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# The Lascar Group and the Strong Types of Hyperimaginaries

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**Abstract** This is an expository note on the Lascar group. We also study the Lascar group over hyperimaginaries and make some new observations on the strong types over those. In particular, we show that in a simple theory Ltp $\equiv$ stp in *real* context implies that for hyperimaginary context.

The Lascar group introduced by Lascar [8], and related subjects, have been studied by many authors (see Lascar and Pillay [9], Casanovas et al. [1], Kim [7], Hrushovski [6], Gismatullin and Newelski [3], and more). Notably in [9] and [1], a new look on the Lascar group is given, and using compact Lie group theory Lascar and Pillay proved that any bounded hyperimaginary is interdefinable with a sequence of finitary bounded hyperimaginaries. Good summaries on the Lascar group are written in Ziegler [11] and Wagner [10]. While this is another short expository note stating known results in [8], [9], [1], and [7], we supply a couple of new observations. We study the Lascar group in a slightly more general context, namely, over hyperimaginaries. The notion of strong types over hyperimaginaries is somewhat subtler even at the level of the definition (see Example 3.5). As a by-product we show that in a simple theory if $\mathrm{Ltp}(a/A) \equiv \mathrm{stp}(a/A)$ for real tuples $a$ and $A$, then the same holds for hyperimaginaries. A question remains whether this holds in any theory.

We work with an arbitrary complete theory $T$ in $\mathcal{L}$, and a fixed large saturated model $\mathcal{M} \models T$ of size $\bar{\kappa}$, as usual. We recall some definitions. Unless noted otherwise, a tuple can have an infinite size ($< \bar{\kappa}$). By a *hyperimaginary* we mean an equivalence class of a type-definable equivalence relation over $\emptyset$. So a hyperimaginary has the form $a/E = a_E$ where $a$ is a tuple from $\mathcal{M}$ and $E(x, y)$ is the $\emptyset$-type-definable equivalence relation on $\mathcal{M}^{|x|}$. We call $a_E$ an *$E$-hyperimaginary*. We say

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the hyperimaginary is *finitary* if $a$ is a finite tuple. In general, we put $|a_E| := |a|$. In the note *arity* means an arity of a *real* tuple.

From now on $a, b, c, \ldots, A, B, \ldots$ denote hyperimaginaries, but $M, N, \ldots$ denote elementary small submodels of $\mathcal{M}$. Clearly any tuple from $\mathcal{M}$ or $\mathcal{M}^{\mathrm{eq}}$ is also a hyperimaginary. We call such a tuple *real* or *imaginary*, respectively. Given a hyperimaginary $c$, $\mathrm{Aut}_c(\mathcal{M})$ denotes the set of all automorphisms of $\mathcal{M}$ fixing $c$ (i.e., fixing the equivalence class setwise). A relation is said to be *over* $c$ or $c$*-invariant* if it is $\mathrm{Aut}_c(\mathcal{M})$-invariant. A hyperimaginary $a$ is said to be *bounded over* $b$ (written $a \in \mathrm{bdd}(b)$) if $a$ has only boundedly many automorphic conjugates over $b$, that is, $|\{f(a) \mid f \in \mathrm{Aut}_b(\mathcal{M})\}| < \bar{\kappa}$. We say that $a$ is *bounded* if $a \in \mathrm{bdd}(\emptyset)$. Similarly, we say that $a$ is *definable* (resp., *algebraic*) over $b$ written $a \in \mathrm{dcl}(b)$ (resp., $a \in \mathrm{acl}(b)$) if $\{f(a) \mid f \in \mathrm{Aut}_b(\mathcal{M})\}$ is a singleton (resp., finite). Two hyperimaginaries $a, b$ are said to be *interdefinable* or *equivalent* if $a \in \mathrm{dcl}(b)$ and $b \in \mathrm{dcl}(a)$. We use nonstandard notation $a \subseteq b$ to denote $a \in \mathrm{dcl}(b)$. We put

$$\mathrm{acl}^{\mathrm{eq}}(b) = \big(\mathrm{acl}(b) \cap \mathcal{M}^{\mathrm{eq}}\big) \cup \{b\}.$$

Notice the difference between $\mathrm{acl}(b)$ and $\mathrm{acl}^{\mathrm{eq}}(b)$; both are somewhat newly introduced when $b$ is a hyperimaginary. In Corollary 3.4 we shall see that $\mathrm{acl}(b)$ and $\mathrm{acl}^{\mathrm{eq}}(b)$ are interdefinable. As is known, the type of $a$ over $b$, $\mathrm{tp}(a/b)$, makes sense, and $p \in S_E(b)$ means that $p$ is a type of some $E$-hyperimaginary over $b$. As usual $a \equiv_b c$ means $c \models \mathrm{tp}(a/b)$. Of course, when we say $a \equiv_b c$, both $a, c$ must be hyperimaginaries for a common $E$, so in general $E$(-complete) types are computed in $\mathcal{M}/E$. For two partial $E$-types $p(x), q(x)$ we write $q \equiv p$ if $p \models q$ and $q \models p$. For the additional introduction to hyperimaginaries the reader may see Ziegler [11].

## 1 Lascar Group

We restate definitions from [8], [9], and [7]. *Throughout we fix a hyperimaginary $A$.* Recall that $\mathrm{Autf}_A(\mathcal{M})$($=$ a subgroup of $\mathrm{Aut}_A(\mathcal{M})$ generated by $\{f \in \mathrm{Aut}_A(\mathcal{M}) \mid f \in \mathrm{Aut}_M(\mathcal{M})$ for some model $M \supseteq A\}$) is a normal subgroup of $\mathrm{Aut}_A(\mathcal{M})$. We recall a fact on Lascar (strong) types.

**Definition 1.1** Let $a, b$ be hyperimaginaries such that $\mathrm{tp}(a/A) = \mathrm{tp}(b/A)$. We define $d_A(a, b)$ to be the least natural number $n(\geq 1)$ such that there are sequences $\boldsymbol{I}_1, \boldsymbol{I}_2, \ldots, \boldsymbol{I}_n$ and hyperimaginaries $a = a_0, a_1, \ldots, a_n = b$ such that $a_{i-1}^\frown I_i$ and $a_i^\frown I_i$ are both $A$-indiscernible for each $1 \leq i \leq n$. (If there is no such $n < \omega$, then we write $d_A(a, b) = \infty$.)

The following is proved in [7] when $a, b, A$ are real. The same proof works for hyperimaginaries.

**Fact 1.2** *The following are equivalent ($a, b$ are $E$-hyperimaginaries):*

(1) $\mathrm{Ltp}(a/A) = \mathrm{Ltp}(b/A)$ *(or write $a \equiv^L_A b$); that is, there is $f \in \mathrm{Autf}_A(\mathcal{M})$ such that $f(a) = b$;*

(2) $d_A(a, b) < \omega$*;*

(3) $\models F(a, b)$ *for any $A$-invariant bounded equivalence relation $F$ coarser than $E$.*

*In conclusion, $x_E \equiv^L_A y_E$ is an invariant bounded equivalence relation over $A$ coarser than $E$ and is the finest among those.*

**Definition 1.3 (Lascar group)** We have $\mathrm{Gal}_{\mathrm{L}}(\mathcal{M}, A) := \mathrm{Aut}_A(\mathcal{M}) / \mathrm{Autf}_A(\mathcal{M})$.

In the rest of the paper, when there is no risk of confusion *we may omit* $A$ for notational convenience, so $A \subseteq M$ for any model mentioned below, and $\mathrm{Aut}(\mathcal{M})$, $\mathrm{Autf}(\mathcal{M})$ below indeed mean $\mathrm{Aut}_A(\mathcal{M})$, $\mathrm{Autf}_A(\mathcal{M})$, respectively. As was said in the introduction, Sections 1 and 2 form a summary of known results from [1], [9], and [11] when $A$ is real. Most of the arguments there go through even when $A$ is a hyperimaginary, and we will repeat some arguments for the sake of completion.

**Remark 1.4** We argue that the Lascar group depends only on $T$ and $A$, and we write $\mathrm{Gal_L}(T, A)$ for $\mathrm{Gal_L}(\mathcal{M}, A)$. We have $|\mathrm{Gal_L}(T, A)| \leq 2^{|T|+|A|}$.

(1) Let $M$ be a (small) elementary submodel of $\mathcal{M}$. For $f, g \in \mathrm{Aut}(\mathcal{M})$, if $f(M) \equiv_M g(M)$, then $f.\mathrm{Autf}(\mathcal{M}) = g.\mathrm{Autf}(\mathcal{M})$ in $\mathrm{Gal_L}(\mathcal{M})$. There is $h$ in $\mathrm{Autf}(\mathcal{M})$ which fixes $M$ pointwise and sends $f(\mathcal{M})$ to $g(\mathcal{M})$. Then since $s := f^{-1}.h^{-1}.g$ fixes $M$ too, $s \in \mathrm{Autf}(\mathcal{M})$. Thus the claim follows.

(2) In (1), we can choose $M$ having size $|T|+|A|$. Since there are at most $2^{|T|+|A|}$ many types over $M$, we have $|\mathrm{Gal_L}(\mathcal{M}, A)| \leq 2^{|T|+|A|}$.

(3) Let $\mathcal{M}' \succ \mathcal{M}$ be saturated, and let $|\mathcal{M}'| > |\mathcal{M}|$. There is a canonical isomorphism from $\mathrm{Gal_L}(\mathcal{M})$ to $\mathrm{Gal_L}(\mathcal{M}')$.

Let $f \in \mathrm{Aut}(\mathcal{M})$. Any two automorphisms of $\mathcal{M}'$ extending $f$ are in the same coset in $\mathrm{Gal_L}(\mathcal{M}')$. This induces a homomorphism $\alpha : \mathrm{Aut}(\mathcal{M}) \to \mathrm{Gal_L}(\mathcal{M}')$. We claim that $\ker(\alpha) = \mathrm{Autf}(\mathcal{M})$. If $f \in \mathrm{Autf}(\mathcal{M})$, then clearly any extension of $f$ in $\mathrm{Aut}(\mathcal{M}')$ is in $\mathrm{Autf}(\mathcal{M}')$. Conversely, let $f' \in \mathrm{Autf}(\mathcal{M}')$ extend $f \in \mathrm{Aut}(\mathcal{M})$. Then for a small model $M \prec \mathcal{M}$, by Fact 1.2, $M \equiv^L f'(M) = f(M)$ in $\mathcal{M}$. Hence there is $g \in \mathrm{Autf}(\mathcal{M})$ such that $f(M) = g(M)$. Then $g^{-1}.f \in \mathrm{Autf}(\mathcal{M})$ since it fixes $M$, so $f \in \mathrm{Autf}(\mathcal{M})$, too.

Let us also write $\alpha : \mathrm{Gal_L}(\mathcal{M}) \to \mathrm{Gal_L}(\mathcal{M}')$ for the induced injection. We claim that $\alpha$ is surjective. Let $g \in \mathrm{Aut}(\mathcal{M}')$. For a small model $M \prec \mathcal{M}$, there is $M' \prec \mathcal{M}$ such that $g(M) \equiv_M M'$. Now there is $f \in \mathrm{Aut}(\mathcal{M})$ sending $M$ to $M'$. Then by (1), $\alpha(f) = g.\mathrm{Autf}(\mathcal{M}')$.

In the rest of this section, for $f, g \in \mathrm{Aut}(\mathcal{M})$, we write $f \approx g$ if $f.\mathrm{Autf}(\mathcal{M}) = g.\mathrm{Autf}(\mathcal{M})$.

We shall endow $\mathrm{Gal_L}(T)$ with a quotient topology to make it a compact (*but not necessarily Hausdorff*) topological group. Let $\pi : \mathrm{Aut}(\mathcal{M}) \to \mathrm{Gal_L}(= \mathrm{Gal_L}(T, A))$ be the canonical projection. Fix a model $M \prec \mathcal{M}$, and let

$$S_M(M) := \{\mathrm{tp}(f(M)/M) \mid f \in \mathrm{Aut}(\mathcal{M})\},$$

equipped with its Stone topology. Then by Remark 1.4(1), $\pi$ factors through the surjection $\mu : \mathrm{Aut}(\mathcal{M}) \to S_M(M)$ sending $f$ to $\mathrm{tp}(f(M)/M)$; that is, there is a canonical surjection $\nu = \nu_M : S_M(M) \to \mathrm{Gal_L}$ such that $\pi = \nu.\mu$. We use these maps. We give $\mathrm{Gal_L}$ the quotient topology under the map $\nu$. This topology is independent from the choice of $M$. It suffices to show this for a model $N(\prec \mathcal{M})$, an elementary extension of $M$ (since any two models have a common extension). Now the map $\nu_N : S_N(N) \to \mathrm{Gal_L}$ factors through the restriction map $S_N(N) \to S_M(M)$ sending $\mathrm{tp}(f(N)/N)$ to $\mathrm{tp}(f(M)/M)$. Since the restriction map is continuous and both $S_N(N)$, $S_M(M)$ are compact Hausdorff, $\nu_M$, $\nu_N$ induce the same quotient topology.

Lascar originally introduced the topology on $\mathrm{Gal_L}$ in terms of ultrafilters. It is known that his and the quotient topology coincide. Contrary to what the reader might expect, the proof that $\mathrm{Gal_L}$ is a topological group is quite subtle. The only known complete and correct proof can be found in [11], and the proof goes through when working over some hyperimaginary $A$.

**Proposition 1.5** $\mathrm{Gal_L}(T, A)$ *is a compact topological group.*

## 2 Quotient Groups of the Lascar Group

We introduce two canonical subgroups of $\mathrm{Gal_L}(T, A)$. Note that $\overline{\{\mathrm{id}\}}$ is a closed normal subgroup of $\mathrm{Gal_L}(T, A)$. The connected component of $\mathrm{Gal_L}(T, A)$ containing id is denoted by $\mathrm{Gal}^0_{\mathrm{L}}(T, A)$; it is also closed and normal.

**Definition 2.1** We have the following:

(1) $\mathrm{Autf_{KP}}(\mathcal{M}, A) := \pi^{-1}(\overline{\{\mathrm{id}\}})$.
(2) $\mathrm{Autf_S}(T, A) := \pi^{-1}(\mathrm{Gal}^0_{\mathrm{L}}(T, A))$.
(3) $\mathrm{Gal_{KP}}(T, A) := \mathrm{Gal_L}(T, A)/\overline{\{\mathrm{id}\}} = \mathrm{Aut}_A(\mathcal{M})/\,\mathrm{Autf_{KP}}(\mathcal{M}, A)$.
(4) $\mathrm{Gal_S}(T, A) := \mathrm{Gal_L}(T, A)/\,\mathrm{Gal}^0_{\mathrm{L}}(T, A) = \mathrm{Aut}_A(\mathcal{M})/\,\mathrm{Autf_S}(\mathcal{M}, A)$.
(5) We say that $T$ is *G-compact* over $A$ if $\mathrm{Gal_L}(T, A)$ is Hausdorff.

KP stands for Kim–Pillay, and S stands for Shelah or "strong." Again, below we may omit the subscript $A$.

**Remark 2.2**

(1) Recall that for topological groups $H \lhd G$, the quotient group $G/H$ is Hausdorff if and only if $H$ is closed in $G$. Hence both $\mathrm{Gal_{KP}}(T)$ and $\mathrm{Gal_S}(T)$ are compact Hausdorff. Moreover, $\mathrm{Gal_S}(T)$ is totally disconnected, so it is a profinite group.

Now $T$ is G-compact if and only if $\{\mathrm{id}\}$ is closed if and only if $\mathrm{Autf}(\mathcal{M}) = \mathrm{Autf_{KP}}(\mathcal{M})$ (then $\mathrm{Gal_{KP}}(T)$ and $\mathrm{Gal_L}(T)$ are canonically isomorphic).

(2) For the rest of this section we endow $\mathrm{Aut}(\mathcal{M})$ with a topology having basic open sets of the form $\{f \in \mathrm{Aut}(\mathcal{M}) \mid f(a) = b\}$ for some real $n$-tuples $a, b \in \mathcal{M}$. (However, the topologies of $\mathrm{Gal_S}, \mathrm{Gal_{KP}}$ are always the quotient topologies obtained from $\mathrm{Gal_L}$.)

(3) Let $\Gamma$ be a subgroup of $\mathrm{Aut}_A(\mathcal{M})$. Fix $F$ an $\emptyset$-type-definable equivalence relation on $\mathcal{M}^\alpha$ ($\alpha$ an arity). We write $\boldsymbol{E}^F_\Gamma$ to denote an equivalence relation such that for $F$-hyperimaginaries $c, d$, we have $\boldsymbol{E}^F_\Gamma(c, d)$ if and only if $d = f(c)$ for some $f \in \Gamma$. So $\boldsymbol{E}^F_\Gamma$ is an equivalence relation on $\mathcal{M}^\alpha/F$ or, equivalently, an equivalence relation on $\mathcal{M}^\alpha$ *coarser* than $F$. When we write $\boldsymbol{E}^=_\Gamma(x, y)$, $=$ means the finest *real* equivalence relation $x = y$. We omit $F$ if $F$ is clear from context. Note that if $\Gamma$ is normal, then $\boldsymbol{E}_\Gamma$ is $A$-invariant, but in general it *need not* be. When $\Gamma = \mathrm{Autf}(\mathcal{M}, A)$ we know that $\boldsymbol{E}_\Gamma$ is $\equiv^L_A$, and $\boldsymbol{E}_{\mathrm{Autf_{KP}}(\mathcal{M},A)}$ is denoted by $\equiv^{KP}_A$.

Notice that $\Gamma$ is closed if and only if $\Gamma = \overline{\Gamma} = \{f \in \mathrm{Aut}_A(\mathcal{M}) \mid$ for each finite real tuple $a \in \mathcal{M}$, $\boldsymbol{E}^=_\Gamma(a, f(a))$, that is, $f$ stabilizes all $\boldsymbol{E}^=_\Gamma$-classes of finite arities$\}$.

**Lemma 2.3**

*(1) Let $H'$ be a closed normal subgroup of* $\mathrm{Gal_L}$*, and let $H = \pi^{-1}(H')$. Then given $F$, $\boldsymbol{E}^F_H$ is a type-definable bounded equivalence relation (over $A$).*
*(2) The restriction map $\pi$ is continuous, so both* $\mathrm{Autf_S}(\mathcal{M})$ *and* $\mathrm{Autf_{KP}}(\mathcal{M})$ *are closed in* $\mathrm{Aut}(\mathcal{M})$.

**Proof** (1) Let $a = u_F \models p(x) \in S_F(A)$ where $u$ is a real tuple of arity $\alpha$. Fix a model $(u \subseteq) M = M_p \prec \mathcal{M}$. Since $H'$ is closed, there is $\Phi(x', M)$ over $M$ type-defining $\nu^{-1}(H')$; that is, $f \in H$ if and only if $\Phi(f(M), M)$ holds. Note that $H$ being a group implies that $\Phi(x', y')$ type-defines an equivalence relation on $\mathrm{tp}(M)$. Moreover, $x' \equiv^L_A M \to \Phi(x', M)$. Thus $\Phi(x', y')$ is a bounded equivalence relation on $\mathrm{tp}(M/A)$. Note also that since $H$ is normal it easily follows that $\Phi(x', y') \equiv \boldsymbol{E}^{=}_H(x', y')$ on $\mathrm{tp}(M/A)$. Then by taking existential quantifiers to $\Phi(x', y')$, clearly $E^{=}_H(x, y)$ is type-definable on $q(x) = \mathrm{tp}(u/A)$ too. Hence $E^F_H(x, y)$ is type-defined by

$$\Psi_p(x, y) \equiv \exists zw\big(E^{=}_H(z, w) \wedge q(z) \wedge q(w) \wedge F(z, x) \wedge F(w, y)\big)$$

on $\mathrm{tp}(a/A) = p(x)$. We put

$$\Psi'_p(x, y) \equiv \big(\Psi_p(x, y) \wedge p(x) \wedge p(y)\big) \vee x \equiv_A y.$$

Therefore, clearly $\boldsymbol{E}^F_H(x, y) \leftrightarrow \bigwedge\{\Psi'_p \mid p(x) \in S_F(A)\}$.

(2) Let $f \in \mathrm{Aut}(\mathcal{M})$, and let $U$ be an open subset of $\mathrm{Gal_L}(T)$ containing $\pi(f) = f.\mathrm{Autf}(\mathcal{M})$. Since $\nu$ is continuous, $\nu^{-1}(U) \subseteq S_M(M)$ contains a basic open neighborhood

$$V_{\varphi(x)} = \big\{\mathrm{tp}\big(g(M)/M\big) \ni \varphi(x) : g \in \mathrm{Aut}(\mathcal{M})\big\}$$

of $\mu(f) = \mathrm{tp}(f(M)/M)$, where $\varphi(x)$ is some formula over $M$. Let $a \in M$ be a finite tuple corresponding to $x$. Then we have simply

$$\mu^{-1}(V_{\varphi(x)}) = \big\{g \in \mathrm{Aut}(\mathcal{M}) : g(a) \models \varphi(x)\big\}.$$

Since $f \in \mu^{-1}(V_{\varphi(x)})$, in particular $\models \varphi(f(a))$ holds. Therefore, a basic open neighborhood $\{h \in \mathrm{Aut}(\mathcal{M}) \mid h(a) = f(a)\}$ of $f$ is contained in $\mu^{-1}(V_{\varphi(x)})$. Hence $\pi$ is continuous. □

Now fix a closed normal subgroup $H' \lhd \mathrm{Gal_L}$, and let $H = \pi^{-1}(H')$. We write $x \equiv^H y$ if $\boldsymbol{E}^{=}_H(x, y)$ holds, which on any arity, due to Lemma 2.3(1), is a bounded type-definable equivalence relation (over $A$).

**Proposition 2.4**

*(1) We have $H = \{f \in \mathrm{Aut}(\mathcal{M}) \mid f$ stabilizes all the $\equiv^H$-classes of any arities$\} = \{f \in \mathrm{Aut}(\mathcal{M}) \mid f$ stabilizes all the $\equiv^H$-classes of finite real arities$\}$.*

*(2) The following are equivalent:*

*(a) $c_0 \equiv^H c_1$ for real tuples;*

*(b) $c'_0 \equiv^H c'_1$ for each corresponding finite subtuple $c'_i$ of $c_i$ $(i = 0, 1)$.*

**Proof** (1) Due to Lemma 2.3(2), $H$ is closed in $\mathrm{Aut}(\mathcal{M})$. Hence it comes from Remark 2.2(3).

(2) Clearly (a) implies (b). Assume that (b) holds. In this proof all the tuples are real. Note that there is $h' \in H$ such that $h'(c'_0) = c'_1$. Hence for any finite $d_0$, there is $d_1 = h'(d_0)$ with $c'_0 d_0 \equiv^H c'_1 d_1$. Thus by compactness there is a sufficiently saturated $M_i$ containing $c_i$ such that for each finite corresponding $b_i \in M_i$, $b_0 \equiv^H b_1$ (*). In particular $\mathrm{tp}(M_0) = \mathrm{tp}(M_1)$, and there is $h \in \mathrm{Aut}(\mathcal{M})$ sending $M_0$ to $M_1$. Let $d/\equiv^H$ with arbitrary finite $d$ be given. We claim that $h$ fixes $d/\equiv^H$, so by (1), $h \in H$, and (a) follows. Due to the saturation of $M_0$ and the boundedness of $\equiv^H$, there is $d' \in M_0$ such that $\mathrm{tp}(d) = \mathrm{tp}(d')$ and $d \equiv^H d'$ holds. Then by (*), $h$ clearly fixes $d'/\equiv^H = d/\equiv^H$. □

In [9], using compact Lie group theory, Lascar and Pillay showed that any bounded hyperimaginary $e$ is equivalent to a sequence of finitary bounded hyperimaginaries. As a corollary of Proposition 2.4, the result can be directly obtained when $\mathrm{Aut}_e(\mathcal{M})$ is a normal subgroup of $\mathrm{Aut}(\mathcal{M})$. (This is observed by Casanovas and Potier [2].)

**Corollary 2.5** *Let $e = c_F$ with real $c$ be a hyperimaginary bounded over $A$. Assume additionally that $\mathrm{Aut}_{eA}(\mathcal{M}) \lhd \mathrm{Aut}_A(\mathcal{M})$. Then there are finitary hyperimaginaries $e_i$ $(i \in I)$ such that $e$ and $(e_i \mid i \in I)$ are interdefinable over $A$.*

**Proof** In the proof we again omit $A$. Let $p(x) = \mathrm{tp}(e)$. We may reset $F$ as $(p(x) \wedge p(y) \wedge F(x, y)) \vee x \equiv y$, so that $F$ as a whole is a type-definable bounded equivalence relation (over $A$). Choose a model $M$ containing $c$. Note that $F$ type-defines a bounded equivalence relation on $\mathcal{M}^{|M|}$ too. Moreover, $\mathrm{Aut}_e(\mathcal{M}) = \{f \in \mathrm{Aut}(\mathcal{M}) :\models F(f(c), c)\}$. Hence $\pi(\mathrm{Aut}_e(\mathcal{M}))$ is a closed subgroup of $\mathrm{Gal_L}$. By the assumption it is normal as well. Hence our corollary follows from Proposition 2.4(2). □

**Corollary 2.6**

*(1) For real $x, y$, $x \equiv^{KP}_A y$ is the finest bounded type-definable equivalence relation over A. Precisely, for real $u_i$ $(i = 0, 1)$ the following are equivalent:*
  *(a) $u_0 \equiv^{KP}_A u_1$ holds;*
  *(b) $u_0 \equiv_A u_1$, and $E'(u_0, u_1)$ holds for any $\emptyset$-type-definable equivalence relation $E'$ such that $u_0/E' \in \mathrm{bdd}(A)$;*
  *(c) $u'_0 \equiv^{KP}_A u'_1$ for each corresponding finite subtuple $u'_i$ of $u_i$.*

*(2)* $\mathrm{Autf_{KP}}(\mathcal{M}, A)$
  $= \{f \in \mathrm{Aut}(\mathcal{M}, A) \mid f$ stabilizes all the bounded $A$-type-definable equivalence classes$\}$
  $= \{f \in \mathrm{Aut}(\mathcal{M}, A) \mid f$ stabilizes all the bounded $A$-type-definable equivalence classes of finite arities$\}$.

*(3) The following are equivalent:*
  *(a) $a \equiv^{KP}_A b$ where $a = u_F, b = v_F$ with real $u, v$;*
  *(b) $F^{KP}_A(u, v)$ holds where*

$$F^{KP}_A(x, y) \equiv \exists z z'\big(z \equiv^{KP}_A z' \wedge F(z, x) \wedge F(z', y)\big) \equiv \exists z\big(F(z, y) \wedge z \equiv^{KP}_A x\big)$$

  *($z \equiv^s_A x$ is of course equality of KP-types over $A$ of* real *tuples); $F^{KP}_A(x, y)$ is a bounded type-definable equivalence relation over $A$ coarser than $F$ and is the finest among those;*
  *(c) $a \equiv_{\mathrm{bdd}(A)} b$.*

**Proof** (1)(a)⇔(b) This comes from the argument in the proof of Lemma 2.3(1). Note that for a type $\Psi(x, M)$ type-defining $\nu^{-1}(\overline{\{\mathrm{id}\}})$, $\Psi(x, y)$ must be the finest bounded type-definable equivalence relation on $\mathrm{tp}(M/A)$.

The equivalence of (1)(c) to others and (2) are due to Proposition 2.4.

(3)(a)⇔(b) Again this is from the argument in the proof of Lemma 2.3(1). Note that if $u' \equiv^{KP}_A v'$ and $F(u, u'), F(v, v')$ with real $u', v'$ so that there is $f \in \mathrm{Autf_{KP}}(\mathcal{M}, A)$ with $v' = f(u')$, then $f(u) \equiv^{KP}_A u$ and $F(f(u), v)$ holds. Hence one can easily verify that $\Psi_p$ there with $H = \mathrm{Autf_{KP}}(\mathcal{M}, A)$ is equivalent to $F^{KP}_A$ above on $p(x) = \mathrm{tp}(a/A)$. Note also that $F^{KP}_A$ is coarser than both $\equiv^s$ and $F$. It remains to show that $F^{KP}_A$ is the finest one as stated. There clearly is a finest one;

let it be $F'$. If $F_A^{KP}(u', v')$ holds, then there is $u''$ such that $F(u', u'') \wedge u'' \equiv_A^{KP} v'$. Hence $F'(u', u'')$ and $F'(u'', v')$; so $F'(u', v')$ must hold.

(c)⇒(b) This is obvious. Note only that due to (1)(b), $u/\equiv_A^{KP}$ is equivalent over $A$ to a *hyperimaginary*.

(b)⇒(c) Assume (b). Let $a_0 \in \mathrm{bdd}(A)$. By compactness it suffices to show $c_0 \equiv_{Aa_0} c_1$. Suppose that $\{a_i \mid i \in I\}$ is the set of all $A$-conjugates of $a_0$. Write $p(x, a_0) = p(x_F, a_0) = \mathrm{tp}(c_0/Aa_0)$; and $p(x, a_i)$ is the conjugate of $p(x, a_0)$ over $A$. Now there is a maximal subset $J \subseteq I$ containing zero such that $\{p(x, a_i) \mid i \in J\}$ is realized by $c_0$. Put $a_J = \langle a_i \mid i \in J\rangle$, and let $p'(xz) = \mathrm{tp}(c_0 a_J/A); q(z) = \mathrm{tp}(a_J/A)$. Consider

$$\bar{E}(x, y) \equiv \exists z\big(p'(x, z) \wedge p'(y, z) \wedge q(z)\big) \vee x_F \equiv_A y_F.$$

Due to the maximality of $J$, $\bar{E}$ is an $A$-type-definable equivalence relation, bounded, and coarser than $F$. Thus by (b), $\bar{E}(c_0, c_1)$ holds. Note that $c_0 \models \bigwedge_{i\in J} p(x, a_i)$, so $c_1 \models p(x, a_0)$, and $c_0 \equiv_{Aa_0} c_1$, as desired. □

**Proposition 2.7** *The following are equivalent.*

*(1) $T$ is $G$-compact over $A$.*

*(2) $\mathrm{Autf}(\mathcal{M}, A)$ is closed in $\mathrm{Aut}_A(\mathcal{M})$, and for each finite arity, $x \equiv_A^L y$ is type-definable over $A$.*

*(3) For any arity, $x \equiv_A^L y$ if and only if $x \equiv_A^{KP} y$.*

*(4) For any arity, $x \equiv_A^L y$ is type-definable.*

*(5) For any $F$, $x_F \equiv_A^L y_F$ if and only if $x_F \equiv_A^{KP} y_F$.*

**Proof** (1)⇒(2),(3) By Lemma 2.3 and Corollary 2.6.

(2)⇒(1) Let $f \in \mathrm{Autf}_{\mathrm{KP}}(\mathcal{M})$. Then by (2) and Corollary 2.6, $f$ fixes all $\equiv^L$-classes of finite arities. Then since $\mathrm{Autf}(\mathcal{M})$ is closed, from Remark 2.2(3), $f \in \mathrm{Autf}(\mathcal{M})$.

(3)⇒(1) Let $f \in \mathrm{Autf}_{\mathrm{KP}}(\mathcal{M})$. Due to (3), for a model $M$, we have $M \equiv^L f(M)$, and there is $g \in \mathrm{Autf}(\mathcal{M})$ such that $f(M) = g(M)$. Thus by Remark 1.4(1), $f \in \mathrm{Autf}(\mathcal{M})$.

(3)⇔(4), (1)⇒(5)⇒(3) This is clear. □

As is well known, any simple theory $T$ is $G$-compact over an arbitrary hyperimaginary.

## 3 Strong Types of Hyperimaginaries

Now we talk about strong types in the hyperimaginary context, which is somewhat more subtle, even the definition, than in the real case. Recall that a *finite* equivalence relation means an equivalence relation having finitely many classes.

**Definition 3.1** We say that two hyperimaginaries $a, b$ have the same *strong* or *Shelah type over* $A$, written $a \equiv_A^s b$ or $\mathrm{stp}(a/A) = \mathrm{stp}(b/A)$, if $\boldsymbol{E}_{\mathrm{Autf_S}(\mathcal{M},A)}(a, b)$ holds.

**Proposition 3.2**

*(1) $\mathrm{Gal}_{\mathrm{L}}^0(T, A)$ is the intersection of all closed (normal) subgroups of $\mathrm{Gal}_{\mathrm{L}}(T, A)$ having finite index.*

*(2) $\mathrm{Autf_S}(\mathcal{M}, A) = \{f \in \mathrm{Aut}_A(\mathcal{M}) \mid f$ stabilizes all strong types over $A$ of finite arities$\}$.*

**Proof** (1) We recall Remark 2.2(1). Clearly $\mathrm{Gal}^0_{\mathrm{L}}$ is contained in any closed subgroup of $\mathrm{Gal}_{\mathrm{L}}$ of finite index. Moreover, $\mathrm{Gal}_{\mathrm{S}}$ is a profinite group. In a profinite group, the identity is the intersection of all normal closed subgroups of finite index. Hence (1) follows.

Then (2) follows from Remark 2.2(3) and Lemma 2.3(2). □

Due to Lemma 2.3, $x \equiv^s_A y$ is bounded type-definable over $A$. We have a more precise collection of formulas type-defining it.

**Proposition 3.3** *Let $u, v$ be real tuples. The following are equivalent:*

*(1) $u \equiv_A v$; and for any $\emptyset$-definable equivalence relation $E$, if $u/E \in \mathrm{acl}(A)$, then $E(u, v)$ holds;*
*(2) $u \equiv^s_A v$;*
*(3) for any $A$-type-definable equivalence relation $E$, if $u/E$ has finitely many conjugates over $A$, then $E(u, v)$ holds;*
*(4) $u \equiv_{\mathrm{acl}(A)} v$;*
*(5) $u \equiv_{\mathrm{acl}^{\mathrm{eq}}(A)} v$;*
*(6) $u'_0 \equiv^s_A u'_1$ for each corresponding finite subtuple $u'_i$ of $c_i$ ($i = 0, 1$).*

**Proof** (1)⇒(2) By Proposition 3.2(1), $\mathrm{Gal}^0_{\mathrm{L}}(T, A) = \bigcap_i H_i$ where $H_i$ is a closed normal subgroup of $\mathrm{Gal}_{\mathrm{L}}(T, A)$ having finite index, so $H_i$ is open as well. Due to the similar argument as in the proof of Lemma 2.3(1), there is a formula $F_i(x', y')$ over $\emptyset$ defining an equivalence relation on $\mathrm{tp}(u)$ and hence on $\mathcal{M}^{|u|}$ by compactness, such that $u/F_i \in \mathrm{acl}(A)$. Since $H_i$ is normal it again follows that on $p(x) = \mathrm{tp}(u/A)$, $F_i(x', y')$ defines $\boldsymbol{E}_{H_i}(x, y)$; in particular, $p(x) \models F_i(x', u') \leftrightarrow \boldsymbol{E}_{H_i}(x, u)$ (the corresponding finite $u' \subseteq u$). Therefore (1)⇒(2) follows.

(2)⇒(3) Let $E(x, y)$ type-define an equivalence relation over $A$ such that $e = u/E \in \mathrm{acl}(A)$. Then $\mathrm{Aut}_{eA}(\mathcal{M})$ is a subgroup of $\mathrm{Aut}_A(\mathcal{M})$ containing $\mathrm{Autf}_A(\mathcal{M})$. Now $E(x, u)$ clearly defines $\boldsymbol{E}_{\mathrm{Aut}_{eA}(\mathcal{M})}(x, u)$ on $\mathrm{tp}(u/A)$. Hence $\pi(\mathrm{Aut}_{eA}(\mathcal{M}))$ is closed (may not be normal), and it has finite index in $\mathrm{Gal}^0_L(T, A)$ as automorphisms in a coset send $e$ to a different $E$-class. Therefore $\mathrm{Gal}^0_L(T, A) \subseteq \pi(\mathrm{Aut}_{eA}(\mathcal{M}))$, and (2)⇒(3) follows.

(3)⇒(1) and (4)⇒(5)⇒(1) This is obvious.

(3)⇒(4) The proof is the same as that of Corollary 2.6(3)(b)⇒(c). This time the $J \subseteq I$ are finite, and $q$ is algebraic.

(1)⇔(6) This follows by compactness. □

**Corollary 3.4** $\mathrm{acl}(A)$ *and* $\mathrm{acl}^{\mathrm{eq}}(A)$ *are interdefinable.*

**Proof** Let $u/E \in \mathrm{acl}(A)$ ($u$ real). It suffices to show that $u/E$ is definable over $\mathrm{acl}^{\mathrm{eq}}(A)$. Due to Proposition 3.3, $u_E \in \mathrm{dcl}(u/\equiv^s)$ and $u/\equiv^s$ is definable over $\mathrm{acl}^{\mathrm{eq}}(A)$. □

**Example 3.5** We give a couple of examples related to Proposition 3.3. In (1) that $u \equiv_A v$ is essential. Let $\mathcal{L} = \{=\}$, and let $u \neq v$ be singletons in the infinite $\mathcal{M}$. By quantifier elimination, the second clause of (1) holds with $A = v$, but $u \not\equiv_v v$ much less $u \equiv^s_v v$.

Also, if $A$ is real as well, then $u \equiv^s_A v$ if and only if for any $A$-definable finite equivalence relation $E$, $E(u, v)$ holds. *This no longer holds in the hyperimaginary context.* Even the right-hand side need not imply $u \equiv_A v$. The difference comes from the fact that in real context any formula in $\mathrm{tp}(u/v)$ is $v$-invariant, but not in general if $v$ is a hyperimaginary. Consider a typical example where Ltp $\not\equiv$ stp, that is, a model $(C; \{U_n(x, y)\}_{0<n\in\omega})$ where $C$ is the unit circle, and $U_n(a, b)$ holds for $a, b \in C$ if and only if the length of the shorter arc from $a$ to $b$ is at most $n^{-1}$. Let $F$ be an equivalence relation on $C$ type-defined by $\{U_n(x, y) \mid 0 < n \in \omega\}$. Choose $u, v \in C$ such that $u_F \neq v_F$. Then for any $v_F$-invariant finite definable equivalence relation $E$ (there are almost no such except trivial one), $E(u, v)$ holds. But not even $u \equiv_{v_F} v$ holds.

The properties of strong types of hyperimaginaries follow in the same manner.

**Proposition 3.6** *Let $a = u_F, b = v_F$ with real $u, v$ be $F$-hyperimaginaries. The following are equivalent.*

*(1) We have $a \equiv^s_A b$.*
*(2) $F^s_A(u, v)$ holds, where $F^s_A(x, y)$ is a bounded type-definable equivalence relation over $A$ coarser than $F$ such that*

$$F^s_A(x, y) \equiv \exists z z'\big(z \equiv^s_A z' \wedge F(z, x) \wedge F(z', y)\big) \equiv \exists z\big(F(z, y) \wedge z \equiv^s_A x\big)$$

*($z \equiv^s_A x$ is equality of strong types over $A$ of* real *tuples).*
*(3) We have $a \equiv_{\mathrm{acl}(A)} b$.*

**Proof** (1)⇔(2) This is by the similar reason as in the proof of Corollary 2.6(3)(a)⇔(b).

(1)⇔(3) This comes from Proposition 3.3. □

**Proposition 3.7** *The following are equivalent:*

*(1)* $\mathrm{Autf}_{\mathrm{KP}}(\mathcal{M}, A) = \mathrm{Autf}_{\mathrm{S}}(\mathcal{M}, A)$*;*
*(2)* $\mathrm{Gal}_{\mathrm{KP}}(T, A)$ *is compact Hausdorff and totally disconnected;*
*(3) $\equiv^{KP}_A$ is equivalent to $\equiv^s_A$ for any arity;*
*(4) $\equiv^{KP}_A$ is equivalent to $\equiv^s_A$ for finite arities;*
*(5) $\equiv^{KP}_A$ is equivalent to $\equiv^s_A$ for any hyperimaginary variables;*
*(6)* $\mathrm{acl}(A)$ *and* $\mathrm{bdd}(A)$ *are interdefinable.*

In Example 3.5, $\mathrm{Autf}_{\mathrm{KP}} \neq \mathrm{Autf}_{\mathrm{S}}$. Some non-G-compact examples (so $\mathrm{Autf} \neq \mathrm{Autf}_{\mathrm{KP}}$) are constructed in [1]. In one example $\equiv^L$ is different from $\equiv^{KP}$ for a finite tuple, while they can be equal for all finite arities in another non-G-compact example. In [11], given any compact Hausdorff topological group $G$ a corresponding theory $T_G$ is constructed so that $\mathrm{Gal}_{\mathrm{L}}(T_G) = G$.

That "Ltp $\equiv$ stp in real (resp., hyperimaginary) context" means for any tuple $c$ and a set $A$ both real (resp., hyperimaginaries), it holds that $\mathrm{Ltp}(c/A) \equiv \mathrm{stp}(c/A)$.

**Proposition 3.8** *Assume that $T$ is simple. Then* Ltp $\equiv$ stp *in real context implies that in hyperimaginary context. (In particular, both low theories and supersimple theories have* Ltp $\equiv$ stp *in hyperimaginary context.)*

**Proof** Assume that Ltp $\equiv$ stp in real context. Let $u_E$ be a hyperimaginary, and let $v, v'$ be real. It suffices to show that $v \equiv^s_{u_E} v'$ implies $v \equiv^L_{u_E} v'$. Choose a model $M$ containing $u$. Now $E$ clearly type-defines an equivalence relation on $M$ as well, and $u/E$ and $M/E$ are interdefinable. Hence there is no harm in supposing that $u$ is some enumeration of the model. Now let a hyperimaginary $e := \mathrm{Cb}(u/u_E)$, and let $F(x, y)$ be $E(x, y) \wedge x \equiv^L y$. Then $u_F$ is hyperdefined by $x \equiv^L_{u_E} u$. As is known

$u_F = \mathrm{Cb}(u/u_E)$: since $F$ is finer than $E$, clearly $u_E \in \mathrm{dcl}(u_F)$. Hence $u \mathop{\smile\!\!\!\!\!\!\mid}_{u_F} u_E$. Also $u_F \in \mathrm{bdd}(u_E)$. Thus $\mathrm{tp}(u/u_F)$ is a Lascar type, and $e \in \mathrm{dcl}(u_F)$. Conversely, note now $u \mathop{\smile\!\!\!\!\!\!\mid}_e u_E$. Since $u_E \in \mathrm{dcl}(u)$, we have $u_E \in \mathrm{bdd}(e)$, so $u_F \in \mathrm{bdd}(e)$ too. Then $\mathrm{tp}(u/e) \equiv \mathrm{tp}(u/\mathrm{bdd}(e)) \models \mathrm{tp}(u/u_F) \models F(x,u)$. Therefore $u_F \in \mathrm{dcl}(e)$. So we can put $e = u_F$.

We claim that $v \equiv^s_e v'$ implies $v \equiv^L_e v'$. Suppose that $v \equiv^s_e v'$. We can clearly assume $u \mathop{\smile\!\!\!\!\!\!\mid}_e v$, Take $v''$ such that $v'' \equiv^L_e v'$ and $v'' \mathop{\smile\!\!\!\!\!\!\mid}_e v$. Now there is $u'$ such that $u'v'' \equiv^s_e uv$. Hence by type amalgamation there is $u'' \models \mathrm{tp}(u/ev) \cup \mathrm{tp}(u'/ev'')$. Then $u''v'' \equiv u''v$, and as $u''(\supseteq e)$ is a model, $v \equiv^L_e v''$ so $v \equiv^L_e v'$, as desired.

By the claim and Proposition 3.7, $\mathrm{bdd}(u_E) = \mathrm{bdd}(u_F) = \mathrm{acl}(u_F)$. Moreover, by the definition of $F$ and our assumption, $u_F$ is definable over $\mathrm{acl}(u_E)$. Hence $\mathrm{bdd}(u_E) = \mathrm{acl}(u_E)$, and by Proposition 3.7 again, Ltp $\equiv$ stp over $u_E$, as wanted. □

**Question** Is Proposition 3.8 true for all $T$?

We express our thanks for the following comment given by an anonymous referee: "Another approach to defining the strong type over a hyperimaginary would be to consider the classical theory as a theory in continuous logic with the discrete metric. In this case, there is no distinction between hyperimaginaries and imaginaries and one could rework material on the Lascar group in this context."

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