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Modular Ax–Lindemann–Weierstrass with Derivatives

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To Anand Pillay on his 60th birthday

Abstract In a recent paper I established an analogue of the Lindemann–Weierstrass part of Ax–Schanuel for the elliptic modular function. Here I extend this to include its first and second derivatives. A generalization is given that includes exponential and Weierstrass elliptic functions as well.

1 Introduction

Schanuel's conjecture (see Lang [9, p. 30]) captures the expected transcendence properties of values of the exponential function. It is open in general but encompasses various known results, such as the Lindemann–Weierstrass theorem: *If algebraic numbers* $a_1, \ldots, a_n$ *are linearly independent over* $\mathbb{Q}$*, then* $\exp a_1, \ldots, \exp a_n$ *are algebraically independent over* $\mathbb{Q}$. Schanuel also made the analogue of his conjecture in the differential field setting (see Ax [1]). This was subsequently proved by Ax in [1]. The result, known as "Ax–Schanuel," encompasses in particular a statement corresponding to the Lindemann–Weierstrass theorem: *Suppose that* W *is an irreducible algebraic variety over* $\mathbb{C}$*, with function field* $\mathbb{C}(W)$*. If* $a_1, \ldots, a_n \in \mathbb{C}(W)$ *are linearly independent over* $\mathbb{Q}$ *modulo constants* (which is to say that there is no nontrivial relation of the form $\sum_{i=1}^{n} q_i a_i = c, q_i \in \mathbb{Q}, c \in \mathbb{C}$), *then the functions* $\exp a_1, \ldots, \exp a_n$ *on* W *are algebraically independent over* $\mathbb{C}$ *(even over* $\mathbb{C}(W)$*).* This statement I call "Ax–Lindemann–Weierstrass" for the exponential function.

Let $j : \mathbb{H} \to \mathbb{C}$ be the modular function, where $\mathbb{H}$ is the complex upper half-plane. In the recent paper Pila [15], I proved a result concerning algebraic dependencies among the compositions of j with algebraic functions. It is an analogue for the j-function of the Ax–Lindemann–Weierstrass statement and asserts, roughly speaking, that all algebraic dependencies among nonconstant functions $j(a_1), \ldots, j(a_n)$, for

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$a_i \in \mathbb{C}(W)$, an algebraic function field, come from modular (i.e., $\mathrm{GL}_2^+(\mathbb{Q})$) relations among the functions a_i.

In this paper I extend this result to include j' and j'', where $'$ denotes differentiation with respect to $\tau \in \mathbb{H}$. It is well known (see, e.g., Bertrand and Zudilin [5]) that $j''' \in \mathbb{Q}(j, j', j'')$, while, by a result of Mahler [10] (or see [5]), the functions $j(\tau), j'(\tau), j''(\tau)$ on $\mathbb{H}$ are algebraically independent over $\mathbb{C}(\tau)$. The result of this paper says that, for compositions of these three functions with algebraic functions, all algebraic dependencies again come from modular relations.

To frame the result, consider an algebraic function field $\mathbb{C}(W)$, where $W \subset \mathbb{C}^m$ is some irreducible algebraic variety. If $a_1, \ldots, a_n \in \mathbb{C}(W)$ have the property that $a_\nu(P) \in \mathbb{H}$ for some $P \in W$, then the compositions $j(a_\nu)$ may be considered simultaneously as functions in some neighborhood of P on W, that is, on $W \cap B$ where B is some open ball containing P. In this situation, $a_1, \ldots, a_n \in \mathbb{C}(W)$ will be called *geodesically independent* if the a_ν are nonconstant and there are no relations of the form

$$a_\nu = g a_\mu$$

where $\nu \neq \mu$ and $g \in \mathrm{GL}_2^+(\mathbb{Q})$ acts by fractional linear transformations. (The $^+$ indicates positive determinant, the condition required for such a transformation to preserve $\mathbb{H}$.)

Theorem 1.1 *Suppose that $\mathbb{C}(W)$ is an algebraic function field and that*

$$a_1, \ldots, a_n \in \mathbb{C}(W)$$

take values in $\mathbb{H}$ at $P \in W$ and are geodesically independent. Then the $3n$-functions

$$j(a_1), \ldots, j(a_n), \qquad j'(a_1), \ldots, j'(a_n), \qquad j''(a_1), \ldots, j''(a_n)$$

(considered as functions on W locally near P) are algebraically independent over $\mathbb{C}(W)$.

Note that, if some a_ν is a constant (in $\mathbb{H}$), then $j(a_\nu)$ is algebraic over $\mathbb{C}$, while if some relation $a_\nu = g a_\mu$ holds where $\nu \neq \mu$ and $g \in \mathrm{GL}_2(\mathbb{Q})^+$, then $j(a_\nu)$ and $j(a_\mu)$ are related by a modular equation and so are algebraically dependent (over $\mathbb{Q}$). So the result is sharp.

This paper is devoted to proving Theorem 1.1, and a generalization which includes the exponential and Weierstrass $\wp$-functions, using an elaboration of the method used in [15]. That method uses o-minimality (see van den Dries [22] or the foundational papers by Pillay and Steinhorn [18], [19] and Knight, Pillay, and Steinhorn [8]) and employs in particular the counting theorem of Pila and Wilkie [16, Theorem 1.10] (as refined in [14, Theorem 3.5]), in contrast to the differential field methods of Ax in [1] (and to the differential geometric arguments of Ax in [2]). To apply this result requires that the restriction $j : F \to \mathbb{C}$ of j to the usual fundamental domain $F = \{z \in \mathbb{H} : |\mathrm{Re}(z)| \leq 1/2, |z| \geq 1\}$ for $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ be *definable in an o-minimal structure over* $\mathbb{R}$ (for the definition see [16]). The well-known q-expansion of $j(\tau)$ (where $q = \exp(2\pi i \tau)$; see, e.g., Serre [20]) shows that $j : F \to \mathbb{C}$ is definable in $\mathbb{R}_{\mathrm{an,exp}}$, a fact first observed by Peterzil and Starchenko in [12] as a consequence of their definability result for $\wp_\tau(z)$ as a function of both variables (for a generalization of Ax–Lindemann–Weierstrass in a different direction see Bertrand and Pillay [4], and for a generalization of Schanuel's conjecture encompassing the values of j, exp, and Weierstrass $\wp$-functions derived from the generalized Grothendieck conjecture on periods see Bertolin [3]).

2 Generalization and Reformulation

By using the functions $a_1, \dots, a_n$ to determine a mapping $W \to W' \subset \mathbb{C}^n$ we can always reduce to the case where $a_1, \dots, a_n$ are the coordinate functions on W. Theorem 1.1 is thus equivalent to the following version, in which $\tau_1, \dots, \tau_n$ are the coordinate functions on $\mathbb{C}^n$, $W \subset \mathbb{C}^n$ is an irreducible algebraic variety defined over $\mathbb{C}$ with $W \cap \mathbb{H}^n \neq \emptyset$, and

$$\overline{\tau_1}, \dots, \overline{\tau_n}$$

are the functions on W induced by $\tau_1, \dots, \tau_n$.

Theorem 2.1 *With the notation (and assumption $W \cap \mathbb{H}^n \neq \emptyset$) as above, suppose that $\overline{\tau_1}, \dots, \overline{\tau_n}$ are geodesically independent. Then the $3n$-functions*

$$j(\overline{\tau_1}), \dots, j(\overline{\tau_n}), \qquad j'(\overline{\tau_1}), \dots, j'(\overline{\tau_n}), \qquad j''(\overline{\tau_1}), \dots, j''(\overline{\tau_n})$$

(defined locally) on W are algebraically independent over $\mathbb{C}(W)$.

Definition 2.2 Let n, m, ℓ be nonnegative integers. Let $X = \mathbb{C}^n \times E_1 \times \cdots \times E_m \times \mathbb{G}^\ell$ where E_i are elliptic curves over $\mathbb{C}$ corresponding to lattices $\Lambda_i \subset \mathbb{C}$ with Weierstrass $\wp$-functions $\wp_i$. Let $\Lambda = \Lambda_1 \oplus \cdots \oplus \Lambda_m \subset \mathbb{C}^m$. Let $U = U_X = \mathbb{H}^n \times \mathbb{C}^m \times \mathbb{C}^\ell$. Let $W \subset \mathbb{C}^{n+m+\ell}$ be an irreducible algebraic variety having a nonempty intersection with U. Let

$$\tau_1, \dots, \tau_n, \qquad z_1, \dots, z_m, \qquad \zeta_1, \dots, \zeta_\ell$$

be the coordinate functions on $\mathbb{C}^{n+m+\ell}$, and let

$$\overline{\tau_1}, \dots, \overline{\tau_n}, \qquad \overline{z_1}, \dots, \overline{z_m}, \qquad \overline{\zeta_1}, \dots, \overline{\zeta_\ell}$$

be their images in $\mathbb{C}(W)$. A subset of the coordinate function, which for simplicity we take to be

$$\overline{\tau_1}, \dots, \overline{\tau_\nu}, \qquad \overline{z_1}, \dots, \overline{z_\mu}, \qquad \overline{\zeta_1}, \dots, \overline{\zeta_\lambda},$$

where $0 \leq \nu \leq n, 0 \leq \mu \leq m, 0 \leq \lambda \leq \ell$, will be called *geodesically independent* if all of the following conditions hold.

1. The functions $\overline{\tau_1}, \dots, \overline{\tau_\nu}$, are nonconstant, and there are no relations of the form $\overline{\tau_a} = g\overline{\tau_b}$ where $a \neq b$ and $g \in \mathrm{GL}_2(\mathbb{Q})^+$. If $\nu = 0$ we consider this condition to be satisfied.
2. The functions $\overline{z_1}, \dots, \overline{z_\mu}$ do not satisfy any system of $\mu - h$ linearly independent equations $\sum_{j=1}^{\mu} \alpha_{ij}\overline{z_j} = c_i, i = 1, \dots, \mu - h$, $h < \mu$, where the $\alpha_{ij}, c_i \in \mathbb{C}$ and the h-dimensional linear subspace L defined by $\sum_{j=1}^{\mu} \alpha_{ij} z_j = 0, i = 1, \dots, \mu - h$, contains $L \cap \Lambda$ as a lattice (i.e., of full rank $2h$). That is, the locus $(\overline{z_1}, \dots, \overline{z_\mu})$ is not contained in a coset of the tangent space of a proper subtorus of $\mathbb{C}^m/\Lambda$. If $\mu = 0$ we consider this condition to be satisfied.
3. The functions $\overline{\zeta_1}, \dots, \overline{\zeta_\lambda}$ are $\mathbb{Q}$-linearly independent modulo constants, that is, there do not exist $q_1, \dots, q_\lambda \in \mathbb{Q}$, not all zero, such that $\sum_{i=1}^{\lambda} q_i\overline{\zeta_i} \in \mathbb{C}$. If $\lambda = 0$ we consider this condition to be satisfied.

Theorem 2.3 *Let the notation (and assumption that $W \cap U \neq \emptyset$) be as above. If the functions*

$$\overline{\tau_1}, \dots, \overline{\tau_\nu}, \qquad \overline{z_1}, \dots, \overline{z_\mu}, \qquad \overline{\zeta_1}, \dots, \overline{\zeta_\lambda}$$

in $\mathbb{C}(W)$ *are geodesically independent, then the* $(3\nu+\mu+\lambda)$*-functions*

$$j(\overline{\tau_1}),\dots,j(\overline{\tau_\nu}),\qquad j'(\overline{\tau_1}),\dots,j'(\overline{\tau_\nu}),\qquad j''(\overline{\tau_1}),\dots,j''(\overline{\tau_\nu}),$$
$$\wp_1(\overline{z_1}),\dots,\wp_\mu(\overline{z_\mu}),\qquad \exp(\overline{\zeta_1}),\dots,\exp(\overline{\zeta_\lambda})$$

(defined locally) on $W\cap U$ *are algebraically independent over* $\mathbb{C}(W)$*.*

Without the j'- and j''-functions, this result was established in [15]. Taking W defined by $\zeta_1=\tau_1$ it establishes in particular the independence of exp from j, j', j'', a result of Mahler [10]. We show that Theorem 2.3 is equivalent to yet another formulation.

Definition 2.4 A *weakly special variety* $W\subset\mathbb{C}^n\times\mathbb{C}^m\times\mathbb{C}^\ell$ is a variety determined by some equations as in Definition 2.2; that is, each equation is of one of the following forms:

1. $\tau_i=c$ where $i\in\{1,\dots,n\}$ and $c\in\mathbb{C}$;
2. $\tau_i=g\tau_j$ where $i,j\in\{1,\dots,n\}$ are distinct and $g\in\mathrm{GL}_2^+(\mathbb{Q})$;
3. $\sum_{j=1}^m\alpha_{ij}z_j=c_i, i=1\dots,k$, where $\alpha_{ij},c_i\in\mathbb{C}$, and the linear space L defined by $\sum_{j=1}^m\alpha_{ij}z_j=0, i=1,\dots,k$, contains $L\cap\Lambda$ as a lattice;
4. $\sum_{k=1}^\ell q_k\zeta_k=c$ where $q_k\in\mathbb{Q}$ are not all zero and $c\in\mathbb{C}$.

A *maximal proper weakly special variety* is a weakly special variety that is not contained in any proper weakly special variety of larger dimension.

In the following, overline variables

$$\overline{\tau_1},\dots,\overline{\tau_\nu},\qquad \overline{z_1},\dots,\overline{z_\mu},\qquad \overline{\zeta_1},\dots,\overline{\zeta_\lambda}$$

continue to denote the functions on the algebraic variety W induced by the coordinates.

Theorem 2.5 *Suppose that* $W\subset\mathbb{C}^{n+m+\ell}$ *is an irreducible algebraic variety with* $W\cap\mathbb{H}^n\times\mathbb{C}^{m+\ell}\neq\emptyset$*,* Y *is a connected component of* $W\cap\mathbb{H}^n\times\mathbb{C}^{m+\ell}$*, the functions*

$$j(\overline{\tau_1}),\dots,j(\overline{\tau_\nu}),\qquad j'(\overline{\tau_1}),\dots,j'(\overline{\tau_\nu}),\qquad j''(\overline{\tau_1}),\dots,j''(\overline{\tau_\nu}),$$
$$\wp_1(\overline{z_1}),\dots,\wp_\mu(\overline{z_\mu}),\qquad \exp(\overline{\zeta_1}),\dots,\exp(\overline{\zeta_\lambda})$$

(defined locally) on Y *are algebraically dependent over* $\mathbb{C}(W)$*, and that* Y *is maximal with respect to these properties. Then* W *is a maximal proper weakly special variety.*

The proof of the equivalence of Theorems 2.3 and 2.5 is a variant of the argument giving the equivalence of Theorems 9.1 and 9.2 in [15].

Proof that Theorem 2.5 implies Theorem 2.3 We show that Theorem 2.5 implies the contrapositive of Theorem 2.3. Suppose that the functions in Theorem 2.3 are algebraically dependent over $\mathbb{C}(W)$. Then, by Theorem 2.5, $W\subset V$ for some maximal proper weakly special variety V. Since the dependence involves the indicated variables only, we may assume that the variety W is a cylinder on these variables; the other variables may be chosen arbitrarily. Then V is defined by equations involving these variables, and so those variables are not geodesically independent. □

Proof that Theorem 2.3 implies Theorem 2.5 Let W be maximal with the properties in the statement of Theorem 2.5. Take a maximal subset of the variables (which for simplicity will be assumed to be an initial segment of each of the sets of variables, as in Definition 2.2) such that the corresponding functions are algebraically independent over $\mathbb{C}(W)$. By Theorem 2.3, each other variable is geodesically dependent on these, so that W is contained in a proper weakly special variety V. By maximality, V is maximal proper weakly special, and $W = V$. □

3 Some Preliminaries

Notation 3.1 We introduce a group G whose real points $g \in G(\mathbb{R})$ act on U as biholomorphic self-maps. The group G is the Cartesian product of groups acting on each factor of U. On the factor corresponding to the variable τ_i, the group factor is $G_{\tau_i} = \mathrm{SL}_2$ acting by fractional linear transformations. On the factor corresponding to z_i with lattice Λ_i with $\mathbb{Z}$-basis λ_i, μ_i, the factor is $G_{z_i} = \mathbb{G}_{\mathrm{a}}^2$, with $t = (u, v) \in \mathbb{R}^2$ acting on $\mathbb{C}$ as translation by $u\lambda_i + v\mu_i$. On the factor corresponding to the variable ζ_i we put $G_{\zeta_i} = \mathbb{G}_a$ with $s \in \mathbb{R}$ acting on $\mathbb{C}$ as translation by $2\pi i s$. Elements of G will be denoted $g = (g_1, \dots, g_n, t_1, \dots, t_m, s_1, \dots, s_\ell)$.

Write $\mathbf{z} = (\tau_1, \dots, \tau_n, z_1, \dots, z_m, \zeta_1, \dots, \zeta_\ell)$, and write

$$\mathbf{\Pi} = (J_1, \dots, J_n, K_1, \dots, K_n, L_1, \dots, L_n, P_1, \dots, P_m, E_1, \dots, E_\ell).$$

The group G acts on the polynomial ring

$$\mathbb{C}(\mathbf{z})[\mathbf{\Pi}]$$

as follows. For $g = (g_1, \dots, g_n, t_1, \dots, t_m, s_1, \dots, s_\ell) \in G$ with $g_i = \left(\begin{smallmatrix} a_i & b_i \\ c_i & d_i \end{smallmatrix}\right)$, and

$$F = F(\mathbf{z}, \dots, J_a, \dots, K_b, \dots, L_c, \dots, P_j, \dots, E_k, \dots) \in \mathbb{C}(\mathbf{z})[\mathbf{\Pi}],$$

set

$$\begin{aligned} F_g = F\big(&g^{-1}\mathbf{z}, \dots, J_\alpha, \dots, (-c_\beta\tau_\beta + a_\beta)^2 K_\beta, \dots, \\ &(-c_\gamma\tau_\gamma + a_\gamma)^4 L_\gamma - 2c_\gamma(-c_\gamma\tau_\gamma + a_\gamma)^3 K_\gamma, \dots, P_j, \dots, E_k, \dots\big). \end{aligned}$$

The reader may verify that $F_{gh} = (F_g)_h$.

We may observe that the functions $j(\tau_i), \wp_j(z_j), \exp(\zeta_k)$ are invariant under $G(\mathbb{Z})$, while the functions $j'(\tau_i), j''(\tau_i)$ transform in a simple manner:

$$\begin{aligned} j'\Big(\frac{a\tau + b}{c\tau + d}\Big) &= (c\tau + d)^2 j'(\tau), \\ j''\Big(\frac{a\tau + b}{c\tau + d}\Big) &= (c\tau + d)^4 j''(\tau) + 2c(c\tau + d)^3 j'(\tau). \end{aligned}$$

The functions j, j', j'', exp are regular on their domains, while the $\wp$-functions are meromorphic, taking values in $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Thus on U we define

$$\pi : U \to \mathbb{C}^{3n} \times \hat{\mathbb{C}}^m \times \mathbb{C}^\ell$$

by

$$\begin{aligned} &\pi(\tau_1, \dots, \tau_n, z_1, \dots, z_m, \zeta_1, \dots, \zeta_\ell) \\ &\quad = \big(j(\tau_1), \dots, j(\tau_n), j'(\tau_1), \dots, j'(\tau_n), j''(\tau_1), \dots, j''(\tau_n), \\ &\qquad \wp_1(z_1), \dots, \wp_m(z_m), \exp(\zeta_1), \dots, \exp(\zeta_\ell)\big). \end{aligned}$$

Suppose that Y is a component of $W \cap U$ on which the component functions of π are algebraically dependent over $\mathbb{C}(W)$. So we have

$$F\big(\pi(Y)\big) = 0$$

for some nonzero F in the polynomial ring in $3n + m + \ell$ variables over $\mathbb{C}(W)$. Clearing denominators and representing elements of $\mathbb{C}[W]$ as the function on W induced by suitable elements of $\mathbb{C}[\mathbf{z}]$, we have a polynomial, which we also denote $F \in \mathbb{C}[\mathbf{z}][\mathbf{\Pi}]$, such that

$$F\big(\ldots, \tau_i, \ldots, z_j, \ldots, \zeta_k, \ldots, j(\tau_i), \ldots, j'(\tau_i), \ldots, \\ j''(\tau_i), \ldots, \wp_j(z_j), \ldots, \exp(\zeta_k), \ldots\big)$$

vanishes identically for points of Y. That F gives a nontrivial algebraic dependence of the functions on W now means that the coefficient polynomials in $\mathbb{C}[\mathbf{z}]$ do not all vanish identically on W: for then the algebraic independence of j, j', j'' shows that F does not vanish identically on $\pi(U)$.

Suppose that $g \in G(\mathbb{Z})$. Then gY is a component of $gW \cap U$, and the transformation rules for the component functions of π imply that

$$F_g\big(\pi(gY)\big) = 0.$$

Suppose that F has coefficients that do not vanish identically on W. We show that F_g has coefficients not vanishing identically on gW. (Note that the transformation factors $(-c_i\tau_i + a_i)$ cannot cause problems as they do not vanish identically on W.) Suppose that F has some terms involving the L_γ-variables with nonzero coefficients. Consider the terms of highest degree in the L_γ-variables. Then the F_g-terms of the same degree in the L_γ-variables have coefficients of the form $P(g^{-1}\mathbf{z})M$ where P is a coefficient of a similar term in F and M is a product of factors of the form $(-c_i\tau_i + a_i)$. If P is not identically vanishing on W, then $P(g^{-1}\mathbf{z})$ is not identically vanishing on gW, and we see that F_g gives a nontrivial relation of the component functions of π on gY. If F has no terms involving L_γ, the same argument applies to terms of highest degree in the K_β-variables. If there are no such terms either, then the coefficients of F_g are just of the form $P(g^{-1}\mathbf{z})$ where $P(\mathbf{z})$ are the terms of F, and these are not all vanishing on W.

4 Rational Points of Definable Sets

Fix an o-minimal expansion $\mathcal{R} = (R, 0, 1, +, -, \cdot, <, \ldots)$ of a real closed field R. By *definable* we will mean, in this section, definable (with parameters) in $\mathcal{R}$, while *semi-algebraic* will mean definable (with parameters) in $(R, 0, 1, +, -, \cdot, <)$. A *definable family* will mean a definable set $Z \subset R^n \times R^m$, considered as the family of sets $Z_x \subset R^n$, where $x \in R^m$.

The field R contains the field $\mathbb{Q}$ of standard rational numbers. Our interest is in rational points and, more generally, algebraic points of some bounded degree, counted according to their height, of (definable) subsets of R^n. For any set $Z \subset R^n$ and $T \geq 1$ we can consider

$$Z(\mathbb{Q}, T) = \big\{(z_1, \ldots, z_n) \in Z : z_i \in \mathbb{Q}, H(z_i,) \leq T, i = 1, \ldots, n\big\}$$

where the *height* $H(a/b) = \max(|a|, |b|)$ for a rational number a/b in lowest terms ($\gcd(a, b) = 1$). We then have the counting function $N(Z, T) = \#Z(\mathbb{Q}, T)$. More generally, for a positive integer k we consider

$$Z(k, T) = \big\{(z_1, \ldots, z_n) \in Z : \big[\mathbb{Q}(z_i) : \mathbb{Q}\big] \leq k, H(z_i) \leq T, i = 1, \ldots, n\big\}$$

where H is the *absolute multiplicative height* on $\overline{\mathbb{Q}}$ (which extends the above height on $\mathbb{Q}$; see Bombieri and Gubler [7, Definition 1.5.4]), and

$$N_k(Z,T) = \#Z(k,T).$$

Thus $Z(1,T) = Z(\mathbb{Q},T)$ and $N_1(Z,T) = N(Z,T)$.

Definition 4.1 **([15])**

1. A *block* (of dimension w and degree d) in $\mathbb{R}^n$ is a connected definable set $W \subset \mathbb{R}^n$ of dimension w, regular of dimension w at every point, such that there is a semialgebraic set $A \subset \mathbb{R}^n$, of dimension w and degree at most d, regular of dimension w at every point, with $W \subset A$.
2. A *block family* (of dimension w and degree d) is a definable family $W \subset \mathbb{R}^n \times \mathbb{R}^m$ such that every fiber $W_x, x \in \mathbb{R}^m$ is a block of dimension w and degree at most d.

Cells of dimension zero are permitted: a point is a block.

The following is a further refinement of the result established in [16]. Examining the proofs of the versions in Pila [14], [15], and the analytic ingredient Pila [13, Proposition 4.1] shows that the proof works in any o-minimal structure over a real closed field (not just over $\mathbb{R}$). It also holds with the *polynomial height* H_k^{poly} (see [15, Definition 3.3]) in place of H.

Theorem 4.2 *Let $Z \subset R^n \times R^m$ be a definable family of sets in R^n, and let $k \geq 1$ and $\epsilon > 0$. There exists a finite set J of block families $W \subset R^n \times R^m \times R^{\ell_W}$ (depending on Z, k, ϵ) in R^n such that*

1. *$W_{(x,y)} \subset Z_x$ for all $(x,y) \in R^m \times R^{\ell_W}$, all $W \in J$;*
2. *$Z_x(k,T)$ is contained in the union of at most $c(Z,k,\epsilon)T^\epsilon$ fibers $W_{(x,y)}$ of $W \in J$, for all $x \in R^m$, $T \geq 1$.*

Definition 4.3 The *algebraic part* of a set Z, denoted $\mathrm{Alg}(Z)$, is the union of all connected positive-dimensional semialgebraic sets contained in Z.

Since the positive-dimensional blocks provided by Theorem 4.2 are contained in the algebraic parts of the corresponding fibers of Z, the result implies that, given k, ϵ,

$$N_k\big(Z - \mathrm{Alg}(Z), T\big) \leq c(Z,k,\epsilon)T^\epsilon$$

for any definable set Z. However, Theorem 4.2 provides more information in the case where Z has "many" rational (or fixed-degree algebraic) points, meaning that $N_k(Z,T) \geq cT^\delta$ for some positive k, c, δ. Then the theorem implies that, for any $\delta' < \delta$ and positive c', for large T, Z contains a block B with $N_k(B,T) \geq c'T^{\delta'}$.

For the purposes of the present paper, in order to consider all the sets and functions involved as sets in real space we shall simply use the real and complex parts of all the complex variables.

For the remainder of the paper, *definable* will mean definable (with parameters) in $\mathbb{R}_{\text{an,exp}}$.

5 Proof of Theorem 2.5

The proof is an elaboration of the proof of [15, Theorem 6.8]. Where the argument is the same, the steps will merely be sketched; more detail will be given where the present argument varies from the argument given in [15].

Proof of Theorem 2.5 We have a component Y of $W \cap U$ on which the component functions of π are algebraically dependent over $\mathbb{C}(W)$. Thus we have $F(\pi(Y)) = 0$ for some $F \in \mathbb{C}[\mathbf{z}][\mathbf{\Pi}]$ as above, whose coefficients (in $\mathbb{C}[\mathbf{z}]$) do not all vanish on W, and Y is maximal with these properties. We may assume that no τ_i is constant on W or the conclusion is immediate.

Suppose that $g \in G(\mathbb{Z})$. As already observed, the components of π on gY are also dependent over $\mathbb{C}(gW)$, a dependence being given by F_g. Moreover, gY is also maximal: for if gY were contained in Y', a component of $W' \cap U$ for some W', then $gW \subset W'$ as gY is Zariski-dense, and if W' has larger dimension than W, then $g^{-1}W'$ strictly contains W, $Y \subset g^{-1}Y'$ is a component of $g^{-1}W' \cap U$, and we contradict the maximality of Y.

The idea of the proof is this. Observe that if $G' \subset G$ is definable, $Y' \subset Y$ is definable, and $F' \subset U$ on which π is definable, then

$$S(G', Y', F') = \left\{g \in G' : \dim(gY \cap F') = \dim(Y) \text{ and } F_g\big(\pi(gY)\big) = 0\right\}$$

is a definable set (the relation $F_g(\pi(gY)) = 0$ needs to be checked only locally on Y but then holds globally) to which we may apply Theorem 4.2. We will find such sets with "many" rational (indeed, integer) points. By Theorem 4.2, we get a positive-dimensional semialgebraic subset. This leads either to a larger set Y, contradicting its supposed maximality, or to identities for the algebraic functions parameterizing W. With enough such identities we show that W has the required form.

We may choose some subset of the variables (so that the induced functions are a transcendence basis for $\mathbb{C}(W)$) such that Y is parameterized by suitable algebraic functions of these variables. As in [15], we can make exchanges among variables that are dependent on each other so that certain types of dependencies are avoided. Specifically, "dependent" τ-variables depend only on "free" τ-variables (and not on any z- or ζ-variables); "dependent" z-variables depend on free τ- and free z-variables (but not on any ζ-variables); and dependent ζ-variables depend on free (τ, z, ζ)-variables.

Following such exchanges, Y is parameterized on the "free" variables

$$\tau_{f,i}, \qquad z_{f,j}, \qquad \zeta_{f,k}$$

by the algebraic functions

$$\tau_{f,a} = \varphi_a(\tau_{f,i}), \qquad z_{d,b} = \theta_{d,b}(\tau_{f,i}, z_{f,j}), \qquad \zeta_{d,c} = \psi_c(\tau_{f,i}, z_{f,j}, \zeta_{f,k}).$$

The parameterizing functions are defined in some connected open region for the free variables. They may be analytically continued, perhaps with some branching, throughout a region bounded by the loci determined by some τ-variable (either free or dependent) becoming real.

By further exchanges among τ-variables if necessary we may assume that U has a boundary where some free τ-variable, say, $\tau_{f,1}$, becomes real. Now the real loci of a dependent variable $\tau_{d,a}$ may coincide with the real locus of $\tau_{f,1}$; otherwise they will intersect in a lower-dimensional set (these loci are real semialgebraic).

We can thus find a small neighborhood $U_{f,1}$ in the $\tau_{f,1}$-plane, centered on its real locus, and neighborhoods $U_{f,i}, i \neq 1$, of the other free variables, away from their real loci, such that the product of that part of the $\tau_{f,1}$-neighborhood with the other neighborhoods is in U and at positive distance from the real loci of any dependent $\tau_{d,a}$-variables whose real loci do not coincide with that of $\tau_{f,1}$, and such that all the parameterizing algebraic functions are bounded and univalent in the product region. We may assume further that the regions $U_{f,i}$ for all the free variables other than $\tau_{f,1}$ are contained in a single fundamental domain for the respective actions (of $\mathrm{SL}_2(\mathbb{Z})$, Λ_i, $\mathbb{Z}$), while the $\tau_{f,1}$-region contains infinitely many fundamental domains, of which we fix one. We take a fundamental region for each dependent variable and let F^* be the product of these fundamental domains. Then π restricted to F^* is definable.

Now we generate various definable subsets of $G(\mathbb{R})$ containing "many" integer points. Let U^* be the product of $U_{f,i}$ over the free variables. Let Y^* be the graph of the parameterizing functions on U^*, a definable subset of Y. Choose relatively prime integers a, c such that a/c is in the real $\tau_{f,1}$-boundary of U^*. Choose $b, d \in \mathbb{Z}$ such that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. Let

$$G^* = \left\{ g \in G : g_{f,1} = \begin{pmatrix} a & b+ta \\ c & d+tc \end{pmatrix}, t \in \mathbb{R}, g_{f,i} = \mathrm{id}, i \neq 1, t_{f,j} = 0, s_{f,k} = 0 \right\}$$

with no restrictions on the group elements corresponding to "dependent" variables.

For large $t \in \mathbb{N}$, the domain $g_{f,1}F_{f,1} \subset U_1^*$, and the graph Y^* over $g_{f,1}F_{f,1}^* \times \prod U_{f,i}$ over all other free variables may be "brought back" (at least partially) into F^* by an element of $\prod G_i(\mathbb{Z})$ on the dependent variables of height bounded by a suitable polynomial in $|t|$. This is established in [15, Section 5]. Accordingly, the definable set

$$S(G^*, Y^*, F^*)$$

has "many" integer points up to height T, for all large T.

Therefore this set contains a positive-dimensional semialgebraic subset, with all points regular; moreover, according to Theorem 4.2, it contains such subsets which (for large T) contain "many" integer points.

We can find a smooth arc of a real algebraic curve $g(t)$ in G, parameterized by t in an open interval containing an integer point t_0, such that $g(t) \in S(G^*, Y^*, F^*)$. Then $g(t)Y$ is parameterized on

$$\tau_{f,i}, \qquad z_{f,j}, \qquad \zeta_{f,k}$$

by

$$\begin{aligned} \tau_{d,a} &= g_{d,a}(t)\varphi_a\big(g_{f,1}(t)^{-1}\tau_{f,1}, \tau_{f,i}, i \neq 1\big), \\ &t_{d,b}(t)\theta_b\big(g_{f,1}(t)^{-1}\tau_{f,1}, \tau_{f,i}, i \neq 1, z_{f,j}\big), \\ &s_{d,c}(t)\psi_c\big(g_{f,1}(t)^{-1}\tau_{f,1}, \tau_{f,i}, i \neq 1, z_{f,j}, \zeta_{f,k}\big), \end{aligned}$$

and we have

$$\begin{aligned} 0 = F\big(&g(t)^{-1}\mathbf{z}, \ldots, j(\tau_a), \ldots, \big(-c_b(t)\tau_b + a_b(t)\big)^2 j'(\tau_b), \ldots, \\ &\big(-c_\gamma(t)\tau_\gamma + a_\gamma(t)\big)^4 j''(\tau_\gamma) - 2c_\gamma(t)\big(-c_\gamma(t)\tau_\gamma + a_\gamma(t)\big)^3 j'(\tau_\gamma), \ldots, \\ &\wp_j(z_j), \ldots \exp(\zeta_k), \ldots\big) \end{aligned} \tag{$*$}$$

holding identically for the free variables in some neighborhood, for real t in an interval about t_0, and so (considering the neighborhood for $\tau_{f,1}$ to be at positive distance from its real line) for all t in some *complex* neighborhood of t_0 as well, and this provides a new "free" variable which we use to try to "enlarge" W.

Suppose that some dependent $\tau^* = \tau_{d,a}$ exists. Suppose that, considered as a function over $\mathbb{C}(\tau_{f,i})$, the function

$$g_{d,a}(t)\varphi_a\big(g_{f,1}(t)^{-1}\tau_{f,1}, \tau_{f,i}, i \neq 1\big)$$

is a nonconstant function of t. Then t is a nonconstant algebraic function of τ^* (over $\mathbb{C}(\tau_{f,i})$).

We now have an algebraic variety W' parameterized locally by

$$\tau_{f,i}, \qquad z_{f,j}, \qquad \zeta_{f,k}, \tau^*$$

on which the component functions of π are algebraically dependent over an *algebraic* extension of $\mathbb{C}(W')$, as witnessed by ($*$) above. The dependence ($*$) is nontrivial, as it restricts to a nontrivial dependence at $t = t_0$, say. But then these functions are algebraically dependent over $\mathbb{C}(W')$ by field theory. Now W' has higher dimension than W and intersects U in a component containing $g(t_0)Y$. This contradicts the maximality of $g(t_0)Y$, which follows from the assumed maximality of Y.

Therefore the functions

$$g_{d,a}(t)\varphi_a\big(g_{f,1}(t)^{-1}\tau_{f,1}, \tau_{f,i}, i \neq 1\big)$$

are constant. From here the argument is the same as in [15], and may be sketched as follows.

First consider some dependent variable $\tau_{d,a}$ which (for some choice of the $\tau_{f,i}, i \neq 1$) does not have its real locus coincident with $\tau_{f,1}$. Then we have chosen $U^*_{d,a}$ to be entirely contained in a single fundamental domain for $G_{d,a}(\mathbb{Z})$. We may therefore assume that $g_{d,a}(t) = 1$ identically. (Just impose this condition on $S(G^*, Y^*, F^*)$.) Now $g_{f,1}(t)$ is certainly nonconstant, and we conclude that φ_a does not depend on $\tau_{f,1}$, so $\tau_{d,a}$ does not depend on $\tau_{f,1}$. The same argument shows that $z_{d,b}, \zeta_{d,c}$ do not depend on $\tau_{f,1}$.

Next consider some dependent $\tau_{d,a}$ which has its real locus coincident with that of $\tau_{f,1}$ (for all choices of the $\tau_{f,i}, i \neq 1$). Fix some choice of $\tau_{f,i}, i \neq 1$. Now we have that $\varphi(\tau) = \varphi_a(\tau, \tau_{f,i}, i \neq 1)$ satisfies an identity (locally, but then globally by analytic continuation)

$$g_{d,a}(t_1)\varphi\big(g_{f,1}(t_1)\tau\big) = g_{d,a}(t_0)\varphi\big(g_{f,1}(t_0)\tau\big), \quad t_1 \neq t_0, t_0, t_1, \in \mathbb{Z}.$$

Rewrite this as

$$\varphi(g\tau) = h\varphi(\tau) \tag{$**$}$$

(the resulting form of g, h is shown in [15]; they occur in positive-dimensional semialgebraic families with "many," i.e., a positive power of height, choices $g, h \in \mathrm{SL}_2(\mathbb{Z})$, and the g are parabolic with fixed point a/c).

Each choice of a rational a/c in the real $\tau_{f,1}$-boundary of U^* gives rise to different such identities, and it follows that φ is a real fractional linear transformation. This is proved in [15] but may also be seen as follows. The two functions ($**$) of τ share the same branch points. So g preserves the set S of branch points of φ. If $\#S \geq 3$, this restricts g to a finite set of possibilities. (An element of $\mathrm{SL}_2(\mathbb{R})$ is determined by its action on 3 points.) It is not possible to have $\#S = 1$. If $\#S = 2$, then for suitable $k, l \in \mathrm{SL}_2(\mathbb{C})$ and $c \in \mathbb{C}$ we have $\psi = cl\varphi(k\tau) = \tau^{p/q}$ as, by Hurwitz's formula, the function must be maximally ramified at the two points. The identity now takes the form $\psi g^* = cl\varphi k[k^{-1}gk] = [lhl^{-1}]cl\varphi k = h^*\psi$, where $g^* = k^{-1}gk, h^* = lhl^{-1}$. Then such g^*, preserving $\{0, \infty\}$, are of the form

$\tau \mapsto \alpha\tau^{\pm 1}$, and then the original g are of the form $k\left(\begin{smallmatrix} 0 & \alpha \\ -1/\alpha & 0 \end{smallmatrix}\right)k^{-1}$ or $k\left(\begin{smallmatrix} \beta & 0 \\ 0 & 1/\beta \end{smallmatrix}\right)k^{-1}$. This is not possible, as the g in ($**$) are parabolic, but the forms above are not. So we are left to consider $\#S = 0$. Then φ is a polynomial. Applying the same argument to φ^{-1} we see that φ is fractional linear. Since φ preserves $\mathbb{H}$, we conclude that $\varphi \in \mathrm{SL}_2(\mathbb{R})$. Then φ cannot depend algebraically on the $\tau_{f,i}, i \neq 1$. Hence $\tau_{d,a} = \varphi\tau_{f,1}$ for some $\varphi \in \mathrm{SL}_2(\mathbb{R})$. Further, as the identities ($**$) in fact occur with $g, h \in \mathrm{SL}_2(\mathbb{Z})$, we can show, as in [15], that $\varphi \in \mathrm{GL}_2^+(\mathbb{Q})$.

Repeating this argument, we see that any dependencies among the τ-variables are of the form $\tau_i = g\tau_j$ for some $g \in \mathrm{GL}_2^+(\mathbb{Q})$, and that (z, ζ)-variables do not depend on τ-variables.

Now we consider the dependencies among the (z, ζ)-variables. We may suppress the τ-variables, as there is no dependence on them, and now we are in exactly the same situation as in [15] in seeking the form of a maximal algebraic W such that a certain algebraic dependency holds on the functions $\wp_j(z_j)$, $\exp(\zeta_k)$. Such W are shown in [15] to be weakly special.

So we see that W is weakly special. On any proper weakly special variety the constituent functions of π are certainly algebraically dependent (over $\mathbb{Q}$). But on all U they are algebraically independent. It follows that W is a maximal proper weakly special variety, as required. □

Remark 5.1 The proof does not require taking the same functions j, j', j'' on each τ_i; we could take $f_i(\tau_i)$, $f_i'(\tau_i)$, $f_i''(\tau_i)$ for any nonconstant modular functions $f_1, \ldots, f_n$.

Remark 5.2 The results of Mahler [10] were extended by Nishioka [11]. Nishioka proves that a function $f(z)$ automorphic for a discontinuous subgroup G of $\mathrm{SL}_2(\mathbb{R})$ with at least three limit points satisfies no algebraic differential equation of second order over $\mathbb{C}(z, e^{uz})$, for any $u \in \mathbb{C}$. I thank the referees for suggesting that I remark on the possibility of extending the present result to such functions. Ullmo and Yafaev have a preprint [21] establishing "Ax–Lindemann–Weierstrass" for any compact Shimura variety, thus including compact Shimura curves. It may well be possible to extend their argument to include derivatives, which might enable one to deal with the case where G is arithmetic. As the methods of [15] and [21] rely on o-minimality and point counting, they seem to be inapplicable to nonarithmetic groups. It seems interesting also to add suitable derivatives to the Ax–Lindemann–Weierstrass results of [21] more generally, as well to the result in Pila and Tsimerman [17] for the moduli space of abelian surfaces. The analogue of Mahler's result for Siegel modular forms is established by Bertrand and Zudilin [6], [5].

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