{\mathrm{Gr}}(\mathcal{A})$ attached to $\mathcal{A}$ (seen as a smooth projective variety over $\overline{\mathbb{Q}}$) by means of the fibering $\pi : \mathcal{A} \longrightarrow C$ and by using a relative generalization (over C) of Theorem 5.1, we may prove the following.

Proposition 7.4 *With the above notation, Conjecture 7.3 holds if and only if the smooth projective variety* $\mathcal{A}$ *over* $\overline{\mathbb{Q}}$ *satisfies* $\mathrm{GPC}^1(\mathcal{A})$.

7.3.2 Consider $f : S \longrightarrow C$ a smooth projective connected surface S over $\overline{\mathbb{Q}}$ fibered over C. Assume for simplicity that f is a smooth morphism (all fibers of f are therefore smooth projective curve) and admits a section o. Then we may introduce the relative Jacobian

$$\mathcal{J} := \mathrm{Jac}(S/C)$$

of S over C. It is an abelian scheme over C. Using the section o, we may define a relative Jacobian embedding

$$j_o : S \hookrightarrow \mathcal{J}.$$

(It is a closed embedding, over S, which maps o to the zero section $0_{\mathcal{J}}$ of $\mathcal{J}$ over C.) Pulling back by j_o establishes a bijection between line bundles $\mathcal{L}$ over $\mathcal{J}$ defining as above sections over C of the dual abelian schemes $\widehat{\mathcal{J}}$[26] and line bundles $\mathcal{M}$ over S, rigidified along o and of degree zero on the fibers of f.

With this notation, we have the following variant[27] of Proposition 7.4.

Proposition 7.5 *The validity of* $\mathrm{GPC}^1(S)$ *is equivalent to the validity of Conjecture 7.3 for* $\mathcal{A} = \mathcal{J}$.

Conjecture 7.3 may be extended to possibly degenerating families of abelian varieties over C (say, with semiabelian bad fibers). This generalized version may be applied to the relative Jacobian of any smooth projective surface fibered over C (say, with semistable fibers) and would imply the validity of GPC^1 for any smooth projective surface and, actually, for any smooth projective variety over $\overline{\mathbb{Q}}$. This approach to GPC^1 through fibrations of surfaces over curves and associated families of Jacobian varieties is very much in the spirit of the classical works of Picard, Poincaré, and Lefschetz which constituted our starting point in Section 1.2.

7.3.3 To avoid technicalities, I prefer not to discuss this in detail and would instead stress the fact that Conjecture 7.3 may be rephrased as an algebraization criterion concerning formal line bundles, satisfying suitable "differential algebraic" and "analytic" conditions, in the spirit of Theorems 4.1 and 4.2 à la Schneider–Lang, as expected in Section 4.4.

Indeed, consider a pair of classes (α, β) as in Conjecture 7.3.

The class α lies in

$$H^1_{\mathrm{dR}}\big(C, (\mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/C), \nabla_{GM})\big) \simeq \mathrm{Ext}^1_{\mathrm{mic}/C}\big(\mathrm{Lie}_C\, \mathbf{E}(\mathcal{A}), \mathrm{Lie}_C\, \mathbf{G}_{m\,S}\big)$$

and defines an extension of vector bundles with (integrable) connections over C, defined over $\overline{\mathbb{Q}}$:

$$0 \longrightarrow \mathrm{Lie}_C\, \mathbf{G}_{m\,C} \longrightarrow (M, \nabla) \longrightarrow \mathrm{Lie}_C\, \mathbf{E}(\mathcal{A}) \longrightarrow 0.$$

It may be interpreted as an extension of "formal commutative D-group schemes over C":

$$0 \longrightarrow \widehat{\mathbf{G}_{m\,C}} \longrightarrow \mathbf{G}_{\mathrm{for}} \longrightarrow \widehat{\mathbf{E}(\mathcal{A})} \longrightarrow 0, \tag{7.6}$$

where $\widehat{\mathbf{G}_{m\,C}}$ (resp., $\widehat{\mathbf{E}(\mathcal{A})}$) denotes the completion of the D-group scheme $\mathbf{G}_{m\,C}$ (resp., $\mathbf{E}(\mathcal{A})$) over C along its unit (resp., zero) section. (Here we use that the base field $\overline{\mathbb{Q}}$ has characteristic zero, so that we have formal exponential maps at our disposal.)

Observe that, by forgetting the D-structure, from (7.6) we deduce an extension of formal groups over C,

$$0 \longrightarrow \widehat{\mathbb{G}_{m\,C}} \longrightarrow G_{\mathrm{for}} \longrightarrow \widehat{E(\mathcal{A})} \longrightarrow 0,$$

which in turn defines a $\mathbb{G}_m$-torsor or, equivalently, a line bundle $\mathcal{N}_{\mathrm{for}}$, on the formal completion $\widehat{E(\mathcal{A})}$.

The class β lies in

$$H^1\big(C^{\mathrm{an}}_{\mathbb{C}}, \mathcal{H}^1_{\mathrm{B}}(\mathcal{A}^{\mathrm{an}}_{\mathbb{C}}/C^{\mathrm{an}}_{\mathbb{C}})\big) \simeq \mathrm{Ext}^1_{\text{Ab-Sheaves}}(\mathrm{Per}_{C_{\mathbb{C}}}\, \mathcal{A}_{\mathbb{C}}, \mathbb{Z}_{C^{\mathrm{an}}_{\mathbb{C}}})$$

and defines an extension of local systems over free $\mathbb{Z}$-modules of finite rank over $C^{\mathrm{an}}_{\mathbb{C}}$:

$$0 \longrightarrow \mathbb{Z}_{C^{\mathrm{an}}_{\mathbb{C}}} \longrightarrow \Gamma \longrightarrow \mathrm{Per}_{C_{\mathbb{C}}}\, \mathcal{A}_{\mathbb{C}} \longrightarrow 0. \tag{7.7}$$

After tensoring with the multiplicative group $\mathbb{G}^{\mathrm{an}}_{m\mathbb{C}}$, we deduce from (7.7) an extension of "commutative D-complex Lie groups" over $C^{\mathrm{an}}_{\mathbb{C}}$:

$$0 \longrightarrow \mathbf{G}^{\mathrm{an}}_{m,C_{\mathbb{C}}} \longrightarrow \Gamma \otimes \mathbf{G}^{\mathrm{an}}_{m,C_{\mathbb{C}}} \longrightarrow \mathbf{E}(\mathcal{A})^{\mathrm{an}}_{\mathbb{C}} \longrightarrow 0. \tag{7.8}$$

This construction is easily seen to establish a one-to-one correspondence between extensions of local systems (7.7) and extension in the analytic category of $\mathbf{E}(\mathcal{A})^{\mathrm{an}}_{\mathbb{C}}$ by $\mathbf{G}^{\mathrm{an}}_{m,C_{\mathbb{C}}}$. When β is the image by $\mathrm{Per}^1_{C_{\mathbb{C}}}$ of some extension class $[\mathcal{E}_{\mathbb{C}}]$, the extension (7.8) is nothing but the analytification $\mathcal{E}^{\mathrm{an}}_{\mathbb{C}}$ of $\mathcal{E}_{\mathbb{C}}$.

Here again the extension (7.8) defines some analytic line bundle $\mathcal{N}^{\mathrm{an}}$ over $E(\mathcal{A})^{\mathrm{an}}_{\mathbb{C}}$, by forgetting the D-structure and part of the group structure on $\Gamma \otimes \mathbb{G}^{\mathrm{an}}_{m,C_{\mathbb{C}}}$.

The equality (7.5)

$$\alpha \otimes_{\overline{\mathbb{Q}}} 1_{\mathbb{C}} = 2\pi i \beta \otimes_{\mathbb{Z}} 1_{\mathbb{C}}$$

expresses the fact that the extension of "commutative formal analytic D-groups" over $C^{\mathrm{an}}_{\mathbb{C}}$ deduced from (7.8) by completion along the zero sections coincides with the analytification of the "commutative formal D-groups" over $C_{\mathbb{C}}$ deduced from (7.6) by extending the base field from $\overline{\mathbb{Q}}$ to $\mathbb{C}$.

Finally, Conjecture 7.3 may be rephrased as asserting the algebraicity of any pair $(\mathcal{N}^{\mathrm{for}}, \mathcal{N}^{\mathrm{an}})$, consisting of a formal line bundle $\mathcal{N}^{\mathrm{for}}$ on the formal completion $\widehat{E(\mathcal{A})}$ of $E(\mathcal{A})$ along its zero section and of some analytic line bundle $\mathcal{N}^{\mathrm{an}}$ over $E(\mathcal{A})^{\mathrm{an}}_{\mathbb{C}}$ such that the associated $\mathbb{G}_m$-torsors $\mathcal{N}^{\mathrm{for}\times}$ and $\mathcal{N}^{\mathrm{an}\times}$ may be endowed with suitably compatible structures of D-group schemes over C and $C^{\mathrm{an}}_{\mathbb{C}}$ (in the respective formal and analytic categories).

Notes

1. We refer the reader to Buium [21] and [25], Pillay [79], Bouscaren [18], and Marker [74] for more systematic presentations, surveys, and additional references.

2. That is, given on the Zariski-open set $U_\alpha \cap U_\beta$ by the quotient of two (nonvanishing over $U_\alpha \cap U_\beta$) homogeneous polynomials of the same degree on $\mathbb{C}^{N+1}$.

3. Conversely, to recover Kodaira and Spencer's version from Lefschetz and Hodge's, one needs to know that any topologically trivial analytic line bundle over X is algebraic: this follows from the algebraicity of the Albanese variety and of the Albanese morphism of X, and from the algebraicity of analytic line bundles over complex abelian varieties. But for the algebraicity of the Albanese morphism, itself a consequence of Chow's theorem (cf. 2.3.1 infra), these results are actually consequences of Hodge theory and of Lefschetz's work on complex abelian varieties.

4. Curiously enough, Siegel points out the relation of Chow's paper with Poincaré's article, but does not seem aware that Chow's theorem may be derived from (2.1).

5. In a more mundane vein, I would simply add that an especially negative assessment by Lefschetz of the approach of Kodaira and Spencer [61] turns out to be well documented (see, e.g., Kohn et al. [63, p. 21]).

6. The precise definition of the map $\alpha \mapsto \alpha^{\mathrm{an}}_{\mathbb{C}}$ actually involves the specific sign conventions used in homological algebra and sheaf cohomology. The "standard" convention used in Deligne [36] indeed introduces a minus sign in the above compatibility relation: $c_{1,\mathrm{dR}}(L)^{\mathrm{an}}_{C} = -2\pi i c_{1,\mathrm{top}}(L^{\mathrm{an}}_{\mathbb{C}})$. In the remainder of the article, we shall generally neglect these delicate problems of signs involved in various "canonical" isomorphisms and their compatibility—although the important sign issue encountered in Section 5.2 (see notably (5.4) and (5.6)) would plead for a more careful treatment, on the model of Berthelot, Breen, and Messing [7, Section V.1].

7. To "algebraize" an analytic connection ∇^{an} over E^{an} by means of the GAGA comparison theorem, identify (algebraic or analytic) connections with (algebraic or analytic) splittings of the Atiyah extension of E, $0 \to \Omega^1_M \otimes E \to J^1_M E \to E \to 0$, defined by the vector bundle $J^1_M E$ of 1-jets of E over M.

8. This occurrence of commutative algebraic groups over $\mathbb{C}$ that are analytically, but not algebraically, isomorphic was first pointed out by Conforto (see Conforto [34], [35] and Severi [94, Appendice]).

9. In other words, for every $t \in \mathbb{C}$, $f(t) = (F_0(t) : \cdots : F_N(t))$.

10. In other words, it is deduced by extension of scalars from K to $\mathbb{C}$ from a formal germ in the formal completion $\widehat{X}_{f(z)}$ of X at the K-rational point $f(z)$.

11. That is, when the “irregularity” $h^{1,0}(X) = h^{0,1}(X)$ of X is positive.

12. This conjecture is mentioned briefly in [46, note (10), p. 102] and with more details in Lang [68, Chapter IV, Historical Note]. We refer the reader to Andre [1, Section 7.5, Chapitre 23], for a “modern” presentation and for variants and generalizations.

13. Notably the original Grothendieck period conjecture for a given smooth projective variety X over $\overline{\mathbb{Q}}$ should imply the conjunction of conjectures $\mathrm{GPC}^k(X^n)$ for all positive integers k and n.

14. Where $\mathrm{Ext}^1_{\text{c-gp}/k}$ and $\mathrm{Ext}^1_{\mathcal{O}_A\text{-mod}}$ stand for “group of 1-extensions” in the category of commutative algebraic groups over k, and of sheaves of $\mathcal{O}_A$-modules, respectively.

15. Recall that $\sigma^{\geq 1}\Omega^\bullet_{\hat{A}/k}$ denotes the “stupid” truncation $0 \to \Omega^1_{\hat{A}/k} \to \Omega^2_{\hat{A}/k} \to \cdots$ of $\Omega^\bullet_{\hat{A}/k}$.

16. Both the above isomorphism $\iota_{E(A)}$ at the level of k-points and this infinitesimal version are special instances of a canonical isomorphism $\iota_{E(A)}$ of fpqc k-sheaves (cf. [75] and [17]).

17. Namely, the elements sent to their opposite by the automorphism of $\mathrm{Hom}_{\mathrm{dRB}}(\mathbb{Z}(0), H^1_{\mathrm{dRB}}(A) \otimes H^1_{\mathrm{dRB}}(A) \otimes \mathbb{Z}(1))$ defined by “switching” the two copies of $H^1_{\mathrm{dRB}}(A)$.

18. This section could be skipped at first reading. It has been included since Proposition 5.4 constitutes an application of the theorem of Schneider–Lang close in spirit to the ones in the previous section, and for comparison with Conjecture 7.3 infra.

19. Quoted in [57, Section 4.1]: *“Un cristal possède deux propriétés caractéristiques : la rigidité et la faculté de croître dans un voisinage approprié. Il y a des cristaux de toute espèce de substances : des cristaux de soude, de soufre, de modules, d’anneaux, de schémas relatifs, etc.”*

20. In other words, its sheaf of regular sections is closed under Lie bracket.

21. Such bases exist: simply write $k(S)$ as a finite-degree extension of $k(X_1, \ldots, X_s)$, and lift the standard basis $(\partial/\partial X_1, \ldots, \partial/\partial X_s)$.

22. As usual, by this we mean a short exact sequence of fppf sheaves over S. Since we work over a base field k of characteristic zero, this is equivalent to the following “geometric” condition, expressed in terms of some algebraic closure $\overline{k}$ of k: for any point $s \in S(\overline{k})$, the diagram

$$0 \longrightarrow G_{2s}(\overline{k}) \xrightarrow{i_s} G_s(\overline{k}) \xrightarrow{p_s} G_{1s}(\overline{k}) \longrightarrow 0,$$

is a short exact sequence of abelian groups.

23. By a "complex Lie group over a complex analytic manifold M," we mean a group object in the category of complex analytic manifolds "smooth" (in the "algebrogeometric" sense, that is, "submersive") over M.

24. In [98], this S-scheme is denoted $\mathbf{R}_{\mathrm{DR}}(M/S, o, N)$.

25. The content of Sections 6.4 and 6.5 is thoroughly discussed, with a slightly different perspective, in [9, Part 3 and Appendix], which constitutes the main reference for these two sections.

26. That is, line bundles rigidified along $\mathcal{J}$, and algebraically equivalent to zero in the fibers of $\mathcal{J}$ over C.

27. This variant is actually simpler than Proposition 7.4: its proof does not require Theorem 5.1 and its relative generalization.

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