Notre Dame Journal of Formal Logic
Volume 52, Number 1, 2011

Characterizing the Join-Irreducible Medvedev Degrees

Paul Shafer

Abstract We characterize the join-irreducible Medvedev degrees as the degrees of complements of Turing ideals, thereby solving a problem posed by Sorbi. We use this characterization to prove that there are Medvedev degrees above the second-least degree that do not bound any join-irreducible degrees above this second-least degree. This solves a problem posed by Sorbi and Terwijn. Finally, we prove that the filter generated by the degrees of closed sets is not prime. This solves a problem posed by Bianchini and Sorbi.

1 Introduction

We present solutions to three problems concerning the Medvedev degrees. A *mass problem* is a set $\mathcal{A} \subseteq \omega^\omega$. For mass problems $\mathcal{A}$ and $\mathcal{B}$, we say that $\mathcal{A}$ *Medvedev reduces* to $\mathcal{B}$ ($\mathcal{A} \leq_{\mathrm{M}} \mathcal{B}$) if there is a Turing functional Φ such that $\Phi(\mathcal{B}) \subseteq \mathcal{A}$. That is, $\Phi(f) \in \mathcal{A}$ for all $f \in \mathcal{B}$. We say that $\mathcal{A}$ and $\mathcal{B}$ are *Medvedev equivalent* ($\mathcal{A} \equiv_{\mathrm{M}} \mathcal{B}$) if $\mathcal{A} \leq_{\mathrm{M}} \mathcal{B}$ and $\mathcal{B} \leq_{\mathrm{M}} \mathcal{A}$. The equivalence class $[\mathcal{A}]$ is called the *Medvedev degree* of $\mathcal{A}$, and the structure $\mathfrak{M} = (2^{\omega^\omega}/\equiv_{\mathrm{M}}, \leq_{\mathrm{M}})$ is called the *Medvedev degrees*. See Sorbi [15] for a good introduction to the theory of the Medvedev degrees.

For $f, g \in \omega^\omega$, let $f \oplus g$ be the function $(f \oplus g)(2n) = f(n)$ and $(f \oplus g)(2n+1) = g(n)$. For $m \in \omega$ and $f \in \omega^\omega$, let $m^\frown f$ be the function $(m^\frown f)(0) = m$ and $(m^\frown f)(n+1) = f(n)$. In general, '$^\frown$' denotes string concatenation. Functions $f \in \omega^\omega$ are interpreted as ω-length strings when appropriate. For a mass problem $\mathcal{A}$, let $m^\frown \mathcal{A} = \{m^\frown f \mid f \in \mathcal{A}\}$. Given mass problems $\mathcal{A}$ and $\mathcal{B}$, let $\mathcal{A} + \mathcal{B} = \{f \oplus g \mid f \in \mathcal{A} \wedge g \in \mathcal{B}\}$ and let $\mathcal{A} \times \mathcal{B} = 0^\frown \mathcal{A} \cup 1^\frown \mathcal{B}$. Then $[\mathcal{A}] + [\mathcal{B}] = [\mathcal{A} + \mathcal{B}]$ is the *join* (i.e., $\leq_{\mathrm{M}}$-least upper bound) of $[\mathcal{A}]$ and $[\mathcal{B}]$, while $[\mathcal{A}] \times [\mathcal{B}] = [\mathcal{A} \times \mathcal{B}]$ is the *meet* (i.e., $\leq_{\mathrm{M}}$-greatest lower bound) of $[\mathcal{A}]$ and $[\mathcal{B}]$. Hence $\mathfrak{M}$ is a lattice. In fact, $\mathfrak{M}$ is a distributive lattice, meaning that join and meet distribute over each other: $\mathbf{a} + (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} + \mathbf{b}) \times (\mathbf{a} + \mathbf{c})$ and

Received March 28, 2010; accepted May 25, 2010; printed December 1, 2010
2010 Mathematics Subject Classification: Primary, 03D30, 03G10, 03B55
Keywords: Medvedev degrees, lattices, Brouwer algebras, intermediate logics
 10.1215/00294527-2010-034

$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c})$. Notation for join and meet appears in the literature variously as $+$, $\times$, as $\vee$, $\wedge$, and confusingly as $\wedge$, $\vee$. We choose the $+$, $\times$ notation to avoid conflict with the logical notation and to match Sorbi and Terwijn [16].

$\mathfrak{M}$ has a least element $\mathbf{0} = [\omega^\omega]$ (and any $\mathcal{A}$ containing a recursive function has this degree), a second-least element $\mathbf{0}' = [\{f \mid f >_{\mathrm{T}} 0\}]$, and a greatest element $\mathbf{1} = [\varnothing]$. (The Medvedev degree $\mathbf{0}'$ has little to do with $0'$, the Turing jump of the 0 function. Here $\mathbf{0}'$ always refers to the second-least Medvedev degree.)

In any lattice, an element $\mathbf{a}$ is called *join-reducible* if there are $\mathbf{x}, \mathbf{y} < \mathbf{a}$ such that $\mathbf{a} = \mathbf{x} + \mathbf{y}$. Otherwise, $\mathbf{a}$ is called *join-irreducible*. Dually, $\mathbf{a}$ is called *meet-reducible* if there are $\mathbf{x}, \mathbf{y} > \mathbf{a}$ such that $\mathbf{a} = \mathbf{x} \times \mathbf{y}$. Otherwise, $\mathbf{a}$ is called *meet-irreducible*. Dyment [3] characterized the meet-reducible Medvedev degrees in the following theorem. Its corollary helps identify meet-irreducible Medvedev degrees.

Theorem 1.1 ([3]) *A Medvedev degree* $\mathbf{a}$ *is meet-reducible if and only if* $\mathbf{a} = [\mathcal{A}]$ *for a mass problem* $\mathcal{A}$ *for which there are r.e. sets* $V_0, V_1 \subseteq \omega^{<\omega}$ *such that*

(i) $(\forall f \in \mathcal{A})(\exists \sigma \in V_0 \cup V_1)(\sigma \subset f)$,

(ii) *the following mass problems are* $\leq_{\mathrm{M}}$*-incomparable:*

$$\{f \in \mathcal{A} \mid (\exists \sigma \in V_0)(\sigma \subset f)\} \text{ and } \{f \in \mathcal{A} \mid (\exists \sigma \in V_1)(\sigma \subset f)\}.$$

Corollary 1.2 ([3]) *If* $\mathcal{A}$ *is a mass problem such that* $\sigma^\frown\mathcal{A} \subseteq \mathcal{A}$ *for all* $\sigma \in \omega^{<\omega}$, *then* $[\mathcal{A}]$ *is meet-irreducible.*

In particular, $\mathbf{0}'$ is meet-irreducible because $\sigma^\frown f >_{\mathrm{T}} 0$ whenever $\sigma \in \omega^{<\omega}$ and $f >_{\mathrm{T}} 0$.

The problem of characterizing the join-irreducible Medvedev degrees was posed in [15]. In Section 2, we prove that $\mathbf{a} \in \mathfrak{M}$ is join-irreducible if and only if $\mathbf{a} = [\omega^\omega - \mathcal{I}]$ for some Turing ideal $\mathcal{I}$.

We have seen that $\mathfrak{M}$ is a distributive lattice with $\mathbf{0}$ and $\mathbf{1}$. In fact, $\mathfrak{M}$ is a Brouwer algebra. A *Brouwer algebra* is a distributive lattice with $\mathbf{0}$ and $\mathbf{1}$ such that for every $\mathbf{a}$ and $\mathbf{b}$ there is a least $\mathbf{c}$ such that $\mathbf{a} + \mathbf{c} \geq \mathbf{b}$. This least $\mathbf{c}$ is denoted by $\mathbf{a} \to \mathbf{b}$. For mass problems $\mathcal{A}$ and $\mathcal{B}$, define $\mathcal{A} \to \mathcal{B} = \{e^\frown g \mid (\forall f \in \mathcal{A})(\Phi_e(f \oplus g) \in \mathcal{B})\}$. Then $[\mathcal{A}] \to [\mathcal{B}] = [\mathcal{A} \to \mathcal{B}]$. A Brouwer algebra is dual to a Heyting algebra, but $\mathfrak{M}$ is proved not to be a Heyting algebra in Sorbi [12].

Brouwer algebras give semantics for propositional logic. For any Brouwer algebra $\mathfrak{B}$, a *valuation* is a function $\nu\colon$ propositional variables $\to \mathfrak{B}$. A valuation ν extends to all propositional formulas φ by defining

$$\begin{aligned}
\nu(\varphi \wedge \psi) &= \nu(\varphi) + \nu(\psi), \\
\nu(\varphi \vee \psi) &= \nu(\varphi) \times \nu(\psi), \\
\nu(\varphi \to \psi) &= \nu(\varphi) \to \nu(\psi), \text{ and} \\
\nu(\neg\varphi) &= \nu(\varphi) \to \mathbf{1}.
\end{aligned}$$

A propositional formula φ is called *valid* in $\mathfrak{B}$ if $\nu(\varphi) = \mathbf{0}$ for every valuation ν. Let $\mathrm{Th}(\mathfrak{B})$ denote the set of propositional formulas valid in $\mathfrak{B}$. The axioms of intuitionistic logic are valid in every Brouwer algebra $\mathfrak{B}$, so $\mathrm{IPC} \subseteq \mathrm{Th}(\mathfrak{B}) \subseteq \mathrm{CPC}$ for every Brouwer algebra $\mathfrak{B}$. Here IPC denotes intuitionistic logic and CPC denotes classical logic. Logics L for which $\mathrm{IPC} \subseteq L \subseteq \mathrm{CPC}$ are called *intermediate logics*.

Providing semantics for propositional logic was one of Medvedev's main motivations behind introducing $\mathfrak{M}$, and he proved $\mathrm{Th}(\mathfrak{M}) = \mathrm{JAN}$ in Medvedev [8]. JAN denotes the logic IPC $+\neg p \vee \neg\neg p$ named after Jankov who studied it in Jankov [5]. In any Brouwer algebra $\mathfrak{B}$, the quotient of $\mathfrak{B}$ by the principal filter generated by $\mathbf{a} \in \mathfrak{B}$ is denoted by $\mathfrak{B}/\mathbf{a}$. The quotient $\mathfrak{B}/\mathbf{a}$ is isomorphic to the interval $[\mathbf{0}, \mathbf{a}]$ which is a Brouwer algebra under the operations inherited from $\mathfrak{B}$. Logics of the form $\mathrm{Th}(\mathfrak{M}/\mathbf{a})$ have been studied in Skvortsova [10], Sorbi [14], and Sorbi and Terwijn [16]. (Skvortsova and Dyment are the same person. Dyment married and became Skvortsova.) The results in Section 3 and Section 4 are motivated by the following question which remains open.

Question 1.3 ([16]) *Is* $\mathrm{Th}(\mathfrak{M}/\mathbf{a}) \subseteq \mathrm{JAN}$ *for all* $\mathbf{a} >_{\mathrm{M}} \mathbf{0}'$*?*

Sorbi and Terwijn's study of Question 1.3 in [16] lead them to ask whether every degree $>_{\mathrm{M}} \mathbf{0}'$ bounds a join-irreducible degree $>_{\mathrm{M}} \mathbf{0}'$ because a "yes" answer to this question implies a "yes" answer to Question 1.3. However, Sorbi and Terwijn conjectured that there is a degree $>_{\mathrm{M}} \mathbf{0}'$ that bounds no join-irreducible degree $>_{\mathrm{M}} \mathbf{0}'$, and we prove that this is correct in Section 3. In Section 4 we provide slight extensions to some of the results in [14], thereby widening the class of degrees $\mathbf{a}$ for which $\mathrm{Th}(\mathfrak{M}/\mathbf{a}) \subseteq \mathrm{JAN}$ is known.

Lastly, in Section 5 we use techniques similar to those used to characterize the join-irreducible degrees to prove that the filter generated by the degrees of mass problems closed in ω^ω is not prime. This problem was posed in Bianchini and Sorbi [2] and in Sorbi [15].

2 Characterizing the Join-Irreducible Medvedev Degrees

A *Turing ideal* is a set $\mathcal{I} \subseteq \omega^\omega$ that is closed downward under $\leq_{\mathrm{T}}$ (i.e., $f \in \mathcal{I} \wedge g \leq_{\mathrm{T}} f \rightarrow g \in \mathcal{I}$) and closed under $\oplus$ (i.e., $f, g \in \mathcal{I} \rightarrow f \oplus g \in \mathcal{I}$). We prove that $\mathbf{a} \in \mathfrak{M}$ is join-irreducible if and only if $\mathbf{a} = [\omega^\omega - \mathcal{I}]$ for some Turing ideal $\mathcal{I}$. We frequently use the following well-known lemma without mention.

Lemma 2.1 (see [1] Section III.2) *In a distributive lattice,* $\mathbf{a}$ *is join-irreducible if and only if for all* $\mathbf{x}$ *and* $\mathbf{y}$*,* $\mathbf{a} \leq \mathbf{x}+\mathbf{y}$ *implies* $\mathbf{a} \leq \mathbf{x}$ *or* $\mathbf{a} \leq \mathbf{y}$*. Dually,* $\mathbf{a}$ *is meet-irreducible if and only if for all* $\mathbf{x}$ *and* $\mathbf{y}$*,* $\mathbf{a} \geq \mathbf{x} \times \mathbf{y}$ *implies* $\mathbf{a} \geq \mathbf{x}$ *or* $\mathbf{a} \geq \mathbf{y}$*.*

Proof Suppose $\mathbf{a}$ is join-irreducible and $\mathbf{a} \leq \mathbf{x}+\mathbf{y}$. Then

$$\mathbf{a} = \mathbf{a} \times (\mathbf{x}+\mathbf{y}) = (\mathbf{a} \times \mathbf{x}) + (\mathbf{a} \times \mathbf{y}).$$

Thus $\mathbf{a} = \mathbf{a} \times \mathbf{x}$ or $\mathbf{a} = \mathbf{a} \times \mathbf{y}$ which means $\mathbf{a} \leq \mathbf{x}$ or $\mathbf{a} \leq \mathbf{y}$. Conversely, if $\mathbf{a}$ is join-reducible, then by definition there are $\mathbf{x}, \mathbf{y} < \mathbf{a}$ with $\mathbf{a} = \mathbf{x}+\mathbf{y}$. The proof for the meet-irreducible case is obtained by dualizing the proof for the join-irreducible case. □

For a mass problem $\mathcal{A}$, let $C(\mathcal{A})$ denote the *Turing upward-closure* of $\mathcal{A}$: $C(\mathcal{A}) = \{f \mid (\exists g \in \mathcal{A})(f \geq_{\mathrm{T}} g)\}$. A mass problem $\mathcal{A}$ is called *Turing upward-closed* if $\mathcal{A} = C(\mathcal{A})$. The identity functional witnesses $C(\mathcal{A}) \leq_{\mathrm{M}} \mathcal{A}$ for any mass problem $\mathcal{A}$, and if $\mathcal{A}$ and $\mathcal{B}$ are mass problems such that $\mathcal{A}$ is Turing upward-closed, then $\mathcal{A} \leq_{\mathrm{M}} \mathcal{B}$ if and only if $\mathcal{B} \subseteq \mathcal{A}$. Our starting point is the following observation.

Lemma 2.2 ([15]) *If* $\mathcal{A}$ *is a mass problem such that* $[\mathcal{A}]$ *is join-irreducible, then* $\omega^\omega - C(\mathcal{A})$ *is a Turing ideal.*

Proof We prove the contrapositive. If $\omega^\omega - C(\mathcal{A})$ is not a Turing ideal, then there are $f, g \notin C(\mathcal{A})$ with $f \oplus g \in C(\mathcal{A})$. This means that $\{f\}, \{g\} \not\geq_{\mathrm{M}} \mathcal{A}$ but $\{f\}+\{g\} \geq_{\mathrm{M}} \mathcal{A}$. Thus $[\mathcal{A}]$ is join-reducible. $\square$

The next lemma is the main step in our characterization.

Lemma 2.3 *If $\mathcal{A}$ is a mass problem such that $[\mathcal{A}]$ is join-irreducible, then $\mathcal{A} \equiv_{\mathrm{M}} C(\mathcal{A})$*

Proof We prove the contrapositive. Suppose $\mathcal{A} \not\equiv_{\mathrm{M}} C(\mathcal{A})$. Then it must be that $\mathcal{A} \not\leq_{\mathrm{M}} C(\mathcal{A})$. We find mass problems $\mathcal{X}$ and $\mathcal{Y}$ such that $\mathcal{X}, \mathcal{Y} \not\geq_{\mathrm{M}} \mathcal{A}$ but $\mathcal{X} + \mathcal{Y} \geq_{\mathrm{M}} \mathcal{A}$. Thus $[\mathcal{A}]$ is join-reducible.

To find $\mathcal{X}$ and $\mathcal{Y}$, first find a sequence $(h_n \mid n \in \omega)$ of functions and a sequence $(e_n \mid n \in \omega)$ of indices such that

(i) $\Phi_{e_n}(h_n) \in \mathcal{A}$ for all $n \in \omega$,
(ii) $\Phi_n(h_{2n}) \notin \mathcal{A}$ and $\Phi_n(h_{2n+1}) \notin \mathcal{A}$ for all $n \in \omega$, and
(iii) $h_n(0) = \langle n, e_0, e_1, \ldots, e_{n-1} \rangle$ for all $n \in \omega$.

We find the desired sequences by iterating the following claim.

Claim 2.4 *If $\mathcal{A} \not\leq_{\mathrm{M}} C(\mathcal{A})$, then for every $e, m \in \omega$ there is an $h \in C(\mathcal{A})$ such that $h(0) = m$ and $\Phi_e(h) \notin \mathcal{A}$.*

Proof of claim Suppose not. Then there are $e, m \in \omega$ such that $h(0) = m$ implies $\Phi_e(h) \in \mathcal{A}$ for all $h \in C(\mathcal{A})$. Thus $h \mapsto \Phi_e(m^\frown h)$ is a reduction witnessing $\mathcal{A} \leq_{\mathrm{M}} C(\mathcal{A})$, a contradiction. $\square$

Suppose we have h_i and e_i for all $i < n$. To find h_n and e_n, let $e = \lfloor n/2 \rfloor$ and let $m = \langle n, e_0, e_1, \ldots, e_{n-1} \rangle$. By the claim, there is an $h_n \in C(\mathcal{A})$ such that $h_n(0) = m$ and $\Phi_e(h_n) \notin \mathcal{A}$. The fact that $h_n \in C(\mathcal{A})$ means that there is an e_n such that $\Phi_{e_n}(h_n) \in \mathcal{A}$.

Now set $\mathcal{X} = \{h_{2n} \mid n \in \omega\}$ and $\mathcal{Y} = \{h_{2n+1} \mid n \in \omega\}$. Then $\Phi_e(\mathcal{X}) \not\subseteq \mathcal{A}$ and $\Phi_e(\mathcal{Y}) \not\subseteq \mathcal{A}$ for each e by item (ii). Hence $\mathcal{X}, \mathcal{Y} \not\geq_{\mathrm{M}} \mathcal{A}$. The following reduction witnesses $\mathcal{X} + \mathcal{Y} \geq_{\mathrm{M}} \mathcal{A}$.

Given h, decompose h as $h = f \oplus g$ and decode $f(0)$ and $g(0)$ as $f(0) = \langle 2n, x_0, x_1, \ldots, x_{2n-1} \rangle$ and $g(0) = \langle 2m + 1, y_0, y_1, \ldots, y_{2m} \rangle$. If either $f(0)$ or $g(0)$ is not of the required form, then output the 0 function (as such an h cannot be in $\mathcal{X} + \mathcal{Y}$). Otherwise, output $\Phi_{x_{2m+1}}(g)$ if $2n > 2m + 1$ and output $\Phi_{y_{2n}}(f)$ if $2m + 1 > 2n$.

Suppose this reduction is applied to some $h = h_{2n} \oplus h_{2m+1} \in \mathcal{X} + \mathcal{Y}$. In this case, $f = h_{2n}$, $g = h_{2m+1}$, and $f(0)$ and $g(0)$ are of the required form by item (iii). So if $2n > 2m + 1$ we output $\Phi_{e_{2m+1}}(h_{2m+1})$ and if $2m + 1 > 2n$ we output $\Phi_{e_{2n}}(h_{2n})$. Both alternatives are in $\mathcal{A}$ by item (i). Thus $\mathcal{X} + \mathcal{Y} \geq_{\mathrm{M}} \mathcal{A}$. $\square$

Theorem 2.5 *A Medvedev degree* $\mathbf{a}$ *is join-irreducible if and only if* $\mathbf{a} = [\omega^\omega - \mathcal{I}]$ *for some Turing ideal $\mathcal{I}$.*

Proof Suppose $\mathbf{a}$ is join-irreducible, and let $\mathcal{A}$ be a mass problem such that $\mathbf{a} = [\mathcal{A}]$. Then $\mathcal{I} = \omega^\omega - C(\mathcal{A})$ is a Turing ideal by Lemma 2.2, $\mathcal{A} \equiv_{\mathrm{M}} C(\mathcal{A})$ by Lemma 2.3, and therefore $\mathcal{A} \equiv_{\mathrm{M}} C(\mathcal{A}) = \omega^\omega - \mathcal{I}$. Hence $\mathbf{a} = [\omega^\omega - \mathcal{I}]$ for the Turing ideal $\mathcal{I}$.

Conversely, suppose $\mathcal{I}$ is a Turing ideal and let $\mathcal{X}$ and $\mathcal{Y}$ be mass problems such that $\mathcal{X}, \mathcal{Y} \not\geq_M \omega^\omega - \mathcal{I}$. We show that $\mathcal{X} + \mathcal{Y} \not\geq_M \omega^\omega - \mathcal{I}$. Observe $\mathcal{X}, \mathcal{Y} \nsubseteq \omega^\omega - \mathcal{I}$ for otherwise the identity functional would witness $\mathcal{X}, \mathcal{Y} \geq_M \omega^\omega - \mathcal{I}$. Let $f \in \mathcal{X} \cap \mathcal{I}$ and let $g \in \mathcal{Y} \cap \mathcal{I}$, thereby making $f \oplus g \in (\mathcal{X} + \mathcal{Y}) \cap \mathcal{I}$. The function $f \oplus g$ is in $\mathcal{X} + \mathcal{Y}$, but it does not compute any member of $\omega^\omega - \mathcal{I}$. Therefore, $\mathcal{X} + \mathcal{Y} \not\geq_M \omega^\omega - \mathcal{I}$. Hence, $[\omega^\omega - \mathcal{I}]$ is join-irreducible. □

Theorem 2.5 is also valid for the *Muchnik degrees* $\mathfrak{M}_w$ in place of $\mathfrak{M}$, a fact first noticed by Terwijn [17]. $\mathfrak{M}_w$ is defined just as $\mathfrak{M}$, but with *Muchnik reducibility* (also called *weak reducibility*) $\leq_w$ in place of $\leq_M$: $\mathcal{A} \leq_w \mathcal{B}$ if for every $f \in \mathcal{B}$ there is a $g \in \mathcal{A}$ such that $f \geq_T g$. $\mathfrak{M}_w$ is a Brouwer algebra with $+$, $\times$, and $\rightarrow$ defined by $[\mathcal{A}]_w + [\mathcal{B}]_w = [\mathcal{A} + \mathcal{B}]_w$, $[\mathcal{A}]_w \times [\mathcal{B}]_w = [\mathcal{A} \times \mathcal{B}]_w$, and $[\mathcal{A}]_w \rightarrow [\mathcal{B}]_w = [\{g \mid (\forall f \in \mathcal{A})(\exists h \in \mathcal{B})(h \leq_T f \oplus g)\}]_w$. The proof of Lemma 2.2 also works for $\mathfrak{M}_w$, and it is easy to check that $\mathcal{A} \equiv_w C(\mathcal{A})$ for any mass problem $\mathcal{A}$ (i.e., the $\mathfrak{M}_w$ analogue of Lemma 2.3 is trivial). This gives the forward direction of Theorem 2.5 for $\mathfrak{M}_w$. The proof of the reverse direction of Theorem 2.5 also works for $\mathfrak{M}_w$.

3 Degrees That Bound No Join-Irreducible Degrees $>_M \mathbf{0}'$

Recall that JAN is the intermediate logic IPC $+ \neg p \vee \neg\neg p$. The results of this section and the next are motivated by Question 1.3: Is $\mathrm{Th}(\mathfrak{M}/\mathbf{a}) \subseteq$ JAN for every $\mathbf{a} >_M \mathbf{0}'$?

$\mathrm{Th}(\mathfrak{M}/\mathbf{0}') =$ CPC because $\mathfrak{M}/\mathbf{0}' \cong [\mathbf{0}, \mathbf{0}'] = \{\mathbf{0}, \mathbf{0}'\}$. In fact, $\mathbf{0}'$ is the only degree for which $\mathrm{Th}(\mathfrak{M}/\mathbf{0}') =$ CPC. This is because if $\mathbf{a} >_M \mathbf{0}'$, then $\mathbf{0}' \rightarrow \mathbf{a} = \mathbf{a}$, hence $\mathbf{0}' \times (\mathbf{0}' \rightarrow \mathbf{a}) = \mathbf{0}'$. Thus, let $p = \mathbf{0}'$ to see that the formula $p \vee \neg p$ is not valid in $\mathrm{Th}(\mathfrak{M}/\mathbf{a})$.

Furthermore, if $\mathbf{a} >_M \mathbf{0}'$, then we cannot have $\mathrm{Th}(\mathfrak{M}/\mathbf{a}) \supsetneq$ JAN. It is an easy check that in any Brouwer algebra $\mathfrak{B}$ with meet-irreducible $\mathbf{0}$ (such as the algebras $\mathfrak{M}/\mathbf{a}$), $\neg p \vee \neg\neg p \in \mathrm{Th}(\mathfrak{B})$ if and only if $\mathbf{1}$ is join-irreducible. However, if $\mathbf{a} >_M \mathbf{0}'$ is join-irreducible, then $\mathrm{Th}(\mathfrak{M}/\mathbf{a}) =$ JAN [14]. Thus, if $\mathbf{a} >_M \mathbf{0}'$ and $\mathrm{Th}(\mathfrak{M}/\mathbf{a}) \supseteq$ JAN, then $\neg p \vee \neg\neg p \in \mathrm{Th}(\mathfrak{M}/\mathbf{a})$ which implies that $\mathbf{a}$ is join-irreducible which implies that $\mathrm{Th}(\mathfrak{M}/\mathbf{a}) =$ JAN. Thus a "no" answer to Question 1.3 must yield a degree $\mathbf{a}$ such that $\mathrm{Th}(\mathfrak{M}/\mathbf{a}) \nsubseteq$ JAN and JAN $\nsubseteq \mathrm{Th}(\mathfrak{M}/\mathbf{a})$.

The following theorem shows that to give a "yes" answer to Question 1.3 it suffices to show that every $\mathbf{a} >_M \mathbf{0}'$ bounds a finite meet of join-irreducible degrees $>_M \mathbf{0}'$.

Theorem 3.1 **([14])** *If* $\mathbf{a}$ *is a degree such that* $\mathbf{a} \geq_M \prod_{i=0}^{n} \mathbf{d}_i$ *for join-irreducible degrees* $\mathbf{d}_i >_M \mathbf{0}'$, $i \leq n$, *then* $\mathrm{Th}(\mathfrak{M}/\mathbf{a}) \subseteq$ JAN.

(The above theorem is stated more generally in [14]. Each degree $\mathbf{d}_i$ for $i \leq n$ is allowed to be either join-irreducible or $\mathfrak{De}$-irreducible. See the parenthetical discussion following Theorem 4.1 for the definition of $\mathfrak{De}$-irreducible and an explanation of why we do not consider such degrees here. Theorem 4.1 is a restatement of [14], Theorem 2.11, which is the main tool used to prove Theorem 3.1.)

The degrees of the mass problems $\mathcal{B}_f = \{g \mid g \not\leq_T f\}$ play an important role in the study of Question 1.3. The following lemma from Sorbi [13] encapsulates the properties of the $[\mathcal{B}_f]$s that we need in this section and the next.

Lemma 3.2 **([13])**

(i) *Every* $[\mathcal{B}_f]$ *is join-irreducible.*

(ii) *Every* $\sum_{i=1}^{n}[\mathcal{B}_{f_i}]$ *is meet-irreducible.*

(iii) *Let* V *and* J *be finite sets and let* U_v *and* I_j *be finite sets for each* $v \in V$ *and* $j \in J$. *Let* $\mathbf{x}_u^v$ *and* $\mathbf{y}_i^j$ *be degrees of the form* $[\mathcal{B}_f]$ *for every* $v \in V$, $u \in U_v$, $j \in J$, *and* $i \in I_j$. *Let* $\mathbf{a} = \sum_{v\in V}\prod_{u\in U_v}\mathbf{x}_u^v$ *and* $\mathbf{b} = \sum_{j\in J}\prod_{i\in I_j}\mathbf{y}_i^j$. *Then* $\mathbf{a} \leq_{\mathrm{M}} \mathbf{b}$ *if and only if*

$$(\forall v \in V)(\exists j \in J)(\forall i \in I_j)(\exists u \in U_v)(\mathbf{x}_u^v \leq_{\mathrm{M}} \mathbf{y}_i^j).$$

(iv) *In the notation of item (iii),*

$$\mathbf{a} \to \mathbf{b} = \sum\left\{\prod_{i\in I_j}\mathbf{y}_i^j \;\middle|\; (\forall v \in V)\left(\prod_{i\in I_j}\mathbf{y}_i^j \not\leq_{\mathrm{M}} \prod_{u\in U_v}\mathbf{x}_u^v\right)\right\}$$

(where the empty join is $\mathbf{0}$*).*

Proof Item (i) is by Theorem 2.5 and item (ii) is by Corollary 1.2. Item (iv) is proved in [13]. Item (iii) follows from item (iv) because $\mathbf{a} \leq_{\mathrm{M}} \mathbf{b}$ if and only if $\mathbf{b} \to \mathbf{a} = \mathbf{0}$. □

In [16] it is asked if every degree $\mathbf{a} >_{\mathrm{M}} \mathbf{0}'$ bounds a join-irreducible degree $>_{\mathrm{M}} \mathbf{0}'$, and it is conjectured that this is not the case based on the evidence provided by the following theorem.

Theorem 3.3 **([16])** *There is a degree* $\mathbf{a} >_{\mathrm{M}} \mathbf{0}'$ *such that* $\mathbf{a} \not\geq_{\mathrm{M}} [\mathcal{B}_f]$ *for every* $f >_{\mathrm{T}} 0$.

Our characterization of the join-irreducible degrees implies that every join-irreducible degree $>_{\mathrm{M}} \mathbf{0}'$ bounds some degree $[\mathcal{B}_f]$ with $f >_{\mathrm{T}} 0$. Thus the conjecture is correct.

Corollary 3.4 (to Theorem 2.5) *If* $\mathbf{a} >_{\mathrm{M}} \mathbf{0}'$ *is join-irreducible, then* $\mathbf{a} \geq_{\mathrm{M}} [\mathcal{B}_f]$ *for some* $f >_{\mathrm{T}} 0$.

Proof If $\mathbf{a}$ is join-irreducible, then, by Theorem 2.5, $\mathbf{a} = [\omega^\omega - \mathcal{I}]$ for some Turing ideal $\mathcal{I}$. If $[\omega^\omega - \mathcal{I}] >_{\mathrm{M}} \mathbf{0}'$, then $\mathcal{I}$ contains some function $f >_{\mathrm{T}} 0$. Thus $\omega^\omega - \mathcal{I} \subseteq \mathcal{B}_f$. Hence $\mathbf{a} = [\omega^\omega - \mathcal{I}] \geq_{\mathrm{M}} [\mathcal{B}_f]$. □

Theorem 3.5 *There is a degree* $\mathbf{a} >_{\mathrm{M}} \mathbf{0}'$ *such that every degree* $\mathbf{x}$ *with* $\mathbf{0}' <_{\mathrm{M}} \mathbf{x} \leq_{\mathrm{M}} \mathbf{a}$ *is join-reducible.*

Proof By Theorem 3.3, let $\mathbf{a} >_{\mathrm{M}} \mathbf{0}'$ be such that $\mathbf{a} \not\geq_{\mathrm{M}} [\mathcal{B}_f]$ for every $f >_{\mathrm{T}} 0$. This $\mathbf{a}$ is the desired degree because, by Corollary 3.4, if $\mathbf{a} \geq_{\mathrm{M}} \mathbf{x}$ for some join-irreducible $\mathbf{x} >_{\mathrm{M}} \mathbf{0}'$, then $\mathbf{a} \geq_{\mathrm{M}} [\mathcal{B}_f]$ for some $f >_{\mathrm{T}} 0$. □

The degree $\mathbf{a}$ satisfying Theorem 3.3 was constructed by diagonalization in [16]. We can give somewhat more concrete examples of degrees satisfying Theorem 3.3 and Theorem 3.5. Recall the following definitions. Functions $f, g >_{\mathrm{T}} 0$ are a *Turing minimal pair* if, for all h, $h \leq_{\mathrm{T}} f, g$ implies $h \leq_{\mathrm{T}} 0$. A function f has *minimal Turing degree* if, for all h, $h <_{\mathrm{T}} f$ implies $h \leq_{\mathrm{T}} 0$. Minimal pairs and minimal degrees exist. In fact, there are continuum many distinct minimal Turing degrees. See Lerman [6], Section II.4 and Section V.2.

Theorem 3.6 *If f and g are a minimal pair, then the degree $\mathbf{a} = [\mathcal{B}_f] \times [\mathcal{B}_g]$ witnesses Theorem 3.5.*

Proof Let f and g be a minimal pair. Then $[\mathcal{B}_f], [\mathcal{B}_g] >_{\mathrm{M}} \mathbf{0}'$ because $f, g >_{\mathrm{T}} 0$. Thus $[\mathcal{B}_f] \times [\mathcal{B}_g] >_{\mathrm{M}} \mathbf{0}'$ because $\mathbf{0}'$ is meet-irreducible by Corollary 1.2. To show that $[\mathcal{B}_f] \times [\mathcal{B}_g]$ bounds no join-irreducible degree $>_{\mathrm{M}} \mathbf{0}'$, it suffices by Corollary 3.4 to show that $[\mathcal{B}_f] \times [\mathcal{B}_g]$ bounds no $[\mathcal{B}_h]$ for $h >_{\mathrm{T}} 0$. This is true because f, g is a minimal pair, so for any $h >_{\mathrm{T}} 0$, either $h \not\leq_{\mathrm{T}} f$ or $h \not\leq g$. Thus, either $h \in \mathcal{B}_f$ or $h \in \mathcal{B}_g$, which means $\mathcal{B}_f \times \mathcal{B}_g$ contains a function $\equiv_{\mathrm{T}} h$. $\mathcal{B}_h$ contains no function $\leq_{\mathrm{T}} h$; therefore, $\mathcal{B}_f \times \mathcal{B}_g \not\geq_{\mathrm{M}} \mathcal{B}_h$. □

We can extend the idea behind Theorem 3.6 to find a degree $\mathbf{a} >_{\mathrm{M}} \mathbf{0}'$ that does not bound any finite meet of join-irreducible degrees $>_{\mathrm{M}} \mathbf{0}'$. Several of our examples in this section and the next are of the form $\left[\bigcup_{i \in \omega} i^\frown \mathcal{D}_i\right]$ for mass problems $\mathcal{D}_i$, $i \in \omega$.

Lemma 3.7 *Let $\mathbf{d} = \left[\bigcup_{i \in \omega} i^\frown \mathcal{D}_i\right]$ where $[\mathcal{D}_i] >_{\mathrm{M}} \mathbf{0}'$ for each $i \in \omega$. Then $\mathbf{d} >_{\mathrm{M}} \mathbf{0}'$.*

Proof Suppose for a contradiction that Φ is a reduction witnessing $\mathbf{d} \leq_{\mathrm{M}} \mathbf{0}'$ (i.e., $\Phi(f) \in \bigcup_{i \in \omega} i^\frown \mathcal{D}_i$ for all $f >_{\mathrm{T}} 0$). Let $\sigma \in \omega^{<\omega}$ be such that $\Phi(\sigma)(0)\downarrow$ and let $i = \Phi(\sigma)(0)$. Then $f \mapsto \Phi(\sigma^\frown f)$ is a reduction witnessing $\mathbf{0}' \geq_{\mathrm{M}} [\mathcal{D}_i]$, contradicting $[\mathcal{D}_i] >_{\mathrm{M}} \mathbf{0}'$. □

Theorem 3.8 *There is a degree $\mathbf{a} >_{\mathrm{M}} \mathbf{0}'$ such that no degree $\mathbf{x}$ with $\mathbf{0}' <_{\mathrm{M}} \mathbf{x} \leq_{\mathrm{M}} \mathbf{a}$ is of the form $\prod_{i=0}^n \mathbf{d}_i$ for join-irreducible degrees $\mathbf{d}_i >_{\mathrm{M}} \mathbf{0}'$, $i \leq n$.*

Proof By Corollary 3.4, it suffices to find a degree $\mathbf{a} >_{\mathrm{M}} \mathbf{0}'$ which is not above any degree of the form $\prod_{i=0}^n [\mathcal{B}_{f_i}]$ where $f_i >_{\mathrm{T}} 0$ for each $i \leq n$. Let $\{g_i \mid i \in \omega\}$ be a countable collection of functions all of distinct minimal Turing degree. Let $\mathcal{A} = \bigcup_{i \in \omega} i^\frown \mathcal{B}_{g_i}$ and put $\mathbf{a} = [\mathcal{A}]$. Lemma 3.7 proves that $\mathbf{a} >_{\mathrm{M}} \mathbf{0}'$.

Now consider any degree $\prod_{i=0}^n [\mathcal{B}_{f_i}]$, where $f_i >_{\mathrm{T}} 0$ for each $i \leq n$. There is a $j \in \omega$ such that $g_j \not\geq_{\mathrm{T}} f_i$ for each $i \leq n$. Thus, for this j, $[\mathcal{B}_{g_j}] \not\geq_{\mathrm{M}} [\mathcal{B}_{f_i}]$ for each $i \leq n$. Therefore, $[\mathcal{B}_{g_j}] \not\geq_{\mathrm{M}} \prod_{i=0}^n [\mathcal{B}_{f_i}]$ because $[\mathcal{B}_{g_j}]$ is meet-irreducible. Clearly, $[\mathcal{B}_{g_j}] \geq_{\mathrm{M}} \mathbf{a}$, so $\mathbf{a} \not\geq_{\mathrm{M}} \prod_{i=0}^n [\mathcal{B}_{f_i}]$ as well. □

For mass problems $\mathcal{A}_i$, $i \in \omega$, the Medvedev degree $\left[\bigcup_{i \in \omega} i^\frown \mathcal{A}_i\right]$ is not in general the greatest lower bound of the degrees $[\mathcal{A}_i]$, $i \in \omega$. Such greatest lower bounds need not even exist. For example, the degrees $[\mathcal{B}_{g_i}]$, $i \in \omega$ from Theorem 3.8 do not have a greatest lower bound. This follows from results in Dyment [4] which studies when countable collections of degrees have least upper bounds and greatest lower bounds.

If $\mathbf{a}$ is a degree such that $\mathbf{a} \not\geq_{\mathrm{M}} \mathbf{d}$ for all join-irreducible $\mathbf{d} >_{\mathrm{M}} \mathbf{0}'$, then $\mathbf{a} \to \mathbf{d} = \mathbf{d}$ for all join-irreducible $\mathbf{d} >_{\mathrm{M}} \mathbf{0}'$. The degree $\mathbf{a}$ constructed in Theorem 3.8 enjoys a similar property.

Theorem 3.9 *There is a degree $\mathbf{a} >_{\mathrm{M}} \mathbf{0}'$ such that $\mathbf{a} \to \prod_{i=0}^n \mathbf{d}_i = \prod_{i=0}^n \mathbf{d}_i$ whenever $\mathbf{d}_i >_{\mathrm{M}} \mathbf{0}'$ and is join-irreducible for each $i \leq n$.*

Proof As in Theorem 3.8, let $\{g_i \mid i \in \omega\}$ be a countable collection of functions all of distinct minimal Turing degree, let $\mathcal{A} = \bigcup_{i \in \omega} i^\frown \mathcal{B}_{g_i}$, and put $\mathbf{a} = [\mathcal{A}]$. Suppose $\mathbf{d}_i >_{\mathrm{M}} \mathbf{0}'$ and is join-irreducible for each $i \leq n$. By Theorem 2.5, for each $i \leq n$, let $\mathcal{I}_i \subseteq \omega^\omega$ be a Turing ideal such that $\mathbf{d}_i = [\omega^\omega - \mathcal{I}_i]$. Thus $\prod_{i=0}^n \mathbf{d}_i = \left[\bigcup_{i=0}^n i^\frown(\omega^\omega - \mathcal{I}_i)\right]$ and

$$\mathbf{a} \to \prod_{i=0}^n \mathbf{d}_i = \left[\left\{e^\frown g \;\middle|\; \left(\forall f \in \mathcal{A}\right)\left(\Phi_e(f \oplus g) \in \bigcup_{i=0}^n i^\frown(\omega^\omega - \mathcal{I}_i)\right)\right\}\right].$$

We now describe a reduction witnessing $\mathbf{a} \to \prod_{i=0}^n \mathbf{d}_i \geq_{\mathrm{M}} \prod_{i=0}^n \mathbf{d}_i$.

Given $e^\frown g$, for each $i \leq n+1$ search for a string $i^\frown \sigma_i$ such that

$$\Phi_e((i^\frown \sigma_i) \oplus g)(0)\downarrow.$$

If there is a $k \leq n$ such that

$$\Phi_e((i^\frown \sigma_i) \oplus g)(0) = \Phi_e((j^\frown \sigma_j) \oplus g)(0) = k$$

for two distinct $i, j \leq n+1$, choose the least such k and output $k^\frown g$. Otherwise, output 0.

Suppose we apply this reduction to $e^\frown g \in \mathcal{A} \to \bigcup_{i=0}^n i^\frown(\omega^\omega - \mathcal{I}_i)$. $\Phi_e(f \oplus g)$ must be total for each $f \in \mathcal{A}$, and for each $i \in \omega$ there is an $f \in \mathcal{A}$ with $f(0) = i$. Thus for each $i \leq n+1$ the search finds a string $i^\frown \sigma_i$ such that $\Phi_e((i^\frown \sigma_i) \oplus g)(0)\downarrow$. Moreover, each $i^\frown \sigma_i$ can be extended to a function in $\mathcal{A}$, so $\Phi_e((i^\frown \sigma_i) \oplus g)(0) \leq n$ for each $i \leq n+1$. Therefore, there is a least $k \leq n$ for which there are distinct $i, j \leq n+1$ with $\Phi_e((i^\frown \sigma_i) \oplus g)(0) = \Phi_e((j^\frown \sigma_j) \oplus g)(0) = k$. The reduction outputs $k^\frown g$, so we must show that $k^\frown g \in \bigcup_{i=0}^n i^\frown(\omega^\omega - \mathcal{I}_i)$ which means we must show that $g \notin \mathcal{I}_k$. Suppose for a contradiction that $g \in \mathcal{I}_k$. The functions g_i and g_j have distinct minimal degree, so either $g \not\leq_{\mathrm{T}} g_i$ or $g \not\leq_{\mathrm{T}} g_j$ ($g >_{\mathrm{T}} 0$ because $\mathbf{a} \not\geq_{\mathrm{M}} \prod_{i=0}^n \mathbf{d}_i$ by Theorem 3.8). For the sake of argument, suppose $g \not\leq_{\mathrm{T}} g_i$. Then $\sigma_i{}^\frown g \not\leq_{\mathrm{T}} g_i$ as well, so $\sigma_i{}^\frown g \in \mathcal{B}_{g_i}$ and $i^\frown \sigma_i{}^\frown g \in \mathcal{A}$. However, $\Phi_e((i^\frown \sigma_i{}^\frown g) \oplus g) \in k^\frown(\omega^\omega - \mathcal{I}_k)$ by the choice of $i^\frown \sigma_i$. This cannot be because $(i^\frown \sigma_i{}^\frown g) \oplus g \in \mathcal{I}_k$; thus anything it computes is also in $\mathcal{I}_k$. □

By Corollary 4.6 below, the degree $\mathbf{a} = \left[\bigcup_{i \in \omega} i^\frown \mathcal{B}_{g_i}\right]$ used to witness Theorem 3.8 and Theorem 3.9 satisfies $\mathrm{Th}(\mathfrak{M}/\mathbf{a}) \subseteq \mathrm{JAN}$ and so does any degree that bounds it. There are, however, degrees $>_{\mathrm{M}} \mathbf{0}'$ that do not bound any degree of the form $\left[\bigcup_{i \in \omega} i^\frown \mathcal{D}_i\right]$ where $[\mathcal{D}_i] >_{\mathrm{M}} \mathbf{0}'$ and is join-irreducible for each $i \in \omega$.

Theorem 3.10 *There is a degree $\mathbf{a} >_{\mathrm{M}} \mathbf{0}'$ such that $\mathbf{a} \not\geq_{\mathrm{M}} \left[\bigcup_{i \in \omega} i^\frown \mathcal{D}_i\right]$ whenever $[\mathcal{D}_i] >_{\mathrm{M}} \mathbf{0}'$ and is join-irreducible for each $i \in \omega$.*

Proof Let $\mathcal{D}_i$ be such that $[\mathcal{D}_i] >_{\mathrm{M}} \mathbf{0}'$ and is join-irreducible for each $i \in \omega$. By Corollary 3.4, for every $i \in \omega$, there is an $f_i >_{\mathrm{T}} 0$ such that $\mathcal{D}_i \geq_{\mathrm{M}} \mathcal{B}_{f_i}$. The mass problem $\mathcal{B}_{f_i}$ is Turing upward-closed for each $i \in \omega$, so $\mathcal{D}_i \subseteq \mathcal{B}_{f_i}$ for each $i \in \omega$. Thus $\bigcup_{i \in \omega} i^\frown \mathcal{D}_i \subseteq \bigcup_{i \in \omega} i^\frown \mathcal{B}_{f_i}$. Hence it suffices to find a degree $\mathbf{a} >_{\mathrm{M}} \mathbf{0}'$ that does not bound any degree of the form $\left[\bigcup_{i \in \omega} i^\frown \mathcal{B}_{f_i}\right]$, where $f_i >_{\mathrm{T}} 0$ for each $i \in \omega$.

We use the same construction used in [16] to prove Theorem 3.3. Build mass problems $\mathcal{A}_s \subseteq \{g \mid g >_{\mathrm{T}} 0\}$ such that $\{g \mid g >_{\mathrm{T}} 0\} - \mathcal{A}_s$ is finite for each $s \in \omega$. Set $\mathcal{A}_0 = \{g \mid g >_{\mathrm{T}} 0\}$. At stage $s+1$, choose $h_s >_{\mathrm{T}} 0$ such that h_s does not compute any of the (finitely many) functions in $\{g \mid g >_{\mathrm{T}} 0\} - \mathcal{A}_s$. If $\Phi_s(h_s)$ is total and $>_{\mathrm{T}} 0$, let $g_s = \Phi_s(h_s)$ and set $\mathcal{A}_{s+1} = \mathcal{A}_s - \{g_s\}$. Otherwise, set $\mathcal{A}_{s+1} = \mathcal{A}_s$. Put $\mathcal{A} = \bigcap_{s \in \omega} \mathcal{A}_s$ and put $\mathbf{a} = [\mathcal{A}]$.

To see $\mathbf{a} >_{\mathrm{M}} \mathbf{0}'$, observe that by construction $\Phi_s(h_s) \notin \mathcal{A}$ for each $s \in \omega$. Now let $f_i >_{\mathrm{T}} 0$ for each $i \in \omega$. We need to show that $\Phi_e(\mathcal{A}) \nsubseteq \bigcup_{i\in\omega} i^\frown \mathcal{B}_{f_i}$ for every index e. To do this, we first show that the functions in $\{g \mid g >_{\mathrm{T}} 0\} - \mathcal{A}$ have distinct Turing degree. Suppose that g_i leaves $\mathcal{A}$ at stage $i+1$ and g_j leaves $\mathcal{A}$ at stage $j+1$ for $i+1 < j+1$ (i.e., at stage $i+1$ we had $\Phi_i(h_i) = g_i >_{\mathrm{T}} 0$, and at stage $j+1$ we had $\Phi_j(h_j) = g_j >_{\mathrm{T}} 0$). Then $g_i \nleq_{\mathrm{T}} g_j$ because otherwise $g_i \leq_{\mathrm{T}} g_j \leq_{\mathrm{T}} h_j$, contradicting that h_j was chosen $\ngeq_{\mathrm{T}} g_i$ at stage $j+1$. Now suppose $\Phi_e(g)$ is total for all $g \in \mathcal{A}$. Fix any $\sigma \in \omega^{<\omega}$ such that $\Phi_e(\sigma)(0)\downarrow$, and let n be such that $\Phi_e(\sigma)(0) = n$. $\mathcal{A}$ is missing at most one function $\equiv_{\mathrm{T}} f_n$, so let $g \in \mathcal{A}$ be such that $\sigma \subset g$ and $g \equiv_{\mathrm{T}} f_n$. Then $\Phi_e(g)(0) = n$, but $\Phi_e(g) \notin n^\frown \mathcal{B}_{f_n}$. Hence $\Phi_e(\mathcal{A}) \nsubseteq \bigcup_{i\in\omega} i^\frown \mathcal{B}_{f_i}$. □

Question 3.11 *Let* $\mathbf{a}$ *be the degree constructed in Theorem 3.10. Does*

$$\mathbf{a} \to \left[\bigcup_{i\in\omega} i^\frown \mathcal{D}_i\right] = \left[\bigcup_{i\in\omega} i^\frown \mathcal{D}_i\right]$$

whenever $[\mathcal{D}_i] >_{\mathrm{M}} \mathbf{0}'$ *and is join-irreducible for each* $i \in \omega$*? Is* $\mathrm{Th}(\mathfrak{M}/\mathbf{a}) \subseteq \mathrm{JAN}$*?*

Finally, we note that the answer to Question 1.3 is "no" for $\mathfrak{M}_{\mathrm{w}}$ in place of $\mathfrak{M}$. Let $f >_{\mathrm{T}} 0$ have minimal Turing degree, and let $\mathbf{a} = [\mathcal{B}_f]_{\mathrm{w}}$. Then, in $\mathfrak{M}_{\mathrm{w}}$, $[\mathbf{0}, \mathbf{a}] = \{\mathbf{0}, \mathbf{0}', \mathbf{a}\}$ and $\mathrm{JAN} \subsetneq \mathrm{Th}(\mathfrak{M}_{\mathrm{w}}/\mathbf{a}) \subsetneq \mathrm{CPC}$.

4 New Degrees Whose Corresponding Logic Is Contained in JAN

We extend Theorem 3.1 by proving that $\mathrm{Th}(\mathfrak{M}/\mathbf{a}) \subseteq \mathrm{JAN}$ for degrees $\mathbf{a}$ such that $\mathbf{a} \geq_{\mathrm{M}} \left[\bigcup_{i\in\omega} i^\frown \mathcal{D}_i\right]$ for some collection of join-irreducible degrees $[\mathcal{D}_i] >_{\mathrm{M}} \mathbf{0}'$, $i \in \omega$.

A propositional formula is called *positive* if the connective '$\neg$' does not appear in it. For a logic L, let L^+ denote the positive formulas in L, and for a Brouwer algebra $\mathfrak{B}$, let $\mathrm{Th}^+(\mathfrak{B})$ denote the set of positive formulas valid in $\mathfrak{B}$. JAN is the maximum intermediate logic L for which $L^+ = \mathrm{IPC}^+$ [5]. This means that $L^+ = \mathrm{IPC}^+$ implies $L \subseteq \mathrm{JAN}$ for any intermediate logic L. Thus $\mathrm{Th}^+(\mathfrak{B}) = \mathrm{IPC}^+$ implies $\mathrm{Th}(\mathfrak{B}) \subseteq \mathrm{JAN}$ for any Brouwer algebra $\mathfrak{B}$.

Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be Brouwer algebras. An injection $f\colon \mathfrak{B}_1 \to \mathfrak{B}_2$ is called a B-*embedding* if it preserves $\mathbf{0}$, $\mathbf{1}$, $+$, $\times$, and $\to$ (and therefore also $\neg$). An injection $f\colon \mathfrak{B}_1 \to \mathfrak{B}_2$ is called a B^+-*embedding* if it preserves $\mathbf{0}$, $+$, $\times$, and $\to$ (but not necessarily $\mathbf{1}$ or $\neg$). If $\mathfrak{B}_1$ B-embeds into $\mathfrak{B}_2$, then $\mathrm{Th}(\mathfrak{B}_2) \subseteq \mathrm{Th}(\mathfrak{B}_1)$, and if $\mathfrak{B}_1$ B^+-embeds into $\mathfrak{B}_2$, then $\mathrm{Th}^+(\mathfrak{B}_2) \subseteq \mathrm{Th}^+(\mathfrak{B}_1)$. Both of these facts are easily checked in light of [9], Theorem VI.2.4. If $\mathbf{a} \leq \mathbf{b}$ are in a Brouwer algebra $\mathfrak{B}$, then $\mathfrak{B}/\mathbf{a}$ B^+-embeds into $\mathfrak{B}/\mathbf{b}$ by the identity. This implies that $\mathrm{Th}^+(\mathfrak{B}/\mathbf{b}) \subseteq \mathrm{Th}^+(\mathfrak{B}/\mathbf{a})$, and it follows that the $\mathbf{a}$ for which $\mathrm{Th}(\mathfrak{B}/\mathbf{a}) \subseteq \mathrm{JAN}$ is upward-closed in any Brouwer algebra $\mathfrak{B}$.

Our goal is to B^+-embed a certain class of Brouwer algebras into $\mathfrak{M}/\mathbf{a}$. For any set X, let $\mathrm{Fr}(X)$ denote the free distributive lattice generated by X and let $\mathbf{0} \oplus \mathrm{Fr}(X)$ denote $\mathrm{Fr}(X)$ with a new bottom element $\mathbf{0}$. The elements of $\mathrm{Fr}(X)$ are all of the form $\sum_{v\in V} \prod_{u\in U_v} \mathbf{x}_u^v$ where V and the U_v are finite sets of indices and the $\mathbf{x}_u^v$ are all in X (see, for example, Balbes and Dwinger [1], Section V.3). For such representations, $\sum_{v\in V} \prod_{u\in U_v} \mathbf{x}_u^v \leq \sum_{j\in J} \prod_{i\in I_j} \mathbf{y}_i^j$ if and only if

$$(\forall v \in V)(\exists j \in J)(\forall i \in I_j)(\exists u \in U_v)(\mathbf{x}_u^v \leq \mathbf{y}_i^j).$$

If $\mathbf{a}, \mathbf{b} \in \mathrm{Fr}(X)$ are such that $\mathbf{a} \not\geq \mathbf{b}$, then $\mathbf{a} \to \mathbf{b}$ exists. To see this, let $\mathbf{a} = \sum_{v \in V} \prod_{u \in U_v} \mathbf{x}_u^v$ and $\mathbf{b} = \sum_{j \in J} \prod_{i \in I_j} \mathbf{y}_i^j$ be representations for $\mathbf{a}$ and $\mathbf{b}$. Then check

$$\mathbf{a} \to \mathbf{b} = \sum \Big\{ \prod_{i \in I_j} \mathbf{y}_i^j \;\Big|\; \big(\forall v \in V\big)\Big(\prod_{i \in I_j} \mathbf{y}_i^j \not\leq \prod_{u \in U_v} \mathbf{x}_u^v\Big)\Big\}.$$

If $\mathbf{a} \geq \mathbf{b}$ are in $\mathrm{Fr}(X)$ for an infinite X, then $\mathbf{a} \to \mathbf{b}$ fails to exist because in this case $\mathrm{Fr}(X)$ has no least element. We see then that $\mathbf{a} \to \mathbf{b}$ exists for every $\mathbf{a}, \mathbf{b} \in \mathbf{0} \oplus \mathrm{Fr}(X)$. If X is finite, then so are $\mathrm{Fr}(X)$ and $\mathbf{0} \oplus \mathrm{Fr}(X)$. Hence, both are Brouwer algebras. Let $\mathrm{Fr}(n)$ denote the free distributive lattice with n generators. The logic $\mathrm{LM} = \bigcap_{n \in \omega} \mathrm{Th}(\mathbf{0} \oplus \mathrm{Fr}(n))$ is called the *Medvedev logic of finite problems*. (LM is usually defined in terms of Brouwer algebras isomorphic to the $\mathbf{0} \oplus \mathrm{Fr}(n)$. See [16] for details.) We take advantage of the fact that $\mathrm{LM}^+ = \mathrm{IPC}^+$ [8].

If X is infinite, then $\mathbf{0} \oplus \mathrm{Fr}(X)$ fails to be a Brouwer algebra only because it lacks a top element. Therefore, the notion of a B^+-embedding makes sense when we allow $\mathfrak{B}_1$ or $\mathfrak{B}_2$ to be $\mathbf{0} \oplus \mathrm{Fr}(X)$. If we let $\mathbf{0} \oplus \mathrm{Fr}(X) \oplus \mathbf{1}$ denote $\mathrm{Fr}(X)$ with a new bottom element $\mathbf{0}$ and a new top element $\mathbf{1}$, then $\mathbf{0} \oplus \mathrm{Fr}(X) \oplus \mathbf{1}$ is always a Brouwer algebra.

For any partial order $(P, \leq_P)$, let $\mathrm{Fr}(P, \leq_P)$ denote the free distributive lattice generated by $(P, \leq_P)$. $\mathrm{Fr}(P, \leq_P)$ is the quotient $\mathrm{Fr}(P)/\!\equiv$ where, for

$$\mathbf{a} = \textstyle\sum_{v \in V} \prod_{u \in U_v} \mathbf{x}_u^v \text{ and } \mathbf{b} = \sum_{j \in J} \prod_{i \in I_j} \mathbf{y}_i^j \text{ in } \mathrm{Fr}(P),$$

$\mathbf{a} \equiv \mathbf{b}$ if and only if $(\mathbf{a} \preceq \mathbf{b}) \wedge (\mathbf{b} \preceq \mathbf{a})$ and $\mathbf{a} \preceq \mathbf{b}$ if and only if

$$\big(\forall v \in V\big)\big(\exists j \in J\big)\big(\forall i \in I_j\big)\big(\exists u \in U_v\big)\big(\mathbf{x}_u^v \leq_P \mathbf{y}_i^j\big).$$

$\mathrm{Fr}(P, \leq_P)$ is always a distributive lattice, and $\mathbf{0} \oplus \mathrm{Fr}(P, \leq_P) \oplus \mathbf{1}$ is always a Brouwer algebra; also see [13].

The following lemmas facilitate our embeddings. Lemma 4.3 is a slight generalization of the claim in the proof of [13], Lemma 2.3 and of [10], Lemma 6. The embedding is done in Theorem 4.4 which is nearly identical to [14], Theorem 2.11. Part of the reason for reproducing the proof here is to (hopefully) correct the notational inconsistencies in the proof in [14]. We restate [14], Theorem 2.11, for reference.

Theorem 4.1 ([14], Theorem 2.11) *Let $\mathbf{d} = \prod_{i=0}^n \mathbf{d}_i$ where $\mathbf{d}_i >_{\mathrm{M}} \mathbf{0}'$ and $\mathbf{d}_i$ is join-irreducible for each $i \leq n$. Then $\mathbf{0} \oplus \mathrm{Fr}(P, \leq_P) \oplus \mathbf{1}$ B-embeds into $\mathfrak{M}/\mathbf{d}$ for every countable partial order $(P, \leq_P)$.*

(The above theorem is stated more generally in [14]. Each degree $\mathbf{d}_i$, for $i \leq n$, is allowed to be either join-irreducible or $\mathfrak{De}$-irreducible. A degree $\mathbf{a}$ is *dense* if it is of the form $[\mathcal{A}]$ where $\mathcal{A}$ is dense in ω^ω, and a degree $\mathbf{d}$ is $\mathfrak{De}$-*irreducible* if $\mathbf{a} \to \mathbf{d} = \mathbf{d}$ for all dense degrees $\mathbf{a}$. We do not consider $\mathfrak{De}$-irreducible degrees in our version of [14], Theorem 2.11, which is Theorem 4.4 below, because in Theorem 4.4 we require that the mass problems $\mathcal{D}_i$ (which play the role of the degrees $\mathbf{d}_i$ in [14], Theorem 2.11) are Turing upward-closed. Mass problems that are Turing upward-closed are dense and hence their degrees are not $\mathfrak{De}$-irreducible.)

Lemma 4.2 ([3]) *If $\mathcal{X} \not\geq_{\mathrm{M}} \mathcal{Y}$ are mass problems, then there is a $\mathcal{W} \subseteq \mathcal{X}$ with $|\mathcal{W}| \leq \omega$ such that $\mathcal{W} \not\geq_{\mathrm{M}} \mathcal{Y}$.*

Proof $\mathcal{X} \not\geq_{\mathrm{M}} \mathcal{Y}$ means that there is no Turing functional Φ such that $\Phi(\mathcal{X}) \subseteq \mathcal{Y}$. Thus, for each functional Φ_e, there must be some function $f_e \in \mathcal{X}$ such that

$\Phi_e(f_e) \notin \mathcal{Y}$. Let $\mathcal{W}$ consist of a choice of one such $f_e \in \mathcal{X}$ for each functional Φ_e. $\square$

Lemma 4.3 *Let $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{F}_i$, for $i \in \omega$, be mass problems such that $\bigcup_{i\in\omega} i^\frown\mathcal{F}_i \leq_{\mathrm{M}} \mathcal{U}+\mathcal{V}$ and $\sigma^\frown\mathcal{U} \subseteq \mathcal{U}$ for all $\sigma \in \omega^{<\omega}$. Then there are mass problems $\mathcal{V}_i$, for $i \in \omega$, such that $\bigcup_{i\in\omega} i^\frown\mathcal{V}_i \equiv_{\mathrm{M}} \mathcal{V}$ and $\mathcal{F}_i \leq_{\mathrm{M}} \mathcal{U}+\mathcal{V}_i$ for each $i \in \omega$.*

Proof Let $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{F}_i$, for $i \in \omega$, be as in the statement of the lemma. Let Φ be such that $\Phi(\mathcal{U}+\mathcal{V}) \subseteq \bigcup_{i\in\omega} i^\frown\mathcal{F}_i$. For each $i \in \omega$, define $\mathcal{V}_i = \{g \in \mathcal{V} \mid (\exists\sigma \in \omega^{<\omega})(\Phi(\sigma \oplus g)(0) = i)\}$. $\mathcal{V} \leq_{\mathrm{M}} \bigcup_{i\in\omega} i^\frown\mathcal{V}_i$ is clear. $\bigcup_{i\in\omega} i^\frown\mathcal{V}_i \leq_{\mathrm{M}} \mathcal{V}$ by the reduction which, given g, searches for a $\sigma \in \omega^{<\omega}$ such that $\Phi(\sigma \oplus g)(0)\downarrow$ and outputs $\Phi(\sigma \oplus g)(0)^\frown g$. To see $i^\frown\mathcal{F}_i \leq_{\mathrm{M}} \mathcal{U}+\mathcal{V}_i$, consider the reduction which, given $f \oplus g$, searches for a $\sigma \in \omega^{<\omega}$ such that $\Phi(\sigma \oplus g)(0) = i$ and outputs $\Phi((\sigma^\frown f) \oplus g)$. If $f \oplus g \in \mathcal{U}+\mathcal{V}_i$, then such a σ is found, $\sigma^\frown f$ is in $\mathcal{U}$, and $\Phi((\sigma^\frown f) \oplus g)$ is in $i^\frown\mathcal{F}_i$. $\square$

Theorem 4.4 *Let $\mathbf{d} = \left[\bigcup_{i\in\omega} i^\frown\mathcal{D}_i\right]$ where $[\mathcal{D}_i] >_{\mathrm{M}} \mathbf{0}'$, $[\mathcal{D}_i]$ is join-irreducible, and $\mathcal{D}_i$ is Turing upward-closed for each $i \in \omega$. Then $\mathbf{0} \oplus \mathrm{Fr}(2^\omega)$ B^+-embeds into $\mathfrak{M}/\mathbf{d}$.*

Proof Let $\mathcal{D}_i$, for $i \in \omega$, be as in the statement of the theorem, let $\mathcal{D} = \bigcup_{i\in\omega} i^\frown\mathcal{D}_i$, and let $\mathbf{d} = [\mathcal{D}]$. Lemma 3.7 proves that $\mathbf{d} >_{\mathrm{M}} \mathbf{0}'$. By Lemma 4.2, let $\mathcal{A} \subseteq \{f \mid f >_{\mathrm{T}} 0\}$ be a countable mass problem such that $\mathcal{A} \not\geq_{\mathrm{M}} \mathcal{D}$. Let $\{f_x \mid x \in 2^\omega\}$ be a collection of functions such that $f_x \,|_{\mathrm{T}}\, f_y$ for all $x, y \in 2^\omega$ with $x \neq y$ and that $f \not\leq_{\mathrm{T}} f_x$ for all $f \in \mathcal{A}$ and $x \in 2^\omega$. Such a sequence can be constructed via standard recursion-theoretic techniques: build a perfect tree whose paths are Turing incomparable and do not compute any functions in $\mathcal{A}$. See, for example, [6], Section II.4. Notice that $\mathcal{B}_{f_x} \leq_{\mathrm{M}} \mathcal{A}$ (because $\mathcal{A} \subseteq \mathcal{B}_{f_x}$) for each $x \in 2^\omega$.

Define $G\colon \mathbf{0} \oplus \mathrm{Fr}(2^\omega) \to \mathfrak{M}$ as follows. Let $G(\mathbf{0}) = \mathbf{0}$ and let $G(x) = [\mathcal{B}_{f_x}]$ on the generators $x \in 2^\omega$ of $\mathrm{Fr}(2^\omega)$. Then extend G to all of $\mathbf{0} \oplus \mathrm{Fr}(2^\omega)$ so that $G\left(\sum_{v\in V}\prod_{u\in U_v} x_u^v\right) = \sum_{v\in V}\prod_{u\in U_v} G\left(x_u^v\right)$. G preserves $\mathbf{0}$, $+$, and $\times$ by definition, and G is injective and preserves $\to$ by Lemma 3.2, items (iii) and (iv). Hence G is a B^+-embedding (this is essentially [13], Corollary 2.5). Now define $H\colon \mathbf{0} \oplus \mathrm{Fr}(2^\omega) \to \mathfrak{M}/\mathbf{d}$ by $H(\mathbf{a}) = G(\mathbf{a}) \times \mathbf{d}$ for all $\mathbf{a} \in \mathbf{0} \oplus \mathrm{Fr}(2^\omega)$. This H is the desired B^+-embedding. By definition, H preserves $\mathbf{0}$, $+$, and $\times$. We must show that H is injective and that H preserves $\to$.

Clearly, $H(\mathbf{a}) = \mathbf{0}$ if and only if $\mathbf{a} = \mathbf{0}$, so to show that H is injective we let $\mathbf{a}, \mathbf{b} \in \mathrm{Fr}(2^\omega)$ be such that $H(\mathbf{a}) \leq_{\mathrm{M}} H(\mathbf{b})$ and show that $\mathbf{a} \leq \mathbf{b}$. Let $\mathbf{a} = \sum_{v\in V}\prod_{u\in U_v} x_u^v$ be a representation for $\mathbf{a}$ and let $\mathbf{b} = \sum_{j\in J}\prod_{i\in I_j} y_i^j$ be a representation for $\mathbf{b}$. $H(\mathbf{a}) \leq_{\mathrm{M}} H(\mathbf{b})$ means that

$$\sum_{v\in V}\prod_{u\in U_v} G\left(x_u^v\right) \times \mathbf{d} \leq_{\mathrm{M}} \sum_{j\in J}\prod_{i\in I_j} G\left(y_i^j\right) \times \mathbf{d}.$$

Therefore,

$$\sum_{v\in V}\prod_{u\in U_v} G\left(x_u^v\right) \times \mathbf{d} \leq_{\mathrm{M}} \sum_{j\in J}\prod_{i\in I_j} G\left(y_i^j\right) = \prod\left\{\sum_{j\in J} G\left(y_{\alpha(j)}^j\right) \,\middle|\, \alpha \in \prod_{j\in J} I_j\right\}$$

where the equality is by distributivity $\big(\prod_{j\in J} I_j$ denotes the Cartesian product of the I_js$\big)$. Hence,

$$\sum_{v\in V}\prod_{u\in U_v} G\big(x_u^v\big)\times \mathbf{d} \leq_{\mathrm{M}} \sum_{j\in J} G\big(y^j_{\alpha(j)}\big) \text{ for each } \alpha\in\prod_{j\in J} I_j.$$

Each $\sum_{j\in J} G\big(y^j_{\alpha(j)}\big)$ is meet-irreducible by Lemma 3.2, item (ii). Also, $\mathbf{d} \not\leq_{\mathrm{M}} \sum_{j\in J} G\big(y^j_{\alpha(j)}\big)$ for each $\alpha\in\prod_{j\in J} I_j$ because $\sum_{j\in J} G\big(y^j_{\alpha(j)}\big) \leq_{\mathrm{M}}[\mathcal{A}]$ but $\mathbf{d}\not\leq_{\mathrm{M}}[\mathcal{A}]$. Thus,

$$\sum_{v\in V}\prod_{u\in U_v} G\big(x_u^v\big) \leq_{\mathrm{M}} \sum_{j\in J} G\big(y^j_{\alpha(j)}\big) \text{ for each } \alpha\in\prod_{j\in J} I_j,$$

and this implies that

$$\sum_{v\in V}\prod_{u\in U_v} G\big(x_u^v\big) \leq_{\mathrm{M}} \prod\Big\{\sum_{j\in J} G\big(y^j_{\alpha(j)}\big)\;\Big|\;\alpha\in\prod_{j\in J} I_j\Big\}.$$

The left-hand side of the above inequality is $G(\mathbf{a})$ and the right-hand side is $G(\mathbf{b})$. G is a B^+-embedding, so we conclude $\mathbf{a}\leq\mathbf{b}$.

If either of $\mathbf{a},\mathbf{b}\in\mathbf{0}\oplus\mathrm{Fr}(2^\omega)$ is $\mathbf{0}$, then clearly $H(\mathbf{a}\to\mathbf{b}) = H(\mathbf{a})\to H(\mathbf{b})$. So as before, let $\mathbf{a}=\sum_{v\in V}\prod_{u\in U_v} x_u^v$ and let $\mathbf{b}=\sum_{j\in J}\prod_{i\in I_j} y_i^j$ be in $\mathrm{Fr}(2^\omega)$. We see $H(\mathbf{a}\to\mathbf{b}) \geq_{\mathrm{M}} H(\mathbf{a})\to H(\mathbf{b})$ because

$$H(\mathbf{a}\to\mathbf{b}) + H(\mathbf{a}) = H((\mathbf{a}\to\mathbf{b})+\mathbf{a}) \geq_{\mathrm{M}} H(\mathbf{b}).$$

To show that $H(\mathbf{a}\to\mathbf{b}) \leq_{\mathrm{M}} H(\mathbf{a})\to H(\mathbf{b})$, we show that if $\mathbf{z}\in\mathfrak{M}$ is such that $H(\mathbf{b})\leq_{\mathrm{M}} H(\mathbf{a})+\mathbf{z}$, then $H(\mathbf{a}\to\mathbf{b})\leq_{\mathrm{M}}\mathbf{z}$. Suppose $H(\mathbf{b})\leq_{\mathrm{M}} H(\mathbf{a})+\mathbf{z}$. That is,

$$\sum_{j\in J}\prod_{i\in I_j} G\big(y_i^j\big)\times\mathbf{d} \leq_{\mathrm{M}} \Big(\sum_{v\in V}\prod_{u\in U_v} G\big(x_u^v\big)\times\mathbf{d}\Big)+\mathbf{z}. \tag{1}$$

Since $\mathbf{a}\to\mathbf{b} = \sum\big\{\prod_{i\in I_j} y_i^j \mid (\forall v\in V)\big(\prod_{i\in I_j} y_i^j \not\leq \prod_{u\in U_v} x_u^v\big)\big\}$, we have

$$\begin{aligned} H(\mathbf{a}\to\mathbf{b}) &= G(\mathbf{a}\to\mathbf{b})\times\mathbf{d} \\ &= \sum\Big\{\prod_{i\in I_j} G\big(y_i^j\big)\;\Big|\;\Big(\forall v\in V\Big)\Big(\prod_{i\in I_j} G\big(y_i^j\big)\not\leq_{\mathrm{M}}\prod_{u\in U_v} G\big(x_u^v\big)\Big)\Big\}\times\mathbf{d}. \end{aligned}$$

It suffices to show that, given $j\in J$, if $\prod_{i\in I_j} G\big(y_i^j\big)$ satisfies

$$\Big(\forall v\in V\Big)\Big(\prod_{i\in I_j} G\big(y_i^j\big)\not\leq_{\mathrm{M}}\prod_{u\in U_v} G\big(x_u^v\big)\Big),$$

then $\prod_{i\in I_j} G\big(y_i^j\big)\times\mathbf{d}\leq_{\mathrm{M}}\mathbf{z}$. Suppose $\prod_{i\in I_j} G\big(y_i^j\big)$ is such a meet. Then we know

$$\Big(\forall v\in V\Big)\Big(\exists u\in U_v\Big)\Big(\prod_{i\in I_j} G\big(y_i^j\big)\not\leq_{\mathrm{M}} G\big(x_u^v\big)\Big).$$

By choosing such a $u\in U_v$, for every $v\in V$, and by appealing to Lemma 3.2, items (i) and (ii), we see that there is an $\alpha\in\prod_{v\in V} U_v$ such that

$$\prod_{i\in I_j} G\big(y_i^j\big)\not\leq_{\mathrm{M}}\sum_{v\in V} G\big(x^v_{\alpha(v)}\big). \tag{2}$$

Distributing $\sum_{v\in V}\prod_{u\in U_v} G(x^v_u)$ in the right-hand side of (1) yields

$$\prod_{i\in I_j} G(y^j_i) \times \mathbf{d} \leq_{\mathrm{M}} \sum_{v\in V} G(x^v_{\alpha(v)}) + \mathbf{z}.$$

The degree $\sum_{v\in V} G(x^v_{\alpha(v)})$ is a finite join of degrees of the form $[\mathcal{B}_f]$ and thus has a representative $\mathcal{U}$ such that $\sigma^\frown\mathcal{U} \subseteq \mathcal{U}$ for all $\sigma \in \omega^{<\omega}$. So, by Lemma 4.3, there are mass problems $\mathcal{Z}_i$ for $i \in I_j$ and $\widehat{\mathcal{Z}}_i$ for $i \in \omega$ such that

$$\mathbf{z} = \Big(\prod_{i\in I_j}[\mathcal{Z}_i]\Big) \times \Big[\bigcup_{i\in\omega} i^\frown\widehat{\mathcal{Z}}_i\Big],$$

$$G(y^j_i) \leq_{\mathrm{M}} \sum_{v\in V} G(x^v_{\alpha(v)}) + [\mathcal{Z}_i] \text{ for each } i \in I_j, \text{ and}$$

$$[\mathcal{D}_i] \leq_{\mathrm{M}} \sum_{v\in V} G(x^v_{\alpha(v)}) + [\widehat{\mathcal{Z}}_i] \text{ for each } i \in \omega.$$

Each $G(y^j_i)$ is join-irreducible, and $G(y^j_i) \nleq_{\mathrm{M}} \sum_{v\in V} G(x^v_{\alpha(v)})$ by (2). Thus $G(y^j_i) \leq_{\mathrm{M}} [\mathcal{Z}_i]$ for each $i \in \omega$, so $\prod_{i\in I_j} G(y^j_i) \leq_{\mathrm{M}} \prod_{i\in I_j}[\mathcal{Z}_i]$. Each $[\mathcal{D}_i]$ is join-irreducible by assumption, and $[\mathcal{D}_i] \nleq_{\mathrm{M}} \sum_{v\in V} G(x^v_{\alpha(v)})$ because the right-hand side is $\leq_{\mathrm{M}}[\mathcal{A}]$ but the left-hand side is not. It follows that $[\mathcal{D}_i] \leq_{\mathrm{M}} [\widehat{\mathcal{Z}}_i]$ for each $i \in \omega$, and so $\widehat{\mathcal{Z}}_i \subseteq \mathcal{D}_i$ for each $i \in \omega$ because each $\mathcal{D}_i$ is Turing upward-closed. Thus, $\bigcup_{i\in\omega} i^\frown\widehat{\mathcal{Z}}_i \subseteq \mathcal{D}$, so $\mathbf{d} \leq_{\mathrm{M}} \big[\bigcup_{i\in\omega} i^\frown\widehat{\mathcal{Z}}_i\big]$. Therefore, $\prod_{i\in I_j} G(y^j_i) \times \mathbf{d} \leq_{\mathrm{M}} \big(\prod_{i\in I_j}[\mathcal{Z}_i]\big) \times \big[\bigcup_{i\in\omega} i^\frown\widehat{\mathcal{Z}}_i\big] = \mathbf{z}$ as desired. □

Corollary 4.5 *If* $\mathbf{a} \geq_{\mathrm{M}} \mathbf{d}$ *are degrees such that* $\mathbf{d} = \big[\bigcup_{i\in\omega} i^\frown\mathcal{D}_i\big]$ *where* $[\mathcal{D}_i] >_{\mathrm{M}} \mathbf{0}'$ *and is join-irreducible for each* $i \in \omega$*, then* $\mathbf{0} \oplus \mathrm{Fr}(2^\omega)$ B^+*-embeds into* $\mathfrak{M}/\mathbf{a}$.

Proof Let $\mathbf{a}$, $\mathbf{d}$, and $\mathcal{D}_i$, for $i \in \omega$, be as in the statement of the corollary. Let $\mathbf{d}_0 = \big[\bigcup_{i\in\omega} i^\frown C(\mathcal{D}_i)\big]$ and notice that $\mathbf{d} \geq_{\mathrm{M}} \mathbf{d}_0$. $\mathcal{D}_i \equiv_{\mathrm{M}} C(\mathcal{D}_i)$ for each $i \in \omega$ by Lemma 2.3, so $\mathbf{d}_0$ satisfies the hypotheses of Theorem 4.4. Thus, $\mathbf{0} \oplus \mathrm{Fr}(2^\omega)$ B^+-embeds into $\mathfrak{M}/\mathbf{d}_0$ which B^+-embeds into $\mathfrak{M}/\mathbf{a}$. □

Corollary 4.6 *If* $\mathbf{a} \geq_{\mathrm{M}} \mathbf{d}$ *are degrees such that* $\mathbf{d} = \big[\bigcup_{i\in\omega} i^\frown\mathcal{D}_i\big]$ *where* $[\mathcal{D}_i] >_{\mathrm{M}} \mathbf{0}'$ *and is join-irreducible for each* $i \in \omega$*, then* $\mathrm{Th}(\mathfrak{M}/\mathbf{a}) \subseteq \mathrm{JAN}$.

Proof The Brouwer algebra $\mathbf{0} \oplus \mathrm{Fr}(n)$ B^+-embeds into $\mathbf{0} \oplus \mathrm{Fr}(2^\omega)$ for each n, and $\mathbf{0} \oplus \mathrm{Fr}(2^\omega)$ B^+-embeds into $\mathfrak{M}/\mathbf{a}$ by Corollary 4.5. Thus, $\mathrm{Th}^+(\mathfrak{M}/\mathbf{a}) \subseteq \bigcap_{n\in\omega} \mathrm{Th}^+(\mathbf{0} \oplus \mathrm{Fr}(n)) = \mathrm{LM}^+ = \mathrm{IPC}^+$. So $\mathrm{Th}(\mathfrak{M}/\mathbf{a}) \subseteq \mathrm{JAN}$. □

Theorem 4.4 can be modified to B-embed $\mathbf{0} \oplus \mathrm{Fr}(2^\omega) \oplus \mathbf{1}$ into $\mathfrak{M}/\mathbf{d}$ for degrees $\mathbf{d}$ as in the statement of Theorem 4.4. However, if $\mathbf{a} \leq \mathbf{b}$ in a Brouwer algebra $\mathfrak{B}$, it is not in general the case that $\mathfrak{B}/\mathbf{a}$ B-embeds into $\mathfrak{B}/\mathbf{b}$. So the proof of Corollary 4.5 fails for B-embedding $\mathbf{0} \oplus \mathrm{Fr}(2^\omega) \oplus \mathbf{1}$. Theorem 4.4 can also be modified to prove a more precise analogue of [14], Theorem 2.11 (restated as Theorem 4.1 above). Let $\mathbf{d} = \big[\bigcup_{i\in\omega} i^\frown\mathcal{D}_i\big]$ where $[\mathcal{D}_i] >_{\mathrm{M}} \mathbf{0}'$, $[\mathcal{D}_i]$ is join-irreducible, and $\mathcal{D}_i$ is Turing upward-closed for each $i \in \omega$. Then $\mathbf{0} \oplus \mathrm{Fr}(P, \leq_P) \oplus \mathbf{1}$ B-embeds into $\mathfrak{M}/\mathbf{d}$ for every countable partial order $(P, \leq_P)$.

5 $\mathfrak{F}_{\mathrm{cl}}$ Is Not Prime

Recall that a filter $\mathfrak{F}$ in a lattice is called *prime* if $\mathbf{a}+\mathbf{b} \in \mathfrak{F} \to \mathbf{a} \in \mathfrak{F} \vee \mathbf{b} \in \mathfrak{F}$ for all $\mathbf{a}$ and $\mathbf{b}$ in the lattice. Theorem 2.5 can be phrased as a characterization of the prime principal filters in $\mathfrak{M}$: a degree $\mathbf{a}$ generates a prime filter if and only if $\mathbf{a} = [\omega^\omega - \mathcal{I}]$ for some Turing ideal $\mathcal{I}$. There is little general theory of the nonprincipal filters in $\mathfrak{M}$, but several specific cases have been studied in Dyment [3], Sorbi [11], Bianchini and Sorbi [2], and Lewis, Shore, and Sorbi [7]. See also [15] for a summary of many of the results appearing in these works. We consider the filters $\mathfrak{F}$ and $\mathfrak{F}_{\mathrm{cl}}$.

Definition 5.1

(i) A degree $\mathbf{a}$ is called *dense* (*closed*) if $\mathbf{a} = [\mathcal{A}]$ for an $\mathcal{A}$ that is dense (closed) in ω^ω.
(ii) $\mathfrak{I}$ denotes the ideal generated by $\{\mathbf{a} \mid \mathbf{a} \text{ is dense}\}$.
(iii) $\mathfrak{F}$ denotes $\mathfrak{M} - \mathfrak{I}$.
(iv) $\mathfrak{F}_{\mathrm{cl}}$ denotes the filter generated by $\{\mathbf{a} \mid \mathbf{a} >_{\mathrm{M}} \mathbf{0} \text{ and is closed}\}$.

The join and meet of two dense degrees is dense [3], and the join and meet of two closed degrees is closed [2]. It follows that $\mathfrak{I} = \{\mathbf{b} \mid (\exists \mathbf{a} \geq_{\mathrm{M}} \mathbf{b})(\mathbf{a} \text{ is dense})\}$ and $\mathfrak{F}_{\mathrm{cl}} = \{\mathbf{b} \mid (\exists \mathbf{a} \leq_{\mathrm{M}} \mathbf{b})(\mathbf{a} >_{\mathrm{M}} \mathbf{0} \text{ and is closed})\}$. The basic properties of $\mathfrak{I}$, $\mathfrak{F}$, and $\mathfrak{F}_{\mathrm{cl}}$ are as follows: $\mathfrak{I}$ is a prime ideal [11], $\mathfrak{F}$ is a prime filter [2], $\mathfrak{I}$ is not principal [3], $\mathfrak{F}$ and $\mathfrak{F}_{\mathrm{cl}}$ are not principal [2], and $\mathfrak{F}_{\mathrm{cl}} \subsetneq \mathfrak{F}$ [2]. Both [2] and [15] ask for a proof that $\mathfrak{F}_{\mathrm{cl}}$ is not prime. We provide a proof of this fact now.

Lemma 5.2 *For any $f \in \omega^\omega$ there are $\mathcal{A}, \mathcal{B} \subseteq \omega^\omega$ such that $\mathcal{A} + \mathcal{B} \geq_{\mathrm{M}} \{f\}$ and, for any closed $\mathcal{C} \subseteq \omega^\omega$, if $\mathcal{A} \geq_{\mathrm{M}} \mathcal{C}$ or $\mathcal{B} \geq_{\mathrm{M}} \mathcal{C}$, then $\mathcal{C}$ contains a recursive function.*

Proof Fix a recursive bijection $\omega \leftrightarrow \omega^{<\omega}$. For $e, n \in \omega$, if

$$\forall m \forall \sigma (\exists \tau \supseteq \sigma)(\Phi_e(n^\frown\tau)(m)\downarrow),$$

then define $\eta(e,n,i) \in \omega^{<\omega}$ by induction on $i \in \omega$ as follows. Let $\eta(e,n,0) = n^\frown\sigma$, where σ is the least string such that $\Phi_e(n^\frown\sigma)(0)\downarrow$. Given $\eta(e,n,i)$, let $\eta(e,n,i+1) = \eta(e,n,i)^\frown 0^\frown\sigma$, where σ is the least string such that

$$\Phi_e(\eta(e,n,i)^\frown 0^\frown\sigma)(i+1)\downarrow.$$

Let $f \in \omega^\omega$. We construct $\mathcal{A}$ and $\mathcal{B}$ such that

(i) if $g \in \mathcal{A}$, then $g(0)$ has the form

$$g(0) = \langle \ell, \langle n_0, x_0, y_0 \rangle, \ldots, \langle n_{\ell-1}, x_{\ell-1}, y_{\ell-1} \rangle \rangle,$$

where $\ell \in \omega$ and $n_i \in \omega$, $x_i \in \{0,1\}$, and $y_i \in \omega$ for each $i < \ell$;
(ii) if $g \in \mathcal{A}$ and $\langle n_e, 0, y_e \rangle$ is in the eth position of $g(0)$, then
  (a) $\exists m \exists \sigma (\forall \tau \supseteq \sigma)(\Phi_e(n_e{}^\frown\tau)(m)\uparrow)$,
  (b) any $h \in \mathcal{B}$ with $h(0) = n_e$ is of the form $h = n_e{}^\frown\sigma^\frown f$, where $|\sigma| = y_e$;
(iii) if $g \in \mathcal{A}$ and $\langle n_e, 1, y_e \rangle$ is the eth position of $g(0)$, then
  (a) $\forall m \forall \sigma (\exists \tau \supseteq \sigma)(\Phi_e(n_e{}^\frown\tau)(m)\downarrow)$,
  (b) any $h \in \mathcal{B}$ with $h(0) = n_e$ is of the form $h = \eta(e, n_e, i)^\frown 1^\frown f$ for some $i \in \omega$;
(iv) the above properties hold with the roles of $\mathcal{A}$ and $\mathcal{B}$ reversed.

We construct $\mathcal{A}$ and $\mathcal{B}$ in stages. The construction is similar to the construction in Lemma 2.3 in that if g goes into $\mathcal{A}$ before h goes into $\mathcal{B}$, then $h(0)$ codes how to recover f from g, and similarly with the roles of $\mathcal{A}$ and $\mathcal{B}$ reversed. Start at stage 0 with $\mathcal{A} = \varnothing$, $\mathcal{B} = \varnothing$, $s = \langle\,\rangle$, and $t = \langle\,\rangle$.

Stage $e + 1$ Set $n_e = e^\frown t$.

Case 1 $\exists m \exists \sigma (\forall \tau \supseteq \sigma)(\Phi_e(n_e{}^\frown \tau)(m)\uparrow)$. Choose such a σ and put $n_e{}^\frown \sigma^\frown f$ in $\mathcal{A}$. Update $s = s^\frown \langle n_e, 0, |\sigma|\,\rangle$.

Case 2 $\forall m \forall \sigma (\exists \tau \supseteq \sigma)(\Phi_e(n_e{}^\frown \tau)(m)\downarrow)$. Put the functions $\eta(e, n_e, i)^\frown 1^\frown f$ in $\mathcal{A}$ for each $i \in \omega$. Update $s = s^\frown \langle n_e, 1, 0\,\rangle$.

Repeat the above procedure with the roles of $\mathcal{A}$ and $\mathcal{B}$ reversed and the roles of s and t reversed. This completes stage $e + 1$. Then go on to stage $e + 2$. This completes the construction.

Suppose $\mathcal{A} \geq_{\mathrm{M}} \mathcal{C}$ where $\mathcal{C}$ is closed. We show that $\mathcal{C}$ contains a recursive function. The proof with $\mathcal{B}$ in place of $\mathcal{A}$ is the same. Let $\Phi_e(\mathcal{A}) \subseteq C$. Consider stage $e + 1$ of the above construction. Case 1 must not have occurred because otherwise $\mathcal{A}$ would contain a function $n_e{}^\frown \sigma^\frown f$ such that $\Phi_e(n_e{}^\frown \sigma^\frown f)$ is not total. Thus case 2 occurred, and so $\mathcal{A}$ contains the function $\eta(e, n_e, i)^\frown 1^\frown f$ for each $i \in \omega$. Let k be the recursive function $k = n_e{}^\frown \sigma_0{}^\frown 0^\frown \sigma_1{}^\frown 0^\frown \sigma_2{}^\frown 0^\frown \cdots$, where $\eta(e, n_e, i) = n_e{}^\frown \sigma_0{}^\frown 0^\frown \cdots {}^\frown 0^\frown \sigma_i$ for each $i \in \omega$ (think of k as the "limit" of the strings $\eta(e, n_e, i)$ as $i \to \infty$). Then $\Phi_e(\eta(e, n_e, i)^\frown 1^\frown f) \in \mathcal{C}$ and $\Phi_e(\eta(e, n_e, i)^\frown 1^\frown f) \restriction i = \Phi_e(k) \restriction i$ for each $i \in \omega$. Thus $\mathcal{C}$ contains the recursive function $\Phi_e(k)$ because $\mathcal{C}$ is closed.

We now describe a uniform procedure for producing f from $g \oplus h \in \mathcal{A} + \mathcal{B}$. First decode $h(0)$ as $h(0) = \langle \ell, \langle n_0, x_0, y_0\,\rangle, \ldots, \langle n_{\ell-1}, x_{\ell-1}, y_{\ell-1}\,\rangle\,\rangle$ and look for $g(0)$ among the n_e. If $\langle g(0), 0, y_e\,\rangle$ appears in $h(0)$ at position e, then output g from position $y_e + 1$ onward as in this case $g = \sigma^\frown f$ for some string σ of length $y_e + 1$. If $\langle g(0), 1, 0\,\rangle$ appears in $h(0)$ at position e, then $g = \eta(e, g(0), i)^\frown 1^\frown f$ for some $i \in \omega$. Compute which i by successively computing the $\eta(e, g(0), j)$, matching them against g, and checking if the next bit of g is 0 (in which case compute $\eta(e, g(0), j + 1)$) or 1 (in which case $j = i$). Output f once i is found.

The number $g(0)$ appears among the n_e coded into $h(0)$ if g went into $\mathcal{A}$ before h went into $\mathcal{B}$. Otherwise, h went into $\mathcal{B}$ before g went into $\mathcal{A}$, so $h(0)$ appears among the n_e coded in $g(0)$. In this case, switch the roles of g and h and apply the above procedure to compute f. □

Theorem 5.3 *$\mathfrak{F}_{\mathrm{cl}}$ is not prime. In fact, if $\mathfrak{G} \subseteq \mathfrak{F}_{\mathrm{cl}}$, $\mathfrak{G} \neq \{\mathbf{1}\}$ is a filter, then $\mathfrak{G}$ is not prime.*

Proof Suppose $\mathfrak{G} \subseteq \mathfrak{F}_{\mathrm{cl}}$ is a filter such that $\mathfrak{G} \neq \{\mathbf{1}\}$. Let $f >_{\mathrm{T}} 0$ be such that $[\{f\}] \in \mathfrak{G}$. Let $\mathcal{A}, \mathcal{B} \subseteq \omega^\omega$ be as in Lemma 5.2 for this f. Let $\mathbf{a} = [\mathcal{A}]$ and $\mathbf{b} = [\mathcal{B}]$. Then $\mathbf{a}, \mathbf{b} \notin \mathfrak{G}$ because $\mathbf{a}, \mathbf{b} \notin \mathfrak{F}_{\mathrm{cl}}$, but $\mathbf{a} + \mathbf{b} \in \mathfrak{G}$ because $\mathbf{a} + \mathbf{b} \geq_{\mathrm{M}} [\{f\}]$. □

If $\mathbf{x}$ and $\mathbf{y}$ are degrees such that $\mathbf{y}$ is closed and $\mathbf{y} \not\leq_{\mathrm{M}} \mathbf{x}$, then there is no dense degree $\mathbf{z}$ such that $\mathbf{y} \leq_{\mathrm{M}} \mathbf{x} + \mathbf{z}$ [7]. Therefore, if $\mathfrak{G} \subseteq \mathfrak{F}_{\mathrm{cl}}$, $\mathfrak{G} \neq \{\mathbf{1}\}$ is a filter, then any degrees $\mathbf{a}$ and $\mathbf{b}$ witnessing that $\mathfrak{G}$ is not prime must both be in $\mathfrak{F} - \mathfrak{G}$.

The results of Section 3 suggest two new filters to study.

Definition 5.4

(i) $\mathfrak{G}$ denotes the filter generated by

$$\{\mathbf{d} \mid \mathbf{d} >_{\mathrm{M}} \mathbf{0}' \text{ and is join-irreducible}\}.$$

(ii) $\mathfrak{H}$ denotes the filter generated by

$$\left\{\left[\bigcup_{i\in\omega} i^\frown \mathcal{D}_i\right] \;\middle|\; (\forall i \in \omega)([\mathcal{D}_i] >_{\mathrm{M}} \mathbf{0}' \text{ and is join-irreducible})\right\}.$$

$\mathfrak{G}$ is exactly the set of all degrees $\mathbf{b}$ for which $\mathbf{b} \geq_{\mathrm{M}} \prod_{i=0}^{n} \mathbf{d}_i$ for some join-irreducible degrees $\mathbf{d}_i >_{\mathrm{M}} \mathbf{0}'$, $i \leq n$, and $\mathfrak{H}$ is exactly the set of all degrees $\mathbf{b}$ for which $\mathbf{b} \geq_{\mathrm{M}} \left[\bigcup_{i\in\omega} i^\frown \mathcal{D}_i\right]$ for some join-irreducible degrees $[\mathcal{D}_i] >_{\mathrm{M}} \mathbf{0}'$, $i \in \omega$.

Theorem 5.5 *$\mathfrak{F}_{\mathrm{cl}} \subsetneq \mathfrak{G} \subsetneq \mathfrak{H} \subsetneq \{\mathbf{a} \mid \mathbf{a} >_{\mathrm{M}} \mathbf{0}'\}$. $\mathfrak{G} \nsubseteq \mathfrak{F}$ (hence also $\mathfrak{H} \nsubseteq \mathfrak{F}$). Neither $\mathfrak{G}$ nor $\mathfrak{H}$ is principal.*

Proof Every closed degree $>_{\mathrm{M}} \mathbf{0}$ bounds a join-irreducible degree $>_{\mathrm{M}} \mathbf{0}'$ [16]. Hence $\mathfrak{F}_{\mathrm{cl}} \subseteq \mathfrak{G}$. $\mathfrak{G} \subseteq \mathfrak{H}$ is clear. To see $\mathfrak{G} \nsubseteq \mathfrak{F}$, observe that every $\mathcal{B}_f$ is dense, so if $f >_{\mathrm{T}} 0$, then $[\mathcal{B}_f] \in \mathfrak{G} - \mathfrak{F}$. This also shows $\mathfrak{G} \nsubseteq \mathfrak{F}_{\mathrm{cl}}$. The degree constructed in Theorem 3.8 witnesses $\mathfrak{H} \nsubseteq \mathfrak{G}$. The degree constructed in Theorem 3.10 witnesses $\{\mathbf{a} \mid \mathbf{a} >_{\mathrm{M}} \mathbf{0}'\} \nsubseteq \mathfrak{H}$. We show that $\mathfrak{G}$ is not principal. The proof for $\mathfrak{H}$ is the same. First, if $\mathcal{A}$ is countable and contains no recursive functions, then there is a function $f >_{\mathrm{T}} 0$ such that $g \nleq_{\mathrm{T}} f$ for all $g \in \mathcal{A}$. Thus $\mathcal{B}_f \leq_{\mathrm{M}} \mathcal{A}$ (as $\mathcal{A} \subseteq \mathcal{B}_f$) for this f. Every $[\mathcal{B}_f]$ for $f >_{\mathrm{T}} 0$ is in $\mathfrak{G}$, so every $[\mathcal{A}]$ where $\mathcal{A}$ is countable and contains no recursive function is in $\mathfrak{G}$. If $\mathfrak{G}$ were principal, it would be generated by a degree $\mathbf{x}$ such that $\mathbf{x} \leq_{\mathrm{M}} [\mathcal{A}]$ for all countable $\mathcal{A}$ not containing a recursive function. By Lemma 4.2, the only such $\mathbf{x}$ are $\mathbf{0}$ and $\mathbf{0}'$. We know $\mathbf{0}$ and $\mathbf{0}'$ are not in $\mathfrak{G}$, so $\mathfrak{G}$ cannot be principal. □

We end with a question.

Question 5.6

(i) *Is $\mathfrak{F} \subseteq \mathfrak{G}$? Is $\mathfrak{F} \subseteq \mathfrak{H}$?*
(ii) *Is $\mathfrak{G}$ prime? Is $\mathfrak{H}$ prime?*
(iii) *Is $\{\mathbf{a} \mid \mathrm{Th}(\mathfrak{M}/\mathbf{a}) \subseteq \mathrm{JAN}\}$ a filter?*

To prove that $\{\mathbf{a} \mid \mathrm{Th}(\mathfrak{M}/\mathbf{a}) \subseteq \mathrm{JAN}\}$ is a filter, it suffices to prove that $\mathrm{Th}(\mathfrak{M}/(\mathbf{a} \times \mathbf{b})) \subseteq \mathrm{JAN}$ whenever both $\mathrm{Th}(\mathfrak{M}/\mathbf{a})$ and $\mathrm{Th}(\mathfrak{M}/\mathbf{b})$ are $\subseteq$ JAN because $\{\mathbf{a} \mid \mathrm{Th}(\mathfrak{M}/\mathbf{a}) \subseteq \mathrm{JAN}\}$ is upward-closed in $\mathfrak{M}$.

References

[1] Balbes, R., and P. Dwinger, *Distributive Lattices*, University of Missouri Press, Columbia, 1974. Zbl 0321.06012. MR 0373985. 23, 29

[2] Bianchini, C., and A. Sorbi, "A note on closed degrees of difficulty of the Medvedev lattice," *Mathematical Logic Quarterly*, vol. 42 (1996), pp. 127–33. Zbl 0845.03020. MR 1373134. 23, 34

[3] Dyment, E. Z., "Certain properties of the Medvedev lattice," *Mathematics of the USSR Sbornik*, vol. 30 (1976), pp. 321–40. Zbl 0382.03028. MR 0432433. 22, 30, 34

[4] Dyment, E. Z., "Exact bounds of denumerable collections of degrees of difficulty," *Akademiya Nauk Soyuza SSR. Matematicheskie Zametki*, vol. 28 (1980), pp. 899–910, 961. Zbl 0498.03036. MR 603223. 27

[5] Jankov, V. A., "Calculus of the weak law of the excluded middle," *Mathematics of the USSR Izvestiya*, vol. 2 (1968), pp. 997–1004. Zbl 0187.26306. MR 0237298. 23, 29

[6] Lerman, M., *Degrees of Unsolvability. Local and Global Theory*, Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1983. Zbl 0542.03023. MR 708718. 26, 31

[7] Lewis, A. E., R. A. Shore, and A. Sorbi, "Topological aspects of the Medvedev lattice," forthcoming. 34, 35

[8] Medvedev, Y. T., "Finite problems," *Doklady Akademii Nauk SSSR (NS)*, vol. 142 (1962), pp. 1015–18. Zbl 0286.02028. MR 0133233. 22, 30

[9] Rasiowa, H., and R. Sikorski, *The Mathematics of Metamathematics*, Monografie Matematyczne, Tom 41. Państwowe Wydawnictwo Naukowe, Warsaw, 1963. Zbl 0122.24311. MR 0163850. 29

[10] Skvortsova, E. Z., "A faithful interpretation of the intuitionistic propositional calculus by means of an initial segment of the Medvedev lattice," *Sibirskiĭ Matematicheskiĭ Zhurnal*, vol. 29 (1988), pp. 171–78, 225. Zbl 0661.03003. MR 936795. 23, 30

[11] Sorbi, A., "On some filters and ideals of the Medvedev lattice," *Archive for Mathematical Logic*, vol. 30 (1990), pp. 29–48. Zbl 0704.03025. MR 1074447. 34

[12] Sorbi, A., "Some remarks on the algebraic structure of the Medvedev lattice," *The Journal of Symbolic Logic*, vol. 55 (1990), pp. 831–53. Zbl 0703.03022. MR 1056392. 22

[13] Sorbi, A., "Embedding Brouwer algebras in the Medvedev lattice," *Notre Dame Journal of Formal Logic*, vol. 32 (1991), pp. 266–75. Zbl 0737.06009. MR 1123000. 25, 26, 30, 31

[14] Sorbi, A., "Some quotient lattices of the Medvedev lattice," *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, vol. 37 (1991), pp. 167–82. Zbl 0702.03021. MR 1155135. 23, 25, 30, 33

[15] Sorbi, A., "The Medvedev lattice of degrees of difficulty," pp. 289–312 in *Computability, Enumerability, Unsolvability. Directions in Recursion Theory*, edited by S. B. Cooper, T. A. Slaman, and S. S. Wainer, vol. 224 of *London Mathematical Society Lecture Notes*, Cambridge University Press, Cambridge, 1996. Zbl 0849.03033. MR 1395886. 21, 22, 23, 34

[16] Sorbi, A., and S. A. Terwijn, "Intermediate logics and factors of the Medvedev lattice," *Annals of Pure and Applied Logic*, vol. 155 (2008), pp. 69–85. Zbl 1156.03026. MR 2455558. 22, 23, 26, 28, 30, 36

[17] Terwijn, S. A., "Constructive logic and the Medvedev lattice," *Notre Dame Journal of Formal Logic*, vol. 47 (2006), pp. 73–82. Zbl 1107.03024. MR 2211183. 25

Acknowledgments

Thanks to my advisor, Richard Shore, for introducing me to the Medvedev degrees and for many helpful discussions about them. Thanks also to Andrea Sorbi and Sebastiaan Terwijn for suggesting several of the problems considered here. This research was partially supported by NSF grants DMS-0554855 and DMS-0852811.

Department of Mathematics
Malott Hall
Cornell University
Ithaca NY 14853
USA
pshafer@math.cornell.edu
http://www.math.cornell.edu/~pshafer/