Invariance and Definability, with and without Equality

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Abstract The dual character of invariance under transformations and definability by some operations has been used in classical works by, for example, Galois and Klein. Following Tarski, philosophers of logic have claimed that logical notions themselves could be characterized in terms of invariance. In this article, we generalize a correspondence due to Krasner between invariance under groups of permutations and definability in $\mathscr{L}_{\infty\infty}$ so as to cover the cases (quantifiers, logics without equality) that are of interest in the logicality debates, getting McGee's theorem about quantifiers invariant under all permutations and definability in pure $\mathscr{L}_{\infty\infty}$ as a particular case. We also prove some optimality results along the way, regarding the kinds of relations which are needed so that every subgroup of the full permutation group is characterizable as a group of automorphisms.

1 Introduction

Permutations and relations may be seen as two ways of encoding information about a domain of objects. In the case of relations, it is pretty obvious that some piece of information is encoded: a difference is made between objects between which the relation holds, and objects between which it does not. This is also true, if slightly less obvious, of permutations. By taking into account a permutation, it is recognized that some differences between objects do not matter: given the purpose at hand, one may as well consider that such and such objects have been interchanged.

Moreover, permutations and relations may be seen as two *dual* ways of encoding information about objects. Permutations tell us which objects are similar (they can be mapped to each other), relations tell us which objects are different (think of a unary relation true of one object and false of another). This duality is at work whenever mathematicians or physicists consider groups of isomorphisms and classes

of invariants.[1] A group of automorphisms is a group of permutations that respect a given structure: permutations are extracted from a domain equipped with relations. A class of invariants consists of a class of relations that respect a given group of permutations: extensions for symbols to be interpreted in the domain of objects under consideration are extracted from permutations.

In logic, this duality has been extensively investigated within model theory. Typical questions include the characterization of structures that can be described in terms of their automorphism groups, or more generally how much automorphism groups say about the structures they come from (see Cameron [4] on model theory and groups). Within the philosophy of logic, the duality between permutations and extensions comes to the forefront in debates regarding the characterization of logical constants. Invariance under all permutations has been taken as the formal output of a conceptual analysis of what it is to be a logical notion. Thus, it has been proposed as a mathematical counterpart to the generality of logic (by Tarski himself in [24]) and to its purely formal nature (by Sher [22] and MacFarlane [19]).

Our aim here is to bring these two traditions closer together by adapting some results in model theory to the cases that are discussed by philosophers of logic interested in invariance as a logicality criterion. As a case in point, we will show how the much quoted theorem by McGee [20, p. 572] about invariance under all permutations and definability in the pure infinitary logic $\mathscr{L}_{\infty\infty}$ can be seen as following from a suitable extension of much earlier work by Krasner [15], [16]. Our motivation for such a generalization is that a conceptual assessment of a given logicality criterion stated in terms of invariance properties should be grounded in a prior evaluation of the significance of the general duality between groups of permutations and sets of relations and in an independent assessment of what invariance under groups of permutations (or other kind functions) can and cannot do in the contexts of interest. From a technical perspective, our starting point will be Krasner's work and his so-called *abstract* Galois theory. Application to cases of interest in the logicality debates involve two generalizations. To cover the interpretation of quantifiers, we will to need to consider structures equipped with second-order relations. To cover scenarios in which the logicality of identity is not presupposed, we will need to consider functions that are not injective, as suggested by Feferman [9].

To the best of our knowledge, Krasner was the first to generalize Galois theory to a logical setting, but others have followed (see, e.g., da Costa and Rodrigues [8] for an explanation, and comparison with Krasner's work, of the contributions of José Sebastião e Silva). Also, model-theoretic work has been done to generalize Galois theory in a first-order finitary setting (see, e.g., Poizat [21] and Casanovas and Farré [7]). In this article, we investigate a second-order infinitary setting.

Our main result (Theorem 12 below) provides a unified perspective on the duality by establishing a correspondence between certain monoids of relations and sets of operations closed under definability in an infinitary language without equality. In Section 2, we recall Krasner's correspondence between classes of relations of possibly infinite arity closed under definability in $\mathscr{L}_{\infty\infty}$ and subgroups of the permutation group. We show how Krasner's correspondence relates to the standard model-theoretic characterization of groups of permutations which are the automorphism groups of a first-order structure (with relations of finite arity) as closed groups in the topology of pointwise convergence. Section 3 is devoted to extending Krasner's correspondence to second-order relations. We prove some optimality results

along the way: What kind of relations are really needed so that every subgroup is the automorphism group of a class of relations? In the first-order case, infinitary relations were necessary. Second-order binary relations suffice to do the job, but second-order unary relations do not. The next section is devoted to relaxing the assumption of injectivity, extending the correspondence to logics without equality. Finally, we draw some lessons in Section 5 for the logicality debate.

2 Abstract Galois Theory and Permutation Groups

2.1 Krasner's abstract Galois theory To begin with, we will recall the fundamentals of Krasner's *abstract* Galois theory, following in particular [21]. The starting point of Krasner is *classical* Galois theory, in which a particular instance of the general duality we have outlined is at play. Classical Galois theory is the study of the duality between the subgroups of the automorphism group of certain types of field extensions, called *Galois extensions*, and the intermediate fields in the field extension.

A field is an algebraic structure with addition, multiplication, inverses, and units. If $k \subseteq K$ are both fields, then we say that $K : k$ is a *field extension*; it is *Galois* if it is algebraic, normal, and separable (definitions may be found in Lang [17]). Given a field extension $K : k$, let G be the Galois group of $K : k$, that is, the group of automorphisms of K fixing the elements of k pointwise. Define the following pair of mappings:

$$\text{Fix}(H) = \{a \in K \mid ha = a \text{ for all } h \in H\},$$
$$\text{Gal}(A) = \{g \in G \mid ga = a \text{ for all } a \in A\},$$

where H is any subset of G, and A is any subset of the domain of K. The fundamental theorem of Galois theory says that, for finite Galois extensions, $\text{Fix}(\text{Gal}(A))$ is the smallest subfield of K including $k \cup A$ and that $\text{Gal}(\text{Fix}(H))$ is the smallest subgroup of G including H. There is thus a one-to-one correspondence between fields K' such that $k \subseteq K' \subseteq K$ and subgroups of G.

According to Krasner, "the true origin of Galois theory [does not lie in] algebra, in the strict sense of the word, but in logic" ([16, p. 163]). This claim is to be substantiated by exhibiting a general duality, between automorphism groups and relations definable over a structure. Consider a domain Ω, S_Ω the full symmetric group on Ω, and a set r of relations on Ω; these relations may include infinite arity relations regarded as subsets of Ω^α, where α is some ordinal number. We now define the following pair of mappings:

$$\text{Inv}(H) = \{R \subseteq \Omega^\alpha \mid hR = R \text{ for all } h \in H, \alpha \leq |\Omega|\},$$
$$\text{Aut}(r) = \{g \in S_\Omega \mid gR = R \text{ for all } R \in r\}.$$

The logic $\mathcal{L}_{\infty\infty}$ is the infinitary generalization of the predicate calculus where formulas are built by means of arbitrarily long conjunctions and disjunctions and by means of arbitrarily long universal and existential quantifier sequences (see, e.g., Karp [13]). The $\mathcal{L}_{\infty\infty}$-closure of r is the set of relations definable in $\mathcal{L}_{\infty\infty}$ from relations in r. The following theorem is essentially shown by Krasner in [15].[2]

Theorem 1   *Let Ω and S_Ω be as above, let $H \subseteq S_\Omega$ be any set of permutations, and let r be any set of relations on Ω of arities at most $|\Omega|$.*

   (1) $\text{Inv}(\text{Aut}(r))$ *is the $\mathcal{L}_{\infty\infty}$-closure of r.*

   (2) $\text{Aut}(\text{Inv}(H))$ *is the smallest subgroup of S_Ω including H.*

Thus, there is a one-to-one correspondence between the subgroups of the full symmetric group S_Ω of Ω and the sets of relations closed under definability in $\mathscr{L}_{\infty\infty}$.

How does this relate with the original correspondence of classical Galois theory? The first half of the fundamental theorem of Galois Theory can be derived from the first half of Theorem 1.[3] Given a field extension $K : k$ and $A \subseteq K$, let r be the set of addition, multiplication, inverse, and the relation $R_a = \{a\}$ for every $a \in k \cup A$. By definition, we have that $\mathrm{Gal}(A) = \mathrm{Aut}(r)$. Now $\mathrm{Inv}(\mathrm{Aut}(r))$ and $\mathrm{Fix}(\mathrm{Gal}(A))$ are not exactly on a par, because Krasner's theory takes sets of *relations* as values instead of, as in the classical Galois theory, sets of *points*. However, the set of fixed points can be extracted from the set of fixed relations by observing that an element $a \in \Omega$ is fixed by a permutation if and only if the singleton relation $\{a\}$ is fixed. Given that $\mathrm{Inv}(\mathrm{Aut}(r))$ is closed under definability in $\mathscr{L}_{\infty\infty}$, it is immediate that this set of fixed points is a subfield of K. To get the first half of the fundamental Galois theorem, one also needs to show that it is included in any subfield extending $k \cup A$. Krasner proves this in [16] by showing that if R_a is definable in $\mathscr{L}_{\infty\infty}$ over r, then a can be reached from elements in $k \cup A$ by means of the field operations.[4]

2.2 Permutation groups and topology Krasner's result shows that for any group of permutations of Ω there is some set of relations r on Ω such that $\mathrm{Aut}(r)$ is exactly that group. This is not true if we restrict the relations to be of finite arity. If r is a set of relations of finite arity, then (Ω, r) is a "standard" first-order relational structure and, obviously, the group $\mathrm{Aut}(r)$ coincides with the automorphism group of the first-order structure (Ω, r). However, it is not true that every subgroup of the full permutation group is the automorphism group of a first-order structure (with only finite arity relations). We will give a detailed proof of the well-known characterization of automorphism groups of first-order structures in terms of topological properties. This is done partly to set up the scene for the next section.

Throughout this article, let Ω be an *infinite* domain. First we need to introduce the topology, the product topology, on the group of permutations that we will be using. Let Ω be equipped with the discrete topology, that is, the topology in which every set is open. Let G be the full symmetric group S_Ω on Ω, that is, the group of all permutations of Ω. The stabilizer $G_{\bar{a}}$ of a finite tuple $\bar{a} \in \Omega^k$ is the set of all permutations in G keeping all elements in $\bar{a}$ fixed; that is,

$$G_{\bar{a}} = \{g \in G \mid g\bar{a} = \bar{a}\}.$$

The symmetric group S_Ω on Ω is a subset of Ω^Ω; let G inherit the product topology from Ω^Ω. That is, a basis for this topology are the (right or left) cosets of stabilizers of finite tuples, which are sets of the following form:

$$G_{\bar{a},\bar{b}} = \{g \in G \mid g\bar{a} = \bar{b}\},$$

where $\bar{a}, \bar{b} \in \Omega^k$.[5]

This is the topology of pointwise convergence. The (topological) closure of a set $A \subseteq G$ consists of all permutations $g \in G$ such that for every k and every $\bar{a} \in \Omega^k$, there is $f \in A$ such that $f(\bar{a}) = g(\bar{a})$.[6]

Given a subgroup H of G, let $O_{\bar{a}}^H = \{h\bar{a} \mid h \in H\}$ be the orbit of $\bar{a}$ under H. Orbits provide a canonical way of "translating" a group of permutations on a domain into relations on that domain. Then $(\Omega, O_{\bar{a}}^H)_{\bar{a} \in \Omega^k}$ is thus a first-order relational structure, called the *canonical structure* of H.

The following well-known folklore result (see, e.g., Hodges [12, Theorem 4.1.4]) characterizes the subgroups of G that are automorphism groups of a first-order structure (with only finite arity relations).

Proposition 2   *Let H be a subgroup of G. The following are equivalent.*

(1) *There is a first-order structure M with domain Ω whose automorphism group is H.*

(2) *H is the automorphism group of the canonical structure of H, $(\Omega, O_{\bar{a}}^{H})_{\bar{a} \in \Omega^{k}}$.*

(3) *H is closed.*

Proof   (1) implies (3): Let $g \in G$ be in the closure of H; that is, for any $\bar{a} \in \Omega^{k}$ there is $h \in \mathrm{Aut}(M)$ which is identical to g on $\bar{a}$. Let R be a relation in M, and let $\bar{a} \in \Omega^{k}$ and $h \in \mathrm{Aut}(M)$ be such that $g\bar{a} = h\bar{a}$; then $R\bar{a}$ iff $Rh\bar{a}$ iff $Rg\bar{a}$ and so R is invariant under g. Similarly, we can show that any constant and function in M is invariant under g; thus $g \in \mathrm{Aut}(M)$, and $\mathrm{Aut}(M) = H$ is closed.

(2) implies (1): This is obvious.

(3) implies (2): Let M be the canonical structure of H. Clearly, $H \subseteq \mathrm{Aut}(M)$ since H respects all its orbits; that is, for all $h \in H$, $\bar{a} \in O_{\bar{b}}^{H}$ iff $h\bar{a} \in O_{\bar{b}}^{H}$. Assume that $g \in \mathrm{Aut}(M)$ and $\bar{a} \in \Omega^{k}$. Then $g\bar{a} \in O_{\bar{a}}^{H}$ and so there is an $h \in H$ such that $g\bar{a} = h\bar{a}$. This shows that g is in the closure of H; but since H is closed we have $\mathrm{Aut}(M) \subseteq H$, proving that H is the automorphism group of M.   □

The proof actually shows that for any subgroup H of G, the automorphism group of the canonical structure of H is the closure of H.

To see that not all subgroups of G are automorphism groups of first-order structures, it is, using the proposition, sufficient to find a subgroup of G which is not closed. Let the support of a permutation g of Ω be the set of points moved by g, that is, the set $\{a \in \Omega \mid ga \neq a\}$. The set H of permutations with finite support is a nonclosed subgroup of G. Take any permutation $g \in G$ and any $\bar{a} \in \Omega^{k}$. We can find $h \in H$ such that $g\bar{a} = h\bar{a}$ by extending the bijection $g \mid \bar{a} : \{\bar{a}\} \to \{g\bar{a}\}$ to a permutation of $A = \{\bar{a}\} \cup \{g\bar{a}\}$. Let then h be this permutation on A and the identity function outside A. This shows that the closure of H is the full group G, and since not every permutation has a finite support, H is not closed.

Hence the use of relations of infinite arity in Krasner's correspondence is indeed necessary. For nonclosed H, if $H = \mathrm{Aut}(M)$, then M includes relations of infinite arity.

There is a connection between the allowed arities of relations in the structures and the topological closure property characterizing automorphism groups as the following proposition shows. To see this, let us define the α-topology, for α an infinite ordinal, on G in the following manner. The basic open sets are of the form

$$G_{\bar{a}, \bar{b}}\{g \in G \mid g\bar{a} = \bar{b}\},$$

where $|\bar{a}| = |\bar{b}| < \alpha$. The ω-topology coincides with the ordinary product topology described above. It should be noted that the β-topology is a refinement of the α-topology if $\alpha \leq \beta$, which means that an open set in the α-topology is also open in the β-topology. If we relax the definition of a first-order structure to include relations of arity less than α and define α-orbits of a subgroup H to be

$$O_{\bar{a}}^{H} = \{h\bar{a} \mid h \in H\},$$

where now $\bar{a} \in \Omega^\alpha$, then the proof of Proposition 2 goes through with the canonical structure of H replaced by $(\Omega, O_{\bar{a}}^H)_{\bar{a} \in \Omega^\alpha}$ and "closed" replaced by "closed in the α-topology," giving the following generalization.

Proposition 3   *Let H be a subgroup of G, and let α be an infinite ordinal. Then the following are equivalent.*

(1) *There is a first-order structure M with domain Ω and relations of arity less than α whose automorphism group is H.*

(2) *H is the automorphism group of the (canonical) relational structure $(\Omega, O_{\bar{a}}^H)_{\bar{a} \in \Omega^\alpha}$.*

(3) *H is closed in the α-topology.*

Krasner's result that every subgroup of G is $\mathrm{Aut}(M)$ for some relational structure M when relations of *infinite* arity are allowed (first half of Theorem 1) follows from Proposition 3. The $(|\Omega|+1)$-topology is the discrete topology; that is, every set is open. If $g \in G$ and $\bar{a}$ is an enumeration of Ω, then

$$G_{\bar{a},g\bar{a}} = \{g\}$$

is a basic open set. Thus every set of permutations is the union of basic open sets, and therefore open, and therefore also closed. Hence, if relations of arity $|\Omega|$ are allowed, every group of permutations will be the automorphism group of some relational (infinitary) structure.

3   Quantifiers Come into Play

3.1 Second-order relations, $\mathcal{L}_{\infty\infty}$, and Krasner's correspondence We now extend Krasner's correspondence to second-order operations, to account for quantifier extensions, which have been the traditional focus of the debates regarding logicality and invariance. In this section, we state the corresponding generalization of Theorem 1 and prove its first part.

A finite second-order relation Q of type $(i_1, \ldots, i_k)$ on Ω is a subset of $\mathcal{P}(\Omega^{i_1}) \times \cdots \times \mathcal{P}(\Omega^{i_k})$ for finite k and finite $i_1, \ldots, i_k$. A second-order structure q on a domain Ω is a set of (finite-ary) first-order and second-order relations on Ω. A permutation g on Ω preserves a second-order relation Q of type $(i_1, \ldots, i_k)$ if $(R^{i_1}, \ldots, R^{i_k}) \in Q$ iff $(gR^{i_1}, \ldots, gR^{i_k}) \in Q$, where R^{i_j}, $1 \le j \le k$, are first-order relations of arities i_j. The mappings Aut and Inv admit a straightforward generalization to the present setting: $\mathrm{Aut}(q)$ is the group of permutations which preserve all first-order and second-order relations in q, and $\mathrm{Inv}(H)$, for $H \subseteq G$, is the set of first-order and second-order relations which are preserved by all permutations in H.

Given a second-order structure q, $\mathcal{L}_{\infty\infty}(q)$ is an interpreted language in the logic $\mathcal{L}_{\infty\infty}$: it is the language whose signature matches the structure q and whose predicate and quantifier symbols are interpreted by the relations in q. The syntax and semantics for symbols interpreted by second-order relations is familiar from generalized quantifier theory (see Lindström [18]). As a case in point, let us recall what are the intended syntactic and semantic clauses for a second-order relation of type (2). Let Q in q be of type (2); $\mathcal{L}_{\infty\infty}(q)$ is then equipped with a matching quantifier symbol $\bar{Q}$. The syntactic clause for $\bar{Q}$ has it that if ϕ is a formula of $\mathcal{L}_{\infty\infty}(q)$, so is $\bar{Q}x_1x_2\,\phi$. The satisfaction clause for $\bar{Q}$ is given by

$$\Omega, q \vDash \bar{Q}x_1x_2\,\phi\,[\sigma]$$

iff
$$\|\phi\|^{\sigma}_{x_1,x_2} \in Q,$$

where σ is an assignment over Ω and

$$\|\phi\|^{\sigma}_{x_1,x_2} = \left\{(a_1, a_2) \in \Omega^2 \mid \Omega, q \vDash \phi \left[\sigma[x_1 := a_1, x_2 := a_2]\right]\right\}.$$

We will need to consider definability in $\mathscr{L}_{\infty\infty}(q)$ for a given q. Without loss of generality, let Q be a second-order relation of type (2). We say that Q is *definable* in $\mathscr{L}_{\infty\infty}(q)$ if and only if there is a sentence $\phi_Q(\bar{R})$ in $\mathscr{L}_{\infty\infty}(q)$ expanded with a binary predicate symbol $\bar{R}$ such that

$$\Omega, q, R \vDash \phi_Q(\bar{R})$$
iff
$$R \in Q,$$

where R is a binary first-order relation on Ω interpreting $\bar{R}$. The $\mathscr{L}_{\infty\infty}$-closure of a second-order structure q is the set of finite first-order and second-order relations on Ω which are definable in $\mathscr{L}_{\infty\infty}(q)$.

We can now state the generalization of Theorem 1 to second-order structures and automorphism groups thereof as follows.

Theorem 4   *Let Ω be a domain, let $G = S_{\Omega}$ be the symmetric group on Ω, let $H \subseteq G$ be any set of permutations, and let q be a second-order structure on Ω.*

(1) $Q \in \mathrm{Inv}(\mathrm{Aut}(q))$ *if and only if Q is definable in $\mathscr{L}_{\infty\infty}(q)$. The same holds for relations R.*

(2) $\mathrm{Aut}(\mathrm{Inv}(H))$ *is the smallest subgroup of G including H.*

We will postpone the discussion and the proof of the second part of the theorem until the next section. Regarding the first part, the proof that any relation definable in $\mathscr{L}_{\infty\infty}(q)$ is invariant under the automorphisms of q is a straightforward induction on the complexity of formulas of $\mathscr{L}_{\infty\infty}(Q)$. What remains to be proved is the following lemma.

Lemma 5   *Let Q be a first-order or second-order relation on Ω. If $Q \in \mathrm{Inv}(\mathrm{Aut}(q))$, then Q is definable in $\mathscr{L}_{\infty\infty}(q)$.*

Proof   For the sake of simplicity, we will give the proof for Q a second-order relation of type (2). So let Q be invariant under the automorphisms of q. We first construct the sentence $\phi_Q(\bar{R})$ which defines it. Let us fix an enumeration $I \to \Omega$ of the elements in Ω and, given a set of variables X of cardinality $|\Omega|$, an enumeration $I \to X$ of the variables in X. Let σ be a partial assignment over X defined by $\sigma(x_i) = a_i$ for $i \in I$.

For any relation S on Ω and for any symbol $\bar{T}$ matching the type of S, let the description $\Delta_S(\bar{T})$ of S by $\bar{T}$ be as follows. If S is a first-order relation of arity k, $\Delta_S(\bar{T})$ is the conjunction of formulas of the form $\bar{T}x_{i_1} \ldots x_{i_k}$ or $\neg \bar{T}x_{i_1} \ldots x_{i_k}$, which are satisfied in Ω, S under σ. Similarly, if S is a second-order relation, say, again of type (2), $\Delta_S(\bar{T})$ is the conjunction of formulas of the form

$$\bar{T}y_1, y_2 \bigvee_{(i,j) \in K} (y_1 = x_i \wedge y_2 = x_j)$$

or

$$\neg \bar{T} y_1, y_2 \bigvee_{(i,j)\in K} (y_1 = x_i \wedge y_2 = x_j)$$

for $K \subseteq I \times I$, which are satisfied in Ω, S under σ. Moreover, let ψ be the following formula

$$\bigwedge_{i,j\in I, i\neq j} x_i \neq x_j \wedge \forall y \bigvee_{i\in I} y = x_i;$$

that is, ψ says that σ is a bijection from X onto Ω.

Now ϕ_Q may be written down as

$$\forall X \left(\left(\bigwedge_{S\in q} \Delta_S(\bar{S}) \wedge \psi \right) \to \bigvee_{U\in Q} \Delta_U(\bar{R}) \right),$$

where $\forall X$ is short for the infinite sequence of universal quantifiers $\forall x_1, \ldots, \forall x_i, \ldots$ quantifying over every variable in X.

We now prove that $\Omega, q, R \vDash \phi_Q(\bar{R})$ iff $R \in Q$. From left to right, if $\phi_Q(\bar{R})$ is true in Ω, q, R, we have in particular that

$$\bigwedge_{S\in q} \Delta_S(\bar{S}) \wedge \psi \to \bigvee_{U\in Q} \Delta_U(\bar{R}) \tag{1}$$

is satisfied under σ. But, by construction, the premise of (1) is satisfied under σ, hence there is a relation $U \in Q$ such that $\Delta_U(\bar{R})$ is satisfied under σ when $\bar{R}$ is interpreted by R. This forces $U = R$; hence $R \in Q$ as required.

From right to left, let R be a first-order relation in Q, and let σ' be a partial assignment on X. Assume that

$$\Omega, q, R \vDash \bigwedge_{S\in q} \Delta_S(\bar{S}) \wedge \psi \ [\sigma'].$$

We need to show that $\Omega, R \vDash \bigvee_{U\in Q} \Delta_U(\bar{R}) \ [\sigma']$.

Since ψ is satisfied under σ', σ' is a bijection; hence there is a permutation f on Ω such that $f \circ \sigma' = \sigma$. Moreover, since $\bigwedge_{S\in q} \Delta_S(\bar{S})$ is satisfied under σ', f is an automorphism of q; hence f preserves Q.

By definition, $\Omega, fR \vDash \Delta_{fR}(\bar{R}) \ [\sigma]$. Since $\sigma = f \circ \sigma'$, this implies $\Omega, R \vDash \Delta_{fR}(\bar{R}) \ \sigma'$. Since f preserves Q and $R \in Q$, we also have $fR \in Q$. Therefore, $\Delta_{fR}(\bar{R})$ is one the disjuncts in $\bigvee_{U\in Q} \Delta_U(\bar{R})$ and the formula is satisfied under σ' as required.   □

McGee [20] proved that, on a given domain, an "operation is invariant under all permutations if and only if it is described by some formula of $\mathcal{L}_{\infty\infty}$" (p. 572). This result follows quite directly from what we have.[7] Consider the empty structure $\emptyset$. $\text{Aut}(\emptyset)$ is the group G of all permutations. By the first part of Theorem 4, $\text{Inv}(G)$ is the $\mathcal{L}_{\infty\infty}$-closure of $\emptyset$, which is exactly McGee's result. It should be noted that the proof of Lemma 5 itself is a rather straightforward generalization of Krasner's proof in [15]. The possibility to describe every subset of Ω^k by means of formulas of $\mathcal{L}_{\infty\infty}$ is sufficient to handle the extension to second-order relations.

3.2 A topology for second-order structures We now turn to the second half of Theorem 4 and ask when a group of permutations is the automorphism group of a structure, in a context where second-order relations are allowed but all relations are finitary. As with Proposition 2 earlier, we base our answer on topological properties of groups of permutations. We will see that if we allow quantifiers of type (2), then all groups of permutations are automorphism groups of some second-order structure. However, the case is quite different when we only allow monadic quantifiers, that is, quantifiers of type $(1, 1, \ldots, 1)$. There is a natural topology such that exactly the closed groups are the automorphism groups of monadic second-order structures.

We say that a second-order structure is *monadic* if all relations are relations on the power set of Ω. A monadic second-order structure on Ω is nothing else than a first-order structure on Ω together with a relational first-order structure on the power set $\mathscr{P}(\Omega)$ of Ω. Let $S_{\mathscr{P}(\Omega)}$ be the symmetric group on $\mathscr{P}(\Omega)$. The symmetric group $G = S_{\Omega}$ on Ω embeds in a natural way in $S_{\mathscr{P}(\Omega)}$, by the mapping $g \mapsto g^*$ where

$$g^* A = \{ga \mid a \in A\}.$$

Let $S_{\mathscr{P}(\Omega)}$ be equipped with the product topology.

The second-order topology on G is defined by letting the basic open sets be

$$G_{\bar{A}, \bar{B}} = \{g \in G \mid g^* A_i = B_i \text{ for all } i\},$$

where $\bar{A} = A_0, \ldots, A_n$ and $\bar{B} = B_0, \ldots, B_n$ are subsets of Ω. As is seen directly from the definition of the basic open sets, the second-order topology on G is the topology induced by the embedding $^* : G \to S_{\mathscr{P}(\Omega)}$.

Lemma 6   *The set $G^* = \{g^* \mid g \in G\}$ is a closed subgroup of $S_{\mathscr{P}(\Omega)}$.*

Proof   Since * respects composition, G^* is a group. To see that it is closed, let $h \in S_{\Omega}$ be in the closure of G^*; that is, for all $\bar{A} = A_0, \ldots, A_n$ there is $g \in G$ such that $hA_i = g^* A_i$ for $i \leq n$. Define $f \in G$ by $fa \in h\{a\}$, which uniquely determines f since the set $h\{a\}$ is a singleton set. There is $g \in G$ such that $h\{a\} = g^*\{a\} = \{ga\}$.

Assume now that $h \notin G^*$. Then there is $A \subseteq \Omega$ such that $hA \neq f^* A$. We may assume that there is $a \in hA \setminus f^* A$. For otherwise $h(A^c) \setminus f^*(A^c) \neq \emptyset$, since the complement $(f^* A)^c$ of $f^* A$ is $f^*(A^c)$ and also that $h(A^c) = (hA)^c$, since there is $g \in G$ such that $g^* A = hA$ and $g^*(A^c) = h(A^c)$ (take $\bar{A} = A, A^c$).

Let $g \in G$ be such that $g^* A = hA$ and $g^*\{f^{-1}a\} = h\{f^{-1}a\}$ (take $\bar{A} = A, \{f^{-1}(a)\}$). Thus $gf^{-1}a = ff^{-1}a = a$ and so $f^{-1}a \in A$, contradicting that $a \notin f^* A$.   □

From this lemma it follows that a set $A \subseteq G$ is closed in the second-order topology if and only if A^* is closed in $S_{\mathscr{P}(\Omega)}$.

By using Proposition 2 and considering a second-order monadic structure on the domain Ω as a first-order structure on $\mathscr{P}(\Omega)$, we can prove the characterization of automorphism groups of monadic second-order structures.

Proposition 7   *Let H be a subgroup of G. The following are equivalent.*

(1) *There is a second-order monadic structure q with domain Ω whose automorphism group is H.*

(2) *H is the automorphism group of the second-order structure*

$$\mathbb{M} = (\Omega, O_{\bar{R}}^{H^*})_{\bar{R} \in (\mathscr{P}(\Omega))^k}.$$

(3) *H is closed in the second-order topology.*

Proof   The implication (2) $\Rightarrow$ (1) is trivial.

To prove that (1) $\Rightarrow$ (3), let q be a second-order structure with $\mathrm{Aut}(q) = H$, let M be the first-order part of q, and let q' be the second-order part. By Proposition 2, $\mathrm{Aut}(M) \subseteq G$ is closed. Now, q' can be seen as a *first-order* structure on the domain $\mathcal{P}(\Omega)$, and by applying Proposition 2 to this structure on $\mathcal{P}(\Omega)$ we get that $\mathrm{Aut}_{\mathcal{P}(\Omega)}(q') \subseteq S_{\mathcal{P}(\Omega)}$ is closed.[8] By the comment after Lemma 6, $\mathrm{Aut}_{\Omega}(q') \subseteq G$ is closed since

$$\mathrm{Aut}_{\Omega}(q')^* = G^* \cap \mathrm{Aut}_{\mathcal{P}(\Omega)}(q'),$$

and both of these subgroups are closed. To conclude the argument it is enough to observe that $\mathrm{Aut}(q) = \mathrm{Aut}(M) \cap \mathrm{Aut}(q')$ and that both of these groups are closed.

For the last implication (3) $\Rightarrow$ (2) it is enough to observe that $H^* \subseteq S_{\mathcal{P}(\Omega)}$ is closed and thus, by Proposition 2, H^* is the automorphism group of

$$\left(\mathcal{P}(\Omega), O_{\bar{R}}^{H^*}\right)_{\bar{R} \in (\mathcal{P}(\Omega))^k},$$

and that $g \in G$ respects Q if and only if $g^* \in S_{\mathcal{P}(\Omega)}$ respects Q, as a first-order quantifier on $\mathcal{P}(\Omega)$. Thus, H is the automorphism group of $\mathbb{M}$.   □

To conclude the discussion of automorphism groups of monadic second-order structures, we prove that not all subgroups of G are closed in the second-order topology, implying that not all subgroups are automorphism groups of such structures, which proves that the second part of Theorem 4 cannot hold if we restrict $\mathrm{Inv}(\cdot)$ to only include monadic quantifiers. The following example of nonclosed subgroup comes from Stoller [23].

Assume that Ω is countable and that $\leq$ well-orders Ω. Say that a permutation p of Ω is a piecewise order isomorphism if there are partitions $(A_i)_{i \leq k}$ and $(B_i)_{i \leq k}$ of Ω such that p restricted to A_i is an order isomorphism from A_i to B_i. Let H be the set of all such piecewise order isomorphisms.

Lemma 8 ([23, p. 220])   *We have that H is a proper dense subgroup of G.*

Proof   For simplicity, assume that $\Omega = \mathbb{N}$ and that the ordering $\leq$ coincides with the usual ordering of the natural numbers.

It is rather straightforward to check that H is a subgroup. That it is proper follows from the following argument. Let $g(m)$ be $(n+1)^2 - (m+1-n^2)$, where $n^2 \leq m < (n+1)^2$. The function g is defined so that it reverses arbitrarily long sequences of natural numbers: $g(k^2 + a) = (k+1)^2 - a - 1$ for $0 \leq a \leq 2k$, which means that g reverses the sequence $k^2, k^2 + 1, \ldots, k^2 + 2k$ of length $2k + 1$. Assume that there are partitions $(A_i)_{i \leq k}$ and $(B_i)_{i \leq k}$ of $\mathbb{N}$ into k blocks such that g restricted to A_i is an order isomorphism from A_i to B_i. By the pigeonhole principle, two elements of the sequence $k^2, k^2 + 1, \ldots, k^2 + 2k$ of $2k + 1$ are in the same block A_i. That contradicts the assumption that g is an order isomorphism from A_i to B_i.

To see that H is not closed, take some $g \in G$. We prove that for any $\bar{A}$ there is $h \in H$ such that g^* and h^* agree on $\bar{A}$, showing that g is in the closure of H. Let the image of $\bar{A} = A_0, \ldots, A_k$ under g be $\bar{B} = B_0, \ldots, B_k$. For any subset N of $[k] = \{0, 1, \ldots, k\}$, let

$$A_N = \bigcap \{A_i \mid i \in N\} \cap \bigcap \{A_i^c \mid i \notin N\},$$

and define B_N similarly. Then, for any $N \subseteq [k]$, g maps A_N to B_N; and also the A_Ns are pairwise disjoint. Clearly, there is an $h \in H$ mapping the A_Ns to the B_Ns. It is now easy to see that h also maps the A_is to the B_is. Thus, the closure of H is G, and H is dense in G. ☐

Let us now prove the second half of Theorem 4 by observing that the generalization of the second-order topology on G where we regard the permutations of G as permuting binary relations on Ω is the discrete topology and thus that any subgroup is closed in this topology.

Proposition 9 *The topology on G given by the basic open sets*

$$G_{\bar{R}, \bar{S}} = \left\{ g \in G \mid g^*(R_i) = S_i \right\},$$

where $\bar{R} = R_0, \dots, R_n$ and $\bar{S} = S_0, \dots, S_n$ are subsets of Ω^2, is the discrete topology.

Proof   Let $<$ well-order Ω, and let $g \in G$ be some permutation of Ω. Let $R = \{g(<)\}$ be the image of $<$ under g. Then the basic open set $G_{<, R}$ is the singleton set $\{g\}$. If $h(<) = R$ and $h \in G$, then by transfinite induction on x over $<$ we see that $h(x) = g(x)$ for every $x \in \Omega$ and thus that $h = g$. ☐

Let now H be any subgroup of G; by the proposition, H is closed in this topology. By an argument similar to the case of monadic second-order structures, H^* is a closed subgroup of the symmetric group of $\mathscr{P}(\Omega^2)$. Thus, by Proposition 2, there is a relational structure on $\mathscr{P}(\Omega^2)$ whose automorphism group is H^*. This structure is a second-order structure on Ω having H as its automorphism group. Thus, we may conclude that any subgroup of G is the automorphism group of a second-order structure with quantifiers of type $(2, 2, \dots, 2)$, proving something slightly stronger than the second half of Theorem 4.

4   Invariance under Similarities

4.1 The setting for equality-free languages   In the present section, we propose an extension of Krasner's correspondence to languages without equality. Our motivations for doing so are twofold. First, we aim at greater generality: we will show that the correspondence in Theorem 4 between groups of bijections and sets of first-order and second-order operations closed under definability is a special case of an even more encompassing correspondence. Second, invariance criteria in which functions are not required to be one-to-one have recently been proposed (see Feferman [9] and Casanovas [5]):[9] the question is how much the situation does, or does not, change when invariance is thus liberalized.

The generalized correspondence is to hold between monoids of similarities and sets of operations closed under definability in $\mathscr{L}_{\infty\infty}^-$. A similarity π on a domain Ω is simply a binary relation $\pi \subseteq \Omega \times \Omega$, such that for all $a \in \Omega$ there is a $b \in \Omega$ with $a \, \pi \, b$ and for all $b \in \Omega$ there is $a \in \Omega$ such that $a \, \pi \, b$. We have that $\mathscr{L}_{\infty\infty}^-$ is the equality-free version of $\mathscr{L}_{\infty\infty}$. For a given signature, its formulas are all the formulas of $\mathscr{L}_{\infty\infty}$ that do not contain the equality symbol. In their recent model-theoretic studies of equality-free languages, Casanovas, Dellunde, and Jansana [6] have shown that structure-preserving similarities play for first-order logic without equality the same role that isomorphisms play for first-order logic with equality.[10] The present extension of Krasner's correspondence pushes the analogy further.

Our extension comes with a twist. In $\mathscr{L}_{\infty\infty}$, on a fixed domain and with param-
eters, all subsets of the domain are definable. This is not the case in $\mathscr{L}^-_{\infty\infty}$. Gen-
eralized quantifiers may add expressive power that remains ineffective even when
parameters are allowed. As a consequence, the proof method used in the proof of
Theorem 4 breaks down. It relied on describing "from below" the action of invariant
quantifiers by describing element-wise the sets and relations they apply to. Such a
strategy is no longer available.

To see this, let us first define invariance under similarity. When $\bar{a}$ and $\bar{b}$ are
sequences of the same length, $\bar{a}\ \pi\ \bar{b}$ denotes the conjunction of all $a_i\ \pi\ b_i$. Also, if
R and S are relations of the same arity, $R\ \pi\ S$ denotes the statement that for all $\bar{a}$
and $\bar{b}$ such that $\bar{a}\ \pi\ \bar{b}, \bar{a} \in R$ iff $\bar{b} \in S$.

- A relation $R \subseteq \Omega^k$ is invariant under the similarity relation π if $R\ \pi\ R$; that
  is, for all $\bar{a}\ \pi\ \bar{b}$ we have $\bar{a} \in R$ iff $\bar{b} \in R$.
- A quantifier Q over the domain Ω is invariant under π if for all relations
  $R_1, \ldots, R_k, S_1, \ldots, S_k$ on Ω such that $R_i\ \pi\ S_i$ we have $\langle R_1, \ldots, R_k \rangle \in Q$
  iff $\langle S_1, \ldots, S_k \rangle \in Q$.

Consider now, as an extreme case, $\Omega = \mathbb{N}$ and the operations $q = \{\top, \bot, Q^E\}$,
where $\top$ and $\bot$ are the two trivial unary relations[11] defined by $\top = \Omega, \bot = \emptyset$
and $Q^E = \{\{0, 2, 4, 6, 8, 10, \ldots\}\}$. One may show that $Q^O = \{\{1, 3, 5, 7, 9, 11, \ldots\}\}$
is invariant under all similarities under which operations in q are invariant. But in
order to describe "from below" the action of Q^O, we would need to be able to define
the set of odd numbers. As can easily be proved by induction, this is not possible
in a language in which $\top$ and $\bot$ are the only relation symbols, even in the presence
of Q^E.

Our alternative strategy will consist in showing that Krasner's correspondence
holds *when the extra expressive power is discarded*. More precisely, we will show
that the desired definability and closure properties hold when attention is limited to
the action of generalized quantifiers on definable sets and relations. Note, however,
that even though Q^O is not definable "from below" in $\mathscr{L}^-_{\infty\infty}(q)$, it is indeed definable
full stop, simply by the formula $Q^E x \neg P x$. Therefore, the question whether a version
of Krasner's correspondence without a restriction to definable relations holds is still
open.

Question 10   Is it true that if q is a set of operations and Π a set of similarities,
then Q is invariant under all similarities keeping all operations in q fixed? That is, is
it true that $Q \in \mathrm{Inv}(\mathrm{Sim}(q))$, if and only if Q is definable in $\mathscr{L}^-_{\infty\infty}(q)$?

We now introduce the setup for our result about quantifiers acting on definable rela-
tions. Given a set of operations q and a quantifier Q, let $Q^{\restriction q}$ be the restriction of Q
to relations definable in $\mathscr{L}^-_{\infty\infty}(q)$ (with parameters):

$$Q^{\restriction q} = \{\langle R_1 \ldots R_k \rangle \in Q \mid R_i \text{ is definable in } \mathscr{L}^-_{\infty\infty}(q, a)_{a \in \Omega}, 1 \leq i \leq k\}.$$

Let $q^{\restriction}$ be all the relations in q and all the restrictions $Q^{\restriction q}$ of quantifiers in q. It is
easy to see that $\mathscr{L}^-_{\infty\infty}(q)$ and $\mathscr{L}^-_{\infty\infty}(q^{\restriction})$ are elementary equivalent. In that sense,
restricting quantifiers to their definable parts does not change the *effective* expressive
power of the logic.

Restriction to definable parts on the side of quantifiers will correspond to a similar
restriction on the side of the invariance condition. Invariance will only be required

with respect to relations that respect the underlying equivalence relation generated by the operations or the similarities under scrutiny. A set of operations q gives us an equivalence relation $\sim_q$ corresponding to definability with parameters in $\mathscr{L}^-_{\infty\infty}(q)$; that is, $a \sim_q b$ iff

$$\bigwedge_{\phi \in \mathscr{L}^-_{\infty\infty}(q)} \forall \bar{x}\big(\phi(a, \bar{x}) \leftrightarrow \phi(b, \bar{x})\big). \tag{2}$$

Please observe that we allow infinite sequences $\bar{x}$.

Likewise, a set of similarities Π generates an equivalence relation by the following condition:

$a \approx_\Pi b$ iff for all $\bar{c} \in \Omega^k$ there exists $\pi \in \Pi$ such that $a, \bar{c}\ \pi\ b, \bar{c}$, for finite k.

Invariance is now parameterized by the equivalence relations $\sim$ that we are considering.

- If π is a similarity, a quantifier Q is $\sim$-invariant under π if for any $\bar{R}, \bar{S} \in \mathcal{P}(\Omega^{k_1}) \times \cdots \times \mathcal{P}(\Omega^{k_l})$, all invariant under $\sim$, if $\bar{R}\ \pi\ \bar{S}$, then $\bar{R} \in Q$ iff $\bar{S} \in Q$.

Please note that any equivalence relation, such as $\sim$, is a similarity and thus it makes sense to say that a relation is invariant under $\sim$, meaning that for all $\bar{a} \sim \bar{b}$, $\bar{a} \in R$ iff $\bar{b} \in R$, or in other words, R is a union of equivalence classes under $\sim$.

As before, a relation is $\sim$-invariant under π if it is invariant under π. This yields the main definitions of this section.

Definition 11

- Let $\mathrm{Inv}(\Pi)$ be the set of all relations R and quantifiers Q on Ω which are $\approx_\Pi$-invariant under all similarities in Π.
- $\mathrm{Sim}(q)$ is the set of similarities π such that all relations and quantifiers in q are $\sim_q$-invariant under π.

Note that when Π is a group H of permutations, then $\mathrm{Inv}(\Pi) = \mathrm{Inv}(H)$, where $\mathrm{Inv}(H)$ uses the old definition of Inv. This follows directly from the fact that $\approx_\Pi$ in this case is equality.

In general, $\mathrm{Sim}(q)$ is not a group, because the similarities need not be invertible. The relevant closure properties for $\mathrm{Sim}(q)$ are the following. A set Π of similarities is a *monoid with involution* if it is closed under composition and taking converses.[12] Moreover, Π is *full* if it is a monoid with involution, $\approx_\Pi \in \Pi$, and closed under taking subsimilarities, that is, such that if $\pi \in \Pi$ and $\pi' \subseteq \pi$ is a similarity, then $\pi' \in \Pi$.

We are now ready to state Krasner's correspondence for the equality-free case, with the aforementioned restriction to the definable parts of quantifiers.

Theorem 12   *Let q be a set of operations, and let Π be a set of similarities. Then*

  (1) *$Q \in \mathrm{Inv}(\mathrm{Sim}(q))$ if and only if $Q^{\restriction q}$ is definable in $\mathscr{L}^-_{\infty\infty}(q)$, and $R \in \mathrm{Inv}(\mathrm{Sim}(q))$ if and only if R is definable in $\mathscr{L}^-_{\infty\infty}(q)$;*

  (2) *$\mathrm{Sim}(\mathrm{Inv}(\Pi))$ is the smallest full monoid including Π.*

As a test case, let q include equality. Then $\sim_q$ is equality and $\mathrm{Sim}(q) = \mathrm{Aut}(q)$. Theorem 12 says that a quantifier Q is in $\mathrm{Inv}(\mathrm{Aut}(q))$ if and only if it is definable in $\mathscr{L}^-_{\infty\infty}(q)$, which is the same as being definable in $\mathscr{L}_{\infty\infty}(q)$. Thus, part (1) of Theorem 4 follows easily from Theorem 12. Also part (2) follows since if H is a subgroup

of the symmetric group of Ω, then $\approx_H$ is nothing but equality and $\text{Aut}(\text{Inv}(H))$ is just $\text{Sim}(\text{Inv}(H))$, which is H by part (2) of Theorem 12.

The plan to prove Theorem 12 is to derive it from Theorem 4, showing how what happens in the equality-free world of the base domain, in terms of similarities and sets of operations, is reflected in a quotient structure, in terms of permutations and sets of operations containing equality.

4.2 Intermediate steps To carry out this plan we need some technical machinery. First, we need to be able to move from the base domain to a quotient structure where the set of operations may be assumed to contain equality, and back. We define two mappings: $\cdot/\sim$ and $\cup$ which will operate both on operations and on similarities and permutations. Let $\sim$ be an equivalence relation on Ω, let $[a] = \{b | a \sim b\}$ be the equivalence class of a, and let $\Omega/\sim$ be the set of all equivalence classes. Given $R \subseteq \Omega^k$, we define the relation $R/\sim$ on $\Omega/\sim$ by

$$R/\sim \, = \big\{ \langle [a_1], \ldots, [a_k] \rangle \in (\Omega/\sim)^k \mid \langle a_1, \ldots, a_k \rangle \in R \big\}.$$

If $R \subseteq (\Omega/\sim)^k$, then

$$\cup R = \big\{ \langle a_1, \ldots, a_k \rangle \in \Omega^k \mid \langle [a_1], \ldots, [a_k] \rangle \in R \big\}.$$

Similarly, given a quantifier Q on Ω, the quantifier $Q/\sim$ on $\Omega/\sim$ is defined by

$$Q/\sim \, = \big\{ \langle R_1, \ldots, R_k \rangle \big| \langle \cup R_1, \ldots, \cup R_k \rangle \in Q \big\},$$

and given a quantifier Q on $\Omega/\sim$, the quantifier $\cup Q$ on Ω is defined by

$$\cup Q = \big\{ \langle \cup R_1, \ldots, \cup R_k \rangle \big| \langle R_1, \ldots, R_k \rangle \in Q \big\}.$$

Thus, given a set q of operations on Ω and an equivalence relation $\sim$ on Ω, we get a set $q/\sim$ of operations on $\Omega/\sim$. Similarly, we can go from a set of operations on $\Omega/\sim$ to a set of operations on Ω by the operation $\cup$.

Now, our main aim in this section will be to show that, under some minimal conditions, $\sim_q$ and $\approx_{\text{Sim}(q)}$ as well as $\sim_{\text{Inv}(\Pi)}$ and $\approx_\Pi$ are the same (see Proposition 17). This will enable us to describe, in the next section, what happens at the level of Ω in terms of what happens at the level of $\Omega/\sim$ (see Proposition 19). To get there, we will prove that, in the definition of the equivalence relation $\sim_q$, one may restrict attention to formulas with only one free variable (see Lemma 13) and that $\cup$ and $\cdot/\sim_q$ behave nicely with respect to $\cdot^{\upharpoonright q}$ (see Proposition 14). On the way, we will also prove that similarities yield permutations on the suitably defined quotient structure and preserve satisfaction in $\mathscr{L}_{\infty\infty}^-$ (see Lemmas 15, 16).

Since there are at most $2^{2^{|\Omega|}}$ many nonequivalent formulas with $|\Omega|$ many free variables, the conjunction in (2) above can be bounded by that number, and thus (2) is a formula of $\mathscr{L}_{\infty\infty}^-(q)$. We now get the following easy lemma.

Lemma 13   *For any a and b in Ω, $a \sim_q b$ iff*

$$\bigwedge_{\phi \in \mathscr{L}_{\infty\infty}^-(q)} \forall x \big(\phi(a, x) \leftrightarrow \phi(b, x) \big),$$

where the conjunction is only over ϕs with two free variables.

Proof   Left to right holds trivially; the other direction follows by setting $\phi(x, y)$ to be the formula $x \sim_q y$ and x to be a. Then we have

$$a \sim_q a \leftrightarrow a \sim_q b,$$

and thus that $a \sim_q b$.   □

Proposition 14   *Let q be a set of operations on Ω. Then*

(1) $R \subseteq \Omega^k$ *is definable in* $\mathcal{L}^-_{\infty\infty}(q, a)_{a\in\Omega}$ *if and only if R is invariant under*
    $\sim_q$;
(2) $Q^{\restriction q} = \cup(Q/\sim_q)$.

Proof   (1) The left-to-right implication is immediate. For the other implication,
assume that R is invariant under $\sim_q$. Let $\phi_R(x_1, \ldots, x_n)$ be the formula

$$\bigvee_{\langle a_1,\ldots,a_n\rangle\in R}\ \bigwedge_{1\leq i\leq n} a_i \sim_q x_i$$

defining R. If $\bar{a} \in R$, $\Omega, q \vDash \phi(\bar{a})$ since $\sim_q$ is reflexive. If $\Omega, q \vDash \phi(\bar{a})$, there is
$\bar{b} \in R$ such that $\bar{a} \sim_q \bar{b}$, implying that $\bar{a} \in R$ since R is invariant under $\sim_q$.

(2) By (1), $R \subseteq \Omega^k$ is definable in $\mathcal{L}^-_{\infty\infty}(q, a)_{a\in\Omega}$ if and only if it is of the form
$\cup R'$, where $R' \subseteq (\Omega/\sim_q)^k$. This holds if and only if $R = \cup(R/\sim_q)$.

Assume that $\bar{R} \in Q^{\restriction q}$. Then R_i is definable in $\mathcal{L}^-_{\infty\infty}(q, a)_{a\in\Omega}$. Thus
$R_i = \cup(R_i/\sim_q)$, proving that $\bar{R} \in \cup(Q/\sim_q)$. On the other hand, if
$\bar{R} \in \cup(Q/\sim_q)$, then each R_i satisfies $R_i = \cup(R_i/\sim_q)$ and thus they are all
definable in $\mathcal{L}^-_{\infty\infty}(q, a)_{a\in\Omega}$. From the definition it also follows that $\cup(Q/\sim_q) \subseteq Q$
and thus $\bar{R} \in Q$.   □

Lemma 15   *Let q be a set of operators, and suppose that $\pi \in \text{Sim}(q)$. Then*

(1) *the relation $\pi/\sim_q$ is a permutation of $\Omega/\sim_q$,*
(2) *if R is invariant under $\sim_q$, then $R\ \pi\ S$, where $S = \pi(R)$, and*
(3) *if $\bar{a}, \bar{b} \in \Omega^\alpha$, $\bar{a}\ \pi\ \bar{b}$ and $\phi(\bar{x})$ is a formula in $\mathcal{L}^-_{\infty\infty}(q)$, then $\Omega, q \vDash \phi(\bar{a}) \leftrightarrow$
    $\phi(\bar{b})$.*

Proof   Let $\sim$ be $\sim_q$.

(1) To prove that $\pi/\sim$ is a function we prove that for all $a, a' \in \Omega$ and all $b, b' \in \Omega$
such that $a \sim a'$, $a\ \pi\ b$, and $a'\ \pi\ b'$, we have $b \sim b'$. We prove, by induction, that
if $\psi(x, \bar{c})$ is a formula of $\mathcal{L}^-_{\infty\infty}(q, \Omega)$ (observe that $\bar{c}$ can be an infinite string), then
$\Omega, q \vDash \phi(b, \bar{c})$ iff $\Omega, q \vDash \phi(b', \bar{c})$.

For the base case take $R \in q$, and suppose that $R(b, \bar{c})$. Let $\bar{d}$ be such that $\bar{d}\ \pi\ \bar{c}$.
Then $R(a, \bar{d})$ since $a, \bar{d}\ \pi\ b, \bar{c}$. Now $R(a', \bar{d})$ since $a \sim a'$, and then we have
$R(b', \bar{c})$ since $a', \bar{d}\ \pi\ b', \bar{c}$.

For the induction steps negation, disjunction, and existential quantification are
easy. Let us thus take $Q \in q$, and suppose that $\Omega, q \vDash Q\bar{x}\phi(b, \bar{c}, \bar{x})$. Then
$[\![\phi(b, \bar{c}, \bar{x})]\!]^\Omega_{\bar{x}} \in Q$ and by the induction hypothesis

$$[\![\phi(b', \bar{c}, \bar{x})]\!]^\Omega_{\bar{x}} = [\![\phi(b, \bar{c}, \bar{x})]\!]^\Omega_{\bar{x}}$$

and thus $\Omega, q \vDash Q\bar{x}\phi(b', \bar{c}, \bar{x})$.

The proof of the bijective property is very similar and left to the reader.

(2) From (1) we have that if $\pi \in \text{Sim}(q)$ and R respects $\sim$, then $R\ \pi\ S$ iff
$S = \pi(R)$. This is because if $\bar{a}\ \pi\ \bar{b}$ and $\bar{b} \in \pi(R)$, then there is $\bar{a}' \in R$ such that
$\bar{a}'\ \pi\ \bar{b}$ and so by (1) $\bar{a} \sim \bar{a}'$ and since R respects $\sim$, $\bar{a} \in R$.

(3) The proof is by induction on ϕ. For the base case we have $\bar{a}\ \pi\ \bar{b}$, where
$\pi \in \text{Sim}(q)$, and $R \in q$. Thus, R is invariant under π and $R\bar{a}$ iff $R\bar{b}$.

For the induction step, negation and (infinite) conjunction are trivial. For the case with (an infinite string of) existential quantifiers, assume that $\Omega, q \models \exists \bar{x} \phi(\bar{a}, \bar{x})$ is true and thus that there are $\bar{c}$ such that $\Omega, q \models \phi(\bar{a}, \bar{c})$. Since π is a similarity relation, there are $\bar{d}$ such that $\bar{a}, \bar{c} \; \pi \; \bar{b}, \bar{d}$ and thus, by the induction hypothesis, we have $\Omega, q \models \phi(\bar{b}, \bar{d})$ and thus $\Omega, q \models \exists \bar{x} \phi(\bar{b}, \bar{x})$.

We are left with the case of a generalized quantifier $Q \in q$. Suppose that $\Omega, q \models Q\bar{x}\phi(\bar{a}, \bar{x})$, that is, that $R = [\![\phi(\bar{a}, \bar{x})]\!]^{\Omega}_{\bar{x}} \in Q$. Since $\pi \in \mathrm{Sim}(q)$, R is invariant under $\sim$ and by (2) $R \; \pi \; S$ where $S = \pi(R)$, we have $S \in Q$. Now, $\pi(R) = S$ is $[\![\phi(\bar{b}, \bar{x})]\!]^{\Omega}_{\bar{x}}$ and thus $\Omega, q \models Q\bar{x}\phi(\bar{b}, \bar{x})$. □

Next we prove that $\approx_{\Pi}$ satisfies similar properties.

Lemma 16   *Let Π be a monoid of similarities with involution, and let $\pi \in \Pi$. Then*

(1) *$\pi/\approx_{\Pi}$ is a permutation of $\Omega/\approx_{\Pi}$,*
(2) *if R is invariant under $\approx$, then $R \; \pi \; S$, where $S = \pi(R)$, and*
(3) *if also $\approx_{\Pi} \in \Pi$, $\bar{a}, \bar{b} \in \Omega^{\alpha}$, $\bar{a} \; \pi \; \bar{b}$ and $\phi(\bar{x})$ is a formula in $\mathscr{L}^{-}_{\infty\infty}(\mathrm{Inv}(\Pi))$, then $\Omega, \mathrm{Inv}(\Pi) \models \phi(\bar{a}) \leftrightarrow \phi(\bar{b})$.*

Proof   Let $\approx$ be $\approx_{\Pi}$.

(1) To prove that $\pi/\approx$ is a function, we prove that for all $a, a' \in \Omega$ and all $b, b' \in \Omega$ such that $a \approx a'$, $a \; \pi \; b$, and $a \; \pi \; b'$, we have $b \approx b'$. This follows directly from Π being closed under composition and taking inverses. The proof of the bijective property is similar and left to the reader.

(2) If $\bar{a} \; \pi \; \bar{b}$ and $\bar{b} \in \pi(R)$, then there is $\bar{a}' \in R$ such that $\bar{a}' \; \pi \; \bar{b}$ and so by (1) $\bar{a} \approx \bar{a}'$, and since R is invariant under $\approx$, $\bar{a} \in R$.

(3) The proof is by induction on ϕ. For the base case we have $\bar{a} \; \pi \; \bar{b}$, where $\pi \in \Pi$, and $R \in \mathrm{Inv}(\Pi)$. Thus, R is invariant under π and $R\bar{a}$ iff $R\bar{b}$.

For the induction step, negation and (infinite) conjunction are trivial. For the case with (an infinite string of) existential quantifiers, assume that $\Omega, \mathrm{Inv}(\Pi) \models \exists \bar{x}\phi(\bar{a}, \bar{x})$ is true and thus that there are $\bar{c}$ such that $\Omega, \mathrm{Inv}(\Pi) \models \phi(\bar{a}, \bar{c})$. Since π is a similarity relation, there are $\bar{d}$ such that $\bar{a}, \bar{c} \; \pi \; \bar{b}, \bar{d}$ and thus, by the induction hypothesis, we have $\Omega, \mathrm{Inv}(\Pi) \models \phi(\bar{b}, \bar{d})$ and $\Omega, \mathrm{Inv}(\Pi) \models \exists \bar{x}\phi(\bar{b}, \bar{x})$.

We are left with the case of a generalized quantifier $Q \in q$. Suppose that $Q\bar{x}\phi(\bar{a}, \bar{x})$, that is, that $R = [\![\phi(\bar{a}, \bar{x})]\!]^{\Omega}_{\bar{x}} \in Q$. If R is invariant under $\approx$, then we are finished since then, by (2) $R \; \pi \; S$ where $S = \pi(R)$, we have $S \in Q$. Now, $\pi(R) = S$ is $[\![\phi(\bar{b}, \bar{x})]\!]^{\Omega}_{\bar{x}}$ and thus $\Omega, \mathrm{Inv}(\Pi) \models Q\bar{x}\phi(\bar{b}, \bar{x})$. To see that R is invariant under $\approx$, take $\bar{c} \in R$ and $\bar{d}$ such that $\bar{c} \approx \bar{d}$. There is $\pi \in \Pi$ such that $\bar{c} \; \pi \; \bar{d}$ and then since $\approx_{\Pi} \in \Pi$, $\pi' = \pi \circ \approx \in \Pi$ and $\bar{a}, \bar{c} \; \pi' \; \bar{a}, \bar{d}$, and thus $\Omega, \mathrm{Inv}(\Pi) \models \phi(\bar{a}, \bar{c})$ implies that $\Omega, \mathrm{Inv}(\Pi) \models \phi(\bar{a}, \bar{d})$ by the induction hypothesis and we have $\bar{d} \in R$. □

In fact, $\approx_{\Pi}$ and $\sim_q$ are very much related as the next proposition shows.

Proposition 17   *Let q be a set of operators, and let Π be a monoid with involution. Then*

(1) *if $\approx_{\Pi} \in \Pi$, then $\approx_{\Pi}$ is the same relation as $\sim_{\mathrm{Inv}(\Pi)}$, and*
(2) *$\sim_q$ is the same as $\approx_{\mathrm{Sim}(q)}$.*

Proof (1) First, $\approx_\Pi \subseteq \sim_{\mathrm{Inv}(\Pi)}$. Assume $a \approx_\Pi b$. For any $c \in \Omega$, there is a $\pi \in \Pi$ such that $ac\ \pi\ bc$, hence, by (3) in Lemma 16 and since Π is full, for every formula $\phi(x,y)$ in $\mathcal{L}^-_{\infty\infty}(\mathrm{Inv}(\Pi))$, we have $\Omega, \mathrm{Inv}(\Pi) \vDash \phi(a,c)$ iff $\Omega, \mathrm{Inv}(\Pi) \vDash \phi(b,c)$. By Lemma 13, this suffices for $a \sim_{\mathrm{Inv}(\Pi)} b$.

Second, $\approx_\Pi \supseteq \sim_{\mathrm{Inv}(\Pi)}$. Assume $a \sim_{\mathrm{Inv}(\Pi)} b$. For any $\bar{c} \in \Omega^k$, define a relation $H^\Pi_{a\bar{c}}$ of arity $k+1$ by

$$H^\Pi_{a\bar{c}} = \{\bar{d} \in \Omega^{k+1} \mid \text{there is } \pi \in \Pi \text{ such that } a\bar{c}\ \pi\ \bar{d}\}.$$

Check that $H^\Pi_{a\bar{c}}$ is invariant under Π. Assume that $\bar{d}\ \pi\ \bar{d}'$ for some $\pi \in \Pi$. If $\bar{d} \in H^\Pi_{a\bar{c}}$, there is a $\pi' \in \Pi$ with $a,\bar{c}\ \pi'\ \bar{d}$. Since Π is closed under composition, $\pi' \circ \pi \in \Pi$, and then $a,\bar{c}\ \pi' \circ \pi\ \bar{d}'$ implies $\bar{d}' \in H_{a\bar{c},\Pi}$. If $\bar{d}' \in H^\Pi_{a\bar{c}}$, there is a $\pi' \in \Pi$ with $a,\bar{c}\ \pi'\ \bar{d}'$. Since Π is closed under composition and taking converses, $\pi' \circ \pi^\smile \in \Pi$, and then $a,\bar{c}\ \pi' \circ \pi^\smile\ \bar{d}$ implies $\bar{d} \in H^\Pi_{a\bar{c}}$. Therefore, $H^\Pi_{a\bar{c}}$ is the interpretation of a predicate symbol P in the language $\mathcal{L}^-_{\infty\infty}(\mathrm{Inv}(\Pi))$. By definition, $\Omega, \mathrm{Inv}(\Pi) \vDash P(a,\bar{c})$. Since $a \sim_{\mathrm{Inv}(\Pi)} b$, this implies $\Omega, \mathrm{Inv}(\Pi) \vDash P(b,\bar{c})$, hence there is a $\pi \in \Pi$ such that $a\bar{c}\ \pi\ b\bar{c}$.

(2) First, $\sim_q \supseteq \approx_{\mathrm{Sim}(q)}$. Assume $a \approx_{\mathrm{Sim}(q)} b$. For any $c \in \Omega$, there is a $\pi \in \Pi$ such that $ac\ \pi\ bc$, hence, by (3) in Lemma 15, for every formula $\phi(x,y)$ in $\mathcal{L}^-_{\infty\infty}(q)$, we have $\Omega, q \vDash \phi(a,c)$ iff $\Omega, q \vDash \phi(b,c)$. By Lemma 13, this suffices for $a \sim_{\mathrm{Inv}(\Pi)} b$.

Second, $\sim_q \subseteq \approx_{\mathrm{Sim}(q)}$. Assume $a \sim_q b$. Let $\bar{c} \in \Omega^k$. We need to find a $\pi \in \approx_{\mathrm{Sim}(q)}$ with $a\bar{c}\ \pi\ b\bar{c}$. It is sufficient to show that $H^{\mathrm{Sim}(q)}_{a\bar{c}}$ is definable in $\mathcal{L}^-_{\infty\infty}(q)$ by some formula φ, because then $\Omega, q \vDash \varphi(a,\bar{c})$ and $a \sim_q b$ implies $\Omega, q \vDash \varphi(b,\bar{c})$, giving the needed π.

$H^{\mathrm{Sim}(q)}_{a\bar{c}}$ is invariant under $\mathrm{Sim}(q)$ since $\mathrm{Sim}(q)$ is closed under compositions. By Proposition 14, $H^{\mathrm{Sim}(q)}_{a\bar{c}}$ is definable in $\mathcal{L}^-_{\infty\infty}(q, \Omega)$ if $H^{\mathrm{Sim}(q)}_{a\bar{c}}$ is invariant under $\sim_q$. Thus, it is sufficient to show that $\sim_q \in \mathrm{Sim}(q)$. Let R be an n-ary relation in q, $d_1 \ldots d_n \in R$ and $d_1 \ldots d_n \sim_q d'_1 \ldots d'_n$. $d_1 \ldots d_n \in R$ and $d_1 \sim_q d'_1$, hence $d'_1 d_2 \ldots d_n \in R$, and by repeating this reasoning, $d'_1 \ldots d'_n \in R$. Let Q be a quantifier in q, and let $\bar{R}$ and $\bar{S}$ be sequences of relations on Ω invariant under $\sim_q$ and of a type appropriate for Q. We need that if $\bar{R} \sim_q \bar{S}$, then $\bar{R} \in Q$ iff $\bar{S} \in Q$. It is sufficient to note that if R is invariant under $\sim_q$ and $R \sim_q S$, then $R = S$. Let $\bar{d} \in R$ and $\bar{e}$ be such that $\bar{e} \sim_q \bar{d}$. We have that $\bar{e} \in R$ since R is invariant under $\sim_q$, hence since $R \sim_q S$, $\bar{d} \in S$. Similarly, let $\bar{d} \in S$ and $\bar{e}$ be such that $\bar{e} \sim_q \bar{d}$. Since $R \sim_q S$, $\bar{e} \in R$, and then $\bar{e} \sim_q \bar{d}$, and the fact that R is invariant under $\sim_q$ implies $\bar{d} \in R$. $\square$

4.3 Krasner's correspondence without equality In this section, we leverage the output of the machinery which we have just set up to prove Theorem 4. We first characterize the sets of similarities under which a set of operations is invariant, relying on Lemma 15.

Lemma 18 *We have that* $\mathrm{Sim}(q)$ *is a full monoid.*

Proof Let $\Pi = \mathrm{Sim}(q)$. To check that Π is closed under taking converses is trivial, and by using (2) of Lemma 15 we can easily see that Π is also closed under compositions.

We prove that if $\pi \in \Pi$ and $\pi' \subseteq \pi$ is a similarity relation, then $\pi' \in \Pi$. It is trivial to check that every relation of q is invariant under π'. For the case of quantifiers, let R π' S be such that R and S are invariant under $\sim_q$. Then R π S because if not we may assume, without loss of generality, that there are $\bar{a}$ π $\bar{b}, \bar{a} \in R$, and $\bar{b} \notin S$. There are $\bar{c}$ such that $\bar{a}$ π' $\bar{c}$ and thus, since both $\bar{a}$ π $\bar{b}$ and $\bar{a}$ π $\bar{c}$, we have by (1) in Lemma 15 that $\bar{b} \sim_q \bar{c}$, contradicting that S is invariant under $\sim_q$. □

Thanks to Proposition 17, we know that $\sim_q$ is the same as $\approx_{\text{Sim}(q)}$ so when working with Inv(Sim(q)), we can use $\sim_q$ instead of $\approx_{\text{Sim}(q)}$. This enables us to prove the following.

Proposition 19   *Let q be a set of operators, and let $\sim$ be $\sim_q$. We have that*

   (1) $\text{Sim}(q)/\sim$ *is* $\text{Aut}(q/\sim)$, *and*
   (2) Inv(Sim(q)) *is the set of all quantifiers Q such that $Q/\sim$ is in* Inv(Aut($q/\sim$)).

Proof   (1) For the left-to-right inclusion, let $\pi \in \text{Sim}(q)$. By Lemma 15, $f = \pi/\sim$ is a bijection on $\Omega/\sim$ and if $Q \in q$, then $R \in Q/\sim$ iff $\cup R \in Q$ iff $\pi(\cup R) \in Q$ iff $\cup f(R) \in Q$ iff $f(R) \in Q/\sim$.

For the other inclusion, let $f \in \text{Aut}(q/\sim)$. Define a π b iff $f([a]) = [b]$. Then π is a similarity relation on Ω and $f = \pi/\sim$. Let $Q \in q$, and if R is invariant under $\sim$, then $R \in Q$ iff $R/\sim \in Q/\sim$ iff $f(R/\sim) \in Q/\sim$ iff $\cup f(R/\sim) \in Q$. But $\cup f(R/\sim) = \pi(R)$. We have proved that $R \in Q$ iff $\pi(R) \in Q$, and thus Q is invariant under π.

(2) Take $Q \in \text{Inv}(\text{Sim}(q))$ and $f \in \text{Aut}(q/\sim)$. By (1) choose $\pi \in \text{Sim}(q)$ such that $f = \pi/\sim$. Then $R \in Q/\sim$ iff $\cup R \in Q$ iff $\pi(\cup R) \in Q$ iff $\pi(\cup R)/\sim \in Q/\sim$, but $\pi(\cup R)/\sim = f(R)$. Note that $\cup R$ is invariant under $\sim_q$, hence under $\approx_{\text{Sim}(q)}$ by Proposition 17, securing the fact that $\cup R \in Q$ iff $\pi(\cup R) \in Q$.

On the other hand, take Q such that $Q/\sim \in \text{Inv}(\text{Aut}(q/\sim))$, R invariant under $\sim$, which is the same as $\approx_{\text{Sim}(q)}$ by Proposition 17, and $\pi \in \text{Sim}(q)$, then $f = \pi/\sim$ is in $\text{Aut}(q/\sim)$. Thus $R \in Q$ iff $R/\sim \in Q/\sim$ iff $f(R/\sim) \in Q/\sim$ iff $\cup f(R/\sim) \in Q$, but $\cup f(R/\sim) = \pi(R)$. □

Proposition 19 describes the relationships between what happens at the level of a given structure, in terms of similarities and sets of operations (possibly without equality), and what happens at the level of the corresponding quotient structure, in terms of permutations and sets of operations (with equality):
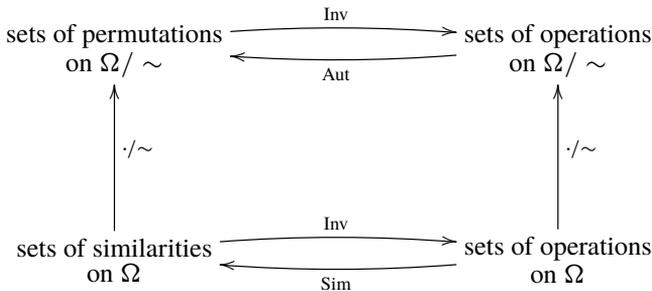
On the one hand, $\mathrm{Sim}(\cdot)/\sim$ and $\mathrm{Aut}(\cdot/\sim)$ lead to the same result, and on the other hand, $\mathrm{Inv}(\mathrm{Sim}(\cdot))$ is $\mathrm{Inv}(\mathrm{Aut}(\cdot/\sim))$ up to taking quotients. From these, we get what was announced as the main theorem of this section.

Theorem 12 *Let q be a set of operations, and let Π be a set of similarities. Then*

(1) $Q \in \mathrm{Inv}(\mathrm{Sim}(q))$ *if and only if $Q^{\restriction q}$ is definable in $\mathscr{L}^-_{\infty\infty}(q)$, and $R \in \mathrm{Inv}(\mathrm{Sim}(q))$ if and only if R is definable in $\mathscr{L}^-_{\infty\infty}(q)$;*

(2) $\mathrm{Sim}(\mathrm{Inv}(\Pi))$ *is the smallest full monoid including Π.*

Proof   (1) By Proposition 19, $Q \in \mathrm{Inv}(\mathrm{Sim}(q))$ iff

$$Q/\sim \in \mathrm{Inv}\big(\mathrm{Aut}(q/\sim)\big).$$

Thus, by Theorem 4, $Q/\sim$ is definable in $\mathscr{L}_{\infty\infty}(q/\sim)$ by a sentence ϕ. Let ϕ' be the translation of ϕ into $\mathscr{L}^-_{\infty\infty}(q)$ where $=$ is replaced by $\sim$.

By induction of formulas, we see that for all $\bar{a} \in \Omega$ and all $\psi(\bar{x})$ in $\mathscr{L}_{\infty\infty}(q/\sim)$,

$$\Omega/\sim \models \psi\big([\bar{a}]\big) \qquad \text{iff} \qquad \Omega \models \psi'(\bar{a}),$$

where, again, ψ' is the translation of ψ into $\mathscr{L}^-_{\infty\infty}(q)$.

It follows that ϕ' defines $\cup(Q/\sim)$ and since $\sim$ is definable in $\mathscr{L}^-_{\infty\infty}(q)$, we have what we wanted.

(2) We prove that for a full monoid Π, we have $\mathrm{Sim}(\mathrm{Inv}(\Pi)) = \Pi$. The general theorem then follows from the fact that every $\mathrm{Sim}(q)$ is a full monoid.

The inclusion $\Pi \subseteq \mathrm{Sim}(\mathrm{Inv}(\Pi))$ is trivial. For the other inclusion, let $H = \Pi/\approx$. It is a group of permutations by Lemma 16. By straightforward checking, we see that

$$\mathrm{Inv}(\Pi)/\approx = \mathrm{Inv}(H).$$

From (1) in Proposition 19, we get that

$$\mathrm{Sim}\big(\mathrm{Inv}(\Pi)\big)/\approx = \mathrm{Aut}\big(\mathrm{Inv}(\Pi)/\approx\big)$$

and thus that this is equal to $\mathrm{Aut}(\mathrm{Inv}(H))$, which by Theorem 4 is nothing else than H. Thus,

$$\mathrm{Sim}\big(\mathrm{Inv}(\Pi)\big)/\approx = \Pi/\approx$$

and by the assumption that Π is full we get that $\mathrm{Sim}(\mathrm{Inv}(\Pi)) = \Pi$. If $\pi \in \mathrm{Sim}(\mathrm{Inv}(\Pi))$, then there is $\pi' \in \Pi$ such that $\pi'/\approx = \pi/\approx$, and then $\pi \subseteq (\approx \circ \pi' \circ \approx)$, and thus $\pi \in \Pi$. $\qquad\square$

5   Concluding Remarks

5.1 Invariance under permutation as a logicality criterion   Let us first go back to languages with equality and groups of permutations. The main benefit of a Krasnerian approach to McGee's result is to show that the correspondence between purportedly logical operations on a domain and the symmetric group on that set is a special case of the general duality which holds between sets of relations and groups of permutations, when sets of relations consist in first-order and second-order relations closed under definability in $\mathscr{L}_{\infty\infty}$. The light thus shed on McGee's result is ambiguous, depending on how the closure condition is looked at. One might insist that groups of permutations are consequently shown to be a sure guide to characterizing sets of relations. When $\mathscr{L}_{\infty\infty}$ provides the logical context, *any* relevant structure may be characterized in terms of invariance under permutation, and therefore the structure consisting of all and only logical relations on a given domain, in particular,

is thus characterizable. Now which group of permutations should be the one giving us all (and only the) logical relations? Since logical notions have generality as their distinguishing feature, we should go for the largest possible group, and take all permutations: this is the gist of the generality argument. Since logic is formal and does not make differences between objects, we should require insensitivity to any possible way of switching objects, and take again all permutations: this is the gist of the formality argument. Of course, several steps are in need of further justification. Maybe identifying "being a logical notion" and "being a general notion" is too quick, because generality remains underspecified. Or maybe "being formal" and "not making distinctions between objects" are not quite the same (see [19] for a thorough discussion of these issues). But our goal here is not to challenge the relevance of the conceptual analysis in terms of generality or indifference to objects. Our point is rather to suggest that these arguments may be deemed to be on the right track *only if we know that logical relations are characterizable in terms of invariants of a group of permutations*. Once this characterizability is granted, both arguments go through, and nicely converge.

Seeing McGee's result as a special case of a more encompassing Krasnerian correspondence reveals $\mathcal{L}_{\infty\infty}$ to be a built-in feature of the framework. Definability in $\mathcal{L}_{\infty\infty}$ is the closure condition on sets of relations which corresponds to working with groups of permutations, rather than with some other kind of transformations. As soon as we choose to explicate logicality (either qua generality or qua formality) in terms of groups of permutation, we let infinitary syntax sneak in. Now the generality and formality arguments tell us which group of permutations we should pick, if we are to single out logical relations by picking one, but they do not tell us that we should single out logical relations by picking a group of permutations to start with. Thus, the fact that the infinitary logic $\mathcal{L}_{\infty\infty}$ ends up being a full-blown logic on this account of logicality might be deemed an artificial effect of the invariance under permutation framework. McGee himself took his result to be an argument in favor of invariance for permutation as a logicality criterion, "inasmuch as every operation on the list is intuitively logical" ([20, p. 567]), the list being the list of basic operations of $\mathcal{L}_{\infty\infty}$. But this presupposes that having reasons to count conjunction as logical automatically counts as having reasons to count infinitary conjunction, and similarly for quantification.

Blaming invariance under permutation for its infinitary upshot is no new criticism (see Feferman [9], [10] and Bonnay [2]), but, in the light of the generalized Krasnerian correspondence, this infinitary burden is seen to be the very consequence of the use of permutations to demarcate the extensions that are admissible for logical constants.

5.2 Invariance under similarity as a logicality criterion Invariance under similarities has been put forward as an alternative logicality criterion by [9].[13] The shift from permutations to similarities makes for greater generality, which is crucial for the generality argument. It also provides a way to escape the commitment of permutations to cardinalities. Every cardinality quantifier, no matter how wild ("there are at least $\aleph_{17}$ many") is permutation-invariant. Shifting to invariance under similarities expels these wild creatures from the paradise of logical quantifiers.[14]

Feferman shows that an operation is definable in first-order logic without equality just in case it is definable in the λ-calculus from homomorphism invariant[15]

monadic quantifiers and asks whether "there is a natural characterization of the homomorphism-invariant propositional operations in general, in terms of logics extending the predicate calculus" ([9, p. 47]). The question is all the more pressing given that the syntactic restriction to monadic quantifiers lacks a proper justification. With reference to an unpublished proof given in Bonnay [3], it has been claimed that invariance under homomorphism corresponds to definability in pure $\mathscr{L}^-_{\infty\infty}$ (see [2], [10]). The proof, however, was faulty, breaking down because of undefinable subsets. Theorem 12 provides a correct generalization. However, it does not answer Feferman's exact question, since the correspondence we eventually get only holds with respect to the action of quantifiers on definable sets. What we know by Theorem 12 is that invariance under homomorphism corresponds to definability in pure $\mathscr{L}^-_{\infty\infty}$ when quantifiers are restricted to their action on definable subsets of the domain. Whether the original claim was correct is still unknown.

The restriction to definable sets notwithstanding, Theorem 12 shows that the shift from permutations to similarities or homomorphisms does not change the picture when it comes to the infinitary explosion of invariance criteria: infinitary syntax is every bit as much hardwired in invariance under similarities as it is in invariance under permutations. Moreover, invariance criteria do not seem well suited to help us adjudicate the status of equality as a logical constant or not. As highlighted by the close parallelism between Theorem 4 and Theorem 12, allowing identity at level of relations exactly corresponds to banning non-injective functions at the level of invariance conditions. Whether there are better grounds on which to adjudicate the issue, or whether it should be considered as primarily a matter of convention remains to be seen.

5.3 Further perspective To conclude, we would like to mention two lines of further research. The first one is somewhat tangential to the main project. We have seen that automorphism groups of monadic second-order structures are characterized as closed groups in a way similar to the first-order case. What do these closed groups tell us about the structures they come from? The question is whether remarkable properties of monadic second-order structures can be reduced to properties of automorphism groups—a famous example in the first-order case is the Ryll-Nardzewski theorem which says that being $\aleph_0$ categorical is such a property.

The second line of research concerns further inquiries into the duality between sets of transformations and sets of relations closed with respect to definability in a given logic. Theorem 12 leaves as an open question whether the restriction to the definable parts of quantifiers is mandatory. More generally, this duality could be extended to other kinds of transformations and less demanding closure conditions. Barwise [1] provides such a result for first-order relations with respect to semiautomorphisms (automorphisms between substructures which are part of a back-and-forth system) and closure under definability in $\mathscr{L}_{\infty\omega}$. This suggests generalizing again to second-order relations. Another tempting move would consist in getting to $\mathscr{L}_{\omega\omega}$ by suitably weakening the conditions on semiautomorphisms. Finally, the connection between properties of the logical syntax and properties of the transformations still remains to be fully understood. It would be nice to achieve even greater generality by connecting explicitly abstract properties of transformations (e.g., the depth of Ehrenfeucht–Fraïssé games) and properties of the syntax for the logic providing the closure condition.

Notes

1. Klein's Erlanger program in [14] and the systematic study of the duality between groups of transformations on a space and geometries on that space is a famous example. It was also a direct inspiration to Tarski [24].

2. The qualification "essentially" is due to the fact that Krasner thinks directly in terms of closure under "logical" operations on relations rather in terms of definability in a formal language.

3. To our knowledge, the question regarding the other half is still open.

4. This amounts to a property of eliminability of quantifiers. Krasner gives a general characterization of structures having this property. See Poizat [21] regarding connections with contemporary model theory.

5. This topology makes G into a topological group, which means that the operations of multiplication (in this case composition) and taking inverses are continuous. As with topological groups in general, all open subgroups are also closed: let H be an open subgroup of G; then the complement of H is the union of all cosets gH of H which are disjoint from H, and thus the complement of H is open.

6. If $\Omega = \{a_0, a_1, \ldots\}$ is countable, then the topology is completely metrizable by the metric $d(g,h) = 1/2^k$ if $g \neq h$ and k is the least number such that $ga_k \neq ha_k$ or $g^{-1}a_k \neq h^{-1}a_k$, and $d(g,g) = 0$. Thus, a sequence of permutations $g_0, g_1, \ldots$ converges to g if and only if, for any k, $g_n a_k = g a_k$ for all sufficiently large n. It should also be noted that G is not a closed subset of Ω^Ω, which means that there are sequences g_i of permutations of Ω converging to a function f which is not a permutation. Let $\Omega = \mathbb{N}$; the sequence

$$(0, 1), (0, 1, 2), (0, 1, 2, 3), \ldots$$

converges to the function $f : n \mapsto n + 1$ which is not a permutation.

7. McGee's operations are actually defined as functions from sequences of sets of assignments to sets of assignments. This difference does not matter in the present context, since McGee's operation may be translated into second-order relations and back in such a way that invariance under sets of permutations is preserved.

8. $\text{Aut}_{\mathscr{P}(\Omega)}(q')$ is the group of all permutations of $\mathscr{P}(\Omega)$ respecting the quantifiers in q'. Compare this to $\text{Aut}_\Omega(q')$, which is the set of all permutations of Ω respecting the quantifiers in q'.

9. Casanovas shows that several nonequivalent ways of defining invariance appear when functions are allowed that are not one-to-one. We are concerned here with what he calls "Feferman invariance," as defined by Feferman in [9] and that we take to be the most natural notion of invariance in that context. See [5] for a detailed examination of the other options, as well as for characterization results for these other options.

10. Casanovas, Dellunde, and Jansana [6] use the term "relativeness correspondence," whereas Feferman [9] goes for "similarity relation." Note also that, instead of similarity

relations, one may work with surjective functions. Any relational composition of surjective maps or inverses of surjective maps is a similarity relation, and any similarity relation can be written as the composition $f \circ g^{-1}$, where $f : C \to B$ and $g : C \to A$ are onto (see [6] for more details).

11. It is useful to assume that there are symbols in the language to express these, in order to guarantee that $\mathcal{L}^{-}_{\infty\infty}(q)$ is not empty.

12. The converse R^{-1} of R is such that $a \ R^{-1} \ b$ iff $b \ R \ a$.

13. Feferman [10], [11] now favors mixed approaches in which invariance only provides necessary conditions for logicality.

14. Feferman had another reason to favor similarities, namely, the fact that they can relate domains of different sizes, and hence force a greater homogeneity for the extensions of quantifiers across domains. We do not deal with across-domain invariance here.

15. Homomorphism invariance is equivalent to invariance under similarities.

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