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On Superstable Expansions of Free Abelian Groups

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Abstract We prove that $(\mathbb{Z}, +, 0)$ has no proper superstable expansions of finite Lascar rank. Nevertheless, this structure equipped with a predicate defining powers of a given natural number is superstable of Lascar rank ω. Additionally, our methods yield other superstable expansions such as $(\mathbb{Z}, +, 0)$ equipped with the set of factorial elements.

1 Introduction

This article fits into the general framework of adding a new predicate to a well-behaved structure and asking whether the obtained structure is still well behaved. A similar line of thought is to impose the desired properties on the expanded structure and ask for which predicates these properties are fulfilled. Even more, one might ask whether there exist proper expansions fulfilling the desired properties.

Many results that belong to the above-mentioned framework have been obtained by various authors. For example, Pillay and Steinhorn [6] proved that there are no (proper) o-minimal expansions of $(\mathbb{N}, \leq)$. On the other hand, Marker [4] proved that there are (proper) strongly minimal expansions of $(\mathbb{N}, s)$, that is, the natural numbers with the successor function. In a more abstract context, Baldwin and Benedikt [1] proved that if $\mathcal{M}$ is a stable structure and I is a *small* set of indiscernibles, then $(\mathcal{M}, I)$ is still stable. Finally, Chernikov and Simon [3] proved the analogous result for not the independence property (NIP) theories, that is, that NIP is preserved after naming a *small* indiscernible sequence.

In this short work, we are mainly interested in (finitely generated) free Abelian groups. We are motivated by the recent addition of torsion-free hyperbolic groups to the family of stable groups (see Sela [8]). In a torsion-free hyperbolic group centralizers of (nontrivial) elements are infinite cyclic, and one is interested in the induced structure on them. It seems that understanding the induced structure on

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these centralizers boils down to understanding whether they are superstable and, if so, whether to calculate their Lascar rank.

Our main result generalizes a theorem in the second-named author's dissertation proving that every Lascar rank 1 expansion of $(\mathbb{Z}, +, 0)$ is a pure group (see Sklinos [9, Theorem 8.2.3]).

Theorem 1 *There are no (proper) superstable finite Lascar rank expansions of* $(\mathbb{Z}, +, 0)$.

We also show that one cannot strengthen the above result any further by proving the following.

Theorem 2 *The theory of* $(\mathbb{Z}, +, 0, \Pi_q)$ *is superstable of Lascar rank* ω, *where* Π_q *denotes the set of powers of a natural number* q.

In fact, our methods can be used to provide other superstable expansions by adding other sets such as sets of the form $\{q^{p^n}\}_{n<\omega}$ for some natural numbers p, q or the set of factorial elements (see Proposition 4.2). On the other hand, if one moves to higher rank free Abelian groups, Theorem 1 is no longer true, and it is not hard to find proper superstable Lascar rank 1 expansions of $(\mathbb{Z}^n, +, 0)$, for $n \geq 2$. The main reason for this is that there exist finite index subgroups of $\mathbb{Z}^n$ (for $n \geq 2$) that are not definable in $(\mathbb{Z}^n, +, 0)$. Still, we record that a superstable finite Lascar rank expansion of $(\mathbb{Z}^n, +, 0)$ is one-based and has Lascar rank at most n.

While checking our results, the second author figured out, based on a talk by Poizat, that Theorem 2 was already proved in [7, Théorème 25]. Nevertheless, given that both approaches are completely different, we felt it was worth recording our result since, as we have already pointed out, it yields distinct examples. Moreover, to our knowledge, Theorem 1 was unknown. The essential tools used to prove it come from geometric stability theory. We combine results from Hrushovski's thesis together with Buechler's dichotomy theorem (see [5, Corollary 2.33]), the characterization of one-based groups by Hrushovski and Pillay, and a result on one-based types due to Wagner (see [10, Section 3.1]).

2 Finite-Rank Expansions

The aim of this section is to study superstable expansions of finite Lascar rank of the structure $(\mathbb{Z}^n, +, 0)$. We assume that the reader is familiar with the general theory of geometric stability (see Pillay [5] and Wagner [10] as references). In addition, we require the following result which characterizes subgroups of finitely generated free Abelian groups.

Fact 2.1 Let G be a subgroup of $\mathbb{Z}^n$. Then there is a basis $(z_1, \dots, z_n)$ of $\mathbb{Z}^n$ and a sequence of natural numbers $d_1, \dots, d_k$ (with d_i dividing d_{i+1} for $i < k$) such that $(d_1 z_1, \dots, d_k z_k)$ forms a basis of G.

One can use Fact 2.1 to prove the following lemma, which we consider as being part of the folklore.

Lemma 2.2 *Let G be a subgroup of $\mathbb{Z}^n$. Then G is definable in* $(\mathbb{Z}, +, 0)$.

Now, we prove Theorem 1.

Proof of Theorem 1 Consider a finite Lascar rank expansion $\mathcal{Z} = (\mathbb{Z}, +, 0, \ldots)$ of $(\mathbb{Z}, +, 0)$, and let $\Gamma \succeq \mathcal{Z}$ be a sufficiently saturated elementary extension. As Γ has finite Lascar rank, its principal generic type is nonorthogonal to a type q of Lascar rank 1, and hence, we can find an $\emptyset$-definable normal subgroup H of infinite index in Γ in a way that Γ/H is $\mathcal{Q}$-internal, where $\mathcal{Q}$ is the family of all $\emptyset$-conjugates of q. In fact, since H is defined without parameters, the subgroup $H \cap \mathbb{Z}$ has infinite index in $\mathbb{Z}$, hence $H \cap \mathbb{Z}$ must be trivial, and so is H. This yields that Γ is $\mathcal{Q}$-internal. On the other hand, as Γ is not ω-stable, by Buechler's dichotomy theorem (see [5, Corollary 2.33]) q must be a one-based type and so are all its conjugates. Thus Γ is one-based by Wagner [11, Corollary 12], and so is the theory of $\mathcal{Z}$. Thus, by the characterization of one-based stable groups in [5, Corollary 4.4.8], every definable subset of $\mathbb{Z}^n$ in the expanded structure is a Boolean combination of cosets of definable subgroups of $\mathbb{Z}^n$ and therefore, any definable set in the theory of $\mathcal{Z}$ is already definable in the theory of $(\mathbb{Z}, +, 0)$ by the previous lemma, as desired. □

We note, in contrast, that not all subgroups of $\mathbb{Z}^n$ are definable in $(\mathbb{Z}^n, +, 0)$. For example, the finite index subgroup $3\mathbb{Z} \oplus 2\mathbb{Z}$ of $\mathbb{Z}^2$ is not definable in $(\mathbb{Z}^2, +, 0)$, and of course any nontrivial infinite index subgroup of $\mathbb{Z}^n$, for $n \geq 2$, is not definable in $(\mathbb{Z}^n, +, 0)$.

Theorem 2.3 *Any finite Lascar rank expansion of* $(\mathbb{Z}^n, +, 0)$ *is one-based and has Lascar rank at most* n.

Proof Consider a finite Lascar rank expansion $\mathcal{Z} = (\mathbb{Z}^n, +, 0, \ldots)$ of $(\mathbb{Z}^n, +, 0)$. A similar argument as in the previous theorem yields that the theory of $\mathcal{Z}$ is one-based. For this, let $\Gamma \succeq \mathcal{Z}$ be a sufficiently saturated model. As it has finite Lascar rank by assumption, the general theory yields the existence of a finite series of $\emptyset$-definable normal subgroups

$$\Gamma = H_0 \trianglerighteq H_1 \trianglerighteq \cdots \trianglerighteq H_{m+1} \trianglerighteq \{0\}$$

such that H_{m+1} is finite and each factor H_i/H_{i+1} is infinite and internal to a family $\mathcal{Q}_i$ of $\emptyset$-conjugates of some type q_i of Lascar rank 1. Since free Abelian groups are torsion-free they do not have any finite (nontrivial) subgroups, and so neither does Γ. This implies that H_{m+1} is trivial. Furthermore, by Fact 2.1 we obtain that no infinite quotient of $\mathbb{Z}^n$ is ω-stable. As all subgroups H_i are $\emptyset$-definable, we deduce that the quotients H_i/H_{i+1} cannot have ordinal Morley rank, and neither do the types from the families $\mathcal{Q}_i$. Whence, we conclude by Buechler's dichotomy theorem that all of them are one-based, and so is Γ again by [11, Corollary 12].

To see that the expansion $\mathcal{Z}$ has Lascar rank at most n, consider the structure $\mathcal{Z}_{\mathrm{proj}}$ given as $(\mathbb{Z}^n, +, 0, P_1, \ldots, P_n)$, where the predicate P_i is interpreted as the projection of $\mathbb{Z}^n$ onto its ith coordinate. It is clear that $\mathcal{Z}_{\mathrm{proj}}$ is interpretable in $(\mathbb{Z}, +, 0)$ and so it has Lascar rank n. On the other hand, since $\mathcal{Z}$ is one-based, it is interpretable in $\mathcal{Z}_{\mathrm{proj}}$ by the characterization of one-based stable groups in [5, Corollary 4.4.8] and thus, it has Lascar rank at most n. □

Remark 2.4 Observe that the proof yields that any superstable finite Lascar rank expansion of $(\mathbb{Z}^n, +, 0)$ is interpretable in the structure $\mathcal{Z}_{\mathrm{proj}}$.

3 Superstable Expansions of $(\mathbb{Z}, +, 0)$

In this section we will see that there are proper superstable expansions of $(\mathbb{Z}, +, 0)$, necessarily, by Theorem 1, of infinite Lascar rank.

Definition 3.1 Let $\mathcal{L}$ be a first-order language, and let $P(x)$ be a unary predicate. We denote by $\mathcal{L}_P$ the first-order language $\mathcal{L} \cup \{P\}$. We say that an $\mathcal{L}_P$-formula $\phi(\bar{y})$ is *bounded* (with respect to P) if it has the form

$$Q_1 x_1 \in P \qquad \cdots \qquad Q_n x_n \in P\, \psi(\bar{x}, \bar{y}),$$

where the Q_i's are quantifiers and $\psi(\bar{x}, \bar{y})$ is an $\mathcal{L}$-formula.

The following theorem is useful for proving Theorem 2; we refer the reader to Casanovas and Ziegler [2] for the proof.

Theorem 3.2 *Let $\mathcal{M}$ be an $\mathcal{L}$-structure, and let $A \subseteq M$. Consider $(\mathcal{M}, A)$ as a structure in the expanded language $\mathcal{L}_P := \mathcal{L} \cup \{P\}$. Suppose that every $\mathcal{L}_P$-formula in $(\mathcal{M}, A)$ is equivalent to a bounded one. Thus, for every $\lambda \geq |\mathcal{L}|$, if both $\mathcal{M}$ and A_{ind} are λ-stable, then $(\mathcal{M}, A)$ is λ-stable.*

Let $\equiv_n$ be the congruence relation modulo n on the integers. Observe that $a \not\equiv_n b$ is equivalent to $a \equiv_n b + 1 \vee a \equiv_n b + 2 \vee \cdots \vee a \equiv_n b + (n-1)$, and hence we get the following remark.

Remark 3.3 Let $\mathcal{L}_{\mathrm{mod}}$ be the language of groups expanded with countably many 2-place predicates. We recall that an $\mathcal{L}_{\mathrm{mod}}$-formula $\phi(\bar{x})$ is equivalent, in $(\mathbb{Z}, +, 0, \{\equiv_n\}_{n<\omega})$, to a finite disjunction of formulas of the form

$$t_1(\bar{x}) = 0 \wedge \cdots \wedge t_k(\bar{x}) = 0,$$
$$r_1(\bar{x}) \neq 0 \wedge \cdots \wedge r_l(\bar{x}) \neq 0,$$
$$s_1(\bar{x}) \equiv_{n_1} 0 \wedge \cdots \wedge s_m(\bar{x}) \equiv_{n_m} 0,$$

where $t_i(\bar{x}), s_i(\bar{x}), r_i(\bar{x})$ are terms in the above language.

Set Π_q to denote the set $\{q^n \mid 1 \leq n < \omega\}$ for some natural number q.

Lemma 3.4 *Let q be a natural number. Let $\bar{b}$ be a tuple in $\mathbb{Z}$, and let $\phi(\bar{x}, y, \bar{z})$ be an $\mathcal{L}$-formula, where $\mathcal{L}$ is the language of groups. Suppose that the set $\Gamma(y) := \{\phi(\bar{b}, y, \bar{\alpha}) \mid \bar{\alpha} \in \Pi_q^{|\bar{z}|}\}$ is consistent with $\mathcal{T}h(\mathbb{Z}, +, 0)$. Then there exists $c \in \mathbb{Z}$ realizing the set $\Gamma(y)$.*

Proof We may assume that $\phi(\bar{x}, y, \bar{\alpha})$ is a formula as in Remark 3.3. If we fix some tuple $\bar{\alpha}_0$ in Π_q, then each disjunctive clause in $\phi(\bar{b}, y, \bar{\alpha}_0)$ asserts that y is equal to some element from a finite list of elements in $\mathbb{Z}$, and y is not equal to any element from a finite list of elements in $\mathbb{Z}$ and y belongs to the intersection of finitely many cosets of fixed subgroups of $\mathbb{Z}$, where these fixed subgroups only depend on ϕ (not $\bar{b}$ or $\bar{\alpha}_0$).

Our assumption that $\Gamma(y)$ is consistent implies that, for each tuple $\bar{\alpha}_0$ in Π_q, we may choose a disjunctive clause in $\phi(\bar{b}, y, \bar{\alpha}_0)$ such that the set of these clauses is again consistent. Note that if one of the chosen clauses involves an equality, then the result holds trivially. So we will assume that no equality is involved in any disjunctive clause of ϕ. On the other hand, the intersection of cosets of subgroups of a group is either empty or a coset of the intersection of the subgroups, thus we may assume that a disjunctive clause that involves congruence modulo relations, involves exactly one.

Next we prove that a finite union of sets of the form

$$\{k_0 + k_1 \cdot \alpha_1 + \cdots + k_s \cdot \alpha_s \mid \alpha_1, \ldots, \alpha_s \in \Pi_q\}$$

cannot cover any coset of any (nontrivial) subgroup of $\mathbb{Z}$. Suppose otherwise that the coset $m + n\mathbb{Z}$ is contained in such a finite union, and observe that we may assume, after subtracting m if necessary, that $m = 0$. Thus, for each set of the above form we can write each given coefficient k_i in base q and obtain a natural number l such that $n\mathbb{Z}$ is covered by finitely many sets of the form

$$\left\{\lambda_0 + \lambda_1 \cdot \alpha_1 + \cdots + \lambda_l \cdot \alpha_l \;\middle|\; \alpha_1, \ldots, \alpha_l \in \Pi_q, 0 \leq |\lambda_0|, \ldots, |\lambda_l| < q\right\}.$$

Assume that l is the biggest number obtained in the above-mentioned fashion. Then, any multiple of n can be written in base q with at most $l + 1$ many summands. Now, let μ be the element $n \cdot (1 + q + q^2 + \cdots + q^{l+1})$, which clearly belongs to $n\mathbb{Z}$. After writing n in base q, we obtain that μ is written in base q as the sum of at least $l + 2$ many summands. Thus, by the uniqueness of the representation of μ in base q, we obtain a contradiction.

Now, the consistency of $\Gamma(y)$ implies that y belongs to the intersection of finitely many cosets of subgroups of $\mathbb{Z}$ and that y is not equal to any element of a finite union of sets of the form

$$\{k_0 + k_1 \cdot \alpha_1 + \cdots + k_s \cdot \alpha_s \mid \alpha_1, \ldots, \alpha_s \in \Pi_q\}.$$

By the previous paragraph, a solution can be found in $\mathbb{Z}$ and this finishes the proof. □

Now we are able to prove the following technical lemma.

Lemma 3.5 *Let q be a natural number. Let $\mathcal{L}$ be the language of groups, and let $P(x)$ be a unary predicate. Let $\mathcal{Z} := (\mathbb{Z}, +, 0, \Pi_q)$ be an $\mathcal{L}_P$-structure.*

Let $\phi(\bar{x}, y, \bar{z})$ be an $\mathcal{L}$-formula. Then there exists $k < \omega$ such that

$$\mathcal{Z} \models \forall \bar{x}\Big(\Big(\forall \bar{z}_0 \in P \ldots \forall \bar{z}_k \in P\ \exists y \bigwedge_{j \leq k} \phi(\bar{x}, y, \bar{z}_j)\Big) \rightarrow \exists y\ \forall \bar{z} \in P \phi(\bar{x}, y, \bar{z})\Big).$$

Proof Since $(\mathbb{Z}, +, 0)$ does not have the finite cover property, we can assign to each formula ϕ a natural number k such that any set of instances of the formula ϕ is consistent if and only if it is k-consistent. By Lemma 3.4, if a set $\{\phi(\bar{b}, y, \bar{\alpha}) \mid \bar{\alpha} \in \Pi_q^{|\bar{z}|}\}$ is consistent, then a solution can be found in $\mathbb{Z}$ and this is enough to conclude. □

The following proposition is an easy corollary of Lemma 3.5. We leave the proof to the reader (see [2, Proposition 2.1]).

Proposition 3.6 *Let q be a natural number. Let $\mathcal{L}$ be the language of groups, and let $P(x)$ be a unary predicate. Let $\mathcal{Z} := (\mathbb{Z}, +, 0, \Pi_q)$ be an $\mathcal{L}_P$-structure. Then every $\mathcal{L}_P$-formula in $\mathcal{Z}$ is bounded.*

As a consequence, we deduce the following.

Corollary 3.7 *Let q be a natural number. Let $\mathcal{L}$ be the language of groups, let $P(x)$ be a unary predicate, and let $(\Gamma, +', 0, \Pi'_q) \equiv (\mathbb{Z}, +, 0, \Pi_q)$ be $\mathcal{L}_P$-structures. Two tuples of Γ realize the same $\mathcal{L}_P$-formulas over any set of parameters $C \subseteq \Gamma$ whenever they realize the same $\mathcal{L}$-formulas over $\Pi'_q \cup C$.*

Proof Let a and b be two tuples realizing the same $\mathcal{L}$-formulas over Π'_q, C. It is easy to see by induction on the number of quantifiers that a and b realize the same formulas of the form

$$Q_1 x_1 \in P \qquad \cdots \qquad Q_n x_n \in P \psi(\bar{x}, \bar{y}),$$

where the Q_i's are quantifiers and $\psi(\bar{x}, \bar{y})$ is an $\mathcal{L}(\Pi'_q \cup C)$-formula. Hence, we conclude by Proposition 3.6. $\square$

Our last task is to prove that the induced structure on the subset of the integers that consists of powers of some natural number, coming from $(\mathbb{Z}, +, 0)$, is tame. Recall that if B is a subset of the domain M, of a first-order structure $\mathcal{M}$, then by the *induced structure on* B we mean the structure with domain B and predicates for every subset of B^n of the form $B^n \cap \phi(M^n)$, where $\phi(x)$ is a first-order formula (over the empty set). We denote this structure by B^{ind}.

Proposition 3.8 *Let q be a natural number. The structure Π_q^{ind} (with respect to $(\mathbb{Z}, +, 0)$) is superstable and has Lascar rank* 1.

The proof is split into a series of lemmas. We first prove some results, which we believe are well known, in the spirit of Diophantine analysis.

Lemma 3.9 *Let q be some natural number. Let $k < n$ be natural numbers such that n is coprime with q, and let $[k]_n$ denote the congruence class of k modulo n. Then $\Pi_q \cap [k]_n = \{q^{m_0 + \varphi(n)\cdot m} : m < \omega\}$, where $\varphi(n)$ is Euler's phi function and m_0 is the smallest natural number for which $q^{m_0} \equiv k \mod n$.*

Proof We first note that if k, n are not coprime, then the intersection of $[k]_n$ with Π_q is empty. The common factor of k and n does not contain a factor of q since n is coprime with q, and it should appear as factor in any element of $k + n \cdot \mathbb{Z}$.

We now assume that k, n are coprime, and we fix k, n, m_0, satisfying the hypothesis of the lemma. We define λ_m recursively as follows:

$$\lambda_0 := \frac{q^{m_0} - k}{n},$$

$$\lambda_{m+1} := \lambda_m \cdot b^{\varphi(n)} + k \cdot \frac{q^{\varphi(n)} - 1}{n}, \quad \text{for } 0 \leq m < \omega.$$

Note that, by Euler's theorem, all the λ_m's are integers. Furthermore, one can easily see, by induction on m, that $\lambda_m \cdot n + k$ is a power of q of the form $q^{m_0 + \varphi(n)\cdot m}$ and therefore $\{\lambda_m \cdot n + k \mid m < \omega\} \subseteq \Pi_q \cap [k]_n$.

In fact, the other inclusion also holds. To see this, let q^l be an arbitrary power of q. We may assume that $l > m_0$, since m_0 is the smallest natural number satisfying the hypothesis. Then we can find some m such that $l = m_0 + \varphi(n)\cdot m + s$ with $s < \varphi(n)$. As $\varphi(n)$ is the order of the multiplicative group $(\mathbb{Z}/n\mathbb{Z})^\times$, we get $q^s \in [1]_n$ only when $s = 0$. Since k, n are coprime, k has a multiplicative inverse modulo n. Therefore,

$$q^l = q^{m_0 + \varphi(n)\cdot m} \cdot q^s \equiv_n k \cdot q^s \equiv_n k \qquad \text{if and only if} \qquad s = 0,$$

and this concludes the proof. $\square$

Remark 3.10 Let q be some natural number. Assume that n is a power of a prime which is not coprime with q. Then the intersection of Π_q with $[k]_n$ is either finite or cofinite in Π_q.

Lemma 3.11 *Let $k_1x_1 + \cdots + k_nx_n = k$ be an equation over the integers, and let $S \subseteq \mathbb{Z}^n$ be its solution set. Then $S \cap \Pi_q^n$ is either empty or a finite union of sets of the form*

$$\begin{aligned}\big\{(q^{\lambda_1}, \ldots, q^{\lambda_n}) \;\big|\; &\lambda_{i_1} > m_1, \ldots, \lambda_{i_k} > m_k,\\ &\lambda_{i_{k+1}} = \alpha_{k+1}\lambda_{i_{j_1}} + m_{k+1},\\ &\vdots\\ &\lambda_{i_n} = \alpha_n\lambda_{i_{j_{n-k+1}}} + m_n\big\},\end{aligned}$$

where $m_1, \ldots, m_n \in \mathbb{Z}$, $\alpha_i \in \{0, 1\}$, and $\{i_{j_1}, \ldots, i_{j_{n-k+1}}\} \subseteq \{i_1, \ldots, i_k\}$.

Proof The proof is by induction. For the base case $n = 1$, we easily see that $k_1x_1 = k$ can either be empty or have a single solution; thus, the solution set is of the required form. Suppose that for every $m < n$ the solution set of any linear equation in m variables is of the required form; we show that the same holds for equations with n variables.

We split the solution set into finitely many subsets according to the finitely many orderings we can put on the n variables. For example, to the ordering $x_1 \leq x_2 \leq \cdots \leq x_n$ corresponds the subset of solutions for which each coordinate takes a greater or equal value to its previous one. We analyze those subsets in parallel. For notational purposes, we analyze the set with the above ordering. Let $\{(q^{\lambda_1(i)}, \ldots, q^{\lambda_n(i)}) \mid i < \omega\}$ be an enumeration of this set. Then

$$q^{\lambda_1(i)}(k_1 + k_2q^{\lambda_2(i)-\lambda_1(i)} + \cdots + k_nq^{\lambda_n(i)-\lambda_1(i)}) = k.$$

We take the following cases.

(*Case 1*) Suppose that the sequence $\lambda_1(i)$ is bounded. Then for each of the finitely many values of $\lambda_1(i)$, we have $k_2q^{\lambda_2(i)-\lambda_1(i)} + \cdots + k_nq^{\lambda_n(i)-\lambda_1(i)} = \frac{k}{q_1^{\lambda}(i)} - k_1$. Using the inductive hypothesis for the linear equation $k_2x_2 + \cdots + k_nx_n = \frac{k}{q_1^{\lambda}(i)} - k_1$, we see that the solution set is contained in a set of the required form.

(*Case 2*) Suppose that the sequence $\lambda_1(i)$ is unbounded. Then k must be zero and $k_1 + k_2q^{\lambda_2(i)-\lambda_1(i)} + \cdots + k_nq^{\lambda_n(i)-\lambda_1(i)} = 0$. Thus, we have

$$q^{\lambda_2(i)-\lambda_1(i)}(k_2 + \cdots + k_nq^{\lambda_n(i)-\lambda_2(i)}) = -k_1.$$

Note that in this case, since $k_1 \neq 0$, we must have that $\lambda_2(i) - \lambda_1(i)$ is bounded. For each of the finitely many values $\lambda_2(i) - \lambda_1(i)$ takes, we continue our analysis in parallel. We have

$$k_3q^{\lambda_3(i)-\lambda_2(i)} + \cdots + k_nq^{\lambda_n(i)-\lambda_2(i)} = \frac{-k_1}{q^{\lambda_2(i)-\lambda_1(i)}} - k_2.$$

At this step and every step thereafter, we take cases according to whether $\lambda_{j+1}(i) - \lambda_j(i)$ is bounded or not. In the case where it is bounded, for each value of the finitely many, a relation of the form $\lambda_{j+1} = \lambda_j + m_j$ is introduced. In the case where it is unbounded, we use the induction hypothesis as our solution set is contained in the solution set of linear equations of the form $k_1x_1 + \cdots + k_mx_m = 0$ and $k_{m+1}x_{m+1} + \cdots + k_nx_n = 0$. □

Lemma 3.12 *Let q be some natural number. Let $\mathcal{N} := (\mathbb{N}, s, \{Q_{k,n}\}_{n<\omega, k<n})$ be a first-order structure where the function symbol s is interpreted as the successor*

function and the predicate $Q_{k,n}$ is interpreted as the set of natural numbers which are residual to k modulo n. Then Π_q^{ind} is definably interpreted in $\mathcal{N}$.

Proof Throughout the proof the symbol s^m will be used to denote $s \circ s \circ \cdots \circ s$ m-times. We also allow m to be negative, in which case s^m denotes the composition of the predecessor function m-times (which is clearly definable).

We first interpret Π_q to be the domain of $\mathcal{N}$. Now let P be a predicate of Π_q^{ind}. By the construction of Π_q^{ind}, we have that P is a subset of the form $\phi(\mathbb{Z}^n) \cap \Pi_q^n$ for some quantifier-free formula ϕ in $(\mathbb{Z}, +, 0, \{\equiv_n\}_{n<\omega})$. Since a quantifier-free formula is a Boolean combination of formulas of the form $t(\bar{x}) = 0$ and $s(\bar{x}) \equiv_l 0$, we only need to interpret in $\mathcal{N}$ solution sets of equations and congruence relations of the above simple form intersected with Π_q^n.

Suppose that $\phi(\bar{x})$ is the equation $t(\bar{x}) = 0$. Then, by Lemma 3.11, the set $\phi(\mathbb{Z}^n) \cap \Pi_q^n$ can be interpreted as a finite union of sets which—for the sake of clarity—can be assumed to have the following form:

$$\bigwedge_{1\leq i<n} x_1 = s^{m_i}(x_{i+1}) \wedge \bigwedge_{1\leq j\leq k} x_1 \neq j.$$

Otherwise, suppose that $\phi(\bar{x})$ is the congruence relation $s(\bar{x}) \equiv_l 0$. If $(r_1, \ldots, r_n)$ is a tuple of integers that satisfy the congruence relation, then any tuple $(q_1, \ldots, q_n)$ for $q_i \in [r_i]_l$ satisfies this relation. Note that we can only have finitely many solutions up to l-congruence. Moreover, we may assume, by the Chinese remainder theorem, that l is a power of a prime number. Thus, by Lemma 3.9 and Remark 3.10, $\phi(\mathbb{Z}^n) \cap \Pi_q^n$ can be interpreted as a finite union of sets of the form

$$\bigwedge_{1\leq i\leq n} Q_{k_i,m_i}(x_i) \wedge \text{“}x_i \text{ is not equal to finitely many elements”}.$$

This finishes the proof. □

Lemma 3.13 *The theory of $\mathcal{N} := (\mathbb{N}, s, \{Q_{k,n}\}_{n<\omega,k<n})$ admits quantifier elimination after adding a constant and a unary function symbol. Moreover, it is superstable and has Lascar rank* 1.

Proof We add a constant to name 1 and a function symbol s^{-1} to name the predecessor function; observe that both are definable in $\mathcal{N}$.

We prove elimination of quantifiers by induction on the complexity of the formula ϕ. It is enough to consider the case where $\phi(\bar{x})$ is a consistent formula of the form $\exists y \psi(\bar{x}, y)$, where $|y| = 1$ and $\psi(\bar{x}, y)$ is a quantifier-free formula. We can clearly assume that ψ is in normal disjunctive form. Thus, since the negation of $Q_{k,n}$ is equivalent to the conjunction $\bigvee_{l\neq k} Q_{l,n}$, it is enough to consider the case where $\psi(\bar{x}, y)$ is a finite conjunction of formulas of the following form:

$$\begin{aligned} &Q_{k,n}(x_i) \wedge Q_{l,m}(y) \wedge x_i = c \wedge y = d \wedge x_i \neq a \wedge y \neq b \\ &\quad \wedge s^p(x_i) = x_j \wedge s^r(x_l) = y \wedge s^f(x_i) \neq x_j \wedge s^g(x_l) \neq y. \end{aligned}$$

Furthermore, we split ψ into a conjunction $\psi_0(\bar{x}, y) \wedge \psi_1(\bar{x})$, where ψ_1 is the conjunction of the atomic formulas of ψ that do not contain y. Clearly we may assume that $\psi_0(\bar{x}, y)$ does not contain instances of the form $y = d$ or $s^g(x_i) = y$. We claim that $\exists y \psi_0(\bar{x}, y)$ is equivalent to $\bar{x} = \bar{x}$. Indeed, the projection of any formula of the form

$$Q_{k,n}(y) \wedge \bigwedge_{1 \le i \le k} s^{g^i}(x) \neq y \wedge \bigwedge_{1 \le j \le l} y \neq d_j$$

is equivalent to $x = x$, thus the claim follows and $\psi(\bar{x}, y)$ is equivalent to $\psi_1(\bar{x})$. So, we obtain the first part of our statement.

Quantifier elimination allows us to prove by an easy counting types argument that the theory is superstable. Fix a set of parameters B. Clearly any nonalgebraic type over B extends the set $\pi(x)$ given by $\{s^n(x) \neq a : a \in B, n \in \mathbb{Z}\}$. Hence, by the elimination of quantifiers, we obtain that any complete nonalgebraic type over B (in one variable) is equivalent to $\pi(x) \cup \pi_0(x)$, where $\pi_0(x)$ is a complete type without parameters. Whence, $|S(B)| = |B| + |S(\emptyset)|$, as desired. In fact, any type without parameters is determined by positive formulas since, as noted before, the formula $\neg Q_{k,n}(x)$ is equivalent to a disjunction of formulas $Q_{l,n}(x)$ for $l \neq k$. In addition, as for any $n \in \mathbb{N}$ the formula $Q_{k,n}(x) \wedge Q_{l,n}(x)$ is inconsistent for distinct $l, k < n$, every complete type contains only one predicate of the form $Q_{k,n}(x)$ for a given n. Thus, it is easy to see that there are continuum many types without parameters; for instance, note that the predicate $Q_{k,2^n}(x)$ splits into $Q_{k,2^{n+1}}(x)$ and $Q_{k+2^n,2^{n+1}}(x)$ when k is odd. Hence $|S(B)| = |B| + 2^\omega$ and whence, the theory is not ω-stable.

Finally, again by quantifier elimination it is easy to see that the only formulas that divide are the algebraic ones. This shows that the theory has Lascar rank 1; the details are left to the reader. $\square$

Now, the proof of Proposition 3.8 follows from Lemmas 3.12 and 3.13. We can prove our second main theorem.

Proof of Theorem 2 It follows from Proposition 3.8 together with Theorem 3.2 that the expanded structure $(\mathbb{Z}, +, 0, \Pi_q)$ is superstable. As it is a proper expansion of $(\mathbb{Z}, +, 0)$, it has infinite Lascar rank by Theorem 1. Whence, it remains to see that it has Lascar rank ω. For this, it is enough to show that any forking extension of the principal generic has finite Lascar rank.

We will work in a sufficiently saturated extension of $(\mathbb{Z}, +, 0, \Pi_q)$, where Π_q is interpreted as Π'_q. Let $p \in S(\emptyset)$ be the generic of the connected component, and let $q = \mathrm{tp}(b/B)$ be an extension of p. Consider a realization a of $p \mid B$, and note by using Lemma 3.13 that Π'_q has Lascar rank 1. Now, working in the theory of $(\mathbb{Z}, +, 0)$, we obtain that $\mathrm{tp}(b/\Pi'_q, B)$ is the principal generic whenever $b \notin \mathrm{acl}(\Pi'_q, B)$. Moreover, if a finite tuple d is algebraic over $\Pi'_q \cup B$ and this is exemplified by some finite tuple $(c_1, \dots, c_n)$ in Π'_q, then we have in $\mathcal{T}h(\mathbb{Z}, +, 0, \Pi_q)$ that $\mathrm{U}(d/B) \le \mathrm{U}(\bar{c}/B) < \omega$ as the set $\Pi'_q \times \overset{n}{\cdots} \times \Pi'_q$ has Lascar rank n. Hence $a \notin \mathrm{acl}(\Pi'_q, B)$ in the sense of $(\mathbb{Z}, +, 0)$ and hence its type over $\Pi'_q \cup B$ is the principal generic. Thus, by Corollary 3.7 we deduce that $p \mid B = \mathrm{tp}(b/B)$ whenever b is not algebraic in the sense of $(\mathbb{Z}, +, 0)$ over $\Pi'_q \cup B$. Therefore, in the case in which $\mathrm{tp}(b/B)$ is a forking extension of p we conclude that $b \in \mathrm{acl}(\Pi'_q, B)$, and so $\mathrm{tp}(b/B)$ has finite Lascar rank, as desired. $\square$

One can see directly that the structure $(\mathbb{Z}, +, 0, \Pi_q)$ has infinite Lascar rank, without using Theorem 1, by showing that the set $\Pi_q + \overset{n}{\cdots} + \Pi_q$ has Lascar rank n. This is left to the reader.

4 Generalizations

In this section we would like to mention a few generalizations, concerning proper superstable expansions of the integers, that follow from our methods. The ideas behind our proof are transparent and clear. First, one reduces the superstability of the expanded structure to the superstability of the induced structure on the new predicate. Second, one needs to understand the induced structure in this new predicate. It seems that this is equivalent to understanding its intersection with arithmetic progressions and with the solution set of linear equations over the integers.

The following example is not very different in nature from the ones we already gave in the previous section. We leave its proof as an exercise to the interested reader.

Example 4.1 Let $(k_1, \ldots, k_m)$ be a sequence of natural numbers, and let

$$\mathrm{SP}_{(k_1,\ldots,k_m)} := \{k_1^{\cdot^{\cdot^{k_m^n}}} \mid n < \omega\}.$$

Then $(\mathbb{Z}, +, 0, \mathrm{SP}_{(k_1,\ldots,k_m)})$ is superstable of Lascar rank ω.

A more interesting example is the subset of the integers consisting of factorial elements, that is, $\mathrm{Fac} := \{n! \mid n < \omega\} \cup \{0\}$.

Proposition 4.2 *The structure* $(\mathbb{Z}, +, 0, \mathrm{Fac})$ *is superstable of Lascar rank* ω.

We first note that the set Fac satisfies the following.

Lemma 4.3 *A finite union of sets of the form*

$$\{k_0 + k_1 \cdot \alpha_1 + \cdots + k_s \cdot \alpha_s \mid \alpha_1, \ldots, \alpha_s \in \mathrm{Fac}\},$$

where $k_0, \ldots, k_s$ *are integers, cannot cover any coset of any (nontrivial) subgroup of* $\mathbb{Z}$.

Proof Suppose otherwise that the coset $m + n\mathbb{Z}$ is contained in such a finite union, and note that we may assume that $m = 0$. By the pigeonhole principle, there are integers $\lambda_0, \ldots, \lambda_l$ determining one of these sets, a prime p greater than $\lambda_0, \ldots, \lambda_l$, and an infinite subset I_0 of $\mathbb{N}$ such that $\{np^k\}_{k \in I_0}$ is contained in the set

$$\{\lambda_0 + \lambda_1 \cdot \alpha_1 + \cdots + \lambda_l \cdot \alpha_l \mid \alpha_1, \ldots, \alpha_l \in \mathrm{Fac}\}.$$

Let $\alpha_1(k), \ldots, \alpha_l(k)$ denote the factorial numbers such that

$$np^k = \lambda_0 + \lambda_1 \cdot \alpha_1(k) + \cdots + \lambda_l \cdot \alpha_l(k).$$

Now, suppose that there is some infinite subset I of I_0 such that for some j, the set $\{\alpha_j(k)\}_{k \in I}$ is finite. Without loss of generality, we may assume that $j = l$. Thus, by the pigeonhole principle, there is some factorial α and some infinite subset I' of I such that

$$np^k = \lambda_0 + \lambda_l \cdot \alpha + \lambda_1 \cdot \alpha_1(k) + \cdots + \lambda_{l-1} \cdot \alpha_{l-1}(k),$$

for k in I'. Hence, after replacing λ_0 by $\lambda_0 + \lambda_l \cdot \alpha$ and I by a suitable infinite subset, and iterating this process, we may assume that for any infinite subset I of I_0 the set $\{\alpha_j(k)\}_{k \in I}$ is unbounded for $1 \le j \le l$. Thus, we can find recursively on j an infinite subset I_j of I_{j-1} such that $\alpha_1(k), \ldots, \alpha_j(k)$ are greater than $p!$ for every k in I_j. In particular, there is a natural number k, in I_l, for which the factorial numbers $\alpha_1(k), \ldots, \alpha_l(k)$ are greater than $p!$. Consequently, as p clearly divides $\lambda_1 \cdot \alpha_1(k) + \cdots + \lambda_l \cdot \alpha_l(k)$, it also divides λ_0, which is a contradiction unless $\lambda_0 = 0$. Therefore, we have shown that the set $\{np^k\}_{k \in I_0}$ is contained in

$$\{\lambda_1 \cdot \alpha_1 + \cdots + \lambda_l \cdot \alpha_l \mid \alpha_1, \ldots, \alpha_l \in \mathrm{Fac}\}.$$

However, this yields a contradiction since for arbitrarily large k, we can find a prime q dividing $\lambda_1 \cdot \alpha_1(k) + \cdots + \lambda_l \cdot \alpha_l(k)$ but not np^k. □

Therefore, a proof similar to the one in Lemma 3.4 gives the following.

Lemma 4.4 *Let $\mathcal{L}$ be the language of groups, and let $P(x)$ be a unary predicate. Let $\mathcal{Z} := (\mathbb{Z}, +, 0, \mathrm{Fac})$ be an $\mathcal{L}_P$-structure. Then every $\mathcal{L}_P$-formula in $\mathcal{Z}$ is bounded.*

We will next prove that the induced structure on Fac comes from equality alone.

Lemma 4.5 *Let $k < n$ be natural numbers, and let $[k]_n$ denote the congruence class of k modulo n. Then* Fac $\cap\, [k]_n$ *is either finite or cofinite in* Fac.

Proof It is easy to see that when k is zero the intersection will be cofinite in Fac, while in any other case the intersection will be finite. □

Given an equation $k_1x_1 + \cdots + k_nx_n = 0$ over the integers and a partition $\mathcal{P} = \{I_j\}_{j \le l}$ of $\{1, \ldots, n\}$, we denote by $X_{\mathcal{P}}$ the set of solutions $(m_1!, \ldots, m_n!)$ such that $m_i = m_k$ if and only if $i, k \in I_j$ for some $j \le l$.

Lemma 4.6 *Let $k_1x_1 + \cdots + k_nx_n = 0$ be an equation over the integers, and let $\mathcal{P} = \{I_j\}_{j \le l}$ be a partition of $\{1, \ldots, n\}$. Then the projection of $X_{\mathcal{P}}$ on its I_j-coordinates is an infinite set if and only if $\sum_{i \in I_j} k_i = 0$.*

Proof Let $\mathcal{P} = \{I_j\}_{j \le l}$ be a partition of $\{1, \ldots, n\}$, and suppose that $\sum_{i \in I_j} k_i = 0$ for some $j \le l$. Clearly, there are infinitely many solutions of the form $(x_1, \ldots, x_n)$ with $x_i = 0$ for $i \notin I_j$ and x_i constant for $i \in I_j$. Hence, we get the result. For the converse, assume for some $k \le l$ that the projection of $X_{\mathcal{P}}$ on its I_k-coordinates yields an infinite set but $\sum_{i \in I_k} k_i$ is nonzero, and let $X_{\mathcal{P}}$ be the set $\{(m_1(t)!, \ldots, m_n(t)!)\}_{t<\omega}$. Set $s_j(t)$ to be the value of every $m_i(t)$ when $i \in I_j$, and note that all $s_j(t)$'s are distinct by the definition of $X_{\mathcal{P}}$. It is clear that

$$\sum_{j \le l} \Big(\sum_{i \in I_j} k_i\Big) \cdot s_j(t)! = 0.$$

Now, let J be the set of subindices $j \le l$ for which $\sum_{i \in I_j} k_i$ is nonzero; note that J is nonempty as $k \in J$ and also that

$$\sum_{j \in J} \Big(\sum_{i \in I_j} k_i\Big) \cdot s_j(t)! = 0.$$

By assumption, this equation holds for all $t < \omega$ and so, by the pigeonhole principle, we can find an enumeration of $J = \{j_1, \ldots, j_r\}$ such that $s_{j_1}(t) > \cdots > s_{j_r}(t)$ for infinitely many values of t. Additionally, for some of these t's we have that $s_{j_1}(t) > |\sum_{i \in I_{j_2}} k_i + \cdots + \sum_{i \in I_{j_r}} k_i|$ and thus

$$\Big|\Big(\sum_{i \in I_{j_1}} k_i\Big) \cdot s_{j_1}(t)!\Big| > \Big|\Big(\sum_{i \in I_{j_2}} k_i\Big) \cdot s_{j_2}(t)! + \cdots + \Big(\sum_{i \in I_{j_r}} k_i\Big) \cdot s_{j_r}(t)!\Big|,$$

which is a contradiction. Hence, we get the result. □

If $k_1x_1 + \cdots + k_nx_n = 0$ is an equation over the integers and $S \subseteq \mathbb{Z}^n$ is its solution set, observe that S is precisely the finite union of all $X_{\wp}$. Therefore, Lemmas 4.5 and 4.6 give that all the induced structure on Fac comes from equality alone, thus $\mathrm{Fac}^{\mathrm{ind}}$ is strongly minimal and Proposition 4.2 follows.

Our article can be seen as opening a pathway for answering the following interesting questions.

Question 4.7

- *(J. Goodrick) Characterize the subsets $\Pi \subset \mathbb{Z}$, for which $(\mathbb{Z}, +, 0, \Pi)$ is superstable.*
- *Characterize the subsets $\Pi \subset \mathbb{Z}$, for which $(\mathbb{Z}, +, 0, \Pi)$ is stable.*

References

[1] Baldwin, J., and M. Benedikt, "Stability theory, permutations of indiscernibles, and embedded finite models," *Transactions of the American Mathematical Society*, vol. 352 (2000), pp. 4937–69. 157

[2] Casanovas, E., and M. Ziegler, "Stable theories with a new predicate," *Journal of Symbolic Logic*, vol. 66 (2001), pp. 1127–40. Zbl 1002.03023. MR 1856732. DOI 10.2307/2695097. 160, 161

[3] Chernikov, A., and P. Simon, "Externally definable sets and dependent pairs, II," *Transactions of the American Mathematical Society*, vol. 367 (2015), pp. 5217–35. Zbl 06429160. MR 3335415. DOI 10.1090/S0002-9947-2015-06210-2. 157

[4] Marker, D., "A strongly minimal expansion of (ω, s)," *Journal of Symbolic Logic*, vol. 52 (1987), pp. 205–7. Zbl 0647.03029. MR 0877867. DOI 10.2307/2273874. 157

[5] Pillay, A., *Geometric Stability Theory*, vol. 32 of *Oxford Logic Guides*, Oxford University Press, New York, 1996. Zbl 0871.03023. MR 1429864. 158, 159

[6] Pillay, A., and C. Steinhorn, "Discrete o-minimal structures," *Annals of Pure and Applied Logic*, vol. 34 (1987), pp. 275–89. 157

[7] Poizat, B., "Supergénérix," *Journal of Algebra*, vol. 404 (2014), pp. 240–70. Zbl 1346.20060. MR 3177894. DOI 10.1016/j.jalgebra.2014.01.026. 158

[8] Sela, Z., "Diophantine geometry over groups, VIII: Stability," *Annals of Mathematics (2)*, vol. 177 (2013), 787–868. Zbl 1285.20042. MR 3034289. DOI 10.4007/annals.2013.177.3.1. 157

[9] Sklinos, R., "Some model theory of the free group," Ph.D. dissertation, University of Leeds, Leeds, United Kingdom, 2011. 158

[10] Wagner, F. O., *Stable Groups*, vol. 240 of *London Mathematical Society Lecture Note Series*, Cambridge University Press, Cambridge, 1997. Zbl 0897.03037. MR 1473226. DOI 10.1017/CBO9780511566080. 158

[11] Wagner, F. O., "Some remarks on one-basedness," *Journal of Symbolic Logic*, vol. 69 (2004), pp. 34–38. Zbl 1073.03018. MR 2039343. DOI 10.2178/jsl/1080938823. 159

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