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New Degree Spectra of Abelian Groups

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Abstract We show that for every computable ordinal of the form $\beta = \delta + 2n+1 > 1$, where δ is zero or a limit ordinal and $n \in \omega$, there exists a torsion-free abelian group having an X-computable copy if and only if X is nonlow$_\beta$.

1 Introduction

The study of the algorithmic nature of classical algebraic objects has a long tradition which goes back to at least Hermann [14] and van der Waerden [27]. We continue the tradition that goes back to Mal'cev [21] and Rabin [24], who initiated the systematic study of computability-theoretic aspects of countably infinite groups.

Definition 1.1 (Rabin [24], Mal'cev [21]) A countably infinite group A is *computably presentable* or simply *computable* if there exists a group B isomorphic to A so that the domain of B is ω and the operation on B is a Turing computable function on two arguments. The group B is called a *computable presentation* or a *computable copy* of A.

Definition 1.1 can be generalized to any algebraic structure. For instance, computable fields (see Metakides and Nerode [23], Ershov and Goncharov [9]) and Boolean algebras (see Goncharov [13]) have been studied extensively. Note that a group is computably presentable exactly if it has a "recursive" presentation in the sense of Higman [15] with decidable word problem. If the word problem is merely computably enumerable (c.e.), then we say that the group is *c.e. presented*.

1.1 Degree spectra Although the central objects of our studies are computable groups, noncomputable presentations appear naturally in many cases. For instance, finitely presented groups with unsolvable word problem have no computable presentation, and infinitely generated c.e. presented torsion-free abelian groups may have no computable copy (see Khisamiev [19]).

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The standard approach of effective algebra to noncomputable algebraic structures uses the following definition. An infinitely countable algebraic structure $\mathcal{A}$ is computable relative to a Turing degree $\mathbf{d}$ (see Soare [26]), or $\mathbf{d}$-*computable*, if the universe of $\mathcal{A}$ can be identified with the natural numbers ω in such a way that the atomic diagram of $\mathcal{A}$ becomes $\mathbf{d}$-computable. For instance, one may speak of structures computable in the halting problem, the natural examples being c.e. presented groups. In the context of groups, this approach is equivalent to Definition 1.1 relativized to $\mathbf{d}$.

The algorithmic nature of an algebraic structure $\mathcal{A}$ is best captured by the following invariant called the *degree spectrum* of $\mathcal{A}$:

$$\mathrm{DegSpec}(\mathcal{A}) := \{\mathbf{d} : \mathcal{A} \text{ is } \mathbf{d}\text{-computable}\},$$

which is the collection of all Turing degrees that can compute a copy of $\mathcal{A}$. The degree spectrum of $\mathcal{A}$ may or may not have a least element under Turing reducibility. If $\mathbf{a}$ is the least element in DegSpec($\mathcal{A}$), then $\mathbf{a}$ is called the *degree* of $\mathcal{A}$ (see Richter [25]).

Even elementary structures such as additive subgroups of the rationals may have no degree. In this case, one uses the Turing jump operator to obtain more information about such structures. Recall that $\mathbf{d}'$ stands for the halting problem for machines with oracle $\mathbf{d}$. Then a Turing degree $\mathbf{a} \geq \mathbf{0}^{(\alpha)}$ is the proper αth jump degree of A if the set $\{\mathbf{d}^{(\alpha)} : \mathbf{d} \in \mathrm{DegSpec}(\mathcal{A})\}$ has $\mathbf{a}$ as its least element and for every $\gamma < \alpha$ the set $\{\mathbf{d}^{(\gamma)} : \mathbf{d} \in \mathrm{DegSpec}(\mathcal{A})\}$ is not a cone under $\leq_T$. (Here α is of course a computable ordinal.) There has been a lot of work on degree spectra and αth jump degrees of various algebraic structures (see, e.g., [25], Downey and Knight [6], Ash, Jockusch, and Knight [2], Jockusch and Soare [16], [17], Knight [20]).

1.2 Degree spectra of torsion-free abelian groups Recently, there has been significant progress in understanding the degree spectra of torsion-free abelian groups. We list all results known in this direction in the next few lines.

It is not difficult to show that, for every $n \leq 3$ and every degree $\mathbf{d} > \mathbf{0}^{(n)}$, there is a torsion-free group G having proper nth jump degree $\mathbf{d}$ (see Downey [5] for $n = 0, 1$ and Melnikov [22] for $n = 2, 3$; discussed in Andersen, Kach, Melnikov, and Solomon [1]). Coles, Downey, and Slaman [4] showed that, in fact, *every* torsion-free abelian group of rank 1 has a first jump degree; their result can be extended to any finite rank (see [22]). The case of higher ordinals remained unresolved until the recent work of Andersen, Kach, Melnikov, and Solomon [1], who showed that, for every computable $\alpha > 3$, each $\mathbf{d} > 0^{(\alpha)}$ can be realized as the proper αth jump degree of a torsion-free abelian group.

In this paper, we continue the investigation of degree spectra of torsion-free abelian groups. We follow Soare [26] in the definition of $\Sigma^0_{\omega+1}$. (Thus, $(\emptyset^{(\omega)})'$ is a $\Sigma^0_{\omega+1}$-complete set.) Andersen, Kach, Melnikov, and Solomon [1, Question 1.6] asked whether for each computable ordinal $\alpha \geq 3$ there exists a torsion-free abelian group having proper αth jump degree $0^{(\alpha)}$. We prove the following result.

Theorem 1.2 *For every computable ordinal β of the form $\delta + 2n + 1 > 1$, where δ is zero or is a limit ordinal and $n \in \omega$, there exists a torsion-free abelian group having an X-computable copy if and only if X is nonlow$_\beta$.*

As an immediate corollary, we obtain the following.

Corollary 1.3 *For every β as in Theorem 1.2, there exists a torsion-free abelian group having $0^{(\beta+1)}$ as its proper $(\beta+1)$th jump degree.*

The nonlow$_\beta$ degree spectra for such computable β's are new to abelian groups in general. We note that such degree spectra cannot be realized in the class of reduced abelian p-groups of small Ulm length (see Kalimullin, Khoussainov, and Melnikov [18]). Theorem 1.2 extends results from the earlier paper [22] where the case of $\beta = 1$ was established.

Proving the theorem requires an implementation of a machinery that has recently been used to study jump degrees of torsion-free abelian groups (see [1]). This technique is based on ideas from classical abelian group theory (see Fuchs [11]) and also on previous work from [1], Downey and Montalbán [8], and Fokina, Knight, Melnikov, Quinn, and Safranski [10]. While the result in [1] gave a uniform and reconstructable coding of a Σ^0_α-set into the isomorphism type of a computable group, the proof of Theorem 1.2 gives an effective coding of a uniformly Σ^0_α-family of finite sets into a computable group (see Proposition 2.1). Coding a family indeed requires new ideas on top of what is already contained in [1] (to be discussed). For instance, we introduce a new technical notion of a σ-shifted component that is central to the proof. We believe that this technical notion may find further applications in the future.

1.3 A framework We note that, in [1] and in the present paper, the choice of primes used in the coding is somewhat irrelevant to the required properties of the constructed groups. For the sake of presentation and future applications, in this paper we use different primes in different locations of the groups we build. The modification makes the groups more rigid. Consequently, we can extract more information from their algebraic structure with less effort (see, e.g., Claim 2.16). For the sake of this modification, we need to formally and inductively define a specific class of groups. Direct sums of groups from this class include most of the torsion-free abelian groups which have been used recently in effective algebra (see, e.g., [8], Downey and Melnikov [7]). We hope that the general approach introduced in this paper may serve as a technical base for future systematic research in the area.

2 Proof of Theorem 1.2

Understanding the proof of Theorem 1.2 requires a background in abelian group theory (see [11], Fuchs [12]). It is expected that the reader is familiar and comfortable with the notions of linear independence, divisible and pure closure, infinite and finite divisibility, notation $p^\infty | x$ and $\frac{z}{p^\infty}$, and so on. A sufficient initial segment of the theory can also be found in [7] and [1]. We use infinitary computable formulae (see Ash and Knight [3]) and assume that the reader is familiar with the foundations of computability theory (see [26]). Without loss of generality, we can restrict ourselves to groups upon the domain ω.

We claim that, to prove Theorem 1.2, it is sufficient to establish the following.

Proposition 2.1 *For every infinite collection $\mathcal{R}$ of finite sets containing the empty set and every computable $\alpha = \delta + 2n + 2 > 2$, where $\delta = 0$ or is a limit ordinal and $n \in \omega$, there exists a torsion-free abelian group $G_{\alpha,\mathcal{R}}$ such that $G_{\alpha,\mathcal{R}}$ has an X-computable presentation if and only if $\mathcal{R}$ has a uniform Σ^X_α-enumeration.*

To see why it is sufficient to prove Proposition 2.1, recall that for every such α there exists a family of finite sets having Σ^X_α-enumeration if and only if X is nonlow$_{\alpha-1}$.

(This fact follows from Wehner [28]. For example, the main result of [28] relativized to $0'$ gives a family that has a $\Sigma^{\mathbf{a}}_1$-enumeration if and only if $\mathbf{a} > 0'$. Thus, the family has a $\Sigma^{X'}_1$-enumeration, or Σ^{X}_2-enumeration, if and only if $X' > 0'$ if and only if X is nonlow (= nonlow$_1$). In this way, we obtain $\beta + 1 = \alpha$ in the notation of Proposition 2.1 and Theorem 1.2.) The rest of the paper is devoted to the proof of Proposition 2.1.

Proof idea. We suppress α in $G_{\alpha,\mathcal{R}}$. We will effectively and uniformly in a given enumeration of $\mathcal{R}$ encode every finite set S of $\mathcal{R}$ into a separate group H_S and then put them together in a direct sum:

$$G_{\mathcal{R}} = \bigoplus_{S\in\mathcal{R}} H_S.$$

We then read S off the isomorphism type of the group by using infinitary logic. The formula that we use, roughly, asks if a certain configuration of elements and relations between them can be found in the group. The *main difficulty* is that the configuration does not have to be within H_S, but can be spread among several components. Thus, we have to be very careful when choosing the isomorphism types of the components, and in fact, the main difficulty of the proof is that not every choice would do the job.

The *first main idea* is coding subsets of S instead of coding a *finite* $S \in \mathcal{R}$ itself. Let $(D_i)_{i\in\omega}$ be the standard effective enumeration of all finite subsets of ω. For every $S \in \mathcal{R}$, we encode

$$\mathrm{Age}_S = \{i : D_i \subseteq S\}$$

into the isomorphism type of a direct component H_S of $G_{\mathcal{R}}$. Observe that this approach does not increase the complexity of the coding in the following sense. Suppose that $E = \{\langle c, i\rangle : c \in E_i\}$ is a Σ^0_α-enumeration of $\mathcal{R}$. Then $E_j \in \mathcal{R}$ is a uniformly Σ^0_α-set, for every j, and the relation $D_i \subseteq E_j$ is Σ^0_α (in the respective indices i and j). Indeed, D_i is fully described by its strong index i, say, $D_i = \{d_1, \dots, d_s\}$, and thus, it is sufficient to check whether $d_k \in E_j$, $k \leq s$. On the other hand, if we can find a Σ^0_α-enumeration of the family that consists of finite unions $\mathrm{Age}_{S_{k(0)}} \cup \cdots \cup \mathrm{Age}_{S_{k(x)}}$, where the $S_{k(i)}$'s range over elements of $\mathcal{R}$ and the x's range over $\mathbb{N}$, then we can uniformly pass to a Σ^0_α-enumeration of $\mathcal{R}$ (to be explained).

The first naive attempt of encoding Age_S would be defining H_S by using a sequence of groups indexed by $i \in \omega$ and encoding Σ^0_α- or Π^0_α-outcomes depending on whether or not $D_i \subseteq S$. (The ordered sequence would then be put together by using the "chain operation," which will be defined later.) Unfortunately, the group encoding the Σ^0_α-outcome depends on the least witness of the Σ^0_α-event (i.e., $G(\Sigma^0_\alpha(n)) \not\cong G(\Sigma^0_\alpha(m))$ when $m \neq n$), and so we get the isomorphism type $G(\Sigma^0_\alpha(n))$ for *some* n.

The *second main idea* allows us to circumvent the difficulty explained above. Recall that S is finite. We take all finite $\sigma \in \omega^{<\omega}$ and produce a "σ-shifted" H_S, denoted $H_{\sigma,S}$, for every such σ. The entries of the string σ will be used to specifically force the corresponding procedures to modify their computations shifting the occurrence of the potential witnesses. The construction is organized so that we necessarily list all possible combinations $G(\Sigma^0_\alpha(n))$ (for all n) among the corresponding components of $H_{\sigma,S}$ *if the true outcome is* Σ^0_α, and also so that the shift does not effect the Π^0_α-outcomes. Note that almost all outcomes are Π^0_α, since S is finite. We then set

$$H_S = \bigoplus_{\sigma \in \omega^{<\omega}} H_{\sigma,S}.$$

The σ-shifts will homogenize the group $G_{\mathcal{R}}$ and will make its isomorphism type independent of the enumeration of $\mathcal{R}$.

Some parts of the verification rely on the machinery developed in [1]. Several adjustments need to be made to the machinery, but these are not crucial. Although we will discuss and illustrate the machinery from [1], we believe that a reader aiming for a deep understanding of the verification may find a detailed study of [1] rather helpful. We will modify the coding components from [1], and this modification will make the verification more accessible. Alternatively, the reader may choose to use the Σ- and Π-coding components from [1] in place of the components defined below. In this case Claim 2.16 has to be replaced by [1, Lemma 4.14]. Both approaches will give a proof of Proposition 2.1, but the second seems to be unsatisfactory since [1, Lemma 4.14] is weaker than Claim 2.16 but is harder to prove.

The proof is organized as follows. To define the Σ^0_α- and Π^0_α-coding components we need to explain the general machinery from [1]; this is done in Section 2.1. In Section 2.1, we also introduce a modification to the machinery which was discussed above. For the sake of this modification, we need to formally define a certain class of groups (Definition 2.2) and a new operation on this class (Definition 2.3). We formally define the Σ^0_α- and Π^0_α-coding components in Section 2.2. The effective content of Section 2.2 is discussed in Section 2.3. Then, in Section 2.4, we prove a proposition which demonstrates a typical application of the machinery. The proposition also gives an idea of how the coding works (locally) in the case of $\alpha = 3$. We will use the proposition in the verification. Only after we describe the machinery and gain some intuition will we finally arrive at a formal definition of $G_{\mathcal{R}}$ in Section 2.5. The verification is contained in Section 2.6.

Although we believe that a good understanding of the construction requires reading the sections linearly, an impatient reader may look at Section 2.1 first and then go to Section 2.5 immediately. After looking at the definition of $G_{\mathcal{R}}$, the reader may go back to Section 2.2 and see the Σ- and Π-coding components involved. The reader may then proceed to the verification.

2.1 Basic operations We call a torsion-free abelian group with a distinguished nonzero element a *rooted* group, and we call the distinguished element the *root* of the group (see [1]). (We will construct an elementary rooted group (G, g) and then remove the root g from its signature.) The (rooted) groups used in [1], [8], and [10] were constructed from smaller groups by using certain elementary operations. To state Definition 2.3 formally, we would like to isolate these operations from concrete contexts.

(1) *Connection by a prime*. For rooted groups (G, g) and (H, h) and a prime q, define

$$\left(H\left(\frac{h+g}{q^\infty}\right)G, h\right)$$

to be the group

$$\left\langle H \oplus G; \frac{h+g}{q^n} : n \in \omega\right\rangle$$

with root h.

(2) *Chain operation*. Given rooted groups $(G_k, g_k)_{k\in\omega}$ and primes $(q_k)_{k\in\omega}$, let

$$\Big(G_k\Big(\frac{g_k+g_{k+1}}{q_k^\infty}\Big)_{k\in\omega} G_{k+1}, g_0\Big)$$

be the group

$$\Big\langle \bigoplus_{k\in\omega} G_k; \frac{g_k+g_{k+1}}{q_k^m} : g_k \in G_k, k, m \in \omega\Big\rangle$$

with root g_0.

(3) *Branching operation*. Given rooted (H, h) and $(G_k, g_k)_{k\in\omega}$ and $J \subseteq \omega$ ($J \neq \emptyset$), let $(H(\frac{h+g_j}{q^\infty})_{j\in J} G_j, h)$ be the group

$$\Big\langle H \oplus \bigoplus_{j\in J} G_j; \frac{h+g_j}{q^m} : h \in H, g_j \in G_j, j \in J, m \in \omega\Big\rangle$$

with root h.

(4) *Prime closure*. Given a rooted group (G, g) and a set of primes P, we define $[(G, g)]_P$, also written $([G]_P, g)$, to be

$$\langle g/p^n : g \in G, p \in P, n \in \omega\rangle \subseteq \mathcal{D}(G),$$

with the same root g. Here $\mathcal{D}(G)$ stands for the divisible closure of G. We write $[G]_p$ instead of $[G]_{\{p\}}$ for a singleton $\{p\}$.

In the following, we call the operations above *elementary*.

Definition 2.2 (Elementary rooted groups) We call a rooted group *elementary* if it can be constructed from rooted groups of the form $([\mathbb{Z}]_P, p^k)$ by using a finite sequence of elementary operations. Here p is a prime, P is a finite set of primes that may or may not contain p, and $k \in \omega$. (These parameters do not have to be the same for different subcomponents. If $P = \emptyset$, then $[\mathbb{Z}]_P = \mathbb{Z}$.)

Suppose that (G, g) is an elementary rooted group. We associate (G, g) with a labeled rooted tree of finite height. The idea is that this tree can later be used to visualize the group or to change the primes that were used in the definition of the group. Formally, we define the labeled tree by induction, as follows.

(i) If $(G, g) = ([\mathbb{Z}]_P, p^k)$, then its tree contains only one node which is labeled by (P, p^k).

(ii) If $(G, g) = (H(\frac{h+g}{q^\infty})U, g)$, then put an edge between the roots of the already defined structural trees of (H, h) and (U, g), and label this edge by $\{q\}$. Declare the root of the structural tree of (U, g) to be the root of the structural tree of (G, g).

(iii) If $(G, g) = (G_k(\frac{g_k+g_{k+1}}{q_k^\infty})_{k\in\omega} G_{k+1}, g_0)$, then connect the root of the already defined structural tree of (G_k, g_k) and the root of the structural tree of (G_{k+1}, g_{k+1}) by an edge and label this edge by $\{q_k\}$. Declare the root of the structural tree of (G_0, g_0) to be the root of the structural tree of (G, g).

(iv) If $(G, g) = (H(\frac{h+g_j}{q^\infty})_{j\in J} G_j, h)$, then for each $j \in J$ connect the root of the structural tree of (H, h) to the root of the structural tree of (G_j, g_j) and label it by $\{q\}$. Declare the root of the structural tree of (H, h) to be the root of the new structural tree.

(v) If $(G, g) = [(H, g)]_S$, then replace every label of the form (P, p^k) by $(P \cup S, p^k)$ and replace every label of the form P by $P \cup S$. (The former correspond to labels on nodes and the latter to labels on edges, and we let P vary.)

Clearly, groups having identical structural trees are isomorphic. Note that an elementary rooted group (G, g) may have more than one structural tree, and these trees may not be isomorphic. Nonetheless, we will always associate a group with some *specifically chosen* structural tree (given by the construction, say), and we call it *the* (structural) *tree* of (G, g). Once the tree is fixed, it makes sense to speak about subcomponents, or blocks, of G, those being groups naturally corresponding to subtrees of the tree.

Definition 2.3 (Prime substitution) Let S be the set of all primes that occur in the labels of the structural tree of an elementary rooted group (G, g), and let $\phi : S \to S'$ be a bijection of S onto a set of primes S'. We define $(G, g)_\phi$ as follows. In each label of the structural tree of (G, g), replace every prime p by $\phi(p)$. Then pass to the elementary group corresponding to the new labeled tree.

It is clear that the class of elementary rooted groups is closed under the operation of prime substitution. All our groups will be direct sums of elementary rooted groups. We can extend the operation of prime substitution to direct sums in the obvious way.

2.2 Groups encoding $\boldsymbol{\Sigma^0_\alpha}$- and $\boldsymbol{\Pi^0_\alpha}$-outcomes We define groups $G(\Pi^0_\beta)$, $G(\Sigma^0_\beta(m))$, and $G(\Sigma^0_\beta)$ by recursion. The definition below is similar to [1, Definition 3.1], but is not the same. More specifically, we use the operation *prime substitution* instead of *prime closure* at intermediate steps.

We use letters p, q, v, u, d with subscripts to denote distinct primes. For the sake of effectiveness, we assume that the subscripts correspond to effective listings of distinct primes. Every ordinal $\beta \leq \alpha$ will be identified with its notation, and hence, every limit ordinal will be associated with a specific effective sequence $(\gamma_i)_{i \in \omega}$ of smaller odd ordinals having β as their supremum. We will be using the operation of prime substitution for which we need to define, by transfinite recursion, bijections from sequences of primes to sets of new fresh primes (see Definition 2.3). To avoid the unnecessary formalism, we will not define these maps formally and will take for granted that it can be done.

Notation 2.4 We write $\phi_{\beta,k}$ for the map indexed by an ordinal β and a number k and assume that $\phi_{\beta,k}$ effectively and uniformly maps primes to new fresh primes which were not used at previous inductive steps. We also assume that the ranges of the $\phi_{\beta,k}$'s do not overlap for different subscripts. Whenever we use any of the $\phi_{\beta,k}$'s for prime substitution, we assume that all the primes in the group under the operation are in the domain of $\phi_{\beta,k}$.

Definition 2.5 In the following, δ always denotes 0 or a limit ordinal.

- For $\beta = 2$, define $G(\Sigma^0_\beta(m))$ to be the group $[\mathbb{Z}]_{p_2}$ with root q_2^m, and let $G(\Pi^0_\beta)$ be the group $[\mathbb{Z}]_{p_2,q_2}$ with root 1.
- For $\beta \geq 3$ of the form $\delta + 2n + 1$, define

$$G(\Sigma^0_\beta) = [\mathbb{Z}]_{p_\beta}\Big(\frac{r + r_k}{q_\beta^\infty}\Big)_{k \in \omega} A_k,$$

where r_k is the root of A_k, the element $r = 1$ in $[\mathbb{Z}]_{p_\beta}$ is the root of $G(\Sigma^0_\beta)$, $A_k = G(\Pi^0_{\beta-1})$ for k even, and $A_k = G(\Sigma^0_{\beta-1}(m))$ for each $k = 2\langle i, m\rangle + 1$.

- For $\beta \geq 3$ of the form $\delta + 2n + 1$, define

$$G(\Pi^0_\beta) = [\mathbb{Z}]_{p_\beta}\Big(\frac{r + r_k}{q^\infty_\beta}\Big)_{k\in\omega} A_k,$$

where r_k is the root of A_k, the element $r = 1$ in $[\mathbb{Z}]_{p_\beta}$ is the root of $G(\Pi^0_\beta)$, and $A_k = G(\Sigma^0_{\beta-1}(m))$ for each $k = \langle i, m\rangle$.

- For $\beta \geq 4$ of the form $\delta + 2n + 2$ and $m \in \omega$, define

$$G\big(\Sigma^0_\beta(m)\big) = H_k\Big(\frac{r_k + r_{k+1}}{v^\infty_{\beta,k}}\Big)_{k\in\omega} H_{k+1},$$

where r_k is the root of H_k, r_0 is declared the root of $G(\Sigma^0_\beta(m))$, and $H_k = [G(\Sigma^0_{\beta-1})]_{\phi_{\beta,k}}$ if $0 \leq k \leq m$ and $H_k = [G(\Pi^0_{\beta-1})]_{\phi_{\beta,k}}$ otherwise. (For future notational convenience, we assume that $\phi_{\beta,0}$ makes r_0 infinitely divisible by p_β.)

- For $\beta \geq 4$ of the form $\delta + 2n + 2$, define

$$G(\Pi^0_\beta) = H_k\Big(\frac{r_k + r_{k+1}}{v^\infty_{\beta,k}}\Big)_{k\in\omega} H_{k+1},$$

where r_k is the root of H_k, r_0 is declared the root of $G(\Pi^0_\beta)$, and $H_k = [G(\Sigma^0_{\beta-1})]_{\phi_{\beta,k}}$ for all k. (For future notational convenience, we assume that $\phi_{\beta,0}$ makes r_0 infinitely divisible by p_β.)

- For $\beta \geq \omega$ a limit ordinal, define

$$G\big(\Sigma^0_\beta(m)\big) = H_k\Big(\frac{r_k + r_{k+1}}{v^\infty_{\beta,k}}\Big)_{k\in\omega} H_{k+1},$$

where r_k is the root of H_k, r_0 is declared the root of $G(\Sigma^0_\beta(m))$, and $H_k = [G(\Sigma^0_{\gamma_k})]_{\phi_{\beta,k}}$ if $0 \leq k \leq m$ and $H_k = [G(\Pi^0_{\gamma_k})]_{\phi_{\beta,k}}$ otherwise. (For future notational convenience, we assume that $\phi_{\beta,0}$ makes r_0 infinitely divisible by p_β.)

- For $\beta \geq \omega$ a limit ordinal, define

$$G(\Pi^0_\beta) = H_k\Big(\frac{r_k + r_{k+1}}{v^\infty_{\beta,k}}\Big)_{k\in\omega} H_{k+1},$$

where r_k is the root of H_k, r_0 is declared the root of $G(\Pi^0_\beta)$, and $H_k = [G(\Pi^0_{\gamma_k})]_{\phi_{\beta,k}}$ for every k. (For future notational convenience, we assume that $\phi_{\beta,0}$ makes r_0 infinitely divisible by p_β.)

2.3 Effective content of Definition 2.5 In Definition 2.5, every alternation of quantifiers corresponds to either an application of the chain operation or the branching operation. We use the branching operation for $\beta = 3$, then the chain operation for $\beta = 4$, and so on. Each group naturally reflects a Σ^0_n-fact. Hence, it is straightforward to establish the following result.

Lemma 2.6 *Suppose that S is a Σ^0_β-set, uniformly in β, where β is a computable ordinal at most α. There exists a uniform procedure which, given e, β, constructs a computable group H_e such that $H_e \cong G(\Sigma^0_\beta)$ (or $G(\Sigma^0_\beta(m))$ for* some *m) if $e \in S$, and $H_e \cong G(\Pi^0_n)$ otherwise.*

Proof The proof proceeds by effective transfinite recursion. Recall that for all computable ordinals less than or equal to α we use a fixed effective notation. Note that the primes which can be used in a given location of a group are specified by a sequence of transformations $(\phi_{\rho_k,k})_k$, where the ρ_k's form a descending sequence of computable ordinals. We can compute the composition of the corresponding maps at every component of the group. The rest follows from the definition of the groups and from the well-known fact that we can choose a presentation of a Σ^0_β-predicate having stable witnesses (i.e., if $\exists x P(x)$, then $\forall y \geq x P(y)$). $\square$

2.4 Illustration of the technique The proposition below illustrates, roughly, how we will extract the Σ^0_3-outcome from a direct sum of $G(\Sigma^0_3)$- and $G(\Pi^0_3)$-subcomponents. Here Σ^0_3 naturally corresponds to "there exists a subcomponent which is infinitely divisible by w." Informally, the proposition says that a "encodes" a Σ^0_3-outcome if and only if it is a linear combination of the h_i's, each of which "encodes" a Σ^0_3-outcome. Proposition 2.7(1) will be used in the verification in a much more general context not necessarily corresponding to Σ^0_3 (to be explained).

Proposition 2.7 *Let u, w, p, q be distinct primes. Suppose that*

$$A \cong \bigoplus_{i \in I}\Big[H_i\Big(\frac{h_i + g_{i,k}}{p^\infty}\Big)_{k \in K_i} G_{i,k}\Big]$$

is a countable group, where $I \subseteq \omega$, $K_i \subseteq \omega$ are nonempty, $(H_i, h_i) \cong ([\mathbb{Z}]_q, 1)$, and either $G_{i,k} \cong ([\mathbb{Z}]_u, w^{k_{i,j}})$ with $k_{i,j} \in \omega$ or $G_{i,k} \cong ([\mathbb{Z}]_{\{u,w\}}, 1)$. Then there exists a computable $\mathcal{L}_{\omega_1\omega}$-formula Ψ of complexity Σ_3 such that $A \models \Psi(a)$ if and only if

(1) $a = \sum_i m_i h_i$, where $m_i \in \mathbb{Z}$, and
(2) for each $m_i \neq 0$ in $a = \sum_i m_i h_i$ there exists k such that $G_{i,k} \cong ([\mathbb{Z}]_{u,w}, 1)$.

The formula can be produced uniformly in the primes u, w, p, q.

Proof We need the following lemma.

Lemma 2.8 *Let $I \subseteq \omega$ be a nonempty set, and let*

$$A = \bigoplus_{i \in I}\Big[H_i\Big(\frac{h_i + g_{i,k}}{p^\infty}\Big)_{k \in K_i} G_{i,k}\Big],$$

where each of the H_i's and $G_{i,k}$'s are prime closures of $\mathbb{Z}$ by nonempty sets of primes not containing p, with roots h_i and $g_{i,k}$, respectively. (Note that we have no further assumptions on the cardinalities of the nonempty sets $K_i \subseteq \omega$.) Assume that a is a nonzero element of A such that $p^\infty | a$. Then

$$a = \sum_i \Big(c_i h_i + \sum_{f \in F_i} d_{i,f}\, g_{i,f}\Big),$$

where the c_i's and $d_{i,f}$'s are rationals and $F_i \subseteq K_i$ are finite sets such that

$$c_i = \sum_{f \in F_i} d_{i,f}$$

for every i.

Remark 2.9 Notice that, for some (possibly, for all) indices i, the coefficients c_i in $a = \sum_i (c_i h_i + \sum_{f \in F_i} d_{i,f} g_{i,f})$ may be zero.

Proof outline of Lemma 2.8 By the definition of A,

$$a = \sum_{i,j} r_{i,j}(h_i + g_{i,j}),$$

where the $r_{i,j}$'s are rationals, for otherwise a would not be infinitely divisible by p. Let $B_i = H_i(\frac{h_i + g_{i,k}}{p^\infty})_{k \in K_i} G_{i,k}$. The groups B_i can be viewed as elementary rooted groups with structural trees T_i of height 1. Taking the restriction of T_i to vertices that occur in $a_i = \sum_j r_{i,j}(h_i + g_{i,j})$ with nonzero coefficients, we obtain a finitely branching tree of height 1. Suppose that $g_{i,j}$ is such that $r_{i,j} \neq 0$. The element

$$a_i' = a_i - r_{i,j}(h_i + g_{i,j})$$

is either 0 or corresponds to a subtree of T_i having fewer successors. In either case, by the inductive hypothesis, there exists a decomposition

$$a_i' = c_i' h_i + \sum_{f \in F_i'} d_{i,f}' g_{i,f}$$

such that $c_i' = \sum_{f \in F_i'} d_{i,f}'$. Then clearly

$$a_i = (c_i' + r_{i,j})h_i + \sum_{f \in F_i'} d_{i,f}' g_{i,f} + r_{i,j} g_{i,j}$$

and $c_i' + r_{i,j} = \sum_{f \in F_i'} d_{i,f}' + r_{i,j}$. □

The formula Ψ states that $a \neq 0$, $q^\infty | a$, and there exists a nonzero x such that $p^\infty | (a + x) \wedge u^\infty | x \wedge w^\infty | x$. We explain why the formula is computable infinitary Σ_3, and then we show that the formula satisfies the desired properties. Note that infinite divisibility by a prime, say, $q^\infty | a$, can be expressed as

$$\bigwedge_{n \in \omega} (\exists y)(q^n y = a),$$

where $q^n a$ is merely an abbreviation for $a + \cdots (n \text{ times}) + a$. The latter is a computable Π_2 infinitary formula in the language of additive groups. Thus, Ψ is an existential projection of a computable Π_2-formula. We conclude that Ψ is computable Σ_3. Note that the formula can be produced uniformly by using *only* primes p, q, u, w as parameters. (In particular, the formula does not depend on any particular choice of the sets K_i.)

We now show that Ψ satisfies Propositions 2.7(1) and 2.7(2). Note that every element of the desired form satisfies the formula trivially. Suppose that a satisfies Ψ. By Lemma 2.8,

$$a + x = \sum_i \Big(c_i h_i + \sum_{f \in F_i} d_{i,f} g_{i,f} \Big),$$

where $F_i \subseteq K_i$ are finite sets and

$$c_i = \sum_{f \in F_i} d_{i,f}$$

for every i. The rest of the proof relies on elementary properties of direct decompositions (see [11], [12]).

Embed the group A into its p-closure

$$[A]_p = \bigoplus_{i \in I}\Big([H_i]_p \oplus \bigoplus_{k \in K_i}[G_{i,k}]_p\Big).$$

Since infinite divisibility is preserved under isomorphic embeddings, we have $[A]_p \models q^\infty|(a+x)$. Suppose that $a = b_a + d_a$, where $b_a \in \bigoplus_i [H_i]_p$ and $d_a \in \bigoplus_{i,k \in K_i}[G_{i,k}]_p$, and similarly suppose that $x = b_x + d_x$. Recall that $u^\infty|x$, and thus, $b_x = 0$. Since $q^\infty|a$, we have $d_a = 0$. Recall that $a + x = \sum_i (c_i h_i + \sum_{f \in F_i} d_{i,f} g_{i,f})$. We conclude that $a = \sum_i c_i h_i$ and $x = \sum_{i,f \in F_i} d_{i,f} g_{i,f}$. By our assumption, $w^\infty|x$, and therefore, $w^\infty|g_{i,f}$ for every i and $f \in F_i$. The latter gives Proposition 2.7(2).

We now prove Proposition 2.7(1). Recall that $c_i = \sum_{f \in F_i} d_{i,f}$, where

$$c_i \in [\mathbb{Z}]_{p,q} \cap [\mathbb{Z}]_{p,u,w} = [\mathbb{Z}]_p.$$

It remains to show $c_i \in \mathbb{Z}$. Since divisibility is preserved under projections onto direct components, it is sufficient to prove that for any choice of integers $m, k > 0$ and each i the element $mh_i/p^k \in [A]_p$ does not belong to A. Suppose that we have $mh_i/p^k \in A$, $p \nmid m$. Then

$$mh_i/p^k = \sum_c \Big(r_c h_c + \sum_j s_{c,j}(h_c + g_{c,j}) + \sum_l \eta_{c,l} g_{c,l}\Big),$$

where $r_c \in [\mathbb{Z}]_q$, $s_{c,j} \in [\mathbb{Z}]_p$, and $\eta_{c,l} \in [\mathbb{Z}]_{u,w}$. Since $h_c, g_{c,j}$ are linearly independent for different choices of c and j,

$$mh_i/p^k = r_i h_i + \sum_j s_{i,j}(h_i + g_{i,j}) + \sum_l \eta_{i,l} g_{i,l}.$$

We have $m/p^k = r_i + \sum_j s_{i,j}$, and for every j there is an l such that $s_{i,j} = -\eta_{i,l}$ (and indeed $l = j$). We conclude that $s_{i,j} \in [\mathbb{Z}]_{u,w} \cap [\mathbb{Z}]_p = \mathbb{Z}$, and thus, $m/p^k \in [\mathbb{Z}]_q \cap [\mathbb{Z}]_p = \mathbb{Z}$, that is, $k = 0$. □

Remark 2.10 If we remove the conjunct $w^\infty|x$ of Ψ from the proof of Proposition 2.7, we will obtain a syntactical necessary and sufficient condition for an element to satisfy Proposition 2.7(1).

In the verification, we will be applying Proposition 2.7 and Remark 2.10 to various subgroups of elementary rooted groups. These applications will require a slight modification of the proposition which will be repeated over and over again in different contexts. We explain this modification below.

We follow the notation from Proposition 2.7. The structural tree of the larger computable group G containing A will possibly have edges connecting the elements playing the roles of g_i and $h_{i,k}$ to some other elements of G outside A. These edges will be labeled by primes, say, by s_i and $v_{i,k}$, respectively. We will have $s_i \neq v_{j,k}$ for any i, j, k, and s_i and $v_{i,k}$ will be unequal to any of the primes playing the roles of u, w, p, q. A typical application of Proposition 2.7 can be described as follows.

- Consider the least (super)group $H \supseteq G$ in the divisible closure of $G \supseteq A$ such that $H \models s_i^\infty | g_j$ and $H \models v_{j,k}^\infty | h_{i,k}$.
- The least pure subgroup S of H containing A will detach as a direct summand of H.
- By the choice of Ψ, we will have $G \models \Psi(g)$ implies $H \models \Psi(g)$ and $g \in S$. (Indeed, $S \models \Psi(g)$.)
- Then we can repeat the proof of Proposition 2.7(1) and get Proposition 2.7(1) but perhaps with $m_i \in \bigcup_i [\mathbb{Z}]_{s_i} \leqq \mathbb{Q}$.
- By Lemma 2.8, the choice of Ψ and $[\mathbb{Z}]_{v_{j,k}} \cap [\mathbb{Z}]_{s_i} = \mathbb{Z}$ will imply that $m_i \in \mathbb{Z}$.

We conclude that Proposition 2.7 goes through in this more general situation. We also note that the formula Ψ stays the same in the situation described above. This is different from [1], where formulae had to be significantly adjusted in every specific application. We will omit the repeated argument above in all actual applications of Proposition 2.7 and Remark 2.10.

2.5 Coding family $\mathcal{R}$ into $G_{\mathcal{R}}$ Given a finite set S, define its *age* to be

$$\mathrm{Age}_S = \{i : D_i \subseteq S\},$$

where $(D_i)_{i \in \omega}$ is the canonical listing of all finite sets of natural numbers.

Recall that $\alpha = \delta + 2n + 2$, where δ is 0 or a limit ordinal and $n \in \omega$. We also fix injective and effective maps $\psi_{\alpha,k}$, $k \in \omega$ (to be used for the operation of substitution), which are different from the $\phi_{\beta,k}$'s and also consistent with Definition 2.5 (i.e., they effectively map the primes used in the corresponding $G(\Sigma)$- or $G(\Pi)$-component H_k to new fresh primes which do not overlap for different k's). In the following, we suppress α in $\psi_{\alpha,k}$. We identify finite strings in $\omega^{<\omega}$ with functions from $\omega^\omega \to \{0, 1\}$ having finite support.

Definition 2.11 Given any finite set S and any finite string $\sigma \in \omega^{<\omega}$, define

$$H_{\sigma,S} = B_{S,\sigma,k} \left(\frac{r_{S,\sigma,k} + r_{S,\sigma,k+1}}{w_{\alpha,k}^\infty} \right)_{k \in \omega} B_{S,\sigma,k+1},$$

where $B_{S,\sigma,k} \cong [G(\Sigma_\alpha^0(\sigma(k)))]_{\psi_k}$ if $k \in \mathrm{Age}_S$, and $B_{S,\sigma,k} \cong [G(\Pi_\alpha^0)]_{\psi_k}$ otherwise. Define

$$H_S = \bigoplus_{\sigma \in \omega^{<\omega}} H_{\sigma,S}.$$

Finally, let

$$G_{\mathcal{R}} = \bigoplus_{S \in \mathcal{R}} H_S.$$

We can effectively choose ψ_k to be consistent with Definition 2.5. Consequently, we have the following.

Lemma 2.12 *There exists a uniform procedure which, given any* Σ_α^0-enumeration *of $\mathcal{R}$, outputs a computable presentation of the group*

$$G_{\mathcal{R}} = \bigoplus_{S \in \mathcal{R}} H_S.$$

Proof Note that $i \in \mathrm{Age}_S$ is a uniformly Σ_α^0-fact. Consequently, we can apply Lemma 2.6, an effective enumeration of $\omega^{<\omega}$, and the fact that the ψ_k's can be effectively and consistently defined. It is crucial that $\mathcal{R}$ consists of finite sets. □

2.6 Reconstructing $\mathcal{R}$ from $G_{\mathcal{R}}$ We use the same notation as in the previous section. According to its definition, the group $G_{\mathcal{R}}$ is of the form

$$G_{\mathcal{R}} = \bigoplus_{S\in\mathcal{R},\sigma\in\omega^{<\omega}} \Big(B_{S,\sigma,k} \Big(\frac{r_{S,\sigma,k} + r_{S,\sigma,k+1}}{w^{\infty}_{\alpha,k}} \Big)_{k\in\omega} B_{S,\sigma,k+1} \Big),$$

where $B_{S,\sigma,k} \cong [G(\Sigma^0_\alpha(\sigma(k)))]_{\psi_k}$ if $k \in \mathrm{Age}_S$, and $B_{S,\sigma,k} \cong [G(\Pi^0_\alpha)]_{\psi_k}$ otherwise. We state a lemma that relies on algebraic and combinatorial techniques developed in [1].

Lemma 2.13 *There exists a uniform collection $(\Phi_k)_{k\in\omega}$ of computable infinitary Σ^c_α-formulae such that $G_{\mathcal{R}} \models \Phi_k(x)$ if and only if*

$$x = \sum_{(S,\sigma)\in I} m_{S,\sigma} r_{S,\sigma,0},$$

where $m_{S,\sigma} \in \mathbb{Z} \setminus \{0\}$, and for some S such that $(S,\sigma) \in I$ we have $k \in \mathrm{Age}_S$. (Equivalently, $r_{S,\sigma,k}$ is the root of $B_{S,\sigma,k} \cong [G(\Sigma^0_\alpha(\sigma(k)))]_{\psi_k}$.)

We first apply Lemma 2.13 to prove Proposition 2.1, and then we prove Lemma 2.13.

Proof of Proposition 2.1 Lemma 2.6 implies that every Σ^0_α-family $\mathcal{R}$ can be uniformly transformed into a computable copy of $G_{\mathcal{R}}$. We prove that if $G_{\mathcal{R}}$ is computable, then $\mathcal{R}$ has a Σ^0_α-enumeration. If we succeed, then a straightforward relativization will prove the proposition.

Given an oracle $\mathcal{Y}$ for the $(\alpha - 1)$th iteration of the Turing jump, enumerate the sets

$$U_x = \big\{k : G_{\mathcal{R}} \models \Phi_k(x)\big\},$$

where $x \in G_{\mathcal{R}}$, $x \neq 0$. If $x = \sum_{(S,\sigma)\in I} m_{S,\sigma} r_{S,\sigma,0}$, then Lemma 2.13 implies that

$$U_x = \bigcup_{S:(S,\sigma)\in I} \mathrm{Age}_S.$$

Acting effectively in $\mathcal{Y}$, we start enumerating a sequence of finite sets

$$\emptyset = D_{i_1} \subseteq D_{i_2} \subseteq \cdots$$

listing indices in U_x. If D_{i_n} has already been defined, then we wait for a $j \in U_x$ such that $D_{i_n} \subset D_j$. While we wait, we keep defining $i_{n+k} = i_n$. If we find such a j and we have already defined $D_{i_{n+k}}$, then we set $i_{n+k+1} = j$. We then repeat the same procedure for i_{n+k+1}, and so on. Notice that the sequence $(D_{i_n})_{n\in\omega}$ has to stabilize on some finite set which is equal to one of the elements in $\{S : (S,\sigma) \in I\}$. Indeed, we may have $D_{i_{n+1}} \supset D_{i_n}$ only if there exists an S in the finite family $\{S : (S,\sigma) \in I\}$ of finite sets such that $S \supseteq D_{i_{n+1}} \supset D_{i_n}$. On the other hand, if D_{i_n} has been defined and there exists at least one S in $\{S : (S,\sigma) \in I\}$ such that $S \supset D_{i_n}$, then there will be a stage at which we will define $D_{i_m} \supset D_{i_n}$. Also notice that if x is not of the form $\sum_{(S,\sigma)\in I} m_{S,\sigma} r_{S,\sigma,0}$, then the sequence consists of only the empty set (by Lemma 2.13). By our assumption, $\emptyset \in \mathcal{R}$. We conclude that, for each $x \neq 0$, the union $C_x = \bigcup_{n\in\omega} D_{i_n}$ of the sequence defined by the procedure above is a $\mathcal{Y}$-c.e. set uniformly in x, and furthermore, $C_x = S$ for some S in $\{S : (S,\sigma) \in I\}$ corresponding to x.

Note that the enumeration of C_x is uniform in x. Thus, we can use the above procedure and define a $\mathcal{Y}$-c.e. enumeration

$$\{C_x : x \in G_{\mathcal{R}}\}$$

of a family of finite sets. Recall that we restrict ourselves to groups upon the domain ω, and thus, C_x is a set of natural numbers indexed by a natural number x. As we have already mentioned, either $C_x = \emptyset$ and then $C_x \in \mathcal{R}$ by our assumption ($\emptyset \in \mathcal{R}$), or $C_x \neq \emptyset$ but still $C_x \in \mathcal{R}$. Also, every set in $\mathcal{R}$ clearly will be listed. This establishes Proposition 2.1 and, hence, Theorem 1.2. □

Proof of Lemma 2.13 We have

$$G_{\mathcal{R}} = \bigoplus_{S\in\mathcal{R},\sigma\in\omega^{<\omega}} \left(B_{S,\sigma,k}\left(\frac{r_{S,\sigma,k} + r_{S,\sigma,k+1}}{w^{\infty}_{\alpha,k}}\right)_{k\in\omega} B_{S,\sigma,k+1}\right),$$

where $B_{S,\sigma,k} \cong [G(\Sigma^0_\alpha(\sigma(k)))]_{\psi_k}$ if $k \in \mathrm{Age}_S$, and $B_{S,\sigma,k} \cong [G(\Pi^0_\alpha)]_{\psi_k}$ otherwise.

Suppose that $x \in G_{\mathcal{R}}$ is not equal to zero.

Claim 2.14 *There exists a formula of complexity Σ^c_3 such that x satisfies the formula if and only if $x = \sum_{(S,\sigma)\in I} m_{S,\sigma} r_{S,\sigma,0}$, where $m_{S,\sigma} \in \mathbb{Z}$ for each (S,σ).*

Proof The formula can be produced using Remark 2.10 with the right choice of primes and index sets in Proposition 2.7. More specifically, the formula says that $x \neq 0$ and $\psi_0(p_\alpha)^\infty | x$, and there exists y such that $\psi_1(p_\alpha)^\infty | y$ and $w^\infty_{\alpha,0} | (x + y)$. □

Given k and $x = \sum_{(S,\sigma)\in I} m_{S,\sigma} r_{S,\sigma,0}$, we would like to check, using a formula, whether $r_{S,\sigma,k}$ is the root of a $G(\Sigma^0_\alpha(\sigma(k)))$-type component. We first prove the following.

Claim 2.15 *Let $x = \sum_{(S,\sigma)\in I} m_{S,\sigma} r_{S,\sigma,0}$, where $m_{S,\sigma} \in \mathbb{Z}$. For every k we can uniformly produce a Σ^c_3-formula Θ such that $\Theta_k(x, y)$ if and only if $y = \sum_{(S,\sigma)\in I} m_{S,\sigma} r_{S,\sigma,k}$. (That is, y is a linear combination of the roots of $B_{S,\sigma,k}$, and furthermore, the corresponding coefficients are equal to those of $r_{S,\sigma,0}$ in x.)*

Proof We prove the claim by induction. The case $k = 0$ is Claim 2.14. Suppose that we have produced $\Theta_{k-1}(x, \cdot)$. Consider the pure subgroup generated by the roots of the $B_{S,\sigma,k-1}$-subcomponents and $B_{S,\sigma,k}$-subcomponents. Define $\Theta_k(x, y)$ to be the formula

$$(\exists z)\big(\Theta_{k-1}(x, z) \wedge w^\infty_{\alpha,k-1} | (y + z) \wedge \psi_k(p_\alpha)^\infty | y\big).$$

By the inductive hypothesis, $z = \sum_{(S,\sigma)\in I} m_{S,\sigma} r_{S,\sigma,k-1}$. Since $\psi_k(p_\alpha)^\infty | y$, we have

$$y = \sum_{(S,\sigma)\in I} t_{S,\sigma} r_{S,\sigma,k},$$

where the $t_{S,\sigma}$'s are rationals. By the inductive hypothesis, we may assume that $\Theta_{k-1}(x, z)$ contains a conjunct of the form $\psi_{k-1}(p_\alpha)^\infty | z$. Consider the pure closure of the subgroup generated by $r_{S,\sigma,k}$ and $r_{S,\sigma,k-1}$ for various S's and σ's. Note that $w^\infty_{\alpha,k-1} | (y + z)$, and thus, by Lemma 2.8 applied to this pure subgroup we have $t_{S,\sigma} = m_{S,\sigma}$. □

Recall that our final goal is to produce a uniform collection $(\Phi_k)_{k\in\omega}$ of computable infinitary Σ^c_α-formulae such that $G_{\mathfrak{R}} \models \Phi_k(x)$ if and only if

$$x = \sum_{(S,\sigma)\in I} m_{S,\sigma} r_{S,\sigma,0},$$

where $m_{S,\sigma} \in \mathbb{Z}\setminus\{0\}$, and for some S the element $r_{S,\sigma,k}$ is the root of $B_{S,\sigma,k} \cong [G(\Sigma^0_\alpha(\sigma(k)))]_{\psi_k}$.

By Claims 2.14 and 2.15, to conclude the proof of the lemma it is sufficient to establish the following.

Claim 2.16 *For every k we can uniformly produce a Σ^c_α-formula Γ_k such that, for every element of the form $x = \sum_{(S,\sigma)\in I} m_{S,\sigma} r_{S,\sigma,k}$, $G_{\mathfrak{R}} \models \Gamma_k$ if and only if for some $m_{S,\sigma} \neq 0$ the corresponding $r_{S,\sigma,k}$ is the root of $B_{S,\sigma,k} \cong [G(\Sigma^0_\alpha(\sigma(k)))]_{\psi_k}$.*

Remark 2.17 The proof of this claim is similar to the proof of [1, Lemma 4.14] and is indeed simpler. For the sake of exposition, we give a full proof.

Proof The proof uses a transfinite induction on "even" α. Before we describe the induction, we do some preliminary syntactical analysis that will be used at all steps of the induction. The idea is that we use infinite divisibility to get access to the smaller components that were employed in the definition of Σ^0_α- and Π^0_α-blocks. This is essentially done by several consecutive applications of Remark 2.10.

We fix k. For every σ and S, the $B_{S,\sigma,k}$-subcomponent is of the form

$$A_{S,\sigma,i}\left(\frac{a_{S,\sigma,i} + a_{S,\sigma,i+1}}{\psi_k(v_{\alpha,i})^\infty}\right)_{i\in\omega} A_{S,\sigma,i+1},$$

where either $A_{S,\sigma,i} \cong G(\Sigma^0_{\alpha-1})$ for all i, or there exists an m such that $A_{S,\sigma,i} \cong G(\Sigma^0_{\alpha-1})$ for $i \leq m$ and $A_{S,\sigma,i} \cong G(\Pi^0_{\alpha-1})$ for $i > m$. The former corresponds to $B_{S,\sigma,k} \cong G(\Pi^0_\alpha)$, and the latter corresponds to $B_{S,\sigma,k} \cong G(\Sigma^0_\alpha(m))$, where m depends on S, σ, and k.

We claim that for every i we can uniformly produce a Σ^c_3-formula $\mathcal{U}_i$ such that $\mathcal{U}_i(x, y)$ holds if and only if

$$y = \sum_{(S,\sigma)\in I} m_{S,\sigma} a_{S,\sigma,i}.$$

Indeed, we may take the formula witnessing Claim 2.14 and replace $w_{\alpha,k-1}$ by $\psi_k(v_{\alpha,i})$ in the formula, and we also replace $\psi_k(p_\alpha)$ by the prime that labels the roots $a_{S,\sigma,i}$ of $A_{S,\sigma,i}$. The same proof will witness that the formula has the desired property. A straightforward syntactical analysis of the formula $\mathcal{U}_i$ shows that it is of the form

$$(\exists z_0)\cdots(\exists z_i)\mathcal{X}_i(x, \bar{z}, y),$$

where $\mathcal{X}_i(x, \bar{z}, y)$ is Π^c_2.

We are ready to consider the basic case $\alpha = 4$ of the induction. We restrict ourselves to the subgroup

$$\bigoplus_{(S,\sigma)} A_{S,\sigma,i},$$

where S ranges over $\mathfrak{R}$ and σ ranges over $\omega^{<\omega}$. By Proposition 2.7, there is a Σ^c_3-formula $\mathcal{W}_{i,3}$ such that $\mathcal{W}_{i,3}(y)$ holds if and only if $y = \sum_{(S,\sigma)} n_{S,\sigma} a_{S,\sigma,i}$, where for each $n_{S,\sigma} \neq 0$ the corresponding $a_{S,\sigma,i}$ is the root of a $G(\Sigma^0_3)$-type component. In fact, this formula can be produced with all possible uniformity. The desired Σ^0_4-formula for the basic case $\alpha = 4$ is

$$(\exists y) \bigvee_i (\exists z_0) \cdots (\exists z_i) \big(\mathcal{X}_i(x, \bar{z}, y) \wedge \neg \mathcal{W}_{i,3}(y)\big).$$

Indeed, the first conjunct inside the parentheses guarantees $m_{\sigma,S} = n_{\sigma,S}$, and the second conjunct says that $B_{S,\sigma,k} \cong [G(\Sigma^0_4(\sigma(k)))]_{\psi_k}$ with $\sigma(k) \leq i$.

Suppose that $\alpha > 4$. By our assumption, α is an "even" successor ordinal. We claim that we can uniformly produce $\Sigma^c_{\alpha-1}$-formulae $\mathcal{W}_{i,\alpha-1}$ such that $\mathcal{W}_{i,\alpha-1}(y)$ holds if and only if $y = \sum_{(S,\sigma)} n_{S,\sigma} a_{S,\sigma,i}$, where for some $n_{S,\sigma} \neq 0$ the corresponding $a_{S,\sigma,i}$ is the root of a $G(\Sigma^0_{\alpha-1})$-type component. If we succeed in producing such formulae, then $(\exists y) \bigvee_i (\exists z_0) \cdots (\exists z_i)(\mathcal{X}_i(x, \bar{z}, y) \wedge \neg \mathcal{W}_{i,\alpha-1}(y))$ will satisfy the desired properties as in the basic case $\alpha = 4$ described above.

We explain how we produce $\mathcal{W}_{i,\alpha-1}(y)$. In the following, we fix k and i, and we sometimes suppress k and i in subscripts. In the next few lines we use Remark 2.10 and Proposition 2.7(1) restricted to the relevant pure subgroups of $G_{\mathfrak{R}}$.

We can produce a Σ^c_3-formula that ensures $y = \sum_{(S,\sigma)} n_{S,\sigma} a_{S,\sigma,i}$, where the $n_{S,\sigma}$'s are integers. (Indeed, this step is not really necessary since $\mathcal{X}_i(x, \bar{z}, y)$ implies $n_{S,\sigma} = m_{S,\sigma}$.) Using primes $\psi_k(p_{\alpha-2})$ and $\psi_k(q_{\alpha-2})$, we can produce a Σ^c_3-formula $\mathcal{Z}$ that holds on z if and only if $z = \sum_{(S,\sigma,s)} l_{S,\sigma,s} d_{S,\sigma,s}$, where the $d_{S,\sigma,s}$'s are the roots of various $G(\Sigma^0_{\alpha-2}(m))$-type and $G(\Pi^0_{\alpha-2})$-type subcomponents of $A_{S,\sigma,i}$. Let $D_{S,\sigma,s}$ denote the subcomponent with root $d_{S,\sigma,s}$. Lemma 2.8 implies that, for every i,

$$\sum_s l_{S,\sigma,s} = n_{S,\sigma}.$$

Furthermore, using a variation of Claim 2.16 with the right choice of primes (e.g., $\psi_k(v_{\alpha-2,j})$), we can produce a uniform sequence of Σ^c_3-formulae $\{\mathcal{F}_j\}_{j\in\omega}$ such that $\mathcal{F}_j(z, c_j) \wedge \mathcal{Z}(y, z)$ holds if and only if $c_j = \sum_{(S,\sigma,s)} l_{S,\sigma,s} k_{S,\sigma,s,j}$, where $k_{S,\sigma,s,j}$ is the root of $K_{S,\sigma,s,j}$ which is the jth subcomponent of $D_{S,\sigma,s}$ that was used in its definition via the chain operation, counting from its root. Adjusting the argument described after the proof of Proposition 2.7, we can ensure that the coefficients $l_{S,\sigma,s}$ are indeed integers. (This is one of the main advantages of our coding when compared with that of [1].)

The subcomponent $K_{S,\sigma,s,j}$ is of a $G(\Sigma^0_{\gamma_j})$-type or of a $G(\Pi^0_{\gamma_j})$-type for some $\gamma_j < \alpha-2$. (Note that $\alpha-2$ could be a limit ordinal.) We can effectively compute (the $\leq \alpha$-notation for) γ_j. By the inductive hypothesis, we can effectively produce the formula $\mathcal{W}_{i,\gamma_j}$. Replacing several primes in $\mathcal{W}_{i,\gamma_j}$ according to the prime substitution that we used in the definition of the group, we can effectively pass to a formula $\mathcal{W}'_{\gamma_j}$ which holds on $c_j = \sum_{(S,\sigma,s)} l_{S,\sigma,s} k_{S,\sigma,s,j}$ if and only if each $k_{S,\sigma,s,j}$ is the root of a $G(\Sigma^0_{\gamma_j})$-type subcomponent.

The desired formula $\mathcal{W}_{i,\alpha-1}(y)$ can be set equal to

$$(\exists z)\Big(\mathcal{Z}(y, z) \wedge \bigwedge_{j\in\omega} (\exists c_j)\big[\mathcal{F}_j(z, c_j) \wedge \mathcal{W}'_{\gamma_j}(c_j)\big]\Big).$$

Indeed, the formula says that there exists a z which is a linear combination, with integer coefficients, of "immediate successors" of (the summands of) y in the structural tree; furthermore, these immediate successors are the roots of $G(\Sigma_{\alpha-2})$-type components (as witnessed by any infinite sequence of $(c_j)_{j\in\omega}$). Each such z satisfies the formula trivially. On the other hand, if z satisfies the formula, then the analysis preceding its definition illustrates that z must be indeed of the desired form. This observation concludes the proof of Claim 2.16. □ □

3 Conclusion

We expect that our machinery can be used to extend Theorem 1.2 to every computable ordinal greater than 0. The "even" ordinals, however, seem to require further technical adjustments and/or new ideas. The issue comes from the algebraic side of the coding, namely, from the embeddability relation between components corresponding to different outcomes. (For more intuition and a discussion of the machinery, see [1].)

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