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Levels of Uniformity

Rutger Kuyper

Abstract We introduce a hierarchy of degree structures between the Medvedev and Muchnik lattices which allow varying amounts of nonuniformity. We use these structures to introduce the notion of the uniformity of a Muchnik reduction, which expresses how uniform a reduction is. We study this notion for several well-known reductions from algorithmic randomness. Furthermore, since our new structures are Brouwer algebras, we study their propositional theories. Finally, we study if our new structures are elementarily equivalent to each other.

1 Introduction

Over the years, the uses of the Medvedev and Muchnik lattices in computability theory have expanded far beyond their applications to intuitionistic logic, as originally intended by Medvedev. These two lattices formalize when a *mass problem*, that is, a set $\mathcal{A} \subseteq \omega^\omega$, is "easier" than another mass problem $\mathcal{B}$. Both say that a mass problem $\mathcal{A}$ is easier than a mass problem $\mathcal{B}$, or that $\mathcal{A}$ reduces to $\mathcal{B}$, if every function in $\mathcal{B}$ computes a function in $\mathcal{A}$. However, the Medvedev lattice imposes an additional restriction, saying that this reduction should be uniform, in the sense that the oracle Turing machine performing the computation should be the same for every function in $\mathcal{B}$. On the other hand, in the Muchnik lattice we can choose a different machine for each function. Thus, the Medvedev and Muchnik lattices are the most uniform and the most nonuniform approach to reducing mass problems.

In practice, there are many reductions between mass problems that turn out to only be Muchnik reductions and not Medvedev reductions. This is especially true in algorithmic randomness, where the fact that the reductions are not uniform can often be shown using a straightforward majority vote argument. Thus, the conclusion is often that the reductions are only Muchnik reductions; that is, that they are highly

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nonuniform. However, this is often not actually the case, but merely results from the fact that we do not yet know of a finer-grained hierarchy between Medvedev and Muchnik reducibility (indeed, in these kind of arguments all one usually needs to know is some finite initial fragment of what one is trying to compute, so it is not as nonuniform as one might expect). This article aims to resolve that problem by introducing exactly such a hierarchy.

Higuchi and Kihara [3] introduced five structures between the Medvedev and Muchnik lattices. One of these is the lattice they call $\mathcal{D}^1_\omega$ and which we will call $\mathcal{M}_1$ in this article. Roughly speaking, in this lattice the reductions do not have to be fully uniform, but the nonuniform choice has to be a Π^0_1-condition. In slightly more detail, a mass problem $\mathcal{A}$ 1-reduces to a mass problem $\mathcal{B}$ if there is a uniformly Π^0_1-sequence $\mathcal{V}_0, \mathcal{V}_1, \dots$ covering $\mathcal{B}$ such that we can compute an element of $\mathcal{A}$ uniformly from an element $f \in \mathcal{B}$ *together* with an $i \in \omega$ such that $f \in \mathcal{V}_i$.

A natural extension of this definition is to replace Π^0_1 by Π^0_n in the informal definition just given. This is what we do in Section 2, which yields the notion of n-reducibility, and the corresponding degree structures $\mathcal{M}_n$ for every $n \in \omega$. In that section, we will also show that the resulting degree structure is always a Brouwer algebra, a lattice with an additional implication operator which can be used to give semantics for propositional logics between intuitionistic logic and classical logic. The Medvedev and Muchnik lattices are also known to be Brouwer algebras. For $\mathcal{M}_1$ this was shown in [3], and it is the only Brouwer algebra among the intermediate degree structures studied by Higuchi and Kihara.

In Section 3, we study maps between $\mathcal{M}_n$ and $\mathcal{M}_m$ for $n \neq m$. We show that the natural surjection from $\mathcal{M}_n$ to $\mathcal{M}_m$ for $m > n$ preserves joins and meets, but not necessarily implications. On the other hand, we show that there are embeddings preserving joins and implications in the other direction.

Next, in Section 4, we introduce the *uniformity* of a pair $(\mathcal{A}, \mathcal{B})$ with $\mathcal{A} \leq_w \mathcal{B}$. This notion tries to capture how uniform a Muchnik reduction exactly is, as motivated above. The uniformity is the least number $n \in \omega$ such that $\mathcal{A} \leq_n \mathcal{B}$, if any such n exists. Thus, the uniformity is the least $n \in \omega$ such that Π^0_n-choices suffice to make the reduction uniform.

We apply this notion of uniformity to algorithmic randomness in Section 5. Here we study the uniformity of some well-known Muchnik reductions from algorithmic randomness. This also allows us to give natural examples separating n-reducibility from m-reducibility for $n \neq m$.

As mentioned above, the structures $\mathcal{M}_n$ are all Brouwer algebras. We study their propositional theories as Brouwer algebras in Sections 6 and 7. While, just as for the Medvedev and Muchnik lattices, their theories are not intuitionistic propositional logic (IPC), we show that there are principal factors of $\mathcal{M}_n$ that do capture exactly IPC if $n \leq 1$ or $n \geq 4$ (a study motivated by Skvortsova's magnificent result stating that such factors exist for the Medvedev lattice). This also allows us to show that the theory of $\mathcal{M}_1$ is exactly Jankov's logic, the deductive closure of IPC and the weak law of the excluded middle $\neg p \vee \neg\neg p$, answering a question of Higuchi and Kihara [4]. The problem remains open for $n = 2$ and $n = 3$.

Finally, in Section 8, we study the first-order theories of the $\mathcal{M}_n$ as lattices. We show that $\mathcal{M}_n$ and $\mathcal{M}_m$ are not elementarily equivalent if $n \neq m$, except for the case $n, m \in \{0, 1\}$, which we do not currently know how to deal with.

Our notation is mostly standard. We use $\oplus$ to denote joins or least upper bounds in lattices, and similarly $\otimes$ to denote meets or greatest lower bounds in lattices. When we write $\Psi(\tau)(n)\downarrow$ for some Turing functional $\Psi : \omega^\omega \to \omega^\omega$, we mean that Ψ halts on input n in at most $|\tau|$ steps with the partial oracle τ (i.e., τ is a finite string of natural numbers). We assume a fixed, computable pairing function $\langle n, m\rangle$. We let Φ_e be the Turing functional with index e. For any set $\mathcal{A} \subseteq \omega^\omega$, we let $C(\mathcal{A})$ be the upwards closure of $\mathcal{A}$ under Turing reducibility; that is, the set of those f such that for some $g \in \mathcal{A}$ we have $f \geq_T g$. We fix effective listings $\{\mathcal{S}_e^n\}_{e\in\omega}$ of all Σ_n^0-classes and $\{\mathcal{P}_e^n = \overline{\mathcal{S}_e^n}\}_{e\in\omega}$ of all Π_n^0-classes. When we say that a sequence $\mathcal{V}_0, \mathcal{V}_1, \dots$ is uniformly Π_n^0, we mean there is a computable sequence $u_0, u_1, \dots$ such that $\mathcal{V}_i = \mathcal{P}_{u_i}^n$. We will denote concatenation of finite strings x and y both by xy and by $x^\frown y$.

For undefined notions from computability theory, we refer to Odifreddi [16], for undefined notions from algorithmic randomness, we refer to Downey and Hirschfeldt [1] and Nies [15], and for more background on the Medvedev lattice we refer to the surveys of Sorbi [20] and Hinman [5].

2 The n-Uniform Degrees

In this section we will introduce the n-uniform degrees and prove some basic results about them.

Definition 2.1 Let $\mathcal{A}, \mathcal{B} \subseteq \omega^\omega$, and let $n \in \omega$. Then we say that $\mathcal{A}$ *n-uniformly reduces to* $\mathcal{B}$ (notation: $\mathcal{A} \leq_n \mathcal{B}$) if there exists a sequence $\mathcal{V}_0, \mathcal{V}_1, \dots$ of uniformly Π_n^0 sets with $\mathcal{B} \subseteq \bigcup_{i\in\omega} \mathcal{V}_i$ and a uniformly computable sequence $e_0, e_1, \dots$ such that for every $i \in \omega$ and every $f \in \mathcal{B} \cap \mathcal{V}_i$, we have $\Phi_{e_i}(f) \in \mathcal{A}$. If both $\mathcal{A} \leq_n \mathcal{B}$ and $\mathcal{B} \leq_n \mathcal{A}$, then we say that $\mathcal{A}$ and $\mathcal{B}$ are *n-uniformly equivalent* (notation: $\mathcal{A} \equiv_n \mathcal{B}$). We let $\mathcal{M}_n = \mathcal{P}(\omega^\omega)/\equiv_n$ and call its elements the *n-uniform degrees*.

For $n = 1$, this structure was introduced by Higuchi and Kihara in [3], as mentioned in the Introduction. Their definition, while different at first sight, is equivalent to ours, as they have shown in [3, Theorem 26].

We will often drop the adjective “uniform” and talk about n-reducibility, n-equivalence, and the n-degrees instead. If $\mathcal{A} \leq_n \mathcal{B}$ and $\mathcal{V}_0, \mathcal{V}_1, \dots$ and $e_0, e_1, \dots$ are as in Definition 2.1 above, then we say that these sequences *witness* that $\mathcal{A} \leq_n \mathcal{B}$.

Remark 2.2 Note that, in Definition 2.1 we can replace Π_n^0 by Σ_{n+1}^0 without changing the concept. Namely, if there are a sequence $\mathcal{U}_i = \bigcup \mathcal{V}_j^i$ covering $\mathcal{B}$ with $\mathcal{V}_i$ uniformly Π_n^0 and a computable sequence $e_0, e_1, \dots$ such that $\Phi_{e_i}(f) \in \mathcal{A}$ for every $f \in \mathcal{B} \cap \mathcal{V}_i$, then it is not hard to see that the sequences $(\mathcal{V}_j^i)_{i,j\in\omega}$ and $(e_i)_{i,j\in\omega}$ witness that $\mathcal{A} \leq_n \mathcal{B}$.

Clearly the relation $\leq_n$ is reflexive on the n-degrees, but let us verify that it is also transitive.

Proposition 2.3 *The relation $\leq_n$ is transitive on $\mathcal{P}(\omega^\omega) \times \mathcal{P}(\omega^\omega)$, hence it induces an ordering on $\mathcal{M}_n$.*

Proof Let $\mathcal{A} \leq_n \mathcal{B}$ be witnessed by $\mathcal{V}_0, \mathcal{V}_1, \dots$ and $e_0, e_1, \dots$, and let $\mathcal{B} \leq_n \mathcal{C}$ be witnessed by $\mathcal{U}_0, \mathcal{U}_1, \dots$ and $s_0, s_1, \dots$. For all $i, j \in \omega$, let $\mathcal{W}_{i,j}$ be as in

Lemma 2.4 below applied to $\mathcal{V}_i$ and Φ_{s_j}. Now, if we let $\mathcal{T}_j$ be the class of functions f for which Φ_{s_j} is total, then $\mathcal{U}_j \cap \mathcal{C} \subseteq \mathcal{T}_j$. So,

$$\Phi_{s_j}(\mathcal{W}_{i,j} \cap \mathcal{U}_j \cap \mathcal{C}) \subseteq \mathcal{V}_i \cap \mathcal{B}.$$

Thus, $\Phi_{e_i}(\Phi_{s_j}(\mathcal{W}_{i,j} \cap \mathcal{U}_j \cap \mathcal{C})) \subseteq \mathcal{A}$. On the other hand, the sequence $(\mathcal{W}_{i,j} \cap \mathcal{U}_j)_{i,j\in\omega}$ covers $\mathcal{C}$, which completes the proof. □

Lemma 2.4 *Let Φ be a Turing functional, and let $\mathcal{T} \subseteq \omega^\omega$ be the class of functions f for which $\Phi(f)$ is total. Then for every $n \geq 1$ and every Π^0_n-class $\mathcal{V}$, we have that $\Phi^{-1}(\mathcal{V})$ is Π^0_n within $\mathcal{T}$; that is, there is a Π^0_n-class $\mathcal{W}$ such that $\Phi^{-1}(\mathcal{V}) = \mathcal{W} \cap \mathcal{T}$. Furthermore, we can find an index for $\mathcal{W}$ uniformly in n and indices for Φ and $\mathcal{V}$.*

Proof Note that Φ^{-1} commutes with unions and intersections (for this we do not even need it to be computable), and that we have that $\Phi^{-1}(\llbracket\sigma\rrbracket)$ is Σ^0_1 uniformly in σ (here, $\llbracket\sigma\rrbracket = \{f \in \omega^\omega \mid \sigma \prec f\}$). In fact, $\Phi^{-1}(\llbracket\sigma\rrbracket)$ is also Π^0_1 within $\mathcal{T}$ uniformly in σ; that is, there are $\mathcal{W}_\sigma$ which are Π^0_1 uniformly in σ such that $\Phi^{-1}(\llbracket\sigma\rrbracket) = \mathcal{W}_\sigma \cap \mathcal{T}$. Indeed, let $\mathcal{W}_\sigma$ be the Π^0_1-class

$$\big\{f \bigm| \forall n\big(\Phi(f \upharpoonright n) \upharpoonright |\sigma|\downarrow \to \Phi(f \upharpoonright n) \upharpoonright |\sigma| = \sigma\big\}\big).$$

From this we can directly construct a $\mathcal{W}$, as required. □

We would like to mention that there is another natural, equivalent definition of n-reducibility, although we will not use this fact in this article. Informally, we have that n-reducibility is the same as reducibility uniformly in the nth jump, which we formalize as follows.

Proposition 2.5 *Let $\mathcal{A}, \mathcal{B} \subseteq \omega^\omega$, and let $n \in \omega$. Then $\mathcal{A} \leq_n \mathcal{B}$ if and only if there exists a Turing functional $\Psi : \omega^\omega \to \omega$ such that for every $f \in \mathcal{B}$, we have that $\Psi(f^{(n)})\downarrow$, and $\Phi_{\Psi(f^{(n)})}(f) \in \mathcal{A}$.*

Proof First, let $\mathcal{A} \leq_n \mathcal{B}$. Fix witnesses $\mathcal{V}_0, \mathcal{V}_1, \dots$ and $e_0, e_1, \dots$ for this fact. Given $f \in \mathcal{B}$, let i be least such that $f \in \mathcal{V}_i$. Note that such an i exists since $\mathcal{V}_0, \mathcal{V}_1, \dots$ covers $\mathcal{B}$. Furthermore, i is computable in $f^{(n)}$. Now let $\Psi(f^{(n)})$ be e_i. Then

$$\Phi_{\Psi(f^{(n)})}(f) = \Phi_{e_i}(f) \in \mathcal{A}.$$

Conversely, let Ψ be as in the statement of the proposition. Let $\mathcal{V}_e$ be the class of those $f \in \omega^\omega$ such that $\Psi(f^{(n)})\downarrow = e$. Then $\mathcal{V}_e$ is Σ^0_{n+1}. Furthermore, for each $f \in \mathcal{V}_e \cap \mathcal{B}$, we have that $\Phi_e(f) \in \mathcal{A}$. Thus, by applying Remark 2.2 to the sequences $\mathcal{V}_0, \mathcal{V}_1, \dots$ and $0, 1, \dots$, we see that $\mathcal{A} \leq_n \mathcal{B}$, as desired. □

Next, we show that 0-reducibility is just Medvedev-reducibility, so the bottom level of our hierarchy of reducibilities is the completely uniform Medvedev-reducibility.

Proposition 2.6 *Medvedev-reducibility and* 0*-reducibility coincide.*

Proof Clearly Medvedev-reducibility implies 0-reducibility. Conversely, if $\mathcal{A} \leq_0 \mathcal{B}$, then by definition there are computable sequences $\sigma_0, \sigma_1, \dots$ and $e_0, e_1, \dots$ such that every $f \in \mathcal{B}$ starts with some string σ_i, and such that if $\sigma_i \subseteq f$ for $f \in \mathcal{B}$, then $\Phi_{e_i}(f) \in \mathcal{A}$. We can then uniformly compute an element of $\mathcal{A}$ from an element $f \in \mathcal{B}$ by computing the least i such that $\sigma_i \subseteq f$ and sending f to $\Phi_{e_i}(f)$. □

As mentioned in the Introduction, for every n the n-uniform degrees form a Brouwer algebra. This is what we prove next. For $n = 0$, the Medvedev lattice was shown by Medvedev [13], and the Muchnik lattice was shown by Muchnik [14]. For $n = 1$, this result is due to Higuchi and Kihara [3, Proposition 16], but the general proof below is our own.

Whenever we have a map $\alpha : \mathcal{P}(\omega^\omega) \to \mathcal{P}(\omega^\omega)$, we say that α *induces* a map from $\mathcal{M}_n$ to $\mathcal{M}_m$ if, whenever $\mathcal{A} \equiv_n \mathcal{B}$, we have that $\alpha(\mathcal{A}) \equiv_m \alpha(\mathcal{B})$. In this case, we implicitly identify α with its induced map from $\mathcal{M}_n$ to $\mathcal{M}_m$ sending the n-degree of $\mathcal{A}$ to the m-degree of $\alpha(\mathcal{A})$.

Proposition 2.7 *For every $n \in \omega$, the n-uniform degrees form a Brouwer algebra.*

Proof The join and meet are induced by the same set operation as in the Medvedev lattice; that is, they are induced by the operations

$$\mathcal{A} \oplus \mathcal{B} = \{f \oplus g \mid f \in \mathcal{A}, g \in \mathcal{B}\}$$

and

$$\mathcal{A} \otimes \mathcal{B} = 0^\frown \mathcal{A} \cup 1^\frown \mathcal{B}$$

for $\mathcal{A}, \mathcal{B} \subseteq \omega^\omega$. The proofs are a straightforward generalization of those for the Medvedev lattice (see, e.g., Sorbi [20, Theorem 1.3]).

For the implication, consider the operation induced by

$$\begin{aligned} \mathcal{A} \to_n \mathcal{B} = \big\{u^\frown e^\frown f \;\big|\; &(\mathcal{P}^n_{\{u\}(i)})_{i\in\omega} \text{ covers } \mathcal{A} \oplus \{f\} \\ &\text{and } \Phi_{\{e\}(i)}\big(\big(\mathcal{A} \oplus \{f\}\big) \cap \mathcal{P}^n_{\{u\}(i)}\big) \subseteq \mathcal{B}\big\}. \end{aligned}$$

Here, $(\mathcal{P}^n_i)_{e\in\omega}$ is an effective enumeration of all Π^0_n-classes, as mentioned in the Introduction.

Then $\mathcal{A} \oplus (\mathcal{A} \to_n \mathcal{B}) \geq_n \mathcal{B}$. For this, consider the sequence $\mathcal{V}_0, \mathcal{V}_1, \ldots,$ where $\mathcal{V}_i$ consists of those $g \oplus (u^\frown e^\frown f)$ such that if $\{u\}(i)\downarrow$, then we have that $g \oplus f \in \mathcal{P}^n_{u(i)}$. Then each $\mathcal{V}_i$ is a Π^0_n-class, and if $g \oplus (u^\frown e^\frown f) \in (\mathcal{A} \oplus (\mathcal{A} \to_n \mathcal{B})) \cap \mathcal{V}_i$, then $\Phi_{\{e\}(i)}(g \oplus f) \in \mathcal{B}$. Thus, if we let k_i be an index for the Turing functional sending $g \oplus (u^\frown e^\frown f)$ to $\Phi_{\{e\}(i)}(g \oplus f)$, then $\mathcal{V}_0, \mathcal{V}_1, \ldots$ and $k_0, k_1, \ldots$ witness that $\mathcal{A} \oplus (\mathcal{A} \to_n \mathcal{B}) \geq_n \mathcal{B}$.

Conversely, if $\mathcal{A} \oplus \mathcal{C} \geq_n \mathcal{B}$, fix u and e such that $\mathcal{P}^n_{u(0)}, \mathcal{P}^n_{u(1)}, \ldots$ and $e(0), e(1), \ldots$ witness this. Then, for every $f \in \mathcal{C}$, we have $u^\frown e^\frown f \in \mathcal{A} \to_n \mathcal{B}$, which shows that $\mathcal{A} \oplus C \geq_n \mathcal{B}$ if and only if $\mathcal{C} \geq_n \mathcal{A} \to_n \mathcal{B}$.

We can now also easily show that $\to_n$ induces a well-defined operation on $\mathcal{M}_n$: if $\mathcal{A}_1 \equiv_n \mathcal{A}_2$ and $\mathcal{B}_1 \equiv_n \mathcal{B}_2$, we have that

$$\begin{aligned} \mathcal{A}_1 \to_n \mathcal{B}_1 \leq_n \mathcal{A}_2 \to_n \mathcal{B}_2 &\Leftrightarrow \mathcal{B}_1 \leq_n \mathcal{A}_1 \oplus (\mathcal{A}_2 \to_n \mathcal{B}_2) \\ &\Leftrightarrow \mathcal{B}_2 \leq_n \mathcal{A}_2 \oplus (\mathcal{A}_2 \to_n \mathcal{B}_2) \\ &\Leftrightarrow \mathcal{A}_2 \to_n \mathcal{B}_2 \leq_n \mathcal{A}_2 \to_n \mathcal{B}_2, \end{aligned}$$

where we use that we already know that $\oplus$ induces a well-defined operation on $\mathcal{M}_n$, as argued above. That $\mathcal{A}_2 \to_n \mathcal{B}_2 \leq_n \mathcal{A}_1 \to_n \mathcal{B}_1$ follows in the same way. □

Let us conclude this section by remarking that instead of just looking at $n \in \omega$, we could also make a version of Definition 2.1 where we look at all ordinals $\alpha < \omega_1^{\mathrm{CK}}$. We expect many of the results in this article hold in this more general setting, but also expect the proofs will get more technical. So, for reasons of clarity, we have decided to restrict ourselves to $n \in \omega$, which we think already covers the most important part.

3 Maps Between $\mathcal{M}_n$ and $\mathcal{M}_m$

In this section, we will show that there are natural maps between $\mathcal{M}_n$ and $\mathcal{M}_m$ for $n, m \in \omega \cup \{w\}$. First, we show that the natural surjection from $\mathcal{M}_n$ to $\mathcal{M}_m$ for $n < m$ preserves the lattice structure, but not the Brouwer algebra structure. This is known for the Medvedev and Muchnik lattices from Muchnik [14].

Note that n-reducibility implies Muchnik-reducibility for every $n \in \omega$. Therefore, let us introduce the following convention: whenever we talk about the set $\omega \cup \{w\}$, we think of it as an ordered set, where the order on ω is the usual ordering, and w is the largest element.

Proposition 3.1 *Let $n, m \in \omega \cup \{w\}$ with $n \leq m$. Then the surjection from $\mathcal{M}_n$ onto $\mathcal{M}_m$ induced by the identity map on $\mathcal{P}(\omega^\omega)$ is a well-defined map preserving $\oplus$ and $\otimes$, but not necessarily $\rightarrow$.*

Proof First, that the map is well defined follows directly from the fact that $\mathcal{A} \leq_n \mathcal{B}$ implies that $\mathcal{A} \leq_m \mathcal{B}$. The preservation of $\oplus$ and $\otimes$ follows directly from the fact that they are induced by the same set operations. That the implication is not preserved follows from the fact proved below in Corollary 5.4 that for every m there are $\mathcal{A}, \mathcal{B}$ such that $\mathcal{A} \leq_m \mathcal{B}$ but $\mathcal{A} \not\leq_n \mathcal{B}$ for all $n < m$; therefore $\mathcal{B} \rightarrow_m \mathcal{A} \equiv_m \omega^\omega$ while $\mathcal{B} \rightarrow_n \mathcal{A} \not\equiv_n \omega^\omega$. Thus, $\mathcal{B} \rightarrow_n \mathcal{A}$ does not contain a computable element and is therefore not even Muchnik-equivalent to $\mathcal{B} \rightarrow_m \mathcal{A}$. □

It is known from Sorbi [19] that there is an embedding of the Muchnik lattice into the Medvedev lattice preserving joins and implications. Higuchi and Kihara [3, Corollary 42] showed that $\mathcal{M}_1$ also embeds into $\mathcal{M}$ as a partially ordered set. In fact, this embedding can be easily seen to preserve joins and implications, as we explain next.

Theorem 3.2 ([3, Corollary 42]) *There is an embedding of $\mathcal{M}_1$ into $\mathcal{M}$ preserving joins and implications, induced by*

$$\alpha(\mathcal{A}) = \{f \mid p(f) \in \mathcal{A} \text{ and } m(f) < \infty\},$$

where $m(f) = |\{i \mid f(i) = 0\}|$ and $p(f)$ is $(f \upharpoonright [k, \infty)) - 1$ for the least k such that $m(f \upharpoonright [k, \infty)) = 0$. (Here, $f \upharpoonright [k, \infty)$ is the function given by $(f \upharpoonright [k, \infty))(n) = f(n + k)$.) Furthermore, $\alpha(\mathcal{A}) \equiv_1 \mathcal{A}$ for all $\mathcal{A}$.

Proof We only prove that joins and implications are preserved, using the fact from the proof of [3, Corollary 42] that the map induced by α is well defined, preserves the order, and satisfies $\alpha(\mathcal{A}) \equiv_1 \mathcal{A}$. Thus, we know that $\alpha(\mathcal{A}) \oplus \alpha(\mathcal{B}) \leq_{\mathcal{M}} \alpha(\mathcal{A} \oplus \mathcal{B})$. Conversely, given $f \in \alpha(\mathcal{A})$ and $g \in \alpha(\mathcal{B})$, we show how to uniformly compute a function $h \in \alpha(\mathcal{A} \oplus \mathcal{B})$. For this, we use an auxiliary number k_s, where we set $k_{-1} = 0$. At stage s, we define $h(2s)$ and $h(2s + 1)$. First, check if either $f(s) = 0$ or $g(s) = 0$. If so, let $h(2s) = h(2s + 1) = 0$, and let $k_s = 0$. Otherwise, let $h(2s) = f(k_{s-1})$, $h(2s + 1) = g(k_{s-1})$, and $k_s = k_{s-1} + 1$. Then it can be directly verified that $h \in \alpha(\mathcal{A} \oplus \mathcal{B})$, and since we computed h uniformly in f and g, we have $\alpha(\mathcal{A}) \oplus \alpha(\mathcal{B}) \equiv_{\mathcal{M}} \alpha(\mathcal{A} \oplus \mathcal{B})$.

For the implication, again we already know that $\alpha(\mathcal{A}) \rightarrow \alpha(\mathcal{B}) \leq_{\mathcal{M}} \alpha(\mathcal{A} \rightarrow \mathcal{B})$ from the fact that the order is preserved. Conversely, $\alpha(\mathcal{A}) \rightarrow \alpha(\mathcal{B}) \equiv_1 \mathcal{A} \rightarrow \mathcal{B}$ because $\alpha(\mathcal{A}) \equiv_1 \mathcal{A}$ and $\alpha(\mathcal{B}) \equiv_1 \mathcal{B}$. Thus, $\alpha(\alpha(\mathcal{A}) \rightarrow \alpha(\mathcal{B})) \equiv_{\mathcal{M}} \alpha(\mathcal{A} \rightarrow \mathcal{B})$. However, $\mathcal{C} \geq_{\mathcal{M}} \alpha(\mathcal{C})$ holds for any $\mathcal{C}$ by sending f to $f + 1$, so we see that $\alpha(\mathcal{A}) \rightarrow \alpha(\mathcal{B}) \geq_{\mathcal{M}} \alpha(\mathcal{A} \rightarrow \mathcal{B})$, as desired. □

We now show that we have such embeddings for all $n, m \in \omega \cup \{w\}$ with $m \leq n$.

Theorem 3.3 *Let $n \in \omega \cup \{w\}$. Then there exists a map $u_n : \mathcal{P}(\omega^\omega) \to \mathcal{P}(\omega^\omega)$ such that for every $m \leq n$, the map $u_{n,m} : \mathcal{M}_n \to \mathcal{M}_m$ induced by u_n is a well-defined embedding preserving joins and implications. Furthermore, $u_n(\mathcal{A}) \equiv_n \mathcal{A}$ for all n and $\mathcal{A}$.*

Proof If $n = w$, then we can use Sorbi's embedding of the Muchnik lattice into the Medvedev degrees mentioned above. Also, if $n = 0$, then there is nothing to be proved, and if $n = 1$, this follows from Theorem 3.2. So, we let $n \in \omega$ with $n \geq 2$.

In the case of the Muchnik lattice, we have a natural representative of the Muchnik degree of $\mathcal{A}$ in the Medvedev lattice: take the Medvedev degree of the upwards closure of $\mathcal{A}$, which is the same as the Medvedev degree of $\bigcup\{\mathcal{B} \subseteq \omega^\omega \mid \mathcal{B} \geq_w \mathcal{A}\}$. In other words, the Muchnik degree of $\mathcal{A}$ contains a maximal mass problem. This is what is used in the embedding of Sorbi for the Muchnik lattice. In our current case it is harder to find a natural representative; we cannot just take $\bigcup\{\mathcal{B} \subseteq \omega^\omega \mid \mathcal{B} \geq_n \mathcal{A}\}$ because $\mathcal{A}$ does not n-reduce to this set in general. This is caused by the fact that, in general, there is not a "universal" sequence of uniform Π^0_n-classes covering $\mathcal{A}$.

Thus, we need to find a different approach which works around this nonexistence of a universal sequence. For this, note that given any set $\mathcal{C} \subseteq \mathcal{P}(2^{\omega^n})$, there is a natural Π^0_n-class $\mathcal{V}$ which covers part of $\mathcal{C}$: take those X such that

$$\forall m_1 \exists m_2 \ldots \forall m_{n-1} \exists m_n \big((m_1, \ldots, m_n) \in X\big).$$

We will show that this is in a certain sense a universal way of making Π^0_n-choices. Furthermore, if we do not take ω^n but ω^{n+1}, by a slight modification we can make a natural Π^0_n-class which uses this extra space to code functions, and if we go up to ω^{n+2}, we can use this extra space to be able to deal with multiple Π^0_n-classes at once. We will use this to define our desired representative.

We now give the full details. Consider $f \in (\omega^{n+1} \times \omega)^\omega$ (where we implicitly identify $(\omega^{n+1} \times \omega)^\omega$ with ω^ω in some computable way). In what follows, when we write $x \in f$, we mean that there is some $i \in \omega$ with $f(i) = x$. We now inductively define when f *is* (ρ, s)*-Σ-valid* and when f *is* (ρ, s)*-Π-valid*, where $1 \leq s \leq n$ and $\rho \in \omega^{n+1-s}$:

(i) f is $(\rho, 1)$-Π-valid if and only if there is a $k \in \omega$ such that for all $m \in \omega$, $(\rho^\frown m, k) \in f$.
(ii) f is $(\rho, 1)$-Σ-valid if and only if there are a $k \in \omega$ and an $m \in \omega$ such that $(\rho^\frown m, k) \in f$.
(iii) f is $(\rho, s+1)$-Π-valid if and only if it is $(\rho^\frown m, s)$-Σ-valid for every $m \in \omega$.
(iv) f is $(\rho, s+1)$-Σ-valid if and only if it is $(\rho^\frown m, s)$-Π-valid for some $m \in \omega$.

First, let us consider how a valid f code functions in a computable way. If f is $(\rho, 3)$-Π-valid, then we know that for each $m \in \omega$ there is an a_m such that f is $(\rho^\frown m^\frown a_m, 1)$-$\Pi$-valid, and therefore there is a k_m such that $(\rho^\frown m^\frown a_m{}^\frown t, k_m) \in f$ for every $t \in \omega$. Given m, let k_m be least (in the order $f(0), f(1), \ldots$) such that $(\rho^\frown m^\frown a_m{}^\frown t, k_m) \in f$ for some $a_m, t \in \omega$. In this case, we let $p_{\text{odd}}(\rho, f) = k_0 k_1 \ldots$. This is computable uniformly in ρ and f.

On the other hand, if f is $(\rho, 2)$-Π-valid, then we know that for each $m \in \omega$ there are k_m and a_m such that $(\rho^\frown m^\frown a_m, k_m) \in f$. Again, let k_m be the least such k_m, and define $p_{\text{even}}(\rho, f) = k_0 k_1 \ldots$. Then this is computable uniformly in ρ and f.

We now let

$$u_n(\mathcal{A}) = \big\{f \;\big|\; \exists i\big(f \text{ is } (i,n)\text{-}\Pi\text{-valid}\big), \text{ and } \forall \rho\big(f \text{ is } (\rho,3)\text{-}\Pi\text{-valid and } f \text{ is } (\rho \upharpoonright 1, n)\text{-}\Pi\text{-valid} \Rightarrow p_{\mathrm{odd}}(\rho, f) \in \mathcal{A}\big)\big\}$$

if n is odd, and we let

$$u_n(\mathcal{A}) = \big\{f \;\big|\; \exists i\big(f \text{ is } (i,n)\text{-}\Pi\text{-valid}\big), \text{ and } \forall \rho\big(f \text{ is } (\rho,2)\text{-}\Pi\text{-valid and } f \text{ is } (\rho \upharpoonright 1, n)\text{-}\Pi\text{-valid} \Rightarrow p_{\mathrm{even}}(\rho, f) \in \mathcal{A}\big)\big\}$$

if n is even. For the remainder of the proof, let us assume that n is odd; the even case proceeds in the same way.

First, we claim that $u_n(\mathcal{A}) \equiv_n \mathcal{A}$. Indeed, to show that $u_n(\mathcal{A}) \leq_n \mathcal{A}$, or in fact even $u_n(\mathcal{A}) \leq_{\mathcal{M}} \mathcal{A}$, send f to the function g given by

$$g\big(\langle i, a_1, \dots, a_n\rangle\big) = \big(i a_1 \cdots a_n, f(a_{n-2})\big).$$

It is easy to verify that this gives an element of $u_n(\mathcal{A})$.

Conversely, consider the classes $\mathcal{V}_\rho$ consisting of those f such that f is $(\rho \upharpoonright 1, n)$-$\Pi$-valid and such that f is $(\rho, 3)$-Π-valid. The former is a Π^0_n-condition, and the latter is Π^0_3. Thus this is a Π^0_n-class uniformly in ρ since $n \geq 2$ and n is odd, hence $n \geq 3$. Also note that it covers $u_n(\mathcal{A})$ because if f is (i,n)-Π-valid, then it is $(\rho, 3)$-Π-valid for some string ρ by definition. Furthermore, for each such f we can uniformly compute an element of $\mathcal{A}$ given a ρ such that $f \in \mathcal{V}_\rho$, by computing $p_{\mathrm{odd}}(\rho, f)$. This shows that $\mathcal{A} \leq_n u_n(\mathcal{A})$ and hence $u_n(\mathcal{A}) \equiv_n \mathcal{A}$.

In particular, if $m \leq n$, then we see that $u_n(\mathcal{A}) \leq_m u_n(\mathcal{B})$ implies that $\mathcal{A} \leq_n \mathcal{B}$. Next, assume that $\mathcal{A} \leq_n \mathcal{B}$. Then also $\mathcal{A} \leq_n u_n(\mathcal{B})$, as shown above. We will show that $u_n(\mathcal{A}) \leq_{\mathcal{M}} u_n(\mathcal{B})$. Fix a computable sequence $\sigma_{i,a_1,\dots,a_n}$ and a computable sequence $e_0, e_1, \dots$ such that $\Phi_{e_i}(\mathcal{V}_i \cap u_n(\mathcal{B})) \subseteq \mathcal{A}$, where

$$\mathcal{V}_i = \bigcap_{a_1 \in \omega} \bigcup_{a_2 \in \omega} \cdots \bigcap_{a_n \in \omega} [\![\sigma_{i,a_1,\dots,a_n}]\!].$$

Given any $f \in u_n(\mathcal{B})$, we show how to uniformly compute an element $\Psi(f)$ of $u_n(\mathcal{A})$. To define $\Phi(f)(m)$, wait until a stage s such that $\sigma_{i,a_1,\dots,a_n} \subseteq f$ for some $i, a_1, \dots, a_n \leq s$ for which no element of $\Phi(f) \upharpoonright m$ begins with $i, a_1, \dots, a_n$ and such that $\Phi_{e_i}(f)(a_{n-2})[s]\downarrow$. If so, let $(i, a_1, \dots, a_n)$ be the least sequence (in some fixed computable well-ordering of ω^{n+1}) for which this holds, and let $f(m) = (i a_1 \cdots a_n, \Phi_{e_i}(f)(a_{n-2}))$. Then $\Psi(f)$ is total because f is in $u_n(\mathcal{B}) \cap \mathcal{V}_i$ for some $i \in \omega$. We can also directly verify that $\Psi(f)$ is (i,n)-Π-valid for this i.

Finally, note that for every $m \in \omega$, letting $\Psi(f)(m) = (i a_1 \cdots a_n, b)$, we have that $b = \Phi_{e_i}(f)(a_{n-2})$ by construction. Furthermore, $\Psi(f)$ is (i,n)-Π-valid if and only if

$$\forall a_1 \exists a_2 \cdots \forall a_n (\sigma_{i,a_1,\dots,a_n} \subseteq f),$$

if and only if $f \in \mathcal{V}_i$. From this we can directly verify that if $\Psi(f)$ is $(\rho,3)$-Π-valid and $(\rho \upharpoonright 1, n)$-$\Pi$-valid, then we have that $p_{\mathrm{odd}}(\rho, \Psi(f)) = \Phi_{e_{\rho(0)}}(f) \in \mathcal{A}$. Thus, $\Psi(f) \in u_n(\mathcal{A})$ and therefore $\mathcal{A} \leq_n \mathcal{B}$ if and only if $u_n(\mathcal{A}) \leq_{\mathcal{M}} u_n(\mathcal{B})$. This also implies that the induced map is well defined for every m.

We have already seen that $u_n(\mathcal{A}) \oplus u_n(\mathcal{B}) \leq_{\mathcal{M}} u_n(\mathcal{A} \oplus \mathcal{B})$. Conversely, given $f \in u_n(\mathcal{A})$ and $g \in u_n(\mathcal{B})$, we show how to construct a function $\Psi(f \oplus g) \in u_n(\mathcal{A} \oplus \mathcal{B})$. What we basically need to use is the fact that the conjunction of two Σ^0_{n+1}-formulas is again a Σ^0_{n+1}-formula. However, we need to make sure we preserve the coding, which takes a little more work.

To define $\Psi(f \oplus g)(\langle 2i, j, a_2, a_4, \dots, a_{n-4}\rangle)$, we let $f(i) = (sb_1 \cdots b_n, x)$. Now we define

$$\begin{aligned}\Psi(f \oplus g)&\big(\langle 2i, j, a_2, a_4, \dots, a_{n-4}\rangle\big)\\ &= \big(\langle s, j\rangle^\frown b_1{}^\frown\langle b_2, a_2\rangle^\frown b_3 \cdots {}^\frown b_{n-3}{}^\frown(2 \cdot b_{n-2})^\frown b_{n-1}{}^\frown b_n, x\big).\end{aligned}$$

To define $\Psi(f \oplus g)(\langle 2i+1, a_2, a_4, \dots, a_{n-4}\rangle)$, we let $g(i) = (sb_1 \cdots b_n, x)$. Now we define

$$\begin{aligned}\Psi(f \oplus g)&\big(\langle 2i+1, j, a_2, a_4, \dots, a_{n-4}\rangle\big)\\ &= \big(\langle j, s\rangle^\frown b_1{}^\frown\langle a_2, b_2\rangle^\frown b_3 \cdots {}^\frown b_{n-3}{}^\frown(2 \cdot b_{n-2}+1)^\frown b_{n-1}{}^\frown b_n, x\big).\end{aligned}$$

Then, if $\rho = (\langle s, t\rangle^\frown a_1{}^\frown\langle a_2, b_2\rangle^\frown \cdots {}^\frown a_{n-3})$, it can be directly verified that $\Psi(f \oplus g)$ is $(\rho, 3)$-Π-valid if and only if f is $(sa_1a_2 \cdots a_{n-3}, 3)$-$\Pi$-valid and g is $(ta_1b_2 \cdots a_{n-3}, 3)$-$\Pi$-valid. Furthermore, in this case we have that

$$p_{\text{odd}}\big(\rho, \Psi(f \oplus g)\big) = p_{\text{odd}}(\rho, f) \oplus p_{\text{odd}}(\rho, g).$$

Now, a straightforward calculation shows that $\Psi(f \oplus g)$ is $(\langle s, t\rangle, n)$-Π-valid if and only if f is (s, n)-Π-valid and g is (t, n)-Π-valid. Combining all of this, we see that $\Psi(f \oplus g) \in u_n(\mathcal{A} \oplus \mathcal{B})$, which is what we needed to show.

Finally, that the implication is preserved can be proved in the same way as in the proof of Theorem 3.2, using the fact that $\mathcal{C} \geq_{\mathcal{M}} u_n(\mathcal{C})$ for every $\mathcal{C}$ as shown above. $\square$

4 Uniformity

Using the n-degrees, we can now introduce the measure of uniformity mentioned in the Introduction.

Definition 4.1 Let $\mathcal{A} \leq_w \mathcal{B}$. Then we say that the *uniformity of* $\mathcal{A}$ *to* $\mathcal{B}$ is the least $n \in \omega$ such that $\mathcal{A} \leq_n \mathcal{B}$, if such an n exists, and w otherwise.

As we will see later, there are cases when the uniformity is not a natural number. However, if $\mathcal{A}$ is a reasonable class, in the sense that it is arithmetical, then it turns out that the uniformity is in fact a natural number. This follows from the following result, which is due to Higuchi and Kihara [3], but the proof given here is our own.

Proposition 4.2 ([3, Proposition 27]) *Let $\mathcal{A} \leq_w \mathcal{B}$ be such that $\mathcal{A}$ is Σ^0_{n+1}. Then the uniformity of $\mathcal{A}$ to $\mathcal{B}$ is at most* $\max(n, 2)$.

Proof Let Φ be a Turing functional. Let $\mathcal{T} \subseteq \omega^\omega$ be the class of functions f for which $\Phi(f)$ is total, which is a Π^0_2-class. From Lemma 2.4 we then see that we have that $\Phi^{-1}(\mathcal{A}) \cap \mathcal{T}$ is $\Sigma^0_{\max(n,2)+1}$.

Now, let $\mathcal{V}_i$ be $\Phi_i^{-1}(\mathcal{A}) \cap \mathcal{T}$, and let $e_i = i$. Then the $\mathcal{V}_i$'s cover $\mathcal{B}$ since we assumed that $\mathcal{A} \leq_w \mathcal{B}$, and $\Phi_e(\mathcal{V}_e) \subseteq \mathcal{A}$. The result now follows, as discussed in Remark 2.2. $\square$

Remark 4.3 Note that, in the proof above, the reduction does not depend on $\mathcal{B}$, but only on $\mathcal{A}$. Thus, given any Σ^0_{n+1}-class $\mathcal{A}$, there is a single reduction which witnesses that $\mathcal{A} \leq_{\max(n,2)} \mathcal{B}$ for every $\mathcal{B}$ with $\mathcal{B} \geq_w \mathcal{A}$.

Corollary 4.4 *If $\mathcal{A}$ is arithmetical, then the uniformity of $\mathcal{A}$ to $\mathcal{B}$ is a natural number.*

In Theorem 5.3 below, we will see that Proposition 4.2 is optimal for $n \geq 2$. For $n = 2$, this follows from the following elegant pair of theorems.

Theorem 4.5 (Jockusch [7, Theorems 5 and 6]) *We have that* $\mathrm{DNC}_2 \equiv_w \mathrm{DNC}_3$, *but* $\mathrm{DNC}_2 \not\leq_{\mathcal{M}} \mathrm{DNC}_3$.

Theorem 4.6 ([4, Corollary 72]) *We have*

$$\mathrm{DNC}_2 \not\leq_1 \mathrm{DNC}_3.$$

Corollary 4.7 *The uniformity of* DNC_2 *to* DNC_3 *is* 2.

Proof This follows from the previous two theorems and the fact that DNC_2 is a Π^0_1-class. □

5 Uniformity and Algorithmic Randomness

To illustrate the definition given in the previous section, we will now study the uniformity of some well-known Muchnik reductions from algorithmic randomness. First, we will study a version of the effective 0-1-law, which is originally due to Kučera [9]. This will be a helpful tool during the remainder of this article.

Theorem 5.1 (Effective 0-1-law, Kučera [9, p. 248], Kautz [8, Lemma IV.2.1]) *Let $n \in \omega$, let $\mathcal{V}$ be a Π^0_n-class of positive measure, and let X be n-random. Then there is a $k \in \omega$ with $X \upharpoonright [k, \infty) \in \mathcal{V}$.*

Proof See, for example, [1, Theorem 6.10.2]. □

Theorem 5.2 *Let $n \in \omega$, let $\mathcal{A}$ be a mass problem, and let n-*Random *be the class of n-randoms. Then $\mathcal{A} \leq_n n$-*Random *if and only if there exists a Π^0_n-class $\mathcal{V}$ of positive measure such that $\mathcal{A} \leq_{\mathcal{M}} \mathcal{V}$.*

Proof First, assume that $\mathcal{A} \leq_n n$-Random, and let this be witnessed by $\mathcal{V}_0, \mathcal{V}_1, \ldots$ and $e_0, e_1, \ldots$. Then some $\mathcal{V}_i$ has positive measure by countable additivity, and $\mathcal{A} \leq_{\mathcal{M}} \mathcal{V}_i$ as witnessed by e_i.

Conversely, let $\mathcal{V}$ be a Π^0_n-class of positive measure, and let Ψ be such that $\Psi(\mathcal{V}) \subseteq \mathcal{A}$. By the effective 0-1-law, we know that for each n-random X there is an $m \in \omega$ such that $X \upharpoonright [m, \infty) \in \mathcal{V}$. Let $\mathcal{V}_m = 2^m{}^\frown\mathcal{V}$; that is, $\mathcal{V}_m$ is the Π^0_n-class consisting of those X such that $X \upharpoonright [m, \infty) \in \mathcal{V}$. Then the $\mathcal{V}_m$ cover the class of n-randoms, as we have just argued. Furthermore, if m is such that $X \in \mathcal{V}_m$, then we can compute an element of $\mathcal{A}$ uniformly from X and m, namely, $\Psi(X \upharpoonright [m, \infty))$. □

That for some Π^0_n-classes $\mathcal{V}$ we have that n-reducibility is optimal, that is, that there are Π^0_n-classes $\mathcal{V}$ and mass problems $\mathcal{A}$ such that $\mathcal{A} \leq_{\mathcal{M}} \mathcal{V}$ but $\mathcal{A} \not\leq_m n$-Random for any $m < n$, will follow from the next theorem.

For any ordinal α which is not 0 or a limit ordinal, we say that f is *α-DNC* (diagonally noncomputable) if f is DNC relative to $\emptyset^{(\alpha-1)}$. Kučera [9] has shown that every n-random set computes an n-DNC function, which lies at the basis of the following theorem.

Theorem 5.3 *Let $n \in \omega$ with $n \geq 1$. Then n-DNC Muchnik-reduces to n-randomness, with uniformity n. Also, $(\omega + 1)$-DNC Muchnik-reduces to $(\omega + 1)$-randomness, with uniformity w.*

Proof Let us first consider $n \in \omega$. From the standard proof of Kučera's result mentioned above (see, e.g., [1, Theorem 8.8.1]), we know that there is a $\Pi_1^{0,\emptyset^{(n-1)}}$-class $\mathcal{V}$ (which is a specific kind of Π_n^0-class) for which n-DNC Medvedev-reduces to $\mathcal{V}$. Now apply Theorem 5.2. Alternatively, for $n \geq 2$ we can use that the collection of n-DNC functions is $\Pi_1^{0,\emptyset^{(n-1)}}$ and apply Proposition 4.2.

On the other hand, let $m < n$, and assume toward a contradiction that n-DNC m-reduces to n-randomness. Fix witnesses $\mathcal{V}_0, \mathcal{V}_1, \ldots$ and $e_0, e_1, \ldots$ for this fact (so $\mathcal{V}_0, \mathcal{V}_1, \ldots$ are uniformly Π_m^0). Without loss of generality, we may assume that every $\mathcal{V}_i$ is a subclass of 2^ω. We show that $\emptyset^{(m)}$ computes an n-DNC function f, which is a contradiction.

Indeed, given $k \in \omega$, to define $f(k)$, look for the first $\sigma \in 2^{<\omega}$ and $i \in \omega$ such that $\mathcal{V}_i$ has strictly positive measure above σ and such that $\Phi_{e_i}(\sigma)(k)\downarrow$. Since the measure of a Π_m^0-class is $\emptyset^{(m)}$-computable, we can do this computably in $\emptyset^{(m)}$. Furthermore, note that such σ and i have to exist by countable additivity. Now $\Phi_{e_i}(\sigma)(k) \neq \{k\}^{\emptyset^{(n-1)}}(k)$ since σ can be extended to an n-random within $\mathcal{V}_i$ (after all, it has positive measure above σ) and $\Phi_{e_i}(\mathcal{V}_i) \subseteq n$-DNC. Thus, f is n-DNC, as desired.

Finally, let us consider $n = w$. The upper bound (i.e., the fact that $(\omega + 1)$-DNC Muchnik-reduces to $(\omega + 1)$-randomness) again follows from the standard proof. On the other hand, if $(\omega + 1)$-randomness m-reduces to $(\omega + 1)$-DNC for some natural number m, then the same argument as above shows that $\emptyset^{(m)}$ computes an $(\omega + 1)$-DNC function, which is again a contradiction. □

The previous theorem now also allows us to separate n-reducibility from m-reducibility for $n \neq m$.

Corollary 5.4 *For every $n \in \omega \cup \{w\}$, there are mass problems $\mathcal{A} \leq_n \mathcal{B}$ such that for no $m < n$, we have $\mathcal{A} \leq_m \mathcal{B}$.*

Recall that n-DNC$_{2^m}$ is the class of n-DNC functions f for which $f(m) \leq 2^m$.

Corollary 5.5 *Let $n \in \omega$ with $n \geq 1$. Then n-DNC$_{2^m}$ Muchnik-reduces to n-randomness, with uniformity n. Furthermore, ω-DNC$_{2^m}$ Muchnik-reduces to ω-randomness, with uniformity w.*

Proof This follows from the same proof as Theorem 5.3. □

Next, let us compare our notion with the notion of layerwise computability introduced by Hoyrup and Rojas [6]. When looking at the uniformity of some $\mathcal{A}$ to n-randomness, we allow arbitrary Π_n^0-classes covering the class of n-randoms to witness this reduction. Instead, we could also decide to only allow the class $\mathcal{V}_n$ to be the nth layer, that is, the complement of $\mathcal{U}_n$ for some fixed universal n-randomness test $\mathcal{U}_0, \mathcal{U}_1, \ldots$. We now show that this is strictly weaker than our notion.

Proposition 5.6 *We do not have that n-DNC_{2^m} reduces layerwise to n-randomness; that is, there is no computable $e_0, e_1, \ldots$ such that $\Phi_{e_i}(\mathcal{V}_i)$ is contained in n-DNC_{2^m} for every $i \in \omega$, where $\mathcal{V}_i$ is the complement of $\mathcal{U}_i$ and $\mathcal{U}_0, \mathcal{U}_1, \ldots$ is some fixed universal n-randomness test.*

Proof Toward a contradiction, assume that such a sequence e_i exists; we show that there is a computable n-DNC_{2^m} function f, which is a contradiction. Given m, to define $f(m)$, look for the first $i \in \omega$ and $k \in \omega$ such that at least measure 2^{-i+1}-many strings σ satisfy $\Phi_{e_i}(\sigma)(m)\downarrow = k$. Such a string must exist, since for every $X \in \mathcal{V}_i$, we have that $\Phi_{e_i}(X)(m)\downarrow \in \{0, \ldots, 2^m\}$ and $\mathcal{V}_i$ has measure at least $1 - 2^{-i}$, so at least measure $\frac{1-2^{-i}}{2^m+1}$-many X must be sent to the same value k, and if i is large enough, then $\frac{1-2^{-i}}{2^m+1}$ is at least 2^{-i+1}. Furthermore, for any i and k such that at least measure 2^{-i+1}-many strings σ satisfy $\Phi_{e_i}(\sigma)(m)\downarrow = k$, we know that

$$\mu\big(\mathcal{V}_i \cap \{X \mid \Phi_{e_i}(X)(m)\downarrow = k\}\big) \geq 1 - (2^{-i} + 1 - 2^{-i+1}) = 2^{-i} > 0,$$

so there is some X in this set and for such X we have that $\Phi_{e_i}(X)$ is n-DNC_{2^m}, so $k \neq \{m\}^{\emptyset^{(n-1)}}(m)$. Therefore, f is n-DNC_{2^m}, as desired. $\square$

Next, let us study the result from Kautz [8, Theorem IV.2.4(iv)] which states that every 2-random degree is hyperimmune, that is, that every 2-random set computes a function which is not computably dominated. Note that the noncomputably dominated functions form a Π^0_3-class, so Proposition 4.2 tells us that the uniformity is at most 3. However, our version of the effective 0-1-law (see Theorem 5.2) tells us that it is even at most 2. We next show that the uniformity is exactly 2.

Theorem 5.7 *The uniformity of the noncomputably dominated functions to the 2-random sets is 2.*

Proof The upper bound follows from the standard proof and our effective 0-1-law, as discussed above. Toward a contradiction, assume that there exists a sequence $\mathcal{V}_0, \mathcal{V}_1, \ldots \subseteq 2^\omega$ of uniformly Π^0_1-classes and uniformly computable $e_0, e_1, \ldots$ which witness that the noncomputably dominated functions 1-reduce to the 2-random sets. Then at least one $\mathcal{V}_i$ has positive measure; without loss of generality, we may assume that this is $\mathcal{V}_0$. Let $q > 0$ be a rational such that $\mathcal{V}_0$ has measure at least q. We will use a majority vote argument to show that there is a computable function which is not computably dominated—a contradiction.

We will define a computable function f and a sequence $\mathcal{V}_0 \supseteq \mathcal{W}_0 \supseteq \mathcal{W}_1 \supseteq \cdots$ of Π^0_1-classes such that every $\mathcal{W}_i$ has measure at least $q(\frac{1}{2} + 2^{-i-2})$ and such that $f(i) \geq \Phi_{e_0}(X)(i)$ for every $X \in \mathcal{W}_i$. In particular, $\mathcal{W} = \bigcap_{i\in\omega} \mathcal{W}_i$ has positive measure, so it is nonempty. Furthermore, if $X \in \mathcal{W} \subseteq \mathcal{V}_0$, then f dominates the function $\Phi_{e_0}(X)$, which is a contradiction.

We let $W_{-1} = \mathcal{V}_0$. At stage s we define $f(s)$ and $\mathcal{W}_s$. Let U_{s-1} be a prefix-free set of strings such that $[\![U_{s-1}]\!] = \overline{\mathcal{W}_{s-1}}$. Look for $n \in \omega$ such that for at least measure $1 - q2^{-s-3}$-many strings $\sigma \in 2^n$, we have that either $\sigma \in U_{s-1}$ or $\Phi_{e_0}(\sigma)(s)\downarrow$. Such an n must exist since $\mathcal{W}_{s-1} \subseteq \mathcal{V}_0$. Once this happens, we let

$$\mathcal{W}_s = \mathcal{W}_{s-1} \setminus [\![\{\sigma \in 2^n \mid \Phi_{e_0}(\sigma)(s)\uparrow\}]\!],$$

and we let $f(s)$ be the maximum of $\Phi_{e_0}(\sigma)(s)$ for those $\sigma \in 2^n$ for which $\Phi_{e_0}(\sigma)(s)\downarrow$. Then $\mathcal{W}_s$ has measure at least

$$\mu(\mathcal{W}_{s-1}) - q2^{-s-3} = q\Big(\frac{1}{2} + 2^{-s-3} + 2^{-s-3}\Big) = q\Big(\frac{1}{2} + 2^{-s-2}\Big),$$

and it is clear that $f(s) \geq \Phi_{e_0}(X)(s)$ for every $X \in \mathcal{W}_s$. $\square$

Next, we consider another result from Kautz [8, Theorem IV.2.4(v)] which states that the 2-random sets even compute 1-generic sets.

Theorem 5.8 *The uniformity of* 1*-genericity to* 2*-randomness is* 2.

Proof Again, that the uniformity is at most 2 follows from the standard proof together with our effective 0-1-law (see Theorem 5.2). If it were the case that 1-genericity 1-reduced to 2-randomness, then the noncomputably dominated functions would also 1-reduce to the 2-random sets because the 1-generic sets uniformly compute functions which are not computably dominated (this follows directly from the proof of Kurtz [10, Corollary 1.1a] that every (weakly) 1-generic set is hyperimmune). This contradicts Theorem 5.7. $\square$

6 Theory of the 1-Uniform Degrees as a Brouwer Algebra

In this section, we study the propositional theory of $\mathcal{M}_1$ as a Brouwer algebra. (For more background on this, we refer to Sorbi [20].)

As for the Medvedev lattice, the theory of $\mathcal{M}_1$ is not IPC because the top element is join-irreducible. We want to study what happens if we look at factors $\mathcal{M}_1/\mathcal{A}$, that is, the quotient of $\mathcal{M}_1$ by the principal filter generated by the degree of $\mathcal{A}$. Skvortsova [18] has shown that there is such a factor of the Medvedev lattice for which the theory is IPC. Her techniques were slightly improved in Kuyper [12]. We will show that these techniques can be adapted to prove that there is a factor of $\mathcal{M}_1$ which has as theory IPC (which implies that the theory of $\mathcal{M}_1$ is Jankov's logic Jan, i.e., IPC plus the weak law of the excluded middle $\neg p \vee \neg\neg p$; see the proof of [12, Corollary 5.3]).

The crucial ingredient we need to adapt Skvortsova's techniques to $\mathcal{M}_1$ is the next proposition.

Proposition 6.1 *Let $\mathcal{A} \in \mathcal{M}_1$ be a Muchnik degree, that is, the degree of some set of functions which is upwards closed under Turing reducibility. Then $\mathcal{A}$ is meet-irreducible. In fact, for all $\mathcal{B}, \mathcal{C} \in \mathcal{M}_1$ we have*

$$\mathcal{A} \to_1 (\mathcal{B} \otimes \mathcal{C}) \geq_1 (\mathcal{A} \to_{\mathcal{M}} \mathcal{B}) \otimes (\mathcal{A} \to_{\mathcal{M}} \mathcal{C}) \geq_{\mathcal{M}} (\mathcal{A} \to_1 \mathcal{B}) \otimes (\mathcal{A} \to_1 \mathcal{C}).$$

Proof Let $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathcal{M}_1$ with $\mathcal{A}$ a Muchnik degree. If $\mathcal{A}$ is the degree of the empty mass problem, then the result is certainly true; so, we may assume this is not the case. It is not hard to see that the meet-irreducibility follows from the second claim. That

$$(\mathcal{A} \to_{\mathcal{M}} \mathcal{B}) \otimes (\mathcal{A} \to_{\mathcal{M}} \mathcal{C}) \geq_{\mathcal{M}} (\mathcal{A} \to_1 \mathcal{B}) \otimes (\mathcal{A} \to_1 \mathcal{C})$$

is also not hard to see; it follows from the fact that a uniform reduction is a special case of an n-reduction. So, it remains to prove that

$$\mathcal{A} \to_1 (\mathcal{B} \otimes \mathcal{C}) \geq_1 (\mathcal{A} \to_{\mathcal{M}} \mathcal{B}) \otimes (\mathcal{A} \to_{\mathcal{M}} \mathcal{C}).$$

Let $u^\frown e^\frown f \in \mathcal{A} \to (\mathcal{B} \otimes \mathcal{C})$. We claim that there exists a $\sigma \in \omega^{<\omega}$ and an $i \in \omega$ such that for every h extending σ we have $(h \oplus f) \oplus f \in \mathcal{P}^1_{\{u\}(i)}$, and such that

$$\Phi_{\{e\}(i)}\big((\sigma \oplus (f \upharpoonright |\sigma|)\big) \oplus \big(f \upharpoonright 2|\sigma|)\big)(0)\downarrow.$$

For now, let us assume that this claim holds and explain how this proves the result. Namely, consider the Π^0_1-classes $\mathcal{W}_{u,e,\sigma,i}$ where all elements of $\mathcal{W}_{u,e,\sigma,i}$ are of the form $u^\frown e^\frown f$, and $u^\frown e^\frown f \in \mathcal{W}_{u,e,\sigma,i}$ if and only if $\{e\}(i)[|\sigma|]\downarrow$, $\{u\}(i)[|\sigma|]\downarrow$,

$$\Phi_{\{e\}(i)}\big((\sigma \oplus (f \upharpoonright |\sigma|)\big) \oplus \big(f \upharpoonright 2|\sigma|)\big)(0)\downarrow,$$

and $(h \oplus f) \oplus f \in \mathcal{P}^1_{\{u\}(i)}$ for every h extending σ. Then, if $u^\frown e^\frown f$ is in both $\mathcal{A} \to (\mathcal{B} \otimes \mathcal{C})$ and $\mathcal{W}_{u,e,\sigma,i}$, compute

$$a = \Phi_{\{e\}(i)}\big((\sigma \oplus (f \upharpoonright |\sigma|)\big) \oplus \big(f \upharpoonright 2|\sigma|)\big)(0).$$

Let b be an index for the functional sending $h_0 \oplus h_1$ to $\Phi_{\{e\}(i)}(((\sigma^\frown h_0) \oplus h_1) \oplus h_1)$ without the first bit. Then it can be directly verified that

$$a^\frown b^\frown f \in (\mathcal{A} \to_{\mathcal{M}} \mathcal{B}) \otimes (\mathcal{A} \to_{\mathcal{M}} \mathcal{C}),$$

where we use the fact that $\mathcal{A}$ is dense because it is upwards closed under Turing reducibility. Furthermore, the $\mathcal{W}_{u,e,\sigma,i}$ cover $\mathcal{A} \to_1 (\mathcal{B} \otimes \mathcal{C})$ by the claim. Thus, this proves the result.

So, let us prove the claim. Toward a contradiction, assume that such σ and i do not exist. Fix a $g \in \mathcal{A}$. Then we also know that $\sigma^\frown g \in \mathcal{A}$ for every string σ, again since $\mathcal{A}$ is dense.

We will now construct an h such that $h \oplus f \geq_T g$ (and hence $h \oplus f \in \mathcal{A}$) and such that $(h \oplus f) \oplus f \notin \bigcup_{i \in \omega} \mathcal{P}^1_{\{e\}(i)}$, which is a contradiction. We construct $h = \bigcup_i \sigma_i$ by a finite extension argument. Let $\sigma_{-1} = \emptyset$. Given σ_{i-1}, to define σ_i we let $\tau \supseteq \sigma_{i-1}$ be the first string such that $[\![(\tau \oplus (f \upharpoonright |\tau|)) \oplus (f \upharpoonright 2|\tau|)]\!] \cap \mathcal{P}^1_{\{u\}(i)} = \emptyset$. Let $\sigma_i = \tau^\frown g(i)$.

Note that such a string τ always exists: namely, if such a string did not exist, we have that $((\sigma_{i-1}{}^\frown g) \oplus f) \oplus f \in \mathcal{P}^1_{\{u\}(i)}$. So, there is some $s \in \omega$ with $\Phi_{\{e\}(i)}((((\sigma_{i-1}{}^\frown g) \oplus f) \oplus f) \upharpoonright 4s)(0)\downarrow$. Since we assumed the claim is false, we then know that there is some $v \in \omega^\omega$ extending $(\sigma_{i-1}{}^\frown g) \upharpoonright s$ for which $(v \oplus f) \oplus f \notin \mathcal{P}^1_{\{u\}(i)}$. Thus, there is some $(\sigma_{i-1}{}^\frown g) \upharpoonright s \subseteq \tau \subseteq v$ such that $[\![(\tau \oplus (f \upharpoonright |\tau|)) \oplus (f \upharpoonright 2|\tau|)]\!]$ is disjoint from $\mathcal{P}^1_{\{u\}(i)}$, as desired.

Note that the entire procedure is f-computable, so $h \oplus f$ computes g. Furthermore, $(h \oplus f) \oplus f \notin \bigcup_{i \in \omega} \mathcal{V}_{\{e\}(i)}$ by construction, which proves the claim and hence the result. □

In the terminology of [18], we have just shown that the set of Muchnik degrees is *canonical* in $\mathcal{M}_1$. This allows us to show the following result. Given a Brouwer algebra B, recall that a *valuation* (relative to B) is a function mapping the propositional variables to elements of a Brouwer algebra. Such a valuation can be canonically extended to the set of all formulas by interpreting $\vee$ as $\otimes$, $\wedge$ as $\oplus$, $\to$ as $\to$, and $\perp$ as 1. We now say that the *propositional theory of B*, written as $\mathrm{Th}(B)$, is the set of all formulas ϕ such that for all valuations v we have that $v(\phi) = 0$. Again, for more background we refer to Sorbi [20].

Theorem 6.2 *Let A be a computably independent set, and let*

$$\mathcal{A} = \{i^\frown f \mid f \geq_T A^{[i]}\}.$$

Then $\mathrm{Th}(\mathcal{M}_1/\mathcal{A}) = \mathrm{IPC}$.

Proof We give a sketch. The proof uses the exact same technique as the proof of [12, Theorem 1.1]. The only real modification is that we need a new proof of the fact that the Muchnik degrees are canonical in $\mathcal{M}_1$, which we have just given.

In a bit more detail, the proof in [12] proceeds as follows.

(1) First, [12, Theorem 3.3] shows that there are certain embeddings of $(\mathcal{P}(I), \supseteq)$ into intervals $[\mathcal{B}, \overline{\mathcal{A}}]_{\mathcal{M}}$. It can be directly verified that the proof given there also works for $\mathcal{M}_1$ (in fact, it works for every $\mathcal{M}_n$).

(2) Next, [12, Corollary 4.3] and [12, Corollary 4.5] extend these embeddings to free Brouwer algebras. This uses the previous fact together with the fact that the set of Muchnik degrees is canonical in the Medvedev lattice. As we have just shown, this is also true in $\mathcal{M}_1$, so these two corollaries also hold in $\mathcal{M}_1$.

(3) Finally, [12, Theorem 1.1] combines these facts with some general lattice-theoretic facts and some computability-theoretic facts. In the proof, certain sets are constructed such that certain equalities hold between certain mass problems in the Medvedev lattice; but of course, if things are Medvedev-equivalent they are certainly 1-equivalent. Therefore, the proof proceeds in the same way for $\mathcal{M}_1$. □

Corollary 6.3 *We have*

$$\mathrm{Th}(\mathcal{M}_1) = \mathrm{Jan}.$$

Proof See the proof of [12, Corollary 5.3]. □

Let us next note that the technique discussed in this section does not work for $n \geq 2$.

Proposition 6.4 *Let $n \in \omega$. If f, g are Δ^0_n, then $C(\{f\}) \otimes C(\{g\}) \equiv_n C(\{f, g\})$.*

Proof If $n \leq 1$, this follows from the fact that the bottom element of $\mathcal{M}_n$ is meet-irreducible. So, assume that $n \geq 2$. Clearly $C(\{f\}) \otimes C(\{g\}) \geq_n C(\{f, g\})$; in fact, this reduction is even a Medvedev reduction since it is just inclusion. For the converse, note that the upper cone of a Δ^0_n-set is Σ^0_{n+1}. By Remark 2.2 above, we therefore see that $C(\{f\}) \otimes C(\{g\}) \leq_n C(\{f, g\})$ by sending $h \in C(\{f, g\}) \cap C(\{f\})$ to $0^\frown h$ and $h \in C(\{f, g\}) \cap C(\{g\})$ to $0^\frown h$. □

Corollary 6.5 *The set of Muchnik degrees is not canonical in $\mathcal{M}_n$ for $n \geq 2$.*

Proof Let f, g be two incomparable Δ^0_2-functions. Then $\mathcal{B} \otimes \mathcal{C} \leq_2 \mathcal{A}$ by the previous proposition. On the other hand, we do not have $\mathcal{B} \leq_w \mathcal{A}$ nor $\mathcal{C} \leq_w \mathcal{A}$ since f and g are incomparable. □

Thus, if we want to study the theory of $\mathcal{M}_n$ for $n \geq 2$, a different technique is needed.

7 Theory of the n-Uniform Degrees as a Brouwer Algebra for $n \geq 4$

In Sorbi and Terwijn [21], it is shown that there are factors of the Muchnik lattice which yield IPC. A different proof is given in Kuyper [11]. By studying how uniform the reductions in that proof are, we show that such factors in fact exist for $\mathcal{M}_n$ with $n \geq 4$.

Theorem 7.1 *Let $\mathcal{A}$ be the class of* 1*-generic Δ^0_2-functions together with the computable functions, and let $n \geq 4$. Then $\mathcal{M}_n/\overline{\mathcal{A}}$ has theory IPC.*

Proof This follows from a careful analysis of the proof for the Muchnik lattice in [11, Theorem 5.8]. In that proof, a function α from $\mathcal{A}$ to $2^{<\omega}$ satisfying certain properties is constructed. First note that we can see such a function as a partial function β from $\omega \to 2^{<\omega}$ instead, identifying functions in $\mathcal{A}$ with their Δ^0_2-indices. Then the graph of β is Σ^0_4.

(i) Checking that e is an index for a Δ^0_2-set, that is, that $\{e\}^{\emptyset'}$ is total, is Π^0_3.

(ii) Checking that $\{e\}^{\emptyset'}$ is 1-generic is Π^0_3. Let $W_0, W_1, \ldots$ be an effective enumeration of the c.e. sets. Then $\{e\}^{\emptyset'}$ is 1-generic if and only if

$$\forall e \exists n \exists s \big(\forall t \geq s \big(\{e\}^{\emptyset'[t]} \upharpoonright n \in \mathcal{W}_e[s]\big) \vee \forall \tau \forall t \geq s \big(\tau \supseteq \{e\}^{\emptyset'[t]} \to \tau \notin \mathcal{W}_e[t]\big)\big).$$

(iii) In the construction we need to check if $\{e_0\}^{\emptyset'} \leq_T \{e_1\}^{\emptyset'}$ for e_0, e_1 which are indices for Δ^0_2-sets, this is Σ^0_4:

$$\exists a \forall n \exists s \forall t \geq s \big(\{a\}^{\{e_1\}^{(\emptyset' \upharpoonright s)[t]}}(n)[t] \downarrow = \{e_0\}^{\emptyset'[t]}(n)[t]\big).$$

(iv) In the construction we need to, given e_0 and finitely many points already defined, do some kind of splitting to find an e_1 satisfying certain properties (this happens in [11, Theorem 4.3]). We can find this index e_1 effectively from e_0 and the points already defined.

Now, any function f is in some $\alpha^{-1}(C(\sigma))$ if and only if f is not in $\mathcal{A}$ (which is Π^0_4) or if there exists an e with $f = \{e\}^{\emptyset'}$ (which is Π^0_3) such that $\beta(e)$ extends σ (which is Σ^0_4, as argued above). Thus, every $\alpha^{-1}(\sigma)$ is Σ^0_5. Using Remark 2.2 it is not hard to see that the meet of two Σ^0_{n+1}-classes is their union in $\mathcal{M}_n$. Furthermore, since each $\alpha^{-1}(C(\sigma))$ is upwards closed, and for upwards closed $\mathcal{A}$ and $\mathcal{B}$ we have that their join is just their intersection and that

$$\mathcal{A} \to_n \mathcal{B} = \big\{f \in \omega^\omega \;\big|\; \forall g \in \mathcal{A}(f \oplus g \in \mathcal{B})\big\}$$

for all $n \in \omega \cup \{w\}$, we see that the Muchnik degrees of $\{\alpha^{-1}(\sigma) \mid \sigma \in 2^{<\omega}\}$ and the $\mathcal{M}_n$-degrees of $\{\alpha^{-1}(\sigma) \mid \sigma \in 2^{<\omega}\}$ for $n \geq 4$ are all pairwise isomorphic. Since these are the only degrees used in the proof in [11], the remainder of the proof is now exactly the same as for the Muchnik lattice. □

Corollary 7.2 *For $n \geq 4$, we have that* $\mathrm{Th}(\mathcal{M}_n) = \mathrm{Jan}$.

Proof See the proof of [12, Corollary 5.3]. □

Thus, we have dealt with the cases $n \leq 1$ and $n \geq 4$. The author currently does not know how to deal with the cases $n = 2$ and $n = 3$. However, we conjecture the following.

Conjecture 7.3 *The propositional theories of $\mathcal{M}_2$ and $\mathcal{M}_3$ are* Jan*; in fact, there are principal factors of $\mathcal{M}_2$ and $\mathcal{M}_3$ which have as propositional theory* IPC.

8 Elementary Equivalence

Finally, in this section we will show that the first-order theories of the n-degrees as lattices (or equivalently, as partially ordered sets) are pairwise different. Recall that a *degree of solvability* is a degree of a singleton $\{f\}$, and that the degrees of solvability ordered by $\leq_{\mathcal{M}}$ are isomorphic to the Turing degrees, as shown by Medvedev [13]. First, we show that Dyment's (Dyment is the maiden name of Skvortsova) definition of the degrees of solvability in the Medvedev lattice from Dyment [2] works in every $\mathcal{M}_n$.

Proposition 8.1 *For every $n \in \omega \cup \{w\}$, the degrees of solvability in $\mathcal{M}_n$ are definable as exactly those $\mathcal{A}$ for which there is a $\mathcal{B} >_n \mathcal{A}$ such that every $\mathcal{C} >_n \mathcal{A}$ satisfies $\mathcal{C} \geq_n \mathcal{B}$.*

Proof For one direction, given a degree of solvability $\{f\}$, let

$$\mathcal{B} = \left\{e^\frown g \;\middle|\; \Phi_e(g) = f \wedge g \not\leq_T f\right\}.$$

Then $\mathcal{B} >_n \mathcal{A}$ for every $n \in \omega \cup \{w\}$, and if $\mathcal{C} >_n \mathcal{A}$ is witnessed by $\mathcal{V}_0, \mathcal{V}_1, \dots$ and $e_0, e_1, \dots$, then we can witness $\mathcal{C} \geq_n \mathcal{B}$ by sending $g \in \mathcal{V}_i \cap \mathcal{C}$ to $e_i{}^\frown g$.

For the converse, let us first consider the case $n = w$. Let $\mathcal{A}$ and $\mathcal{B}$ be as in the statement of the proposition. Then $\mathcal{B} \not\leq_w \mathcal{A}$ so there is an $f \in \mathcal{A}$ such that letting $\mathcal{C} = \{f\}$, we have $\mathcal{C} \not\geq_w \mathcal{B}$. Furthermore, we clearly have $\mathcal{C} \geq_w \mathcal{A}$. Now, if $\mathcal{C} > \mathcal{A}$, then we would have $\mathcal{C} \geq_w \mathcal{B}$ by our assumption on $\mathcal{B}$, a contradiction. Thus $\mathcal{A} \equiv_w \mathcal{C}$, and $\mathcal{A}$ is therefore a degree of solvability.

Finally, let us consider the converse direction for $n \in \omega$, in which case we follow the proof of [2] (see, e.g., [20, Theorem 2.3]). That is, given $\mathcal{A}$ which is not a degree of solvability and $\mathcal{B} \not\leq_n \mathcal{A}$, we construct a $\mathcal{C} >_n \mathcal{A}$ with $\mathcal{C} \not\geq_n \mathcal{B}$. We construct $\mathcal{C}$ as a set of the form $\{x_i{}^\frown f_i \mid i \in \omega\}$ with $f_i \in \mathcal{A}$, which ensures that $\mathcal{C} \geq_{\mathcal{M}} \mathcal{A}$ and hence $\mathcal{C} \geq_n \mathcal{A}$. Our remaining requirements are therefore

(a) $P_{e,u}$: $\exists j \in \omega.\{e\}(j)\uparrow \vee \{u\}(j)\uparrow \vee \Phi_{\{e\}(j)}(\mathcal{C} \cap \mathcal{P}^n_{\{u\}(j)}) \not\subseteq \mathcal{B}$,
(b) $R_{e,u}$: $\exists j \in \omega.\{e\}(j)\uparrow \vee \{u\}(j)\uparrow \vee \Phi_{\{e\}(j)}(\mathcal{A} \cap \mathcal{P}^n_{\{u\}(j)}) \not\subseteq \mathcal{C}$.

The construction is now as in the Medvedev case, so we refer to [20, Theorem 2.3] for the details. □

For the Medvedev lattice, the following result is from [2, Theorem 3.5]; we generalize it to our setting.

Proposition 8.2 *For every $n \in \omega$, the Muchnik degree $\mathcal{B}$ of $\mathcal{A}$ (i.e., the degree of $C(\mathcal{A}) = \{f \in \omega^\omega \mid \exists g \in \mathcal{A}(f \geq_T g)\}$) is definable by the formula $\phi(\mathcal{A}, \mathcal{B})$ given by*

$$\begin{aligned}&\forall\{f\}\big(\{f\} \geq_n \mathcal{B} \to \{f\} \geq_n \mathcal{A}\big)\\ &\quad\wedge \forall\mathcal{C}\big(\forall\{f\}\big(\{f\} \geq_n \mathcal{C} \to \{f\} \geq_n \mathcal{A}\big) \to \mathcal{C} \geq_n \mathcal{B}\big),\end{aligned}$$

where we use the result from Proposition 8.1 that the degrees of solvability are definable. In particular, Muchnik reducibility is definable by

$$\psi(\mathcal{A}_0, \mathcal{A}_1) = \forall B_0, B_1\big(\phi(\mathcal{A}_0, \mathcal{B}_0) \wedge \phi(\mathcal{A}_1, \mathcal{B}_1) \to \mathcal{B}_0 \leq_n \mathcal{B}_1\big).$$

Proof First, given $\mathcal{A}$, let $\mathcal{B}$ be the Muchnik degree of $\mathcal{A}$. Then clearly $\{f\} \geq_n \mathcal{B}$ implies that $\{f\} \geq_n \mathcal{A}$; in fact, $\{f\} \geq_w \mathcal{B}$ implies that $\{f\} \geq_{\mathcal{M}} \mathcal{B}$. Next, let $\mathcal{C}$ be such that for every f with $\{f\} \geq_n \mathcal{C}$ we have $\{f\} \geq_n \mathcal{A}$. Then in particular, for every $f \in \mathcal{C}$ we have $f \geq_T g$ for some $g \in \mathcal{A}$, and therefore $f \in \mathcal{B}$. Thus $\phi(\mathcal{A}, \mathcal{B})$ holds.

Conversely, given $\mathcal{A}$, $\mathcal{B}$ with $\phi(\mathcal{A}, \mathcal{B})$, let $\mathcal{C}$ be the Muchnik degree of $\mathcal{A}$. We will show that $\mathcal{B} \equiv_n \mathcal{C}$. First, if $f \in \mathcal{B}$, then $\{f\} \geq_n \mathcal{B}$, so $\{f\} \geq_n \mathcal{A}$ by the first conjunct of $\phi(\mathcal{A}, \mathcal{B})$. So, $f \geq_T g$ for some $g \in \mathcal{A}$, and therefore we see that

$\mathcal{B} \subseteq \mathcal{C}$; thus $\mathcal{C} \leq_{\mathcal{M}} \mathcal{B}$. Finally, apply the second conjunct of $\phi(\mathcal{A}, \mathcal{B})$ to $\mathcal{C}$ to obtain $\mathcal{C} \geq_n \mathcal{B}$. □

In Proposition 6.4, we gave a sufficient condition so that the meet of two Muchnik degrees $C(\{f\})$ and $C(\{g\})$ is n-equivalent to $C(\{f, g\})$. We now give an example of a situation in which this is not the case.

Proposition 8.3 *Let $n \in \omega$ with $n \geq 1$, let X be weak n-random, let Y be n-random, and let X and Y be Turing incomparable (in particular, these conditions are all satisfied if $X \oplus Y$ is n-random). Then $C(\{X\}) \otimes C(\{Y\}) \not\leq_n C(\{X, Y\})$.*

Proof Assume toward a contradiction that $C(\{X\}) \otimes C(\{Y\}) \leq_n C(\{X, Y\})$ is witnessed by $\mathcal{V}_0, \mathcal{V}_1, \ldots$ and $e_0, e_1, \ldots$. Since the $\mathcal{V}_i$ cover $C(\{X\})$, we know there is some $i \in \omega$ with $X \in \mathcal{V}_i$. Fix such an i. Determine $s \in \omega$ such that $\Phi_{e_i}(X)[s]\downarrow$. Note that then $\Phi_{e_i}(X)[s] = 0$ because X does not compute Y. Also, since X is weakly n-random we know that $\mu(V_i \cap [\![X \restriction s]\!]) > 0$, because X is not in any Π^0_n-class of measure 0. So, by the effective 0-1-law (see Theorem 5.1) applied to Y, we know that $Y \restriction [k, \infty) \in V_i \cap [\![X \restriction s]\!]$ for some $k \in \omega$. But then $X \restriction s \subseteq Y \restriction [k, \infty)$ so $\Phi_{e_i}(Y \restriction [k, \infty))(0)\downarrow = 0$, and therefore $\Phi_{e_i}(Y \restriction [k, \infty))\downarrow \in 0^\frown C(\{X\})$. Thus $Y \equiv_T Y \restriction [k, \infty) \geq_T X$, which is a contradiction. □

We now show that $\mathcal{M}_n$ and $\mathcal{M}_m$ are not elementarily equivalent for almost all $n \neq m$; we have to exclude the case $n = 0$ and $m = 1$.

Theorem 8.4 *Let $n, m \in \omega \cup \{w\}$ with $n < m$ and $m \geq 2$. Then $\mathcal{M}_n$ and $\mathcal{M}_m$ are not elementarily equivalent.*

Proof First, if $m = w$, then this follows from the fact that $\leq_n$ and $\leq_w$ do not coincide, as shown in Corollary 5.4, together with the fact shown in Proposition 8.2 above that Muchnik reducibility is definable in $\mathcal{M}_n$. That is, take the formula ϕ which says that there are $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A} \not\leq \mathcal{B}$ while $\mathcal{A} \leq_w \mathcal{B}$. Then ϕ holds with $\leq$ interpreted as $\leq_n$, but clearly not with $\leq$ interpreted as $\leq_w$.

Next, let $m \in \omega$. By Shore and Slaman [17], we know that the jump is definable in the Turing degrees, so in particular the Δ^0_n-degrees are definable in the Turing degrees. Since the degrees of solvability are definable, as shown in Proposition 8.1 above, we can therefore express the statement "for all $f, g \in \Delta^0_m$, we have that $C(\{f\}) \otimes C(\{g\})$ is a Muchnik degree" by a first-order formula ϕ. Then ϕ holds in $\mathcal{M}_m$ by Proposition 6.4. On the other hand, since the Muchnik degree of $C(\{f\}) \otimes C(\{g\})$ is given by $C(\{f, g\})$, we see that ϕ does not hold in $\mathcal{M}_n$ by Proposition 8.3. □

We should note that the result of Shore and Slaman on the definability of the jump used above is very complex, and that it is probably too strong a tool for the simple thing we wish to prove. However, the author currently does not know of an easier example separating the first-order theories of these lattices.

We conclude with the following open question concerning the single case excluded in Theorem 8.4.

Question 8.5 Are $\mathcal{M}_0$ and $\mathcal{M}_1$ elementarily equivalent?

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University of Wisconsin–Madison
Department of Mathematics
480 Lincoln Drive
Madison, Wisconsin 53706
USA
mail@rutgerkuyper.com
http://rutgerkuyper.com/