Notre Dame Journal of Formal Logic
Volume 59, Number 2, 2018

Nonreduction of Relations in the Gromov Space to Polish Actions

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Abstract We show that in the Gromov space of isometry classes of pointed proper metric spaces, the equivalence relations defined by existence of coarse quasi-isometries or being at finite Gromov–Hausdorff distance cannot be reduced to the equivalence relation defined by any Polish action.

1 Introduction

Gromov [3], [4, Chapter 3] described a space, denoted here by $\mathcal{M}_*$, whose points are isometry classes of pointed complete proper metric spaces. It is endowed with a topology which resembles the Tychonoff topology of $\mathbf{R}^{\mathbf{N}}$, or the compact open topology on the space of continuous functions $C(\mathbf{R})$. It also supports several equivalence relations of geometric interest, like the relation of being at finite Gromov–Hausdorff distance, E_{GH}, and the relation of being (coarsely) quasi-isometric, E_{QI}.

The following concepts relate the complexity of two equivalence relations on topological spaces, E over X and F over Y. A map $\theta : X \to Y$ is called (E, F)-*invariant* if $xEx' \implies \theta(x)F\theta(x')$ (θ induces a mapping $\bar{\theta} : X/E \to Y/F$). It is said that E is *Borel reducible* to F, denoted by $E \leq_B F$, if there is an (E, F)-invariant Borel mapping $\theta : X \to Y$ such that $xEx' \Leftrightarrow \theta(x)F\theta(y)$ ($\bar{\theta} : X/E \to Y/F$ is injective). If $E \leq_B F$ and $F \leq_B E$, then E is said to be *Borel bireducible* with F, and is denoted by $E \sim_B F$. If the map θ can be chosen to be continuous, then the terms *continuously reducible* and *continuously bireducible* are used, with notation $\leq_c$ and $\sim_c$.

For an example of an equivalence relation, let G be a Polish group acting continuously on a Polish space X (a Polish action). We then let E_G^X denote the orbit equivalence relation whose equivalence classes are exactly the G-orbits. For instance,

Received February 18, 2015; accepted October 5, 2015
First published online January 8, 2018
2010 Mathematics Subject Classification: Primary 03E15, 54H05; Secondary 54H20, 54E50
Keywords: Gromov space, Gromov–Hausdorff metric, quasi-isometry
 10.1215/00294527-2017-0026

Hjorth's theory of turbulence (see [5], [6]) is valid for relations defined by Polish actions. The following is our main result.

Theorem 1.1 *For any Polish group G and any Polish G-space X, we have $E_{\mathrm{GH}} \not\leq_B E_G^X$ and $E_{\mathrm{QI}} \not\leq_B E_G^X$.*

The theory of turbulence is extended in Álvarez López and Candel [1] to more general equivalence relations on Polish spaces, and it is applied to E_{QI} and E_{GH}. This is a nontrivial extension by Theorem 1.1.

The proof of Theorem 1.1 uses the following. Let E_1 be the equivalence relation on $\mathbf{R}^{\mathbf{N}}$ consisting of the pairs (x, y), with $x = (x_n)$ and $y = (y_n)$, such that there is some $N \in \mathbf{N}$ so that $x_n = y_n$ for all $n \geq N$ (the relation of eventual agreement). We have $E_1 \not\leq_B E_G^X$ for any Polish group G and any Polish G-space X (see Kechris and Louveau [7, Theorem 4.2]; see also [5, Theorem 8.2] for a different proof).

On the other hand, let E_{K_σ} be the equivalence relation on $\prod_{n=2}^{\infty}\{1,\dots,n\}$ consisting of the pairs (x, y), with $x = (x_n)$ and $y = (y_n)$, such that $\sup_n |x_n - y_n| < \infty$. We have $E \leq_B E_{K_\sigma}$ for any K_σ equivalence relation E (see Rosendal [9, Theorem 17, Proposition 19],[1] and therefore $E_1 \leq_B E_{K_\sigma}$ because E_1 is K_σ (see [9], Gao [2, Exercise 8.4.3]); in particular, $E_{K_\sigma} \not\leq_B E_G^X$ for any Polish group G and every Polish G-space X. Therefore, Theorem 1.1 follows by showing that $E_{K_\sigma} \leq_B E_{\mathrm{GH}}$ and $E_{K_\sigma} \leq_B E_{\mathrm{QI}}$ (see Proposition 4.1).

The relations E_{GH} and E_{QI} resemble the equivalence relation E_{ℓ_∞} on $\mathbf{R}^{\mathbf{N}}$ defined by the action of ℓ_∞ on $\mathbf{R}^{\mathbf{N}}$ by translations,[2] or the equivalence relation E_∞ on $C(\mathbf{R})$ defined by the action of $C_b(\mathbf{R})$. Thus Proposition 4.1 has some analogy with the property $E_{K_\sigma} \sim_B E_{\ell_\infty}$ (see [9, Proposition 19]); in particular, $E_1 \leq_B E_{\ell_\infty}$ (see also [2, Theorem 8.4.2]). It also has some similarity with the property $E_{K_\sigma} \leq_B E_\infty$, which follows because $E_{\ell_\infty} \leq_c E_\infty$; this reduction can be realized by the map $\mathbf{R}^{\mathbf{N}} \to C(\mathbf{R})$, assigning to each element its canonical continuous piecewise affine extension that is constant on $(-\infty, 0]$.

2 The Gromov Space

Let M be a metric space, and let d_M, or simply d, be its distance function. The *Hausdorff distance* between two nonempty subsets $A, B \subset M$ is given by

$$H_d(A, B) = \max\bigl\{\sup_{a\in A}\inf_{b\in B} d(a,b), \sup_{b\in B}\inf_{a\in A} d(a,b)\bigr\}.$$

Observe that $H_d(A, B) = H_d(\overline{A}, \overline{B})$ and $H_d(A, B) = 0$ if and only if $\overline{A} = \overline{B}$. Also, it is well known and easy to prove that H_d satisfies the triangle inequality, and its restriction to the family of nonempty compact subsets of M is finite-valued, and moreover complete if M is complete.

Let M and N be arbitrary nonempty metric spaces. A metric on $M \sqcup N$ is called *admissible* if its restrictions to M and N are d_M and d_N, where M and N are identified with their canonical injections in $M \sqcup N$. The *Gromov–Hausdorff distance* (or *GH distance*) between M and N is defined by

$$d_{\mathrm{GH}}(M, N) = \inf_d H_d(M, N),$$

where the infimum is taken over all admissible metrics d on $M \sqcup N$. It is well known that $d_{\mathrm{GH}}(M, N) = d_{\mathrm{GH}}(\overline{M}, \overline{N})$, where $\overline{M}$ and $\overline{N}$ denote the completions of M and

N, $d_{\mathrm{GH}}(M,N)=0$ if $\overline{M}$ and $\overline{N}$ are isometric, d_{GH} satisfies the triangle inequality, and $d_{\mathrm{GH}}(M,N)<\infty$ if $\overline{M}$ and $\overline{N}$ are compact.

There is also a pointed version of d_{GH} which satisfies analogous properties. The (*pointed*) *Gromov–Hausdorff distance* between two pointed metric spaces (M,x) and (N,y) is defined by

$$d_{\mathrm{GH}}(M,x;N,y)=\inf_d \max\bigl\{d(x,y),H_d(M,N)\bigr\}, \tag{1}$$

where the infimum is taken over all admissible metrics d on $M\sqcup N$.

A metric space, or its distance function, is called *proper* (or *Heine–Borel*) if every open ball has compact closure. This condition is equivalent to the compactness of the closed balls, which means that the distance function to a fixed point is a proper function. Any proper metric space is complete and locally compact, and its cardinality is not greater than the cardinality of the continuum. Therefore it may be assumed that their underlying sets are subsets of $\mathbf{R}$. With this assumption, it makes sense to consider the set $\mathcal{M}_*$ of isometry classes, $[M,x]$, of pointed proper metric spaces, (M,x). The set $\mathcal{M}_*$ is endowed with a topology introduced by Gromov [4, Section 6], [3], which can be described as follows.

For a metric space X, two subspaces $M,N\subset X$, two points $x\in M$ and $y\in N$, and a real number $R>0$, let $H_{d_X,R}(M,x;N,y)$ be given by

$$H_{d_X,R}(M,x;N,y)=\max\Bigl\{\sup_{u\in B_M(x,R)} d_X(u,N),\ \sup_{v\in B_N(y,R)} d_X(v,M)\Bigr\}.$$

Then, for $R,r>0$, let $U_{R,r}\subset\mathcal{M}_*^2$ denote the subset of pairs $([M,x],[N,y])$ for which there is an admissible metric d on $M\sqcup N$ so that

$$\max\bigl\{d(x,y),H_{d,R}(M,x;N,y)\bigr\}<r.$$

Let $\Delta\subset\mathcal{M}_*^2$ denote the diagonal.

Lemma 2.1 *The following properties hold:*

(i) $\bigcap_{R,r>0}U_{R,r}=\Delta$;
(ii) *each* $U_{R,r}$ *is symmetric;*
(iii) *if* $R\le S$, *then* $U_{R,r}\supset U_{S,r}$ *for all* $r>0$;
(iv) $U_{R,r}=\bigcup_{s<r}U_{R,s}$ *for all* $R,r>0$; *and*
(v) $U_{S,r}\circ U_{S,s}\subset U_{R,r+s}$, *where* $S=R+2\max\{r,s\}$.

Proof Items (i)–(iv) are elementary. To prove (v), let $[M,x],[N,y]\in\mathcal{M}_*$ and $[P,z]\in U_{S,r}(N,y)\cap U_{S,s}(M,x)$. Then there are admissible metrics d on $M\sqcup P$ and $\bar d$ on $N\sqcup P$ such that $d(x,z)<r$, $r_0:=H_{d,S}(M,x;P,z)<r$, $\bar d(y,z)<s$ and $s_0:=H_{\bar d,S}(N,y;P,z)<s$. Let $\hat d$ be the admissible metric on $M\sqcup N$ such that

$$\hat d(u,v)=\inf\bigl\{d(u,w)+\bar d(w,v)\bigm| w\in P\bigr\}$$

for all $u\in M$ and $v\in N$. Then

$$\hat d(x,y)\le d(x,z)+\bar d(z,y)<r+s.$$

For each $u\in B_M(x,R)$, there is some $w\in P$ such that $d(u,w)<r_0$. Then

$$d_P(z,w)\le d(z,x)+d_M(x,u)+d(u,w)<r+R+r_0<S.$$

So there is some $v\in N$ such that $\bar d(w,v)<s_0$, and we have

$$\hat d(u,v)\le d(u,w)+\bar d(w,v)<r_0+s_0.$$

Hence $\hat{d}(u, N) < r_0 + s_0$ for all $u \in B_M(x, R)$. Similarly, $\hat{d}(v, M) < r_0 + s_0$ for all $v \in B_N(y, R)$. Therefore $H_{\hat{d},R}(M, x; N, y) \leq r_0 + s_0 < r + s$. Then $[N, y] \in U_{R,r+s}(M, x)$. □

By Lemma 2.1, the sets $U_{R,r}$ form a base of entourages of a metrizable uniformity on $\mathcal{M}_*$. Endowed with the induced topology, $\mathcal{M}_*$ is what is called the *Gromov space* in this article. It is well known that $\mathcal{M}_*$ is a Polish space (see, e.g., Gromov [4] or Petersen [8]); in particular, a countable dense subset is defined by the pointed finite metric spaces with $\mathbf{Q}$-valued metrics.

3 Equivalence Relations on the Gromov Space

Recall the following terminology. A map between metric spaces $\phi : M \to N$ is called *bi-Lipschitz* if there is some $\lambda \geq 1$ such that

$$\lambda^{-1} d_M(u, v) \leq d_N\big(\phi(u), \phi(v)\big) \leq \lambda\, d_M(u, v)$$

for all $u, v \in M$. The term λ*-bi-Lipschitz* may be also used in this case. A subset $A \subset M$ is called a *net*[3] (resp., *separated*) if there is some $C \geq 0$ such that $d_M(x, A) \leq C$ for all $x \in M$ (resp., there is some $\delta > 0$ such that $d_M(x, y) \geq \delta$ if $x \neq y$). The term C*-net* (resp., δ*-separated*) may also be used in this case. There always exist separated nets (see [1, Lemma 9.4]). A (*coarse*) *quasi-isometry* of M to N is a bi-Lipschitz bijection $\phi : A \to B$ for some nets $A \subset M$ and $B \subset N$. The existence of a quasi-isometry of M to N is equivalent to the existence of a finite sequence of metric spaces $M = M_0, \ldots, M_{2k} = N$ such that $d_{\mathrm{GH}}(M_{2i-2}, M_{2i-1}) < \infty$ and there is a bi-Lipschitz bijection $M_{2i-1} \to M_{2i}$ for all $i \in \{1, \ldots, k\}$. A *pointed* (*coarse*) *quasi-isometry* is defined in the same way, by using a pointed bi-Lipschitz bijection between nets that contain the distinguished points. The existence of a pointed quasi-isometry has an analogous characterization involving pointed Gromov–Hausdorff distances and pointed bi-Lipschitz bijections.

The following equivalence relations are considered on $\mathcal{M}_*$.

- The *canonical relation*, E_{can}, is defined by varying the distinguished point; that is, E_{can} consists of the pairs of the form $([M, x], [M, y])$ for any proper metric space M and all $x, y \in M$.
- The *Gromov–Hausdorff relation*, E_{GH}, consists of the pairs $([M, x], [N, y])$ with $d_{\mathrm{GH}}(M; N) < \infty$, or, equivalently, $d_{\mathrm{GH}}(M, x; N, y) < \infty$.
- The *Lipschitz relation*, E_{Lip}, consists of the pairs $([M, x], [N, y])$ such that there is a bi-Lipschitz bijection $M \to N$. If M and N are separated, this is equivalent to the existence of a pointed bi-Lipschitz bijection $(M, x) \to (N, y)$.
- The *quasi-isometric relation*, E_{QI}, consists of the pairs $([M, x], [N, y])$ such that there is a quasi-isometry of M to N, or, equivalently, there is a pointed quasi-isometry of (M, x) to (N, y). By the above observations, E_{QI} is the smallest equivalence relation over $\mathcal{M}_*$ that contains $E_{\mathrm{GH}} \cup E_{\mathrm{Lip}}$.

Since $E_{\mathrm{can}} \subset E_{\mathrm{GH}} \cap E_{\mathrm{QI}}$, it follows that $\mathcal{M}_*/E_{\mathrm{GH}}$ can be identified with the set of classes of proper metric spaces modulo finite GH distance, and $\mathcal{M}_*/E_{\mathrm{QI}}$ can be identified with the set of quasi-isometry types of proper metric spaces.

4 Nonreduction to Polish Actions

As indicated in Section 1, Theorem 1.1 results from the following.

Proposition 4.1 *We have $E_{K_\sigma} \le_c E_{\mathrm{GH}}$ and $E_{K_\sigma} \le_c E_{\mathrm{QI}}$.*

Proof Let us prove first that $E_{K_\sigma} \le_c E_{\mathrm{QI}}$, which is more difficult. Consider the metric d on $\mathbf{R}^2$ defined by

$$d\big((u,v),(u',v')\big) = \begin{cases} |v| + |u-u'| + |v'|, & \text{if } u \neq u', \\ |v-v'|, & \text{if } u = u'. \end{cases}$$

This is the metric of an $\mathbf{R}$-tree. For each $x = (x_n) \in \prod_{n=2}^{\infty}\{1,\dots,n\}$ and $n \ge 2$, let

$$P^{\pm}_{x,n} = \Big(\sum_{i=2}^{n} e^{i^2}, \pm e^{x_n}\Big) \in \mathbf{R}^2,$$
$$M_{x,n} = \{P^+_{x,n}, P^-_{x,n}\},$$

and let $M_x = \bigcup_{n=2}^{\infty} M_{x,n}$, equipped with the restriction d_x of d. Given any $x = (x_n) \in \prod_{n=2}^{\infty}\{1,\dots,n\}$, if $A \subset M_x$ is C-net for some $C \ge 0$, it easily follows that

$$e^{n^2} \ge C \Longrightarrow A \cap M_{x,n} \neq \emptyset, \tag{2}$$
$$(e^{n^2} \ge C \quad \& \quad e^{x_n} > C/2) \Longrightarrow M_{x,n} \subset A. \tag{3}$$

Let $\theta : \prod_{n=2}^{\infty}\{1,\dots,n\} \to \mathcal{M}_*$ be defined by $\theta(x) = [M_x, P^+_{x,2}]$.

Claim 4.2 *We have that θ is continuous.*

With the notation of Section 2, given $x = (x_n) \in \prod_{n=2}^{\infty}\{1,\dots,n\}$ and $R, r > 0$, we have to prove that $\theta^{-1}(U_{R,r}(\theta(x)))$ is a neighborhood of x in $\prod_{n=2}^{\infty}\{1,\dots,n\}$. Take some integer $n_0 \ge 2$ so that $e^2 + \sum_{i=2}^{n_0} e^{i^2} + e^{n_0} > R$. Hence $B_{M_x}(P^+_{x,2}, R) \subset \bigcup_{n=2}^{n_0} M_{x,n}$. Let $\mathcal{N}(x,n_0)$ be the open neighborhood of x in $\prod_{n=2}^{\infty}\{1,\dots,n\}$ consisting of the elements $y = (y_n)$ such that $y_n = x_n$ if $n \le n_0$. Then $P^{\pm}_{x,n} = P^{\pm}_{y,n}$ for $2 \le n \le n_0$ and $y \in V$, obtaining $d(P^+_{x,2}, P^+_{y,2}) = 0$ and $H_{d,R}(M_x, P^+_{x,2}; M_y, P^+_{y,2}) = 0$ for the isometric inclusion of M_x and M_y in $\mathbf{R}^2$ with d. Thus $\theta(\mathcal{N}(x,n_0)) \subset U_{R,r}(\theta(x))$, completing the proof of Claim 4.2.

Claim 4.3 *We have $(\theta\times\theta)(E_{K_\sigma}) \subset E_{\mathrm{Lip}}$, and therefore $(\theta\times\theta)(E_{K_\sigma}) \subset E_{\mathrm{QI}}$.*

This claim can be easily proved as follows. Let $(x,y) \in E_{K_\sigma}$ for $x = (x_n)$ and $y = (y_n)$ in $\prod_{n=2}^{\infty}\{1,\dots,n\}$. Thus there is some $C \ge 0$ such that $|x_n - y_n| \le C$ for all n. Consider the pointed bijection $\phi : (M_x, P^+_{x,2}) \to (M_y, P^+_{y,2})$ defined by $\theta(P^{\pm}_{x,n}) = P^{\pm}_{y,n}$. Then, with $\lambda = e^C$, we have

$$d_x(P^+_{x,n}, P^-_{x,n}) = 2e^{x_n} \le 2e^{y_n + C} = \lambda d_y(P^+_{y,n}, P^-_{y,n}) = \lambda\, d_y\big(\phi(P^+_{x,n}), \phi(P^-_{x,n})\big),$$

and, similarly,

$$d_x(P^+_{x,n}, P^-_{x,n}) \ge \frac{1}{\lambda}\, d_y\big(\phi(P^+_{x,n}), \phi(P^-_{x,n})\big).$$

On the other hand, for $P \in M_{x,m}$ and $Q \in M_{x,n}$ with $m < n$,

$$
\begin{aligned}
d_x(P,Q) &= e^{x_m} + \sum_{i=m+1}^{n} e^{i^2} + e^{x_n} \\
&\le e^{y_m+C} + \sum_{i=m+1}^{n} e^{i^2} + e^{y_n+C} \\
&\le \lambda\Big(e^{y_m} + \sum_{i=m+1}^{n} e^{i^2} + e^{y_n}\Big) \\
&= \lambda\, d_y\big(\phi(P),\phi(Q)\big),
\end{aligned}
$$

and, similarly,

$$
d_x(P,Q) \ge \frac{1}{\lambda}\, d_y\big(\phi(P),\phi(Q)\big).
$$

Thus ϕ is a λ-bi-Lipschitz bijection, completing the proof of Claim 4.3.

Claim 4.4 *We have* $(\theta\times\theta)^{-1}(E_{\mathrm{QI}}) \subset E_{K_\sigma}$.

To prove this assertion, take some $x = (x_n)$ and $y = (y_n)$ in $\prod_{n=2}^{\infty}\{1,\dots,n\}$ such that $(\theta(x),\theta(y)) \in E_{\mathrm{QI}}$. Then, for some $C \ge 0$ and $\lambda \ge 1$, there are C-nets, $A \subset M$ and $B \subset M(y)$ with $P^+_{x,2} \in A$ and $P^+_{y,2} \in B$, and there is a pointed λ-bi-Lipschitz bijection $\phi : (A, P^+_{x,2}) \to (B, P^+_{y,2})$.

Claim 4.5 *If* $e^{n^2} \ge C$, $\frac{1}{n}e^{2n+1} > \lambda$ *and* $e^{(n+2)^2-(n+1)^2} > 3\lambda$, *then* $\phi(M_{x,n} \cap A) \subset M_{y,n}$.

Assume the conditions of this claim. Then $A \cap M_{x,m} \ne \emptyset$ for all $m \ge n$ by (2). Furthermore, for $2 \le k < \ell \le n$,

$$
\begin{aligned}
d_y\big(\phi(M_{x,n}\cap A),\phi(M_{x,n+1}\cap A)\big) &\ge \frac{1}{\lambda}\, d_x(M_{x,n}\cap A, M_{x,n+1}\cap A) \\
&> \frac{1}{\lambda}\, e^{(n+1)^2} \\
&> n e^{(n+1)^2-2n-1} \\
&= n e^{n^2} \\
&\ge 2e^n + \sum_{i=3}^{n} e^{i^2} \\
&\ge e^{y_k} + \sum_{i=k+1}^{\ell} e^{i^2} + e^{y_\ell} \\
&= d_y(P',Q')
\end{aligned}
$$

for all $P' \in M_{y,k}$ and $Q' \in M_{y,\ell}$. On the other hand, for $2 \le k < \ell$ with $\ell \ge n+2$,

$$
\begin{aligned}
d_y\big(\phi(P),\phi(Q)\big) &\le \lambda\, d_x(M_{x,n}\cap A, M_{x,n+1}\cap A) \\
&< \lambda(e^{(n+1)^2} + 2e^{n+1}) \\
&< \lambda 3 e^{(n+1)^2} \\
&< e^{(n+2)^2}
\end{aligned}
$$

$$\begin{aligned}&\le e^{\ell^2}\\ &< e^{y_k} + \sum_{i=k+1}^{\ell} e^{i^2} + e^{y_\ell}\\ &= d_y(M_{y,k}, M_{y,\ell}),\end{aligned}$$

for all $P \in M_{x,n} \cap A$ and $Q \in M_{x,n+1} \cap A$. Therefore, either

$$\phi(M_{x,n} \cap A) \subset M_{y,n} \qquad \text{and} \qquad \phi(M_{x,n+1} \cap A) \subset M_{y,n+1}, \tag{4}$$

or

$$\phi\big((M_{x,n} \cup M_{x,n+1}) \cap A\big) \subset M_{y,m} \tag{5}$$

for some m. In the case (5), we have

$$\begin{aligned}2e^m &= d_y\big(\phi(M_{x,n} \cap A), \phi(M_{x,n+1} \cap A)\big)\\ &\ge \frac{d_x(M_{x,n} \cap A, M_{x,n+1} \cap A)}{\lambda}\\ &> \frac{e^{(n+1)^2}}{\lambda},\end{aligned}$$

giving $m > (n+1)^2 - \ln(2\lambda)$. Applying this to $n+1$ and $n+2$, we get that either

$$\phi(M_{x,n+1} \cap A) \subset M_{y,n+1} \qquad \text{and} \qquad \phi(M_{x,n+2} \cap A) \subset M_{y,n+2}, \tag{6}$$

or

$$\phi\big((M_{x,n+1} \cup M_{x,n+2}) \cap A\big) \subset M_{y,m'} \tag{7}$$

for some $m' > (n+2)^2 - \ln(2\lambda)$. If (5) and (7) hold, then $m = m'$ and

$$\phi\big((M_{x,n} \cup M_{x,n+1} \cup M_{x,n+2}) \cap A\big) \subset M_{y,m},$$

which is a contradiction because ϕ is a bijection whereas

$$\#\big((M_{x,n} \cup M_{x,n+1} \cup M_{x,n+2}) \cap A\big) \ge 3 > 2 = \# M_{y,m}.$$

If (5) and (6) hold, then $n+1 = m > (n+1)^2 - \ln(2\lambda)$, which contradicts the condition $e^{(n+2)^2-(n+1)^2} > 3\lambda$. So (4) must be true, showing Claim 4.5.

From Claim 4.5, it easily follows that

$$\phi(M_{x,n} \cap A) = M_{y,n} \cap B \tag{8}$$

for n large enough. Suppose first that $M_{x,n} \subset A$ for such an n, and therefore $M_{y,n} \subset B$ by (8). Thus

$$2e^{y_n} = d_y(P^+_{y,n}, P^-_{y,n}) = d_y\big(\phi(P^+_{x,n}), \phi(P^-_{x,n})\big) \ge \frac{d_x(P^+_{x,n}, P^-_{x,n})}{\lambda} = \frac{2e^{x_n}}{\lambda},$$

giving $y_n \ge x_n - \ln\lambda$. Similarly, $y_n \le x_n + \ln\lambda$, obtaining $|x_n - y_n| \le \ln\lambda$.

Now, assume that $M_{x,n} \not\subset A$ for such an n; in particular, $C > 0$. Then $M_{y,n} \not\subset B$ by (8). So $e^{x_n}, e^{y_n} \le C/2$ by (3), giving $x_n, y_n \le \ln(C/2)$, and thus $|x_n - y_n| \le \ln(C/2)$.

Hence $|x_n - y_n| \le \max\{\ln\lambda, \ln(C/2)\}$ for all n large enough, and therefore $\sup_n |x_n - y_n| < \infty$, obtaining that $(x, y) \in E_{K_\sigma}$. This completes the proof of Claim 4.4.

Claims 4.2, 4.3, and 4.4 show that θ realizes the reduction $E_{K_\sigma} \le_c E_{\mathrm{QI}}$.

A similar argument with a slight modification of the definition of $M(x)$, using $P^\pm_{x,n} = (\sum_{i=2}^n e^{i^2}, \pm x_n)$, shows that $E_{K_\sigma} \le_B E_{\mathrm{GH}}$. □

Remark 4.6 In Claim 4.2, θ is in fact a topological embedding, as shown in the following argument. First, let us prove that θ is injective. Suppose that $\theta(x) = \theta(y)$ for some $x = (x_n)$ and $y = (y_n)$ in $\prod_{n=2}^{\infty}\{1,\dots,n\}$. This means that there is a pointed isometry $\phi : (M_x P_{x,2}^+) \to (M_y, P_{y,2}^+)$. We get $\phi(M_{x,n}) = M_{y,n}$ for all $n \geq 2$ by Claim 4.5 with $A = M_x$, $B = M_y$, $C = 0$, and $\lambda = 1$; in fact, the argument can be simplified in this case. Hence, for each $n \geq 2$,

$$2e^{x_n} = d_x(P_{x,n}^+, P_{x,n}^-) = d_y\big(\phi(P_{x,n}^+), \phi(P_{x,n}^-)\big) = d_y(P_{y,n}^+, P_{y,n}^-) = 2e^{y_n},$$

giving $x_n = y_n$. Thus $x = y$.

Finally, let us prove that $\phi^{-1} : \phi(\prod_{n=2}^{\infty}\{1,\dots,n\}) \to \prod_{n=2}^{\infty}\{1,\dots,n\}$ is continuous at $\phi(x)$ for every $x = (x_n) \in \prod_{n=2}^{\infty}\{1,\dots,n\}$. With the notation of the proof of Claim 4.2, we have to check that, for all $n_0 \geq 2$, there is some $R, r > 0$ so that $\phi^{-1}(U_{R,r}(\theta(x))) \subset \mathcal{N}(x, n_0)$. Let $y = (y_n) \in \prod_{n=2}^{\infty}\{1,\dots,n\}$ such that $\theta(y) \in U_{R,r}(\theta(x))$ for some $R, r > 0$ to be determined later. Then there is a metric d' on $M_x \sqcup M_y$, extending d_x and d_y, such that $d'(P_{x,2}^+, P_{y,2}^+) < r$ and $H_{d',R}(M_x, P_{x,2}^+; M_y, P_{y,2}^+) < r$. Since $e^n < e^{(n+1)^2}$ for all $n \geq 2$, we can take R such that

$$e^2 + \sum_{i=2}^{n_0} e^{i^2} + e^{n_0} < R < e^2 + \sum_{i=2}^{n_0+1} e^{i^2},$$

and therefore $B_{M_x}(P_{x,2}^+, R) = \bigcup_{n=2}^{n_0} M_{x,n}$ and $B_{M_y}(P_{y,2}^+, R) = \bigcup_{n=2}^{n_0} M_{y,n}$. So, for each $P_{x,n}^{\pm}$ with $2 \leq n \leq n_0$, there is some $\widehat{P}_{x,n}^{\pm} \in M_y$ such that $d(P_{x,n}^{\pm}, \widehat{P}_{x,n}^{\pm}) < r$; in particular, we can take $\widehat{P}_{x,2}^{\pm} = P_{y,2}^{\pm}$. Let $\widehat{M}_{x,n} = \{\widehat{P}_{x,n}^+, \widehat{P}_{x,n}^-\}$ for $2 \leq n \leq n_0$. Choose r such that $r < 1$ and $e^n + r < e^{(n+1)^2}$ for $2 \leq n \leq n_0$. So $\widehat{M}_{x,n} = M_{y,n}$ for $2 \leq n \leq n_0$. Then, by the triangle inequality,

$$2e^{x_n} = d_x(P_{x,n}^+, P_{x,n}^-) \leq d_y(\widehat{P}_{x,n}^+, \widehat{P}_{x,n}^-) + 2r = d_y(P_{y,n}^+, P_{y,n}^-) + 2r = 2e^{y_n} + 2r,$$

giving $e^{x_n} \leq e^{y_n} + r$. Similarly, we get $e^{x_n} \geq e^{y_n} - r$. Thus $|e^{x_n} - e^{y_n}| \leq r$, obtaining $x_n = y_n$ because $r < 1$. Therefore $y \in \mathcal{N}(x, n_0)$, as desired.

Remark 4.7 According to Claim 4.3, the map θ of the proof of Proposition 4.1 also gives the reduction $E_{K_\sigma} \leq_c E_{\mathrm{Lip}}$. An analogous property is satisfied with another point of view: considering Polish metric spaces as the elements of the space of closed subspaces of some universal Polish metric space, like the Urysohn space, the relation given by the existence of bi-Lipschitz bijections is Borel bireducible with E_{K_σ} (see [9, Theorem 24]).

Notes

1. Recall that a subset of a topological space is called K_σ when it is a countable union of compact subsets.

2. Recall that $\ell_\infty \subset \mathbf{R}^{\mathbf{N}}$ is the linear subspace of bounded sequences, and $C_b(\mathbf{R}) \subset C(\mathbf{R})$ is the linear subspace of bounded continuous functions.

3. This term is used by Gromov [4, Definition 2.14] with this meaning. Other authors use it with other meanings.

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Acknowledgments

Álvarez López's and Candel's work was supported by Spanish Ministry of Science and Innovation grants MTM2011-25656 and MTM2008-02640.

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